Absence of Phase Transitions and Preservation of Gibbs Property Under Renormalization

Scientific Talk

Oskar Vavtar

Supervisor: Dr. Evgeny A. Verbitskiy

Leiden University,
Mathematical Institute

June 21, 2024

Preliminaries

Consider finite alphabet $\mathcal{A}$ and write $\Omega = \mathcal{A}^{\mathbb{Z}^d}$.

Definition (Interactions and Hamiltonians)

(1) **Interaction** is a collection of maps $\Phi = (\Phi_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$, where

$$\Phi_\Lambda(\omega) \;=\; \Phi_\Lambda(\omega(x) : x \in \Lambda), \quad \omega \in \Omega.$$

We say Φ is **uniformly absolutely convergent** (UAC), $\Phi \in \mathscr{B}^1(\Omega)$, if

$$\sup_{x \in \mathbb{Z}^d} \sum_{\Lambda \ni x} \|\Phi_\Lambda\|_\infty \;<\; \infty.$$

(2) For $\Phi \in \mathscr{B}^1(\Omega)$, we consider **Hamiltonians** $\mathcal{H} = (\mathcal{H}_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$,

$$\mathcal{H}_\Lambda(\omega) \;=\; \sum_{\Delta \cap \Lambda \neq \emptyset} \Phi_\Delta(\omega), \quad \omega \in \Omega.$$

Preliminaries

Definition (Specification and Gibbs measure)

(1) For $\Phi \in \mathscr{B}^1(\Omega)$, **specification** $\gamma = (\gamma_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$, is given so that $\gamma_\Lambda(\cdot|\cdot) : \mathcal{F}_\Lambda \times \Omega_{\Lambda^c} \to (0,1)$,

$$\gamma_\Lambda(\omega_\Lambda|\xi_{\Lambda^c}) \;=\; \frac{1}{\mathcal{Z}_\Lambda^\xi} \exp(-\mathcal{H}_\Lambda(\omega_\Lambda \xi_{\Lambda^c})),$$

where $\mathcal{Z}_\Lambda^\xi$ is the normalization constant.

(2) $\mu \in \mathcal{M}_1(\Omega)$ is a **Gibbs measure** on Ω consistent with Φ ($\mu \in \mathcal{G}_\Omega(\Phi)$) if for each $\Lambda \Subset \mathbb{Z}^d$,

$$\mu(\omega_\Lambda|\omega_{\Lambda^c}) \;=\; \gamma_\Lambda^\Phi(\omega_\Lambda|\omega_{\Lambda^c}) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Preliminaries

Remark

(i) For $\Lambda \Subset \mathbb{Z}^d$ and ξ_{Λ^c} fixed, $\gamma_\Lambda(\cdot|\xi_{\Lambda^c})$ is a probability measure on $\Omega_\Lambda = \mathcal{A}^\Lambda$. This allows for construction of Gibbs measures via weak limits.

(ii) While $\mathcal{G}_\Omega(\Phi) \neq \emptyset$, we don't necessarily have that $|\mathcal{G}_\Omega(\Phi)| = 1$. How and when this happens is an important subject in statistical mechanics.

Characterization of Gibbsianity

Proposition

Let $\mu \in \mathcal{M}_1(\Omega)$. The following are equivalent:

(i) μ is Gibbs

(ii) μ has the following properties:

  (a) **uniform non-nullness**: $\forall \Lambda \Subset \mathbb{Z}^d\ \exists \alpha_\Lambda, \beta_\Lambda \in (0,1)$ s.t.

$$\alpha_\Lambda \ \leq \ \mu_\Lambda(\omega_\Lambda | \xi_{\Lambda^c}) \ \leq \ \beta_\Lambda, \quad \forall \omega, \xi \in \Omega$$

