

# Quaternions

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## Problem

An artist has given you two keyframe rotation matrices of an object. At time step 0, the artist wants the object rotated by  $45^\circ$  CCW around vector  $(2, 1, 4)$  and at time step 100, the object should be rotated by  $123^\circ$  CW around vector  $(0, 1, 1)$ . Use quaternions to determine the axis and angle of rotation required to achieve this rotation in 100 steps. Show all work.

*All calculations are rounded to 5 decimal places.*

## Solution

A vector  $\vec{v} = (2, 1, 4)$  is formed using the information given above. Also, the angle  $45^\circ$  CCW becomes  $\theta_0 = \frac{45}{2} = 22.5$ .

$$q_0 = (\cos(\theta_0), \sin(\theta_0) \times |\vec{v}|)$$

$$||\vec{v}|| = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{21} = 4.58258$$

$$|\vec{v}| = \frac{\vec{v}}{||\vec{v}||} = \frac{(2, 1, 4)}{4.58258} = (0.43644, 0.21822, 0.87287)$$

$$\cos(\theta_0) = 0.92388$$

$$\sin(\theta_0) = 0.38268$$

$$q_0 = (0.92388, 0.38268 \times 0.43644, 0.38268 \times 0.21822, 0.38268 \times 0.87287)$$

$$q_0 = 0.92388 + 0.16702i + 0.08351j + 0.33403k$$

A vector  $\vec{w} = (0, 1, 1)$  is formed using the information given above. Also, the angle  $123^\circ$  CW becomes  $\theta_{100} = -\frac{123}{2} = -61.5$ .

$$q_{100} = (\cos(\theta_{100}), \sin(\theta_{100}) \times |\vec{w}|)$$

$$||\vec{w}|| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2} = 1.41421$$

$$|\vec{w}| = \frac{\vec{w}}{||\vec{w}||} = \frac{(0, 1, 1)}{1.41421} = (0, 0.70711, 0.70711)$$

$$\cos(\theta_{100}) = 0.47716$$

$$\sin(\theta_{100}) = -0.87882$$

$$q_{100} = (0.47716, 0, -0.87882 \times 0.70711, -0.87882 \times 0.70711)$$

$$q_{100} = 0.47716 + 0i - 0.62142j - 0.62142k$$

Next, to find the quaternion required  $q_\delta$ , the quaternion  $q_\Delta$  needs to be calculated by multiplying quaternions  $q_{100}$  and the inverse of  $q_0$ . This is because  $q_{100}pq_{100}^{-1} = q_\Delta(q_0pq_0^{-1})q_\Delta^{-1}$ . This can be simplified to  $q_\Delta q_0 = q_{100}$ , therefore  $q_\Delta = q_{100}q^{-1}$ . Since  $q_0$  is a unit quaternion, its magnitude is 1, so only taking the conjugate  $q_0^*$  is required, as  $q_0^{-1} = \frac{q_0^*}{||q_0||} = \frac{q_0^*}{1} = q_0^*$ .

$$q_0^* = 0.92388 - 0.16702i - 0.08351j - 0.33403k$$

$$q_{100} = 0.47716 + 0i - 0.62142j - 0.62142k$$

$$q_\Delta = q_0^* \times q_{100}$$

$$\begin{aligned} q_\Delta &= 0.44084 + 0i - 0.57412j - 0.57412k \\ &\quad - 0.07970i - 0i^2 + 0.10379ij + 0.10379ik \\ &\quad - 0.03985j - 0ji + 0.05189j^2 + 0.05189jk \\ &\quad - 0.15939k - 0ki + 0.20757kj + 0.20757k^2 \\ &= 0.44084 - 0.57412j - 0.57412k - 0.07970i \\ &\quad + 0.10379k - 0.10379j - 0.03985j - 0.05189 \\ &\quad + 0.05189i - 0.15939k - 0.20757i - 0.20757 \\ q_\Delta &= 0.18138 - 0.23538i - 0.71776j - 0.62972k \end{aligned}$$

Now that  $q_\Delta$  is found,  $\theta_\Delta$  and  $\vec{v}_\Delta$  can be found. Then the quaternion  $q_\delta$  can be formed by using  $\frac{\theta_\Delta}{100}$  and  $\vec{v}_\Delta$ .

$$(q_\delta)^{100} = q_\Delta$$

$$\theta_\delta = \frac{\theta_\Delta}{100}$$

$$\theta_\Delta = \arccos(0.18138) = 79.54985$$

$$\theta_\delta = \frac{79.54985}{100}$$

$$\theta_\delta = 0.79550$$

$$\sin(79.54985) = 0.98341$$

$$\begin{aligned} \vec{v}_\delta &= \left( -\frac{0.23538}{0.98341}, -\frac{0.71776}{0.98341}, -\frac{0.62972}{0.98341} \right) \\ \vec{v}_\delta &= (-0.23935, -0.72987, -0.64034) \end{aligned}$$

Finally, the quaternion  $q_\delta$  with the required angle and axis can be formed:

$$\cos(0.79550) = 0.99990$$

$$\sin(0.79550) = 0.01388$$

$$q_\delta = (0.99990, 0.01388 \times -0.23935, 0.01388 \times -0.72987, 0.01388 \times -0.64034)$$

$$q_\delta = 0.99990 - 0.00332i - 0.01013j - 0.00889k$$