Quaternions

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9th October 2019

Problem

An artist has given you two keyframe rotation matrices of an object. At time step 0, the artist wants the object rotated by 45° CCW around vector $(2, 1, 4)$ and at time step 100, the object should be rotated by 123° CW around vector $(0, 1, 1)$. Use quaternions to determine the axis and angle of rotation required to achieve this rotation in 100 steps. Show all work.

All calculations are rounded to 5 decimal places.

Solution

A vector $\vec{v} = (2, 1, 4)$ is formed using the information given above. Also, the angle 45° CCW becomes $\theta_0 = \frac{45}{2} = 22.5$.

$$q_0 = (cos(\theta_0), sin(\theta_0) \times |\vec{v}|))$$

$$||\vec{v}|| = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{21} = 4.58258$$

$$|\vec{v}| = \frac{\vec{v}}{||\vec{v}||} = \frac{(2, 1, 4)}{4.58258} = (0.43644, 0.21822, 0.87287)$$

$$cos(\theta_0) = 0.92388$$

$$sin(\theta_0) = 0.38268$$

$$q_0 = (0.92388, 0.38268 \times 0.43644, 0.38268 \times 0.21822, 0.38268 \times 0.87287)$$

$$q_0 = 0.92388 + 0.16702i + 0.08351j + 0.33403k$$

A vector $\vec{w} = (0,1,1)$ is formed using the information given above. Also, the angle 123° CW becomes $\theta_{100} = -\frac{123}{2} = -61.5$.

$$q_{100} = (cos(\theta_{100}), sin(\theta_{100}) \times |\vec{w}|))$$

$$||w|| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2} = 1.41421$$

$$|w| = \frac{|w|}{||w||} = \frac{(0,1,1)}{1.41421} = (0, 0.70711, 0.70711)$$

$$cos(\theta_{100}) = 0.47716$$

$$sin(\theta_{100}) = -0.87882$$

$$q_{100} = (0.47716, 0, -0.87882 \times 0.70711, -0.87882 \times 0.70711)$$

$$q_{100} = 0.47716 + 0i - 0.62142j - 0.62142k$$

Next, to find the quaternion required q_δ, the quaternion q_Δ needs to be calculated by multiplying quaternions q_{100} and the inverse of q_0. Since q_0 is a unit quaternion, its magnitude is 1, so only taking the conjugate q_0^* is required, as $q_0^{-1} = \frac{q_0^*}{||q_0||} = \frac{q_0^*}{1} = q_0^*$.

$$q_0^* = 0.92388 - 0.16702i - 0.08351j - 0.33403k$$

$$q_{100} = 0.47716 + 0i - 0.62142j - 0.62142k$$

$$q_\Delta = q_0^* \times q_{100}$$

$$\begin{aligned} q_\Delta = {} & 0.44084 + 0i - 0.57412j - 0.57412k \\ & - 0.07970i - 0i^2 + 0.10379ij + 0.10379ik \\ & - 0.03985j - 0ji + 0.05189j^2 + 0.05189jk \\ & - 0.15939k - 0ki + 0.20757kj + 0.20757k^2 \end{aligned}$$

$$\begin{aligned} &= 0.44084 - 0.57412j - 0.57412k - 0.07970i \\ &+ 0.10379k - 0.10379j - 0.03985j - 0.05189 \\ &+ 0.05189i - 0.15939k - 0.20757i - 0.20757 \end{aligned}$$

$$q_\Delta = 0.18138 - 0.23538i - 0.71776j - 0.62972k$$

Now that q_Δ is found, θ_Δ and $\vec{v}_\Delta$ can be found. Then the quaternion q_δ can be formed by using $\frac{\theta_\Delta}{100}$ and $\vec{v}_\Delta$.

$$(q_\delta)^{100} = q_\Delta$$

$$\theta_\delta = \frac{\theta_\Delta}{100}$$

$$\theta_\Delta = arccos(0.18138) = 79.54985$$

$$\theta_\delta = \frac{79.54985}{100}$$

$$\theta_\delta = 0.79550$$

$$sin(79.54985) = 0.98341$$

$$\vec{v}_\delta = (-\frac{0.23538}{0.98341}, -\frac{0.71776}{0.98341}, -\frac{0.62972}{0.98341})$$
$$\vec{v}_\delta = (-0.23935, -0.72987, -0.64034)$$

Finally, the quaternion q_δ with the required angle and axis can be formed:

$cos(0.79550) = 0.99990$

$sin(0.79550) = 0.01388$

$$q_\delta = (0.99990, 0.01388 \times -0.23935, 0.01388 \times -0.72987, 0.01388 \times -0.64034)$$

$$q_\delta = 0.99990 - 0.00332i - 0.01013j - 0.00889k$$