Activity 27 – Introduction to Proof

divisibility statements and other proofs using PMI

(1) Verify the base cases in inductive proofs of the following. (Assume the universe of discourse is $\mathbb{N}$.)

(a) $3 \mid x^3 + 5x - 6$

(b) $5 \mid 2x^5 - 7x + 10$

(c) $5 \mid 6^x + 4$

(d) $11 \mid 12^x + x^{11} - x + 10$

(2) In the problems that follow you will often need the binomial coefficients. Write out Pascal's triangle up to the 7th row here.

(3) Use induction to prove that

$$\forall n \in \mathbb{N}, 3 \,|\, n^3 - n$$

(4) Use induction to prove that

$$\forall n \in \mathbb{N}, 5 \,|\, n^5 - n$$

(5) Use induction to prove that

$$\forall n \in \mathbb{N}, 7 \,|\, n^7 - n$$

(6) Find a counterexample to

$$\forall n \in \mathbb{N}, 4 \,|\, n^4 - n$$

(7) Use induction to prove that

$$\forall n \in \mathbb{N}, 5 \,|\, n^5 - 5n^3 + 9n.$$

(8) Use induction to prove that

$$\forall n \in \mathbb{N}, 7 \,|\, 8^n - 1.$$

(9) A *Charmichael number* is a composite integer that, nevertheless satisfies Fermat's little theorem. The smallest Charmichael number is 561. So, for every x, with $0 \leq x < 561$, it follows that $x^{561} - x \equiv 0 \pmod{561}$.

While number like 2^{561} are far too large for your calculator, they can be computed in sage in a tiny fraction of a second. Use a loop in sage to verify that 561 is indeed a Carmichael number. (Recall that % is the sage "mod" operator.)

(10) Suppose n is a natural number such that $(x+1)^n \equiv x^n + 1 \pmod{n}$.

Use induction to show that

$$\forall x \in \mathbb{N},\ x^n \equiv x \pmod{n}.$$