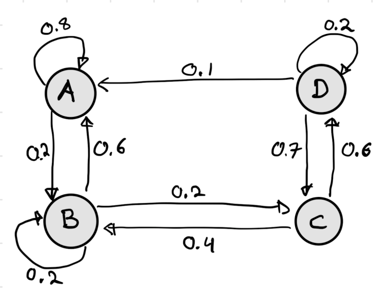
* A Markov Chain is a sequence of events where the probabilities of the future ONLY depend on the present

* Transition matrix P, where P_{xy} shows the probability of moving from state x to state y.

* For a MC to be useful in long term modelling:
  - irreducible: if it is possible to eventually get from any state to any other state in the system
  - aperiodic: if it does not return to itself only at fixed, regular intervals (cycles)

* stationary distribution (π): probability distribution that remains unchanged by the transition matrix ($\pi P = P$)
  - if a MC is aperiodic and irreducible $\Rightarrow$ it's distribution will eventually settle down and converge to this stationary distribution regarless of where it started

* reversibility: a chain is reversible if the probability of being in state X and moving to y is the same as being in y and moving to X
  - if reversible $\Rightarrow$ is a stationary distribution



$$\begin{array}{c|c} & \overleftrightarrow{\quad Y \quad} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ X \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.6 & 0.2 & 0.2 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0.1 & 0 & 0.7 & 0.2 \end{pmatrix} \end{array} = P$$

$$0.645 \cdot 0.2 = 0.204 \cdot 0.6$$
$$0.129 \overset{?}{=} 0{,}1224$$

yes, it's irreducible

A 4 1 2 3 5 6 → period is 1

B " " → " "

⋮

D " " → " "

⇒ chain is aperiodic with period 1 in every state

$10 \cdot X = 0.65$

$X = 0.65/10$

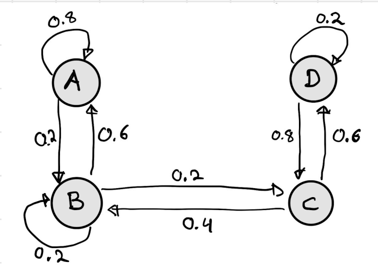
* find stationary distribution $P\pi = P$

$$\begin{bmatrix} \pi(a) \\ \pi(b) \\ \pi(c) \\ \pi(d) \end{bmatrix}^T \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.6 & 0.2 & 0.2 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0.1 & 0 & 0.7 & 0.2 \end{pmatrix} = \begin{bmatrix} 0.8\pi(a) + 0.6\pi(b) + 0.1\pi(d) \\ 0.2\pi(a) + 0.2\pi(b) + 0.4\pi(c) \\ 0.2\pi(b) + 0.7\pi(d) \\ 0.6\pi(c) + 0.2\pi(d) \end{bmatrix}^T$$

$(1 \times 4) \cdot (4 \times 4) = (1 \times 4)$

$$\begin{bmatrix} 0.8\pi(a) + 0.6\pi(b) + 0.1\pi(d) \\ 0.2\pi(a) + 0.2\pi(b) + 0.4\pi(c) \\ 0.2\pi(b) + 0.7\pi(d) \\ 0.6\pi(c) + 0.2\pi(d) \end{bmatrix}^T = \begin{bmatrix} \pi(a) \\ \pi(b) \\ \pi(c) \\ \pi(d) \end{bmatrix}^T$$

$$\begin{bmatrix} -0{,}2\pi(a) + 0.6\pi(b) + 0.1\pi(d) \\ 0{,}2\pi(a) - 0.8\pi(b) + 0.4\pi(c) \\ 0{,}2\pi(b) - \pi(c) + 0.7\pi(d) \\ 0{,}6\pi(c) - 0{,}8\pi(d) \end{bmatrix} \rightarrow \begin{pmatrix} -0.2 & 0.6 & 0 & 0.1 \\ 0.2 & -0.8 & 0.4 & 0 \\ 0 & 0.2 & -1 & 0.7 \\ 0 & 0 & 0.6 & -0.8 \end{pmatrix}$$



