Activity 22 – Introduction to Proof

set operations

(1) Calculate the union and intersection of $\{1, 2, 3, 5\}$ and $\{1, 3, 4, 5\}$.

(2) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{2, 4, 6, 8, 10\}$. Calculate the following:

(a) $A \setminus B \ =$

(b) $B \setminus A \ =$

(c) $A \triangle B \ =$

(d) $A \cup B \ =$

(e) $A \cap B \ =$

(f) $(A \cup B) \setminus (A \cap B) \ =$

(3) Suppose $U = \{1, 2, \ldots 10\}$. Let $A = \{1, 4, 6, 8, 9, 10\}$. What is $\overline{A}$?

(4) (Exercise 2 in §4.3 of GIAM) In a standard deck of playing cards one can distinguish sets based on face-value and/or suit. Let $A, 2, \ldots 9, 10, J, Q$ and K represent the sets of cards having the various face-values. Also, let $\heartsuit$, $\spadesuit$, $\clubsuit$ and $\diamondsuit$ be the sets of cards having the possible suits. Find the following

(a) $A \cap \heartsuit$

(b) $A \cup \heartsuit$

(c) $J \cap (\spadesuit \cup \heartsuit)$

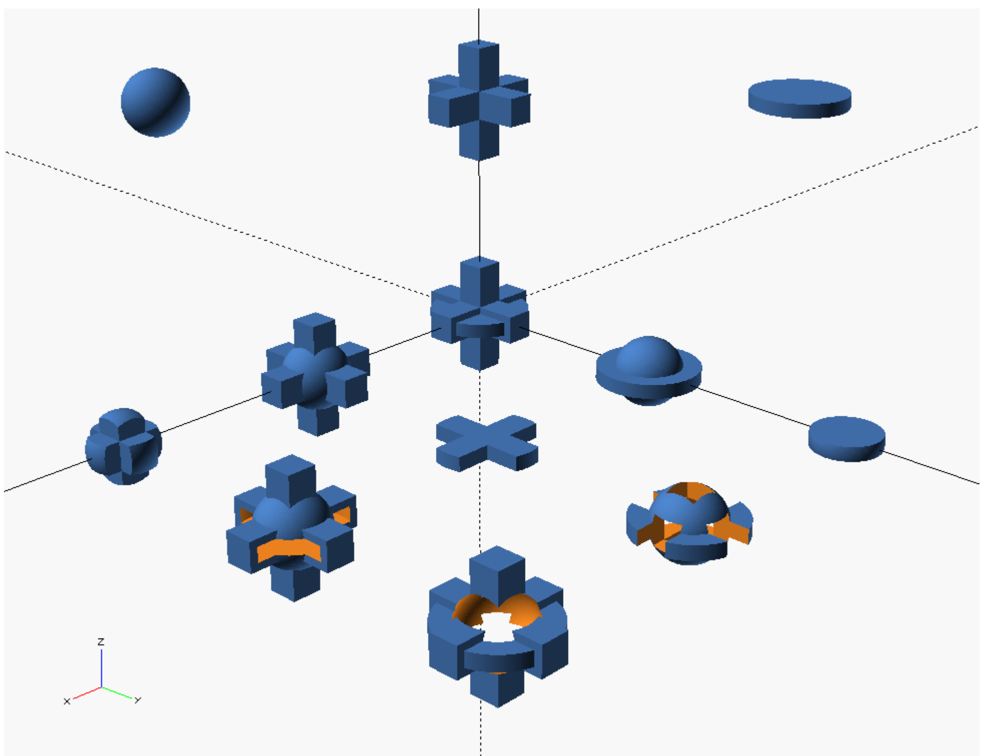
(d) $K \cap \heartsuit$

(e) $A \cap K$

(f) $A \cup K$

(5) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{2, 4, 6, 8, 10\}$. (The same sets as in problem 2.) Calculate all of the elements in DeMorgan's law: $\overline{A \cup B}$ and $\overline{A}$ and $\overline{B}$ and $\overline{A} \cap \overline{B}$.

