Notes for Research Project on Ranking

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Chapter 1

Background

Here, we give some background on our problem. We start by listing some of the basic properties of the matrices we will be considering. Then, we briefly consider two popular ranking methods: the Perron eigenvalue method and the logarithmic least squares method.

1.1 Relevant Matrices

1.1.1 Positive matrices

Theorem 1.1.1 (Perron-Frobenius [1] [3]). *Let $A > 0$. Then:*

1. *There is a positive (column) vector w such that $Aw = \rho(A)w$. We call w the* right Perron vector *of A. Similarly, there is a positive (row) vector v such that $vA = \rho(A)v$. We call v the* left Perron vector *of A.*

2. *The right Perron vector w is orthogonal to all other eigenvectors in right eigenspace; similarly, v is orthogonal to all other eigenvectors in left eigenspace.*

3. *$\rho(A)$ is an algebraically (and hence geometrically) simple eigenvalue of A.*

4. *$\rho(A)$ is the unique eigenvalue of maximum modulus.*

5. *$\rho(A)$ is bounded above by the maximum row sum of A and from below by the minimum row sum of A.*

As an immediate corollary, we can state limiting behavior of positive matrices A in terms of the right and left Perron vectors:

Corollary 1.1.2 ([3]). *Let w and v be the right and left Perron vectors of $A > 0$. Assume w and v are scaled so that $wv = 1$ Then:*

$$\lim_{m\to\infty}\left(\frac{A}{\rho(A)}\right)^m = vw$$

Lastly, we note:

Theorem 1.1.3 ([1]). *Let $A > 0$. The largest eigenvalue of A, $\rho(A)$, increases as any element a_{ij} increases.*

1.1.2 Positive reciprocal matrices

Definition 1.1.4. A is a positive reciprocal matrix if $a_{ij} > 0$ and $a_{ji} = \frac{1}{a_{ij}}$.

Theorem 1.1.5 ([1])**.** *The eigenvalues of a positive reciprocal matrix satisfy:*

$$\sum_{\substack{j,k \\ j \neq k}} \lambda_j \lambda_k = 0$$

1.1.3 Positive consistent (reciprocal) matrices

Definition 1.1.6. A matrix A is said to be consistent if $a_{ij}a_{jk} = a_{ik}$ for all i, j, k. In other words, a positive reciprocal matrix is consistent when pairwise dominance relations are transitive.

Comment: Using the above definition, along with the fact that $a_{ji} = 1/a_{ij}$, the consistency condition becomes $a_{ij}a_{jk}a_{ki} = 1$. So, one can also define consistency in a graph theoretic sense: if we think of our matrix A as describing a weighted adjacency matrix, then consistency means that the product of the edge weights in any 3-cycle is 1.

Comment: Note that all positive consistent matrices are necessarily reciprocal matrices. Assuming A is positive, the definition of consistency implies $a_{ii} = 1$, which, in turn, implies $a_{ij} = a_{ii}/a_{ji} = 1/a_{ji}$. Thus, a positive, consistent matrix is necessarily a consistent, positive reciprocal matrix (but the converse is not necessarily true).

Fact: If A is a consistent positive reciprocal matrix, then the right and left Perron vectors of A, w and v, are reciprocals of each other up to a multiplicative factor. That is, there exists $c \in \mathbb{R}$ such that:

$$w_i = c\left(\frac{1}{v_i}\right)$$

Next, we note that consistency can be characterized in terms of the largest eigenvalue of a reciprocal matrix:

Theorem 1.1.7 ([1])**.** *Let A be a positive reciprocal matrix. Then A is consistent if and only if $\rho(A) = n$.*

Consistent, positive reciprocal matrices also satisfy:

Theorem 1.1.8 ([1])**.** *Let A be a consistent, positive reciprocal matrix. Then:*

$$A^k = n^{k-1}A$$

A natural question that arises is how to measure consistency in matrices that aren't consistent. Several metrics have been proposed - we mention a popular "consistency index" below defined by Saaty [1]:

Definition 1.1.9. The Consistency Index (C.I.) of a positive reciprocal matrix A is defined as:

$$\text{C.I.} = \frac{\rho(A) - n}{n - 1}$$

Open Conjecture: Let $A \in M_n$ be a positive, reciprocal matrix with $n \geq 4$. The right and left Perron vectors of A are reciprocals of each other (up to a multiplicative factor) if and only if A is consistent. (Note: for $n = 2$, any positive reciprocal matrix is consistent. For $n = 3$, it has been proven that regardless of consistency, the right and left eigenvectors are reciprocals of each other. As far as I can tell, the conjecture is still unproven for $n \geq 4$.)

