Mixed volume and Bernstein's Theorem

Week 7

The first part of the exercise list consists of theoretical exercises on the mixed volume, as well as a computational exercise on computing mixed volumes in `Oscar` and `HC.jl`. The second part focuses on using Bernstein's Theorem for bounding the number of solutions to polynomial systems. References are given from the book

[CLO2] D. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry.* 2th edition. Springer. 2005.

Exercise 1 ([CLO2] §7.4, Ex 6)**.** Consider the Newton polytope of the polynomial

$$f(x, y) = a\,x^3y^2 + b\,x + c\,y^2 + d,$$

where $a, b, c, d \in \mathbb{C} \setminus \{0\}$. Compute the volume (area) by applying Proposition (4.6) in [CLO2] §7.4, that says that the volume of an n-dimensional lattice polytope P is given by

$$\mathrm{Vol}_n(P) = \frac{1}{n} \sum_{\mathcal{F} \text{ facet}} a_{\mathcal{F}} \,\mathrm{Vol}'_{n-1}(\mathcal{F}).$$

Check the answer by also computing the area via the usual formula for trapezoids.

Exercise 2. Find the mixed volume for the following pairs (P_1, P_2) of lattice polytopes in $\mathbb{R}^2$ in two different ways. First, using the formula as the coefficient of $\lambda_1\lambda_2$ in the homogeneous polynomial $V_2(\lambda_1 P_1 + \lambda_2 P_2)$, and then using $\mathrm{MV}_2(P_1, P_2) = \mathrm{Vol}_2(P_1 + P_2) - \mathrm{Vol}_2(P_1) - \mathrm{Vol}_2(P_2)$.

(a) $P_1 = P_2 = \mathrm{Conv}(\{(0, 0), (0, 1), (1, 0)\})$.

(b) $P_1 = \mathrm{Conv}(\{(0, 0), (1, 1), (1, 0), (2, 1)\})$, $P_2 = \mathrm{Conv}(\{(0, 0), (0, 1), (1, 1)\})$.

Exercise 3 ([CLO2] §7.4, Ex 7(b))**.** Let P be an n-dimensional lattice polytope. Show that:

(a) $\lambda_1 P + \cdots + \lambda_n P = (\lambda_1 + \cdots + \lambda_n)P$.

(b) $\mathrm{MV}(P, \ldots, P) = n!\,\mathrm{Vol}_n(P)$. (Compare with your answer to Exercise 2(a).)
Hint: Use (a) and find the coefficient of $\lambda_1\lambda_2 \cdots \lambda_n$ in $(\lambda_1 + \cdots + \lambda_n)^n$.

(c) If $n = 1$, show that $\mathrm{MV}(P)$ is simply the length of P.

Exercise 4. Show that the mixed volume is *monotonic* in the sense that if P_1, P_1', P_2 are lattice polytopes in $\mathbb{R}^2$ such that $P_1 \subseteq P_1'$, then

$$\mathrm{MV}(P_1, P_2) \leq \mathrm{MV}(P_1', P_2).$$

You can provide your own proof, or follow these steps:

- Show that for all $\lambda_1, \lambda_2 \geq 0$, it holds $\lambda_1 P_1 + \lambda_2 P_2 \subseteq \lambda_1 P_1' + \lambda_2 P_2$.
- Show that in $\mathrm{Vol}_2(\lambda_1 P_1 + \lambda_2 P_2)$, the coefficient of λ_1^2 is $\mathrm{Vol}_2(P_1)$ and of λ_2^2 is $\mathrm{Vol}_2(P_2)$.
- Use the previous two results to show the desired result.

Exercise 5 (From the exam in January 2023)**.** This exercise should be solved by hand.

(a) Show that for any two segments $\overline{p_1q_1}$, $\overline{p_2q_2} \subseteq \mathbb{R}^2$ (so $p_1 \neq q_1$, $p_2 \neq q_2$) it holds

$$\mathrm{MV}(\overline{p_1q_1}, \overline{p_2q_2}) = \mathrm{Vol}_2(\overline{p_1q_1} + \overline{p_2q_2}).$$

For what configurations of the points p_1, q_1, p_2, q_2 does it hold that $\mathrm{MV}(\overline{p_1q_1}, \overline{p_2q_2}) = 0$? (Recall that $\overline{p_1q_1} + \overline{p_2q_2}$ is the Minkowski sum of the two segments).

(b) Consider the following points in $\mathbb{R}^2$

$$p_1 = (0, 1), \quad q_1 = (0, 3), \qquad p_2 = (0, 0), \quad q_2 = (2, 2)$$

and the segments $\overline{p_1q_1}, \overline{p_2q_2}$. Draw by hand the polytope $\overline{p_1q_1} + \overline{p_2q_2}$. Explain your approach. What is the volume of $\overline{p_1q_1} + \overline{p_2q_2}$?