(6) (Exercise 3 in §4.3 of GIAM) The following is a screenshot from the computational geometry program OpenSCAD (very handy for making models for 3-d printing...) In computational geometry we use the basice set operations together with a few other types of transformations to create interesting models using simple components. Across the top of the image below we see 3 sets of points in $\mathbb{R}^3$, a ball, a sort of 3-dimensional plus sign, and a disk. Let's call the ball A, the plus sign B and the disk C. The nine shapes shown below them are made from A, B and C using union, intersection and set difference. Identify them!



(7) What interval would the following infinite union be equal to?

$$\bigcup_{n=2}^{\infty} [1, n]$$

(8) Let D_n be the set of all positive integers that are divisors of n,

$$D_N \;=\; \{x \in \mathbb{Z}^+ \mid \; x \,|\, n\}.$$

What is the following infinite intersection?

$$\bigcap_{n=1}^{\infty} D_n$$

(9) Here's a proof of the commutative property for $\cup$:

Theorem. *For all sets A and B, $A \cup B = B \cup A$.*

Proof: We need to show that $x \in A \cup B \iff x \in B \cup A$. Suppose A and B are sets and x is a particular but arbitrary element of the universe of discourse.

	$x \in A \cup B$	Given
$\iff$	$x \in A \vee x \in B$	Definition of union
$\iff$	$x \in B \vee x \in A$	Commutative property of disjunction
$\iff$	$x \in B \cup A$	Definition of union

Q.E.D.

Write a proof of the associative property for intersection.

(10) Here's a proof of the associative law for union. (It may be instructive to compare this to your answer on the last question.)

Theorem. *For all sets A, B, and C, $(A \cup B) \cup C = A \cup (B \cup C)$.*

Proof: We need to show that $x \in (A \cup B) \cup C \iff x \in A \cup (B \cup C)$. Suppose A, B and C are sets and x is a particular but arbitrary element of the universe of discourse.

	$x \in (A \cup B) \cup C$	Given
$\iff$	$x \in (A \cup B) \vee x \in C$	Definition of union
$\iff$	$(x \in A \vee x \in B) \vee x \in C$	Definition of union
$\iff$	$x \in A \vee (x \in B \vee x \in C)$	Associative property of disjunction $\iff$ $x \in A \vee$
$\iff$	$x \in A \cup (B \cup C)$	Definition of union

Q.E.D.

Now you try one! Prove the distributive law of union over intersection.

(11) Do a quick review of your last two proofs with the following in mind: On either side of a logical connector ($\wedge$ and $\vee$) we must find logical *statements* (like $x \in A$ or $x \in B$). On either side of a set connector ($\cap$ and $\cup$) we must find *sets*!

(12) Without referring back to the lecture video, do the activity of assembling the two column proof that $A \triangle B = (A \cup B) \setminus (A \cap B)$ from the scrambled steps from page 184.

$= (A \cap \overline{B}) \cup (B \cap \overline{A})$	identity law
$= (A \cup B) \cap \overline{(A \cap B)}$	def. of relative difference
$(A \cup B) \setminus (A \cap B)$	Given
$= ((A \cap \overline{A}) \cup (A \cap \overline{B})) \cup ((B \cap \overline{A}) \cup (B \cap \overline{B}))$	distributive law
$= (A \setminus B) \cup (B \setminus A)$	def. of relative difference
$= (A \cap \overline{(A \cap B)}) \cup (B \cap \overline{(A \cap B)})$	distributive law
$= A \triangle B$	def. of symmetric difference
$= (A \cap (\overline{A} \cup \overline{B}) \cup (B \cap (\overline{A} \cup \overline{B}))$	DeMorgan's law
$= (\emptyset \cup (A \cap \overline{B})) \cup ((B \cap \overline{A}) \cup \emptyset)$	complementarity