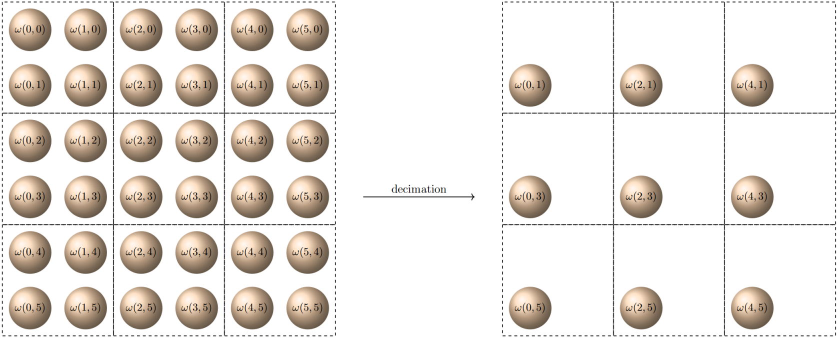
  (b) **quasilocality**: writing $\mathbb{B}_n = [-n,n]^d \cap \mathbb{Z}^d$, $\forall \Lambda \Subset \mathbb{Z}^d$,

$$\sup_{\omega} \sup_{\xi,\zeta} \left| \mu(\omega_\Lambda | \omega_{\mathbb{B}_n \setminus \Lambda} \xi_{\mathbb{B}_n^c \setminus \Lambda}) - \mu(\omega_\Lambda | \omega_{\mathbb{B}_n \setminus \Lambda} \zeta_{\mathbb{B}_n^c \setminus \Lambda}) \right| \ \to \ 0.$$

Source of all problems: Renormalization Group

We want to consider a probability kernel T from Ω to another space Ω'.

Example: decimation



Source of all problems: Renormalization Group

Question: If $\omega \sim \mu$, with μ Gibbs (consistent with $\mathcal{H}$), what about the law of ω'.

In finite volume:

$$\begin{tikzcd} \mathcal{H} \arrow[r, mapsto, "\mathcal{R}"] \arrow[d, mapsto] & \mathcal{H}' \\ \mu \arrow[r, mapsto, "T"] & \mu' \arrow[u, mapsto] \end{tikzcd}$$

Problem: In finite volume, μ' might not be Gibbsian at all, so we cannot speak of $\mathcal{H}'$,[1] $\mathcal{R}$ doesn't make sense.

[1]Fails to be consistent with any quasilocal specification.

Fuzzy Gibbs measures

Consider $\Omega = \mathcal{A}^{\mathbb{Z}^d}$, with $\mathcal{A}$ finite, as before.

Let $\mathcal{B}$ be another alphabet, with $|\mathcal{B}| \leq |\mathcal{A}|$. Write $\Sigma = \mathcal{B}^{\mathbb{Z}^d}$.

We consider a surjection $\pi : \mathcal{A} \to \mathcal{B}$, which we call a **fuzzy map**.[2]

Definition

A fuzzy Gibbs measure ν on Σ is defined as

$$\nu \;=\; \mu \circ \pi^{-1},$$

where μ is some Gibbs measure on Ω.

Question: when is ν Gibbsian?

[2] π induces a surjection $\Omega \to \Sigma$ which we denote by the same letter

Hidden phase transitions

We can partition Ω w.r.t. π as follows:

pick $\sigma \in \Sigma$ and define
$$\Omega_\sigma \;=\; \pi^{-1}(\sigma);$$
we call sets $\{\Omega_\sigma : \sigma \in \Sigma\}$ **fibres**.

Definition (Hidden phase transition)

We say that a **hidden phase transition** occurs on Ω_σ if
$$|\mathcal{G}_{\Omega_\sigma}(\Phi)| \;>\; 1.$$
If $|\mathcal{G}_{\Omega_\sigma}(\Phi)| = 1$ for all $\sigma \in \Sigma$, we talk about **absence of hidden phase transitions**.

Hidden phase transitions

Proposition (Sufficient condition)

In the absence of hidden phase transitions, $\nu = \mu \circ \pi^{-1}$ is Gibbsian.

The following conjecture (stated informally here) was established by van Enter, Fernández and Sokal:

Conjecture (van Enter-Fernández-Sokal hypothesis, [vEFS93],[Ber20])

The fuzzy Gibbs measure is not Gibbsian *if and only if*

(i) $\exists \sigma \in \Sigma : |\mathcal{G}_{\Omega_\sigma}(\Phi)| > 1$, i.e., a hidden phase transition occurs, and

(ii) one can pick different phases of $\mathcal{G}_{\Omega_\sigma}(\Phi)$ by varying boundary conditions.

