Exercises Week 5: Numerical algebraic geometry

Exercise 1. The following system describes the intersection of an ellipse and an elliptic curve:

$$\begin{aligned} 2x^2 + y^2 - 4 &= 0 \\ x^3 - 3x - y^2 + 3 &= 0\,. \end{aligned} \tag{1}$$

(a) Write down the straight line homotopy (with the γ-trick) between the system (1) and the total degree start system, and sketch the corresponding Davidenko differential equation.
(b) Use `HomotopyContinuation.jl` to solve the system (1).
(c) Use certification to determine how many real solutions you have found approximations of, and how many of these are positive.
(d) Does the above constitute a *proof* that (1) has precisely one positive solution?

Exercise 2. High school math tells us that $\mathbf{V}(x^2 - 5x + 4) = \{1, 4\}$, $\mathbf{V}(-x^2 + 2x) = \{0, 2\}$ and $\mathbf{V}(x^2 + 1) = \{\pm i\}$. However, something goes wrong if we solve these systems using a straight line homotopy from the total degree system, and force `HomotopyContinuation.jl` to pick $\gamma = 1$ for the γ-trick by writing

```
solve([x^2-5*x+4], start_system = :total_degree, gamma = 1)
solve([-x^2+2*x], start_system = :total_degree, gamma = 1)
solve([x^2+1], start_system = :total_degree, gamma = 1).
```

Explain what happens, and suggest better values of $\gamma \in \mathbb{C} \setminus \{0\}$ (trial and error is fine).
Hint: Use the applet `https://geogebra.org/m/vftzrnnx` to visualize the homotopies, and `https://www.geogebra.org/3d/kbr92fbt` to visualize the discriminant of $\mathcal{F}(2)$.

Exercise 3 (Certification)**.** In Exercise 2 in Week 3, we used elimination to solve the system

$$\begin{aligned} x^2 + y^2 + z^2 - 1 &= 0 \\ x^2 + y^2 + z^2 - 2x &= 0 \\ 2x - 3y - z &= 0\,. \end{aligned}$$

We obtained the following approximate solutions:

`(0.500000,0.554950,-0.6648529)` and `(0.500000,0.0450490,0.864852)`.

Use certification to show that they correspond to two distinct and real true solutions.

If time permits, you can also try to certify the solutions found in Exercise 4 from Week 4.

Exercise 4 (Overdetermined systems)**.** Why is homotopy continuation not immediately applicable to systems with more equations than variables? Can this be circumvented? If you come up with a strategy, try it on the following system (or simply check how `HC.jl` deals with it):

$$xz - y^2 = 0, \quad y - z^2 = 0, \quad x - yz = 0, \quad x + y + z + 1 = 0\,.$$

Exercise 5 (More on the γ-trick)**.** Consider the straight line homotopy (with the γ-trick)

$$H(t, x) = t\,F(x) + (1 - t)\,\gamma\,G(x)\,,$$

between the target system

$$F(x) = \begin{bmatrix} x^2 - y^2 - 1 \\ 2x^2 + y^2 - 8 \end{bmatrix}$$

and the corresponding total degree start system $G(x)$. Find a specific $\gamma \in \mathbb{C} \setminus \{0\}$ such that at least one of the paths encounter a singular Jacobian at $t = 1/2$. Note that this corresponds to finding a γ such that there is a solution of the system

$$H\left(\tfrac{1}{2}, x\right) = 0\,, \qquad \det\Big(\frac{\partial H}{\partial x}\left(\tfrac{1}{2}, x\right)\Big) = 0\,.$$

Check what happens if you force `HomotopyContinuation.jl` to use this particular γ by running

```
solve(F, start_system = :total_degree, gamma = ...).
```

