Activity 31 – Introduction to Proof

equivalence relations

(1) (Exercise 1 in GIAM section 6.3) Consider the relation A defined by

$$\mathsf{A} = \{(x, y) \mid x \text{ has the same astrological sign as } y\}.$$

Verify that A is reflexive, symmetric, and transitive.

(2) Consider the generic "equivalence mod m" relation:

$$x \mathsf{R} y \quad \Longleftrightarrow \quad x \mod m = y \mod m.$$

Show that R is reflexive, symmetric and transitive.

(3) In the book we defined the function $sf(n)$ which returns n divided by the largest perfect square that divides n. For example $sf(20) = 5$ since 4 is the largest perfect square that divides 20, and $20/4 = 5$

Use the sf function to define a relation:

$$x\mathsf{S}y \quad \iff \quad sf(x) = sf(y)$$

Show that S is an equivalence relation on $\mathbb{N}$.

(4) Continuing with the relation S defined in problem 4, notice that for any number x that actually is a perfect square, $sf(x) = 1$. Given this, what is $\overline{1}$?

(5) (Still working with the relation from problem 4.) Many different natural numbers can be used as a "label" for the equivalence classes under S. For instance $\overline{2}$, $\overline{8}$ and $\overline{18}$ all denote the same equivalence class – let's follow the convention of always using the smallest such label. Characterize these reduced representatives for the equivalence classes.

(6) Define a relation Q on the set of all finite sets by

$$A \mathsf{Q} B \iff |A| = |B|$$

What are the equivalence classes under Q?

(7) (Exercise 3 in GIAM section 6.3.)

Define a relation A on the set of all words by

$$w_1 \mathsf{A} w_2 \iff w_1 \text{ is an anagram of } w_2.$$

Show that A is an equivalence relation. (Words are anagrams if the letters of one can be re-arranged to form the other. For example, 'ART' and 'RAT' are anagrams.)

(8) (Exercise 4 in GIAM section 6.3.)

The two diagrams below both show a famous graph known as the Petersen graph. The picture on the left is the usual representation which emphasizes its five-fold symmetry. The picture on the right highlights the fact that the Petersen graph also has a three-fold symmetry. Label the right-hand diagram using the same letters (A through J) in order to show that these two representations are truly isomorphic.