Activity 16 – Introduction to Proof

indirect arguments

(1) Prior to starting a proof by contradiction you must first determine the logical negation of the statement you are hoping to prove. Find negations of the following.

(a) There are infinitely-many natural number p such that p is prime and $p+2$ is prime.

(b) $\forall n \in \mathbb{N},\ n^3$ is even $\implies$ n is even

(c) $\forall n \in \mathbb{N},\ (n \geq 2) \implies (\exists p, q \in \text{Primes},\ p+q = 2n$

(2) Proofs by contradiction and by contraposition are considered equivalent because, if you have a proof of one sort, you can recast it as the other sort. If we are trying to prove a universal conditional sentence $\forall x, P(x) \implies Q(x)$, a proof by contradiction consists in showing that $\forall x, (P(x) \wedge \neg Q(x)) \implies c$, while a proof by contraposition consists of showing that $\forall x, \neg Q(x) \implies \neg P(x)$.

Use a truth table to determine whether $(P \wedge \neg Q) \implies c$ and $\neg Q \implies \neg P$ are equivalent.

(3) Devise a proof by contraposition and a proof by contradiction for item (b) in problem 1.

(4) Show, by contradiction, that there is no smallest positive real number.

(5) Show that $\forall x, y \in \mathbb{R}, (x \in \mathbb{Q} \ \wedge \ xy \notin \mathbb{Q}) \implies y \notin \mathbb{Q}$.

(6) Prove that $\forall x, y \in \mathbb{Z},\ (x > 1) \implies (x \nmid y \ \vee \ x \nmid y + 1)$

(7) Let's do one that's not an indirect argument. (Just to mix things up a little.) The squares of natural numbers can only have certain values mod 4. In particular, $\forall x \in \mathbb{N},\ x^2 \mod 4 = 0$ or $x^2 \mod 4 = 1$. You can prove this by looking at two possibilities: either x is even, or x is odd.

(8) A *Pythagorean triple* is a set of three natural numbers, a, b and c, that satisfy the Pythagorean theorem: $a^2 + b^2 = c^2$. Prove that, in a Pythagorean triple, at least one of a and b is even. (Use contradiction – it will involve the statement we just proved in problem 7.