Exercise 6 (The Euclidean distance problem)**.** Let $\mathbf{V}(f) \subseteq \mathbb{R}^2$ be a smooth curve, and let $u \in \mathbb{R}^2 \setminus \mathbf{V}(f)$ be some point outside the curve. The goal of this exercise will be to devise a numerical algebraic geometry approach to determining which $x \in \mathbf{V}(f)$ is the closest to u, i.e., determining the *global minimum* of the optimization problem

$$\min_{x \in \mathbf{V}(f)} \|x - u\|^2 .$$

The key idea is that the global minimum will be contained in the set of *critical points* of the optimization problem, which are the solutions to the polynomial system

$$\begin{aligned} f(x) &= 0 \\ \nabla f(x) - \lambda(x-u) &= 0 , \end{aligned} \tag{2}$$

where $\lambda \neq 0$ is a new auxiliary variable (often referred to as a *Lagrange multiplier*), which is introduced to encode that $x - u$ is parallel to ∇f (i.e., orthogonal to the tangent space $T_x\mathbf{V}(f)$).

We consider the optimization problem for

$$f(x_1, x_2) = x_1^2 x_2^2 - 3x_1^2 - 3x_2^2 + 5 , \qquad u = (1, 2) .$$

(a) Find the critical points by solving (2) with homotopy continuation.
(b) Evaluate $\|x - u\|^2$ at the real critical points, and determine which one gives the lowest value.
(c) Plot $\mathbf{V}(f)$ together with u and the real critical points you found (e.g., using Maple).
(d) Discuss to what extent we can *prove* that the value found in (b) is the global minimum.

Exercise 7 (27 lines on a cubic)**.** A classical result from algebraic geometry states that every smooth cubic surface in $\mathbb{C}^3$ contains exactly 27 lines. A particularly nice example, where all the 27 lines are *real*, is the *Clebsch surface*, which is the variety $\mathbf{V}(f) \subseteq \mathbb{C}^3$ given by

$$\begin{aligned} f(x, y, z) = 81(x^3 + y^3 + z^3) &- 189(x^2y + x^2z + y^2x + y^2z + xz^2 + yz^2) + 54\,xyz \\ &+ 126(xy + xz + yz) - 9(x^2 + y^2 + z^2) - 9(x + y + z) + 1 . \end{aligned}$$

The goal of this exercise is to explicitly find these 27 real lines.

Setting up the system: A complex line L in $\mathbb{C}^3$ can be parametrized by

$$\mathbb{C} \to L, \qquad t \mapsto p + tv,$$

where $p = (p_1, p_2, p_3) \in L$ is an arbitrary but fixed point on the line, and $v = (v_1, v_2, v_3) \in \mathbb{C}^3 \setminus \{0\}$ gives the direction. Note that the choices of p and v are not unique. However, if we pick a random affine hyperplane in $\mathbb{C}^3$, it will intersect each of the 27 lines precisely once with probability one. For instance, you can try this hyperplane:

$$7 + p_1 + 3p_2 + 5p_3 = 0 . \tag{3}$$

Similarly, if we impose a random affine relation in v_1, v_2 and v_3, there will be unique v satisfying this for each of the 27 lines with probability one. For instance, you can try

$$11 + 3v_1 + 5v_2 + 7v_3 = 0 . \tag{4}$$

The line parametrized by $p + tv$ is contained in the surface $\mathbf{V}(f)$ if and only if

$$f(p + tv) = f(p_1 + tv_1, p_2 + tv_2, p_3 + tv_3) = 0 \quad \text{for all } t \in \mathbb{C} .$$

Note that $f(p + tv)$ can be viewed as a polynomial in t with coefficients $c_0, c_1, c_2, c_3 \in \mathbb{C}[p, v]$:

$$f(p + tv) = c_0(p, v)t^3 + c_1(p, v)t^2 + c_2(p, v)t + c_3(p, v)$$

and that $f(p + tv)$ vanishes for all $t \in \mathbb{C}$ if and only if

$$c_0(p, v) = 0 , \quad c_1(p, v) = 0 , \quad c_2(p, v) = 0 , \quad c_3(p, v) = 0 . \tag{5}$$

Thus, to find the 27 lines contained in the Clebsch surface, we should find all pairs $(p, v) \in \mathbb{C}^3 \times \mathbb{C}^3$ satisfying the equations (3), (4) and (5).

