Tutorial: 1D Finite Elements

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The objective of this example sheet is to reinforce theoretical concepts learned in the lectures via a simple one-dimensional example. We will go through all the important steps of developing a finite element solution, including the derivation of the weak form, the definition of the discrete system and the calculation of the stiffness matrix and nodal force vector by assembling element contributions. The advantage of considering a one-dimensional problem is that most calculations can be done analytically (at least for a small number of elements). You are however encouraged to implement the finite element solution into your own code to consider larger numbers of elements.

We consider an elastic bar fixed to a support at $x = 0$ and $x = L$, and subjected to a uniform distributed axial load b (force per length) (Fig. 1). We write A the constant cross-section area of the bar and E its Young modulus. We aim to calculate the fields of displacement $u(x)$, strain $\varepsilon(x)$ and stress $\sigma(x)$ in the bar due to the applied loading.

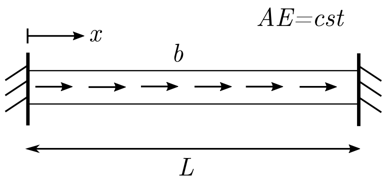

Figure 1: Axially-loaded bar.

The governing equations for this problem are the following:

$$
\begin{aligned}
0 &= A\frac{d\sigma}{dx} + b && \text{(mechanical equilibrium)},\\
\sigma &= E\varepsilon && \text{(Hooke's law)},\\
\varepsilon &= \frac{du}{dx} && \text{(kinematics)},
\end{aligned}
$$

together with the Dirichlet boundary conditions $u(0) = 0$ and $u(L) = 0$. Combining the three equations leads to a second-order ODE for the displacement:

$$
AE\frac{d^2u}{dx^2} + b = 0. \tag{1}
$$

Eq. (1) together with the boundary conditions constitute the **strong form** of the boundary-value problem.

The analytical solution is readily found by integrating the ODE (1) twice, identifying the two integration constants using the two boundary conditions. The solution is:

$$
u(x) = \frac{bL^2}{2AE}\frac{x}{L}\left(1 - \frac{x}{L}\right). \tag{2}
$$

Questions

1. Show that the weak form of the boundary-value problem is:

$$AE \int_0^L \frac{du}{dx}\frac{d\hat{u}}{dx} dx = \int_0^L b\hat{u}\ dx, \quad \forall \hat{u} \text{ s.t. } \hat{u}(0) = 0 \text{ and } \hat{u}(L) = 0.$$

2. Consider an approximate solution of the form $u^h(x) = \sum_{i=1}^{N} U_i \tau_i(x)$, where U_i are the nodal values and $\tau_i(x)$ are a-priori specified shape functions. Using Galerkin's method, show that the nodal values are solutions of an algebraic system $\mathbf{AU} = \mathbf{B}$, with:

$$\begin{aligned} A_{ij} &= AE \int_0^L \frac{d\tau_i}{dx}\frac{d\tau_j}{dx} dx \\ B_i &= \int_0^L b\tau_i\ dx \end{aligned}$$

3. Consider a mesh of the interval $]0, L[$ with N_1 *linear* elements of equal size h. Calculate the local stiffness matrix $\mathbf{A}^e$ and local nodal force vector $\mathbf{B}^e$. (Hint: they are the same for all the elements in this case.)

4. Calculate the global stiffness matrix $\mathbf{A}$ and global nodal force vector $\mathbf{B}$ for 2 elements and 3 nodes.

5. Calculate the nodal displacements, accounting for the Dirichlet boundary conditions.

6. Calculate the strain and stress in each element.

7. Answer Questions 1-6 again, this time assuming that the bar is subjected to prescribed tensile force P at $x = L$ (Neumann boundary condition):

$$A\sigma(L) = P \Rightarrow \frac{du}{dx}(L) = \frac{P}{AE}.$$

Advanced questions

1. Solve the same boundary-value problems using two *quadratic* elements of equal size and 5 nodes.

2. Write a Matlab program that calculates the finite element solution of the bar problem for an arbitrary number of linear and/or quadratic elements. Use Gauss-Legendre integration method with one integration point (linear elements) or two integration points (quadratic elements).

Answers

3. Local stiffness matrix and local nodal force vector:

$$A^e_{ij} = \frac{AE}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$B^e_i = \frac{bh}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

4. Global stiffness matrix and global nodal force vector:

$$A_{ij} = \frac{2AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$B_i = \frac{bL}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

5. $U_1 = U_3 = 0, \quad U_2 = \frac{bL^2}{8Ae}$

6. $\varepsilon^1 = \frac{U_2}{h}$, $\varepsilon^2 = -\frac{U_2}{h}$, $\sigma^e = E\varepsilon^e$.

7. The stiffness matrix is unchanged. The global nodal force vector becomes:

$$B_i = \frac{bL}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix}$$

Nodal displacements:

$$U_1 = 0, \quad U_2 = \frac{3}{8}\frac{bL^2}{AE} + \frac{PL}{2AE}, \quad U_3 = \frac{bL^2}{2AE} + \frac{PL}{AE}.$$