(c) Consider the set of polynomial systems $f_1 = 0, f_2 = 0$ in $\mathbb{C}[x, y]$ such that $\mathrm{NP}(f_1) = \overline{p_1q_1}$ and $\mathrm{NP}(f_2) = \overline{p_2q_2}$, where p_1, q_1, p_2, q_2 are as in (ii).
- What are the monomials of f_1? And of f_2?
- Use (i) and (ii) to deduce the generic number of roots in $(\mathbb{C}^*)^2$ for systems in this set.

Exercise 6 ([CLO2] §7.4, Ex 10)**.** Let $P, Q, P_1, \ldots, P_r$ be polytopes in $\mathbb{R}^n$.

(a) If $\lambda > 0$ and $p_0 \in P$, show that $(1-\lambda)p_0 + \mathrm{Aff}(\lambda P + Q) = \mathrm{Aff}(P + Q)$.
Hint: $(1-\lambda)p_0 + \lambda p + q = \lambda(p+q) - \lambda(p_0+q) + p_0 + q$.

(b) Conclude that $\dim(\lambda P + Q) = \dim(P + Q)$.

(c) Prove that $\dim(\lambda_1 P_1 + \cdots + \lambda_r P_r)$ is independent of the λ_i, provided they are all positive.

Exercise 7. Open the accompanying Jupyter notebooks and learn how to compute volume and mixed volume of polytopes with `Oscar` and `HC.jl`. Check your answer from Exercise 1 and 2.

Exercise 8. Consider the following system with variables x, y and coefficients $a = (a_1, \ldots, a_6)$:

$$\begin{aligned} 0 &= a_1\, x^2 y + a_2\, x + a_3\, y \\ 0 &= a_4\, x y^2 + a_5\, x + a_6\, y. \end{aligned}$$

(a) What is the Bézout bound for the number of solutions in $\mathbb{C}^2$? What are the BKK bounds for $(\mathbb{C}^*)^2$ and $\mathbb{C}^2$, respectively? (Use `Julia` for the mixed volume computations.)

(b) Use `HC.jl` to solve the system for a random choice of $a \in \mathbb{R}^6$, using the following optional inputs to the `solve` command: (i) `start_system=:total_degree`;
(ii) `start_system=:polyhedral`; (iii) `start_system=:polyhedral, only_non_zero=true`.

(c) Think a bit about what the advantages/disadvantages of the respective options could be.

Exercise 9. Let $f_1 = x_1^4 x_2 + x_2^2 - 2x_2 + 1$ and $f_2 = -x_1^4 x_2 - x_2^2 + 3x_2 + 5$. Solve both $f_1 = f_2 = 0$ and $f_1 = f_1 + f_2 = 0$ with `HC.jl`, using the option `start_system=:polyhedral`. How many paths are traced in each case? Compare with Exercise 4.

Exercise 10 (Adapted from the exam in January 2022)**.** Consider the following system of polynomial equations:

$$0 = z + y^2 + x\, z^2 - x\, y\, z^2, \qquad 0 = x\, z - y + 1, \qquad 0 = x^2 z - y + 2.$$

(a) Compute the mixed volume of the Newton polytopes.

(b) Prove computationally that the system has less solutions in $(\mathbb{C}^*)^3$ than the answer of (a). Is this contradicting Bernstein's theorem?

(c) Find a new polynomial system with the same support as above (involving the same monomials in each equation, but with different coefficients), such that the BKK bound is attained.

Exercise 11 (Euclidean distance degree)**.** Recall from Exercise 6, Week 5 that for a smooth curve $\mathbf{V}(f) \subseteq \mathbb{R}^2$ and a point $u \in \mathbb{R}^2 \setminus \mathbf{V}(f)$, the critical points to the optimization problem $\min_{x \in \mathbf{V}(f)} \|x - u\|^2$ are (x_1, x_2) for all solutions (x_1, x_2, λ) of the system

$$f(x) = \nabla f(x) - \lambda(x - u) = 0. \tag{1}$$

The number of complex solutions for generic $u \in \mathbb{R}^2$ is called the *Euclidean distance (ED) degree* of $\mathbf{V}(f)$. This notion was introduced in a 2016 paper by Draisma et al., and plays a big role in many parts of applied algebra – for instance in the algebraic study of computer vision.

Let $f = x_1^2 x_2^2 - 3x_1^2 - 3x_2^2 + 5$. We saw in Week 5 that the system (1) then becomes

$$x_1^2 x_2^2 - 3x_1^2 - 3x_2^2 + 5 = 2x_1 x_2^2 - 6x_1 - \lambda(x_1 - u_1) = 2x_1^2 x_2 - 6x_2 - \lambda(x_2 - u_2) = 0.$$

(a) Use Bézout's and Bernstein's theorems to give upper bounds on the ED degree of $\mathbf{V}(f)$.

(b) Compute the ED degree by computing a parametric Gröbner basis over the field $\mathbb{Q}(u_1, u_2)$.

