Activity 6 – Introduction to Proof

irrational numbers

(1) Which of the following pairs of numbers are relatively prime? Do as many as you can "by hand," but feel free to use sage for the tougher ones.

52 & 143	7 & 112	2457 & 1265
59 & 69	10000 & 101	5183 & 4189
256 & 243	13706 & 14861	999 & 1000

(2) True or False: A number and its immediate successor (i.e. n and $n+1$) are always relatively prime. Explain your answer. (Hint: it may be better to think about $n \mod d$ rather than $d|n$ for this.)

(3) True or False: A number and its successor's successor (i.e. n and $n+2$) are never relatively prime. Explain your answer.

(4) Play around (with sage or your calculator) and find a criterion for when the squareroot of an integer is irrational.

(5) Recall the lemma we used as part of the proof of the irrationality of $\sqrt{2}$:

$$\forall x \in \mathbb{Z}, \quad \text{if } x^2 \text{ is even, then } \ x \text{ is even.}$$

This can be restated using the divisibility symbol as:

$$\forall x \in \mathbb{Z}, \quad \text{if } 2 \,|\, x^2 \text{ then } \ 2 \,|\, x.$$

State the lemma that we'd need for making a similar proof (to that of Hippassus) which shows that $\sqrt{3}$ is irrational.

(6) A similar change (substituting 4 for 2) gives:

$$\forall x \in \mathbb{Z}, \quad \text{if } 4 \,|\, x^2 \text{ then } \ 4 \,|\, x.$$

but this lemma is false! Provide a counterexample and explain what this has to do with your answer to # 4.

(7) A variant of Hippasus' proof involves slightly weakening the "relatively prime" restriction on the numerator and denomenator of a fraction. The variant would start with:

Suppose to the contrary that $\sqrt{2} = \dfrac{a}{b}$ *where* a *and* b *are integers which are not both even.*

How would you end this version of the proof? The middle part of the argument would stay the same. You just need to finish the sentence that begins with "Finally, we have arrived at the desired absurdity because..."

(8) Try writing a proof that $\sqrt{3}$ is irrational using the lemma

$$\forall x \in \mathbb{Z}, \text{ if } 3 \,|\, x^2 \text{ then } \ 3 \,|\, x.$$

and the variant approach from the previous problem.