Activity 26 – Introduction to Proof

formulas for sums and products

(1) Complete the following tables about sums of squares.

| $n$ | $\sum_{j=1}^{n} j^2$ |
|---|---|
| 1 | 1 |
| 2 | |
| 3 | |
| 4 | |

| $n$ | $\frac{n(n+1)(2n+1)}{6}$ |
|---|---|
| 1 | 1 |
| 2 | |
| 3 | |
| 4 | |

(2) The base case for an inductive proof that

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

really just consists of noticing that the first rows in the two tables above are identical. Write the base case formally.

(3) The inductive step in a proof that

$$\sum_{j=1}^{n} j^2 \;=\; \frac{n(n+1)(2n+1)}{6}$$

involves verifying the following equality

$$(k+1)^2 \;+\; \frac{n(n+1)(2n+1)}{6} \;=\; \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

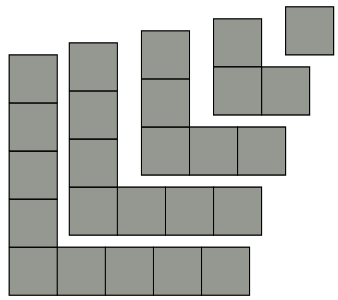
or (after simplifying the right-hand side)

$$(k+1)^2 \;+\; \frac{n(n+1)(2n+1)}{6} \;=\; \frac{(k+1)(k+2)(2k+3)}{6}$$

Do it! (Try to take advantage of the factored form as much as possible.)

(4) Put the last problems together and write a complete inductive proof of the formula for the sum of squares.

(5) The image below illustrates that a perfect square can be decomposed into a sum of odd numbers. (The L-shaped things are a visual representation for odd numbers. Do you see why?)



Notice that the number of terms in the sum is the quantity that is squared, so

$$\forall n \in \mathbb{N}, \sum_{i=0}^{n} 2i + 1 \;=\; (n+1)^2$$

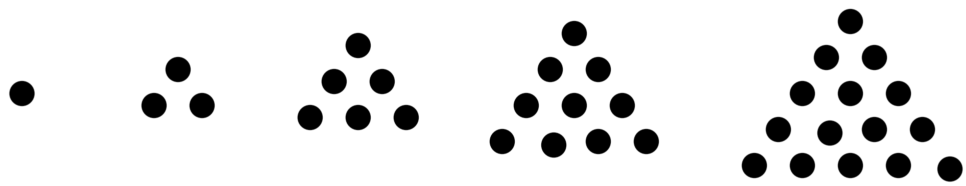
(Since a sum that goes from 0 to $n$ has $n + 1$ terms.)

What will we need to add to both sides of the inductive hypothesis?

Work out the algebraic part in the inductive step:

$$(n+1)^2 + \underline{\hspace{2cm}} \;=\; ((n+1)+1)^2$$

(6) The triangular numbers are the numbers that can be expressed as triangular arrays of points.



Let $T_n$ represent the $n$-th triangular number. So $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, *et cetera.*

Each triangular number can be thought of as a sum, which allows us to deduce a formula for $T_n$. What is it?

(7) Find a formula for the sum

$$\sum_{j=1}^{n} T_j,$$

and prove it using mathematical induction.

(8) A *hexagonal number* is one that can be represented by a hexagonal array of dots. A formula for the hexagonal numbers is

$$H_n = 3n^2 - 3n + 1.$$

Find a formula for the sum

$$\sum_{j=1}^{n} H_j$$

and prove it using mathematical induction.

(9) (Exercise 5 in GIAM §5.2) Prove the following formula for a product.

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i}\right) = \frac{1}{n}$$

(10) Here is a formula for the sum of the first $n$ fourth powers

$$\frac{n \cdot (n+1) \cdot (2n+1) \cdot (3n^2+3n-1)}{30}.$$

Prove it using induction.