Coursework 3: Graph Algorithms and Complexity Theory

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Tutorial Session: Thursday 1pm

Explanation: A *hamiltonian path* in an undirected graph is a path that contains all vertices of the graph (without repetition). Similarly, a *hamiltonian cycle* is a cycle that contains all vertices of the graph. A graph with hamiltonian path is *traceable*, and a graph with hamiltonian cycle is *hamiltonian*.

1. Specify decision problems HP and HC dealing with hamiltonian paths and hamiltonian cycles in undirected graphs.

   HP:
   Input: An undirected graph $G = (V, E)$.
   Question: Does G contain a Hamiltonian Path?

   HC:
   Input: An undirected graph $G = (V, E)$.
   Question: Does G contain a Hamiltonian Circuit?

2. Show HP $\leq_m^p$ HC by completing the following tasks:

   (a) Construct a polynomial transformation f from HP to HC.
   Let $G = (V, E')$ be an input for a Hamiltonian Path. Let $f(G) = G' = (V', E)$, where $V' = V \cup \{v\} v \notin V$ and $E' = E \cup \{\{v, w\} | w \in v\}$. This is in polynomial time as adding a vertex takes constant time and adding edges $\{v, w\}$ for all $w \in V$ takes $|V|$ amount of time. In total this operation takes $|V| + c$ amount of time, which is polynomial.

   (b) Show for all graphs G that $G \in Y_{HP} \Rightarrow f(G) \in Y_{HC}$.
   Proof: Let $G \in HP$. Given a path $p = (v_1, \ldots, v_n)$ connect the vertex v_n to v_1 through v', making a path $p' = (v_1, \ldots, v_n, v', v_1)$, which is a cycle. The vertices v_n and v_1 will always be able to traverse to v' as v' is connected to all vertices. $f(G)$ is a HC as it starts and ends at the vertex v_1.

(c) Show for all graphs G that $f(G) \in Y_{HC} \Rightarrow G \in Y_{HP}$.
Let $f(G) \in HC$. Given a path $p' = (v_1, \ldots, v_{n-1}, v_n, v_1)$ removing v_n, v_1 gives $p'' = (v_1, \ldots, v_{n-1})$. The result has to be a path as v_n will always be the bridge between the last node of a hamiltonian path and the first.

3. Show HC $\leq_m^p$ HP by completing the following tasks:

   (a) Construct a polynomial transformation f from HC to HP.
   Let $G = (V, E)$ be an input for a Hamiltonian Cycle. Let $f(G) = G' = (V', E')'$, where $v \in V$ is a vertex in G, $V' = V \cup \{v', s, t\}$ where $v', s, t \notin V$ and $E' = E \cup \{\{v', w\} | \{v, w\} \in E\} \cup \{\{t, v'\}, \{s, v\}, \{v, v'\}\}$. This is in polynomial time as adding 3 vertices takes constant time and adding edges $\{\{v', w\} | \{v, w\} \in E\} \cup \{\{t, v'\}, \{s, v\}, \{v, v'\}\}$ $|V| + c$ amount of time, which is polynomial.

   (b) Show for all graphs G that $G \in Y_{HC} \Rightarrow g(G) \in Y_{HP}$.
   Given a cycle $c = (v_1, \ldots, v_n, v_1)$, it is known that there is a path where $(v_1, u, \ldots, y, v_1)$ which traverses all the vertices and end up at the same vertex v_1. However, if you create a copy v' of v and then connect s to v and t to v', as s, t being the endpoints this must translate to a hamiltonian path. making cycle $c' = (s, v, u, \ldots, y', v', t)$, which is a hamiltonian path.

   (c) Show for all graphs G that $g(G) \in Y_{HP} \Rightarrow G \in Y_{HC}$.
   Given a path in $f(G)$, it is guaranteed that s, t are the endpoints. Therefore, ignoring the first and last edge of the path will give you the path $c' = (v, u, \ldots, y', v)$ which has to be a cycle.