$$\begin{array}{c|c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.6 & 0.2 & 0.2 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0.8 & 0.2 \end{pmatrix} \end{array} = P$$

yes, irreducible

↳ means all states have the same period

A	1	2	3	4	5
C	–	2	3	4	5
D	1	2	3	4	5

→ period 1

$$\begin{pmatrix} -0.2 & 0.2 & 0 & 0 \\ 0.6 & -0.8 & 0.2 & 0 \\ 0 & 0.4 & -1 & 0.6 \\ 0 & 0 & 0.8 & -0.8 \end{pmatrix}$$

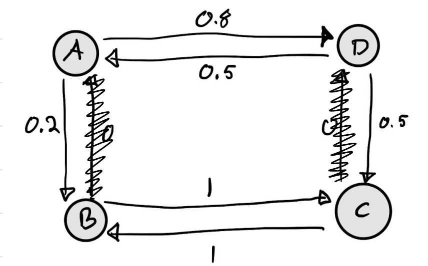
solve and get stationary distribution

$\Rightarrow [0.25 \; 0.25 \; \cdots \; 0.25]$

$$\pi(x)P(x,y) = \pi(y)P(y,x)$$

$$0.25 \cdot 0.2 = 0.25 \, 0.6$$

$\neq \Rightarrow$ not reversible



$$
\begin{array}{c|cccc}
 & A & B & C & D \\
\hline
A & 0 & 0.2 & 0 & 0.8 \\
B & 0 & 0 & 1 & 0 \\
C & 0 & 1 & 0 & 0 \\
D & 0.5 & 0 & 0.5 & 0
\end{array} = P
$$

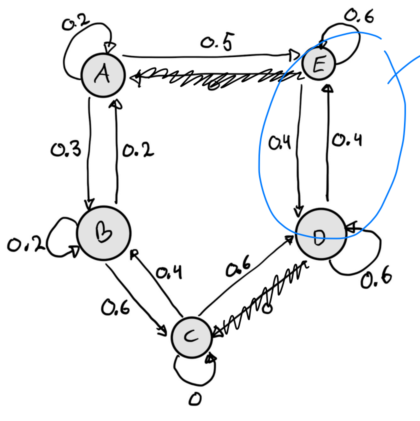
not irreducible

not reversible

$$
\left.\begin{array}{ll}
A & - \ 2 \\
B & - \ 2 \\
C & - \ 2 \\
D & - \ 2
\end{array}\right\} \text{period of } 2
$$

$\Rightarrow$ **NOT** periodic

will get a stationary distribution here



$$
\begin{array}{c|ccccc}
 & A & B & C & D & E \\
\hline
A & 0.2 & 0.3 & 0 & 0 & 0.5 \\
B & 0.2 & 0.2 & 0.6 & 0 & 0 \\
C & 0 & 0.4 & 0 & 0.6 & 0 \\
D & 0 & 0 & 0 & 0.6 & 0.4 \\
E & 0 & 0 & 0 & 0.4 & 0.6
\end{array} = P
$$

not irreducible

$$
\left.\begin{array}{llllll}
A & 1 & 2 & 3 & 4 & 5 \ldots \\
B & 1 & 2 & 3 & \ldots & \\
C & - & 2 & 3 & 4 & \ldots \\
D & 1 & 2 & & & \\
E & 1 & 2 & & &
\end{array}\right\} \text{period of}
$$