Construction of conditional measures

Goal: construct distribution of ω, conditional on $\pi(\omega) = \sigma$

Definition

Given $B \subseteq \Sigma$ measurable with $\nu(B) > 0$, define

$$\mu^B \;=\; \mu(\cdot|\pi^{-1}(B)).$$

One can consider a net of conditional measures μ^B (on Ω), indexed with pairs (V, B), where V is an open neighbourhood of σ and $B \subseteq V : \nu(B) > 0$.

Write $\overline{\mathfrak{M}}_\sigma$ for accumulation points of the above net, as open neighbourhoods (V) "approach" σ.

Tjur points

Definition

If $|\overline{\mathfrak{M}}_\sigma| = 1$ for a given $\sigma \in \Sigma$, denote by μ^σ the only member of $\overline{\mathfrak{M}}_\sigma$, the limit of the corresponding net. In this case, we say that σ is a **Tjur point**.

One can restate the previously presented conjecture as follows:

Conjecture (van Enter-Fernández-Sokal hypothesis, [Ber20])

The fuzzy Gibbs measure is Gibbsian *if and only if* $|\overline{\mathfrak{M}}_\sigma| = 1$ for all $\sigma \in \Sigma$, i.e., all points are Tjur.

Proposition (Berghout, Verbitskiy, [Ber20])

Direction $(\Leftarrow)$ holds.

Tjur points: sufficient condition revisited

Proposition

$\overline{\mathfrak{M}}_\sigma \neq 0$, each member is a probability measure supported on Ω_σ. If $\mu \in \mathcal{G}_\Omega(\Phi)$, then

$$\overline{\mathfrak{M}}_\sigma \;\subseteq\; \mathcal{G}_{\Omega_\sigma}(\Phi).$$

Corollary

Absence of phase transitions implies Gibbsianity of $\nu = \mu \circ \pi^{-1}$.

Remark

By demonstrating the absence of phase transitions, we not only obtain Gibbsianity of the fuzzy Gibbs measure, but also verify that the example doesn't contradict the unproven direction of the van Enter-Fernández-Sokal hypothesis.

Classical Potts model

Write E^d for the (nearest-neighbour) edge set of $\mathbb{Z}^d$ and

$$\mathsf{E}_\Lambda = \Big\{ \langle x, y \rangle \in \mathsf{E}^d : x, y \in \Lambda \Big\}, \quad \partial \mathsf{E}_\Lambda = \Big\{ \langle x, y \rangle \in \mathsf{E}^d : x \in \Lambda, y \notin \Lambda \Big\}.$$

Definition (Interaction of Potts model)

The interaction of q-state Potts model $\Phi_{\beta,q}$ is given by

$$\Phi_{\Lambda;\beta,q}(\omega) = \begin{cases} 2\mathbb{1}_{\{\omega(x)\neq\omega(y)\}} - 1, & \Lambda = \{x, y\} : x \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

Hamiltonians are thus given by

$$\mathcal{H}_{\Lambda;\beta,q}(\omega) = \beta \sum_{\langle x,y \rangle \in \mathsf{E}_\Lambda \cup \partial \mathsf{E}_\Lambda} (2\mathbb{1}_{\{\omega(x)\neq\omega(y)\}} - 1).$$

Classical Potts model: phase transition

Write $\Omega = \{1, \ldots, q\}^{\mathbb{Z}^d}$.

Theorem

For each $q \geq 2$ and $d \geq 2$, there exists $\beta_c(d, q) \in (0, \infty)$, such that

(i) for $\beta < \beta_c(d, q)$, $|\mathcal{G}_\Omega(\Phi_{\beta,q})| = 1$,

(ii) for $\beta > \beta_c(d, q)$, $\mathcal{G}_\Omega(\Phi_{\beta,q})$ contains q distinct mutually singular measures .

Mutually singular measures in (ii) are precisely measures $\mu_{\beta,q}^{\mathbb{Z}^d,1}, \ldots, \mu_{\beta,q}^{\mathbb{Z}^d,q}$, corresponding to constant boundary conditions $1, \ldots,$ q.