Comment: While it may not be helpful, we note the above conjecture can be alternatively stated in the language of graph theory: Let A be the weighted adjacency matrix of the complete weighted digraph on n vertices for $n \geq 4$. The products of the edge weights of every 3-cycle is 1 if and only if the right and left Perron vectors of A are reciprocals (up to a multiplicative factor) of each other.

1.2 Two ranking methods

1.2.1 Perron eigenvalue method

Saaty advocates use of the Perron eigenvalue method in which the ranking is given by the Perron vector of A. Saaty makes two key points to justify this method. The first is that, when A is consistent, the Perron vector w satisfies:

$$a_{ij} = \frac{w_i}{w_j}$$

This can be easily proved. Since every elements of A can be determined from the first row of A, A has rank one and exactly one nonzero eigenvalue. Using the aforementioned fact that $A^2 = nA$, and denoting the columns of A as $[a_1, a_2, \ldots, a_n]$ we can see:

$$\begin{aligned} A^2 &= A[a_1, a_2, \ldots, a_n] \\ &= [Aa_1, Aa_2, \ldots, Aa_n] \\ &= [na_1, na_2, \ldots, na_n] \\ &= nA \end{aligned}$$

which tells us the columns of A are all scalar multiples of the (unique) dominant eigenvector. So, for column a_k, the Perron vector w satisfies, for some scalar c:

$$\begin{aligned} w_i &= ca_{ik} \\ w_j &= ca_{jk} \end{aligned}$$

which, if we take the ratio and use the consistency definition, yields the desired result:

$$a_{ij} = \frac{w_i}{w_j}$$

The second key justification Saaty makes in favor of using the Perron eigenvalue method concerns perturbation. Namely, he shows that small changes in the entries of the matrix result in small changes in the Perron vector. While these two arguments seem to be the most compelling, Saaty and others also have other arguments, including empirical arguments, on why the Perron vector should be used in ranking.

Asymmetry in rankings derived from right and left Perron vectors

While Saaty and others have traditionally used the right Perron vector for ranking, there seems to be no justification for why the left eigenvector should not be used in place of the right eigenvector. Larger values in the right Perron vector signify a higher ranking for that item whereas smaller values in the left Perron vector signify a higher ranking for that item. In the case where A is consistent, given the above mentioned reciprocal relationship between the right and left Perron vectors, both vectors will lead to the same ranking. However, if A is inconsistent, then the right and left Perron vectors can lead to different rankings. In a recent survey of the Analytic Hierarchy Process, the author listed this issue as the largest theoretical dispute in the field. We give an example below from [4] to that illustrates the rank reversal issue:

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 3 & 1/3 & 1/2 \\ 1/3 & 1 & 1/6 & 2 \\ 3 & 6 & 1 & 1 \\ 2 & 1/2 & 1 & 1 \end{pmatrix} \end{array}$$

The right Perron vector is: $w = [0.184; 0.152; 0.436; 0.227]$, which leads to the ranking: $c > d > a > b$ while the left Perron vector is $v = [0.248, 0.338, 0.105, 0.259]$, which leads to the different ranking of $c > a > d > b$.