$1 \Rightarrow$ aperiodic

$$F[x] = \begin{cases} 0, & x \le 0 \\ \dfrac{e^{x^2}-1}{e-1} & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$$

bounded $[0,1]$

uniform distribution $\Rightarrow y = 1$

CDF $\to$ PDF (derivate)

$$f(x) = \left(\frac{e^{x^2}-1}{e-1}\right)' = \frac{2xe^{x^2}}{e-1}$$

$$M = f(1) = \frac{2e}{e-1}$$

$$F(x) = 20\,x e^{20-\frac{1}{x}}$$

$f(x) \le M\,g(x)$ for all x

$$f(x) = \frac{20\,(x+1)\,e^{20-\frac{1}{x}}}{x}$$

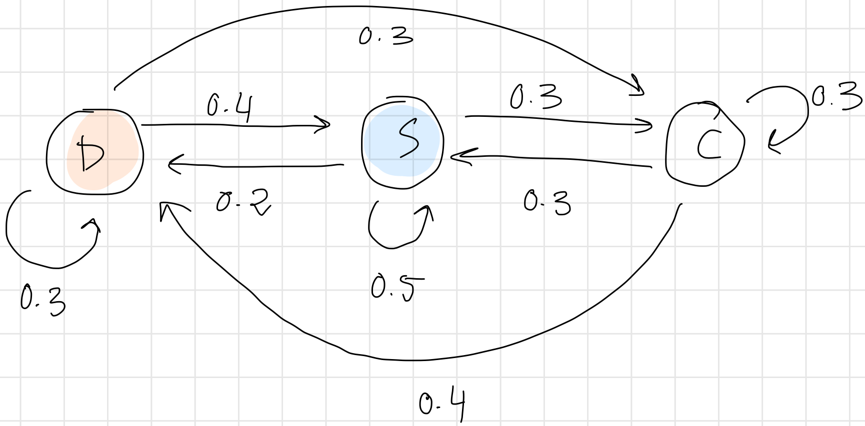
uniform $= \dfrac{1}{b-a}$

$\Rightarrow g(x) = 20$

$$\frac{f(x)}{g(x)} = \frac{\cancel{20}\,(x+1)\,e^{20-\frac{1}{x}}}{x\ \cancel{20}} = \frac{(x+1)\,e^{20-1/x}}{x}$$

$M = 21$

$$f(1/20) = \frac{\cancel{20}\cdot\left(\frac{1}{\cancel{20}}+1\right)e^{20-20}}{1/20} = \frac{1+20}{1/20} = 21\cdot 20$$



① in S, P that it will be in D after 2 steps

$$\left.\begin{array}{lll} S & S & D \quad 0.5 \cdot 0.2 = 0.1 \\ S & C & D \quad 0.3 \cdot 0.4 = 0.12 \\ S & D & D \quad 0.2 \cdot 0.3 = 0.06 \end{array}\right\} 0.28$$

② in S, P that it will be in D after 2 steps being in D for the first time

$$\left.\begin{array}{lll} S & S & D \quad 0.5 \cdot 0.2 = 0.1 \\ S & C & D \quad 0.3 \cdot 0.4 = 0.12 \end{array}\right\} 0.22$$

$$\boxed{h_i = 1 + \sum_{k=1} p_{ik} h_k}$$

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.3 & 0.4 & 0.3 \\ 2 & 0.2 & 0.5 & 0.3 \\ 3 & 0.4 & 0.3 & 0.3 \end{array}$$

we are in state 1 and want to go to state 2

$$\begin{cases} h_1 = 0 \\ h_2 = \cancel{0.2 h_1} + 0.5 h_2 + 0.3 h_3 + 1 \\ h_3 = \cancel{0.4 h_1} + 0.3 h_2 + 0.3 h_3 + 1 \end{cases}$$

$1 - 0.3$

$$\begin{cases} 1 = 0.5 h_2 - 0.3 h_3 \Rightarrow 0.8 h_2 = h_3 \\ 1 = -0.3 h_2 + 0.7 h_3 \end{cases}$$

$$1 = 0.5 h_2 - 0.3 (0.8 h_2)$$
$$1 = 0.5 h_2 - 0.24 h_2$$
$$1 = 0{,}26 h_2 \Rightarrow h_2 = 3{,}846$$

↑ answer

$$1 = -0.3 (3{,}846) + 0.7 h_3$$
$$2.15 = 0.7 h_3$$
$$\Rightarrow h_3 = 3{,}077$$