Exercise 12 (Bézout's theorem)**.** Let $d_1, \dots, d_n \in \mathbb{N}$ and consider the following system of n equations in n variables $x = (x_1, \dots, x_n)$ with parameters $c_{i,\alpha}$:

$$f_i := \sum_{\substack{\alpha \in \mathbb{N}^n \\ \alpha_1 + \cdots + \alpha_n \leq d_i}} c_{i,\alpha} x^\alpha = 0 \quad \text{for } 1 \leq i \leq n.$$

Use the $\mathbb{C}^n$ version of Bernstein's theorem to prove that the number of complex solutions is $d_1 \cdots d_n$ for generic coefficients $c_{i,\alpha}$. *Hint:* Use that the standard n-simplex

$$\Delta_n := \mathrm{Conv}(\{(0, \dots, 0), (1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}) \subseteq \mathbb{R}^n$$

has volume $\frac{1}{n!}$ (see [CLO2] §7.4 Ex 3 for a strategy for proving this).

Exercise 13 ([CLO2], §7.5, Ex 5)**.** Use Bernstein's theorem to deduce a statement about the number of solutions in $(\mathbb{C}^*)^n$ of a generic system of polynomial equations $f_1 = \cdots = f_n = 0$ when the Newton polytopes of the f_i are all equal. (This was the case considered by Khovanskii.)

Exercise 14 (From the exam in January 2022)**.** This exercise consists of proving the linearity of the mixed volume in a simpler setting, and then applying it to polynomials of a specific form.

(a) Let P_1, P_1', P_2 be lattice polytopes in $\mathbb{R}^2$. Show that

$$\mathrm{MV}_2(\lambda P_1 + \lambda' P_1', P_2) = \lambda\, \mathrm{MV}_2(P_1, P_2) + \lambda'\, \mathrm{MV}_2(P_1', P_2)$$

for all $\lambda, \lambda' \geq 0$. *Hint:* Consider the polynomial representing $\mathrm{Vol}_2(\lambda P_1 + \lambda' P_1' + \lambda_2 P_2)$ and consider the coefficients of $\lambda\lambda_2$ and $\lambda'\lambda_2$.

(b) Consider the lattice polytopes in $\mathbb{R}^2$ given as $C = \mathrm{Conv}(\{(0,0),(1,0),(0,1),(1,1)\})$ and $E = \mathrm{Conv}(\{(0,0),(1,0)\})$. Show that for any 2-dimensional lattice polytope T in $\mathbb{R}^2$ that is invariant after permuting x and y, it holds

$$\mathrm{MV}_2(C, T) = 2\, \mathrm{MV}_2(E, T).$$

(c) For a 2-dimensional lattice polytope T in $\mathbb{R}^2$ that is invariant after permuting x and y, show that $\mathrm{MV}_2(C, T) = 2\alpha$, where $\alpha = \max\{b \mid (a,b) \in T\} - \min\{b \mid (a,b) \in T\}$.

Hint: You may want to consider Bernstein's theorem.

Additional exercises. The following exercises are used in the theory notes.

Exercise 15 ([CLO2] §7.4, Ex 12)**.** Using the notation in [CLO2], for a vector $\nu \in \mathbb{R}^n$ and a polytope P, we let

$$a_P(\nu) = -\min_{p \in P}(\nu \cdot p), \qquad P_\nu := P \cap \{\nu \cdot x + a_P(\nu) = 0\}.$$

Then P_ν is a face of P. Consider now polytopes $P, Q, P_1, \dots, P_r$ in $\mathbb{R}^n$ and $\lambda, \lambda_1, \dots, \lambda_r \in \mathbb{R}_{>0}$.

(a) Show that $(\lambda P)_\nu = \lambda(P_\nu)$ and $a_{\lambda P}(\nu) = \lambda\, a_P(\nu)$.

(b) Show that $(P + Q)_\nu = P_\nu + Q_\nu$ and $a_{P+Q}(\nu) = a_P(\nu) + a_Q(\nu)$.

(c) Conclude that $(\lambda_1 P_1 + \cdots + \lambda_r P_r)_\nu = \lambda_1 (P_1)_\nu + \cdots + \lambda_r (P_r)_\nu$ and $a_{\lambda_1 P_1 + \cdots + \lambda_r P_r}(\nu) = \lambda_1\, a_{P_1}(\nu) + \cdots + \lambda_r\, a_{P_r}(\nu)$.

(d) Prove Proposition (4.3) in [CLO2] §7.4, stating that every face P' of $P_1 + \cdots + P_r$ can be expressed as $P' = P_1' + \cdots + P_r'$, where P_i' a face of P_i for each $1 \leq i \leq r$.

Exercise 16 ([CLO2] §7.4, Ex 14)**.** Let $P_1, \dots, P_r$ be polytopes in $\mathbb{R}^n$ such that $P_1 + \cdots + P_r$ has dimension n. Using Exercises 6 and 15, show that for any positive reals $\lambda_1, \dots, \lambda_r$, the polytopes $\lambda_1 P_1 + \cdots + \lambda_r P_r$ all have the same inward pointing facet normals.