This assignment is due **August 1, 2025**. You may work in teams of up to four people. If you work as a team, choose a team leader to sign up your team on the homework 2 groups page of Bruin Learn. Please submit one html or pdf file. Upload your solutions to Bruin Learn. Show all your work. You may use the modules `numpy`, `pandas`, `scipy`, and `matplotlib`.

1. The matrix

$$P = \begin{pmatrix} 0.84 & 0.13 & 0.03 & 0 \\ 0.10 & 0.77 & 0.09 & 0.04 \\ 0.10 & 0.20 & 0.65 & 0.05 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

denotes the one-year transition probabilities for firms with investment grade, speculative, junk, and defaulted unsecured debt. The rows represent the current states, and the columns represent the states after one year. For example, the probability that a firm with investment grade debt transitions to a firm whose debt is considered junk after one year is 3%, while the probability that a firm whose debt is considered junk transitions to a firm with speculative debt is 20%. Notice that we are supposing firms do not transition out of default.

(a) Compute P^2. Interpret your results. Are there any assumptions attached to your conclusion?

(b) Find the eigenvalues and eigenvectors of P.

(c) Use (b) to diagonalize P.

(d) Analytically compute $\lim_{n\to\infty} P^n$ using (c).

x_i	1	2	3	4	5
y_i	2.13	-0.12	-1.19	1.01	-4.59

2. Define an inner product on the vector space of functions from $\{1, 2, 3, 4, 5\}$ to $\mathbb{R}$ as

$$\langle f, g\rangle = \sum_{i=1}^{5} f(x_i)g(x_i).$$

(a) Orthogonalize the first two elements of the basis $(1, x, x^2, x^3, x^4)$ using the inner product defined above.

(b) Project the function f with the xy-coordinates in the table onto the two orthogonal basis elements you found in (a), and then rewrite your solution in the form $a + bx$. Due to the Best Approximation theorem, the function $g(x) = \alpha + \beta x$ that minimizes

$$\|f - g\|^2 = \sum_{i=1}^{5} [f(x_i) - g(x_i)]^2 = \sum_{i=1}^{5} [y_i - g(x_i)]^2$$

is your result. Note: This only holds on the domain $\{1, 2, 3, 4, 5\}$.

(c) Define

$$\mathcal{L}_2(\alpha, \beta) = \sum_{i=1}^{5} (y_i - \alpha - \beta x_i)^2.$$

Minimize $\mathcal{L}_2$ with respect to α and β to analytically verify (b). Remember to check whether your solution is a minimum or a maximum.

(d) Your results in (b) and (c) minimized the square error, which is highly sensitive to outliers. Another approach is to minimize the absolute error. Use `minimize` in `scipy` to find α and β that minimize

$$\mathcal{L}_1(\alpha, \beta) = \sum_{i=1}^{5} |y_i - \alpha - \beta x_i|.$$

Because $\mathcal{L}_1$ is hard to optimize, `minimize` may fail to converge. As long as your coefficients are reasonable, your solution is acceptable.

(e) Use `scatter` to graph the points in the table. Also, use `plot` to graph the lines corresponding to the coefficients you found in (b) and (d) on the same figure. Make sure to add a legend so that it is easy to interpret your results.

3. A convertible note is a debt instrument that gives the owner the right, but not the obligation, to convert her debt into a predetermined number of shares of common stock. For this problem, we will suppose that the owner has the right to convert her note to one share of common stock.

Supposing the underlying stock does not pay dividends, the owner may only convert her shares to common stock at maturity, and the time until maturity is a multiple of 0.5, then—under the Black-Scholes assumptions—the value of the convertible note is

$$P = S\Phi(d_1) - 100e^{-rT}\Phi(d_2) + 100e^{-rT} + \sum_{k=1}^{2T} \frac{c}{2} e^{-rk/2},$$

where S is the value of one share of common stock, c is the coupon rate paid semiannually, r is the continuously compounded discount rate, T is the time until maturity, σ is the volatility of returns of the underlying common stock,

$$d_1 = \frac{\ln\left(\frac{S}{100}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \qquad \text{and} \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

The function Φ is the standard normal cdf, which can be accessed in Python via `norm.cdf` within `scipy.stats`.

(a) Write a Python function that gives the price of the convertible note as a function of c, r, S, T, and σ. Evaluate it when $c = 5\%$, $r = 8\%$, $S = 65$, $T = 5$, and $\sigma = 35\%$. Make sure that r and σ are written in decimal form when using the pricing formula. The formula above assumes c is *not* converted to a decimal.

(b) Use your function in part (a) to compute the numerical partial derivatives of the pricing function with respect to S and r. As before, evaluate it when $c = 5\%$, $r = 8\%$, $S = 65$, $T = 5$, and $\sigma = 35\%$.

(c) Unlike options where we normally assume there is no credit risk, the discount rate r for a convertible note reflects the credit of the corresponding firm. Credit risk is closely related to the stock price. When the stock price goes up, r goes down because it is easier for the firm to raise capital. In contrast, when the stock price goes down, r goes up. After careful analysis, you discover that if the stock goes from S_0 to S_1, the discount rate goes from r_0 to

$$r_1 \approx 4\% + \left(\frac{S_0}{S_1}\right)^{0.25} (r_0 - 4\%).$$

Write a Python function that returns r_1 given S_0, S_1 and r_0. Then compute the numerical partial derivative of your function with respect to S_1. Suppose $S_0 = S_1 = 65$ and $r_0 = 8\%$.

(d) In part (b), you computed the partial derivative of the convertible note with respect to S. However, as mentioned previously, when the equity changes we also expect the discount rate to change. Use your results from (b) and (c) to find the total derivative of the convertible note price with respect to the value of the underlying equity. In this case, the total derivative is

$$\frac{dP}{dS} = \frac{\partial P}{\partial S} + \frac{\partial P}{\partial r}\frac{\partial r}{\partial S}.$$

Note: We are ignoring the effect of a change in σ due to a change in the stock price.

(e) Convertible arbitrage involves purchasing a convertible note, and then taking a short position in the underlying equity. Let us say we own 100 convertible notes, how many shares of the stock should we short to hedge out the first order exposure we found in (d)?

(f) Even with the first order equity exposure removed from the position, we still have second order effects, i.e. convexity of the convertible note with respect to the underlying stock. Create a table of the loss or gain for the total position if $S = 35, 45, 55, 65, 75, 85$, or 95. Assume that r changes as described in (c), and the other variables are as specified previously.

(g) The convexity effect observed in (f) dissipates over time. Create a new table with the same underlying stock values as (f), but assume $T = 2$ when calculating the total loss or gain.