1.2.2 Logarithmic least squares method

The Perron vector is not the only continuous vector-valued function of positive reciprocal matrices that yields the correct scale when the matrix is consistent. Another popular method that also meets these criteria, introduced in [5], is the geometric mean method, or more commonly known as logarithmic least squares method (LLSM). This approach minimizes the multiplicative error; that is, if p denotes the LLSM vector, then this method aims to minimize e_{ij} below:

$$a_{ij} = \frac{p_i}{p_j} e_{ij}$$

That is, LLSM assumes the multiplicative error and aims to minimize the sums of these errors:

$$\sum_{i,j=1}^{n} \left(\ln a_{ij} - \ln \frac{p_i}{p_j} \right)^2$$

The normalized LLSM vector is given by:

$$p_i = \frac{\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}}$$

The list of relative pros and cons of Perron eigenvalue method versus the LLSM remains long and is a topic of debate.

Chapter 2

Ranking with Offense and Defense

Consider a network of players in a series of one-on-one competitions, the outcomes of which result in a score for each player relative to every other. The scores s_{uv} represent pairwise dominance of the form "player u is s_{uv} times better than player v." Naturally these are inverse relationships.

This can by modeled by a complete directed graph, where the edge (u, v) is weighted by the score s_{uv}. Additionally each node has a directed loop of weight $s_{uu} = 1$. We will refer to the adjacency matrix of this graph as the *dominance matrix* of the network.

Given a dominance matrix A of a network, the goal is to construct a one-dimensional hierarchy of players in the network, called a ranking. As mentioned, one possible ranking is a vector r such that $r(u)/r(v)$ is a good approximation of $A(u, v)$. This intuitively suggests that, as the $(u, v)^{th}$ entry in A represents the relative strength of member u to member v, an appropriate ranking should respect this relationship.

In this work, we suggest a new notion of ranking. Namely, we generalize the idea of dominance and consider facets of strength: offensive strength and defensive strength.

Definition 2.0.1. Let A be a dominance matrix of a network. Let $\omega, \delta : V \to \mathbb{R}$ be vectors assigning offensive and defensive strength, respectively, to each player. Then a *total hierarchical ranking* $R : V \times V \to \mathbb{R}$ is given by

$$R(u, v) = c(\omega(u)\delta(v) - \delta(u)\omega(v)), \tag{2.0.1}$$

for some constant c.

Theorem 2.0.2. *Let A be a dominance matrix of a network, and let B be the matrix $B(u, v) = \log(A(u, v))$. Finally, let f be a solution to $fB = \lambda f$ for some eigenvalue λ of B. Then $\Re(\lambda f^* f)$ gives a total hierarchical ranking, where f^* denotes the conjugate transpose of f.*

Proof. We first note that the matrix B is skew-symmetric, i.e., $B = -B^T$, as A is a positive reciprocal matrix. Thus, since the eigenvalues of B are all of the form $\lambda = \pm i\mu$, f must be a complex vector of the form $f = \psi + i\phi$. Now we compute,

$$\begin{aligned}(\lambda f^* f)(u, v) &= i\mu((\psi + i\phi)(u)(\psi + i\phi)(v)) \\ &= i\mu(\psi(u)\psi(v) + \phi(u)\psi(v) + i\psi(u)\phi(v) - i\phi(u)\psi(v)) \\ &= -\mu(\psi(u)\phi(v) - \phi(u)\psi(v)) + i\mu(\psi(u)\psi(v) + \phi(v)\phi(v))\end{aligned}$$

By taking $c = -\mu$, this matrix satisfies (2.0.1). $\square$

2.1 Analyzing a total hierarchicial ranking

By Theorem 2.0.2, a total hierarchical ranking can be given in terms of some eigenfunction of the matrix B. In this section, we explore how the choice of eigenfunction affects the quality of the ranking.

2.1.1 The score of a ranking

One possibility is simply measuring the spectral (or some other) norm $||R - A||$. Another idea, inspired by the LLSM, is to measure error in each component.

2.1.2 Maximal rankings

One option: the eigenfunction f minimizing $||B - \lambda f^* f||$.

Chapter 3

PageRank and Heat Kernel PageRank

(This chapter is forthcoming)

3.1 PageRank

Definition 3.1.1.

$$\alpha \sum_{k=0}^{\infty} (1-\alpha)^k \frac{A^k e}{e^T A^k e}$$

3.2 Heat Kernel PageRank

Definition 3.2.1.

$$e^t \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{A^k e}{e^T A^k e}$$

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