

Activity 24 – Introduction to Proof
Russell’s paradox

Set Theory review questions

- (1) Use the basic set-theoretic equalities to develop a two-column proof of the following.

$$(A \cap C) \cap (A \cup B) = A \cap C$$

- (2) There are three individuals named Abraham, Balthazar and Cincinnatus. One of them is a knight (who always makes truthful statements), one is a knave (who always makes false statements) and one is a kneither (whose statements can be either true or false).

Abraham says “Cincinnatus is a knave.”

Balthazar says “Abraham is a knight.”

Cincinnatus says “I am a kneither.”

State – **with proof** – who is the knight, who is the knave and who is the kneither?

- (3) Let A be the set $\{\emptyset, 1, 2, \{3, 4, 5\}\}$. Mark the following statements True or False.

a) $\emptyset \subseteq A$

b) $\emptyset \in A$

c) $\{1\} \subseteq A$

d) $\{3, 4, 5\} \subseteq A$

e) $\{\emptyset\} \subseteq A$

f) $\{1, 2\} \in A$

g) $\{3, 4, 5\} \in A$

h) $\{1, 2\} \subseteq A$

- (4) What interval would the following infinite union be equal to?

$$\bigcup_{n=2}^{\infty} [1/n, 1 + 1/n]$$

(5) Use a Venn diagram to solve the following:

Irving's Used Cars currently has 71 vehicles on hand. There are 29 whose engines won't start. There are 21 that need bodywork. And there are 17 that need deep cleaning because they smell very bad. Three of the smelly cars also need bodywork and engine repairs. There are a total of 11 cars that need both engine and body repairs (including the three smelly ones). There is 1 car that stinks and has a bad engine but its body is in good shape, and there is 1 car that stinks and has lots of rust but its engine sounds great.

How many cars are currently suitable for sale?

Questions regarding Russell's paradox

- (6) Verify that $(A \implies \neg A) \wedge (\neg A \implies A)$ is a logical contradiction by filling out a truth table.

- (7) Would it be possible to write a book that catalogued all those books (and only those books) that do not refer to themselves?

- (8) In a Star Trek (TOS) episode an evil robot has taken over their spaceship. Captain Kirk and Mr. Spock approach it and say the following lines

Kirk: You know, everything Mr. Spock says is a lie.

Spock: Actually, everything Captain Kirk says is the truth.

Why does the robot subsequently explode?

- (9) Is there any problem with defining n to be “the smallest positive integer that cannot be described in fewer than fourteen words” ?
Describe the paradox.

- (10) Alan Turing, the famous british mathematician and computer scientist, used an argument similar to the one in Russell's paradox to show the impossibility of solving the halting problem. A solution to the halting problem would be a program that could be used to determine whether a given program (with its associated input) will enter into an infinite loop or will halt in a finite amount of time. Assume that `Check(P,I)` is such a program. The `Check` program takes a program `P` and its input `I` as its input and either outputs `True` (if that program applied to that input will halt) or `False` (if that program applied to that input will loop forever). Turing's clever argument involved thinking about applying `Check` where `P` and `I` are both the same – in other words ask whether program `P`, given its own code listing as input would halt or not. Turing then defined a new program, let's call it `Strange`, as follows:

```
Strange(x):
    if Check(x,x):
        while True:
            print "Aaargh! I'm in an infinite loop!"
    else
        print "Cool, I'm done."
    return
```

Try to figure out what will happen if we run `Strange(Strange)`.