Activity 25 – Introduction to Proof

mathematical induction

Outline of proof by mathematical induction:

We want to prove a statement that is universally quantified over the natural numbers (or something similar):

$$\forall n \in \mathbb{N},\ P_n.$$

The proof consists of 2 parts:

(1) Prove that P_0 is true.

(2) Prove that for some (particular but arbitrarily chosen) $k \in \mathbb{N}$ $P_k \implies P_{k+1}$.

Step 1 is known as the *base case* (a.k.a. *basis*).

Step 2 is known as the *inductive step*.

The parts of the conditional sentence in step 2 are called the *inductive hypothesis* and the *inductive conclusion* (respectively).

(1) Suppose we are trying to prove the following statement:

$$\forall n \in \mathbb{N}, \sum_{i=0}^{n} 2i+1 = (n+1)^2.$$

Match the expressions with the terminology.

inductive step

$$\sum_{i=0}^{0} 2i+1 = (0+1)^2$$

inductive conclusion

$$\sum_{i=0}^{k+1} 2i+1 = (k+2)^2$$

inductive hypothesis

$$\sum_{i=0}^{k} 2i+1 = (k+1)^2$$

base case

(2) Referring to the previous problem, what would you call the part of the proof where one showed that

$$\forall k \in \mathbb{N}, \left(\sum_{i=0}^{k} 2i+1 = (k+1)^2 \right) \implies \left(\sum_{i=0}^{k+1} 2i+1 = (k+2)^2 \right).$$

(3) There is a category of mathematical sentences known as "postage stamp problems" that can be proved using induction. Consider the following:

> Any amount of postage greater than 7¢ can be created using some number of 3¢ and some number of 5¢ stamps.

(a) Write out the ways to use 3's and 5's to get all the possible postages between 10 and 20.

(b) What would the base case in an inductive proof of this statement be?

(4) Consider the statement

$$\forall n \in \mathbb{N}, \sum_{j=0}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Verify the base case for proving this statement by induction.

(5) (Exercise 2 in §5.1 in GIAM) What is wrong with the following inductive proof of "all horses are the same color."?

(P.S. I hope it's clear that this can't be a true theorem, we're being asked to figure out what's wrong with an argument that appears to prove something that we know isn't true.)

Theorem. *Let H be a set of n horses, all horses in H are the same color.*

Proof: We proceed by induction on n.

Basis:

Suppose H is a set containing 1 horse. Clearly this horse is the same color as itself.

Inductive step:

Given a set of k+1 horses H we can con- struct two sets of k horses. Suppose $H = \{h_1, h_2, h_3, ...h_{k+1}\}$. Define $H_a = \{h_1, h_2, h_3, ...h_k\}$ (i.e. H_a contains just the first k horses) and $H_b = \{h_2, h_3, h_4, ...h_{k+1}\}$ (i.e. H_b con- tains the last k horses). By the inductive hypothesis both these sets contain horses that are "all the same color." Also, all the horses from h_2 to h_k are in both sets so both H_a and H_b contain only horses of this (same) color. Finally, we conclude that all the horses in H are the same color.

Q.E.D.

Hint: Look carefully at the early stages.

(6) It's relatively common that the base case in an inductive proof is true for vacuous reasons. (This can feel relatively unsettling, so I usually do two things: (1) check that the smallest non-vacuous statement is also true, and (2) pay careful attention in the inductive step to seeing that the vacuous case implies the first non-vacuous case.) A common situation involves sums and products that have no entries. Suppose $EMPTY$ represents a sum having no terms and $SOME$ is some other sum. What is $SOME + EMPTY$? Explain why an empty sum is equal to 0.

(7) This exercise is about understanding an empty product. Like an empty sum, we define an empty product in terms of how it behaves. We think of a product as a list of numbers, and multiplying two such lists is really just concatenating the lists. For example, if $a = 3 \cdot 5 \cdot 7$ and $b = 2 \cdot 4 \cdot 6$, then $ab = 3 \cdot 5 \cdot 7 \cdot 2 \cdot 4 \cdot 6$. What if one of those lists had no members? Let $PROD_n$ be the product of the numbers from 1 to n. (In other words, $PROD_n$ is the same thing as $n!$). How many multiplicands are in $PROD_n$? Is it clear that $PROD_0$ is an empty product? Consider $PROD_0 \cdot ANY$, where ANY is any other product. What must be the value of an empty product?

(8) A *taxicab path* is a piece-wise linear path, with pieces that are either horizontal or vertical, that run between the points in the plane whose coordinates are integers. Suppose we wished to count all of the taxicab paths from $(0, 0)$ to (n, n), where $n \in \mathbb{N}$. It's necessary to add an additional restriction or the answer will always be infinity. A *reasonable taxicab path* always moves towards the goal, so it will consist of line segments that go in the positive x direction or positive y direction only. Play with this question and try to discover a formula for the number of reasonable taxicab paths from $(0, 0)$ to (n, n). What does your formula say about the base case: a reasonable taxicab path from $(0, 0)$ to $(0, 0)$?