(a) Use the commands `subs` and `coefficients` in `Julia` to find $c_0, c_1, c_2, c_3 \in \mathbb{C}[p, v]$.
(b) Solve the system consisting of (3), (4) and (5) with homotopy continuation. Use certification to check that 27 distinct and real solutions are found.

(c) Plot the surface together with some of the lines you found (for instance with the free tool `https://www.math3d.org/8Uq3KNQMi`).

(d) Try to solve the system again, but now use Gröbner bases and elimination in `Oscar` instead.

Exercise 8 (Method of moments)**.** Suppose we are observing a random variable X whose distribution is a mixture of two Gaussians, in the sense that its probability density function is

$$f(x) = \lambda \frac{1}{\sqrt{2\pi v}} \exp\left(-\tfrac{1}{2v}(x-a)^2\right) + (1-\lambda)\frac{1}{\sqrt{2\pi w}} \exp\left(-\tfrac{1}{2w}(x-b)^2\right).$$

for unknown means $a, b \in \mathbb{R}$, variances $v, w \in \mathbb{R}_{>0}$ and mixing coefficient $\lambda \in (0,1)$, and that we want to estimate these parameters from a sample $x_1, x_2, \ldots, x_N \in \mathbb{R}$.

One way of approaching this problem is the *method of moments*. The idea is that for each integer $r > 0$, the rth *moment* $m_r := \mathbb{E}[X^r]$ is a polynomial in the parameters:

$$\begin{aligned}
m_1 &= \lambda a + (1-\lambda)b \\
m_2 &= \lambda(a^2+v) + (1-\lambda)(b^2+w) \\
m_3 &= \lambda(a^3+3av) + (1-\lambda)(b^3+3bw) \\
m_4 &= \lambda(a^4+6a^2v+3v^2) + (1-\lambda)(b^4+6b^2w+3w^2) \\
m_5 &= \lambda(a^5+10a^3v+15av^2) + (1-\lambda)(b^5+10b^3w+15bw^2) \\
m_6 &= \lambda(a^6+15a^4v+45a^2v^2+15v^3) + (1-\lambda)(b^6+15b^4w+45b^2w^2+15w^3) \\
&\vdots
\end{aligned} \tag{6}$$

If N is large enough, the law of large numbers tells us that the rth *sample moment*

$$\widehat{m}_r := \frac{1}{N}\sum_{i=1}^{N} x_i^r$$

is a good approximation of m_r. Hence, we can estimate the five parameters by plugging in the sample moments $\widehat{m}_1, \ldots, \widehat{m}_5$ in (6) and solving for $a, b \in \mathbb{R}$, $v, w \in \mathbb{R}_{>0}$ and $\lambda \in (0,1)$.

The was done (by hand!) for the first time in 1894 by Karl Pearson, for data on the body length to forehead ratio for a population of $N = 1000$ shore crabs, collected in Naples by zoologist Raphael Weldon, to test the hypothesis that there were two distinct local subspecies of the shore crab. The data and Pearson's inferred distributions are illustrated in the figure below. The goal of this exercise is to use homotopy continuation to replicate Pearson's analysis.

(a) Download the raw data from Absalon, and compute the sample moments $\widehat{m}_1, \widehat{m}_2, \ldots, \widehat{m}_5$.

(b) Substitute each m_r by $\widehat{m}_r$ in (6), and solve the system using homotopy continuation.

(c) Use certification to decide which solutions are statistically relevant.

(d) Discuss how to decide which of the solutions you found in (c) is the most reasonable one.

(e) Use this solution to recreate the figure below (e.g., using Maple).