Fuzzy Potts model

Let $1 < s < q$ and $\mathrm{r} = (r_1, \ldots, r_s)$, such that $r_1 + \ldots + r_s = q$.

Definition

Fuzzy Potts map $\pi_{\mathrm{r}} : \{1, \ldots, q\} \to \{1, \ldots, s\}$ is given by

$$\pi_{\mathrm{r}}(a) = \begin{cases} 1 : & 1 \leq a \leq r_1, \\ 2 : & r_1 + 1 < a \leq r_1 + r_2, \\ \ldots & \\ n : & r_1 + \ldots + r_{n-1} < a \leq r_1 + \ldots r_n, \\ \ldots & \\ s : & r_1 + \ldots + r_{s-1} < a \leq q. \end{cases}$$

Fuzzy Gibbs measure corresponding to $\mu_{\beta,q}^{\mathbb{Z}^d,\xi}$ is given by

$$\nu_{\beta,q}^{\mathbb{Z}^d,\xi} = \mu_{\beta,q}^{\mathbb{Z}^d,\xi} \circ \pi_{\mathrm{r}}^{-1}.$$

Fuzzy Potts model: Gibbsianity

Write $r^* = \min(\{r_1, \ldots, r_s\} \cap \mathbb{N}_{\geq 2})$

Theorem (Häggström, [Häg03])

Let $d \geq 2$, $q \geq 3$ and $\xi \in \{\emptyset, 1, \ldots, \mathrm{q}\}$; consider fuzzy Potts measure $\mu_{\beta,q}^{\mathbb{Z}^d,\xi}$.

(i) For each $\beta < \beta_c(d, r^*)$, $\nu_{\beta,q}^{\mathbb{Z}^d,\xi}$ is a Gibbs measure.

(ii) For each $\beta > \frac{1}{2} \log \frac{1+(r^*-1)p_c(d)}{1-p_c(d)}$, $\nu_{\beta,q}^{\mathbb{Z}^d,\xi}$ is *not* a Gibbs measure.[a]

[a] $p_c(d)$ = critical probability for Bernoulli percolation on $\mathbb{Z}^d$

Goal: provide an alternative proof of (i), using absence of hidden phase transitions.

Idea of alternative proof

Want to show: for each $\sigma \in \{1, \ldots, s\}^{\mathbb{Z}^d}$, $|\mathcal{G}_{\Omega_\sigma}(\Phi_{\beta,q})| = 1$.

Notice:

$$\Omega_\sigma = \prod_{x \in \mathbb{Z}^d} \pi^{-1}(\sigma(x));$$

write, for $j = 1, \ldots, s$,

$$\mathsf{A}_j = \pi^{-1}(j) = \{r_1 + \ldots + r_{j-1} + 1, \ldots, r_1 + \ldots + r_j\}$$

and

$$U_j = \left\{x \in \mathbb{Z}^d : \sigma(x) = j\right\}.$$

Then,

$$\Omega_\sigma = \prod_{x \in \mathbb{Z}^d} \begin{cases} \mathsf{A}_1, & x \in U_1, \\ \ldots & \\ \mathsf{A}_s, & x \in U_s \end{cases} =: \bigotimes_{j=1}^{s} \mathsf{A}_j^{U_j}.$$

Idea of alternative proof

It is enough to show that:

(i) If β is such that

$$|\mathcal{G}_{\mathsf{A}_j^{\mathbb{Z}^d}}(\Phi_{\beta,|\mathsf{A}_j|})| \;=\; 1, \quad \forall j = 1, \ldots, s,$$

then

$$|\mathcal{G}_{\mathsf{A}_j^{U_j}}(\Phi_{\beta,|\mathsf{A}_j|})| \;=\; 1, \quad \forall j = 1, \ldots, s.$$

(ii) If

$$|\mathcal{G}_{\mathsf{A}_j^{U_j}}(\Phi_{\beta,|\mathsf{A}_j|})| \;=\; 1, \quad \forall j = 1, \ldots, s,$$

then

$$|\mathcal{G}_{\otimes_j \mathsf{A}_j^{U_j}}(\Phi_{\beta,q})| \;=\; 1.$$

Idea of alternative proof

Clear: enough to show above for $s = 2$, induction takes care of the rest. Thus sufficient to prove:

Proposition (Part I)

Let $U \subset \mathbb{Z}^d$ and $q \in \mathbb{N}_{\geq 2}$. For $\beta < \beta_c(d, q)$,

$$|\mathcal{G}_{\{1,\ldots,q\}^U}(\Phi_{\beta,q})| = 1.$$

Proposition (Part II)

Let $\mathbb{Z}^d = U \sqcup V$ and $\mathsf{A} \cap \mathsf{B} = \emptyset$. If β is such that

$$|\mathcal{G}_{\mathsf{A}^U}(\Phi_{\beta,|\mathsf{A}|})| = |\mathcal{G}_{\mathsf{B}^V}(\Phi_{\beta,|\mathsf{B}|})| = 1,$$

then

$$|\mathcal{G}_{\mathsf{A}^U \otimes \mathsf{B}^V}(\Phi_{\beta,|\mathsf{A}|+|\mathsf{B}|})| = 1.$$

Spin-flip dynamics: general model

Idea: Pick initial configuration $\omega_0 \in \{-1,+1\}^{\mathbb{Z}^d}$ according to some Gibbs measure and randomly flip spins as time runs.

Question: Having obtained $(\omega_t)_{t\geq 0}$, when is $\mathrm{Law}(\omega_t)$ Gibbsian?

Spin-flip dynamics: general model

Let $\Omega_* = \{-1, +1\}^{\mathbb{Z}^d}$.

Pick $\mu \in \mathcal{G}_{\Omega_*}$ and draw $\omega_0 \sim \mu$.

Obtain $(\omega_t)_{t\geq 0}$ by flipping random spin at random times; dynamics are induced by rates $c(x, \omega)$, which are *translation invariant* in the second coordinate and depend on the second coordinate through a finite set, admitting an uniform bound on its diameter.

This induces a semigroup $(S(t))_{t\geq 0}$ acting on $\mathcal{M}_1(\Omega_*)$, so that

$$\mu S(t) \;=\; \mathrm{Law}(\omega_t).$$

High-temperature dynamics model

We now assume that $c(x, \omega)$ does no longer depend on neither x nor ω.

In other words: each $(\omega_t(x))_{t \geq 0}$ evolves independently, is flipped according to an exponential rate $c > 0$.

Tranditional approach: instead of $\mathrm{Law}(\omega_t)$, study

$$\hat{\mu}_t \ := \ \mathrm{Law}(\omega_0, \omega_t).$$

If $\hat{\mu}_t$ is Gibbsian on $\Omega = \Omega_* \times \Omega_*$, consistent with Hamiltonians $\hat{\mathcal{H}}^{(t)}$, one can show that $\mu S(t)$ is Gibbsian by verifying that

$$|\mathcal{G}_{\Omega_*}(\hat{\mathcal{H}}^{(t)}(\cdot, \eta))| \ = \ 1, \quad \forall \eta \in \Omega_*.$$

Two theorems

We will say that $\Phi \in \mathscr{B}^1(\Phi)$ (or its associated Hamiltonian) is *high-temperature* if

$$\sup_{x \in \mathbb{Z}^d} \sum_{\Lambda \ni x} (|\Lambda| - 1) \sup_{\omega, \tilde{\omega}} |\Phi_\Lambda(\omega) - \Phi_\Lambda(\tilde{\omega})| \ < \ 2.$$

Φ being high-temperature implies $|\mathcal{G}(\Phi)| = 1$ (via Dobrushin's Uniqueness Condition).

Theorem ([vEFdHR02])

Assuming dynamics as above and μ either high-temperature or infinite-temperature (i.i.d.), then $\mu S(t)$ is Gibbsian for all $t \geq 0$.

Two theorems

Write $\mu_{\beta,h}$ for some Gibbs measure for Ising model, i.e., consistent with

$$\mathcal{H}_\Lambda(\omega) \;=\; -\beta \sum_{\langle x,y\rangle \in \mathsf{E}_\Lambda \cup \partial \mathsf{E}_\Lambda} \omega(x)\omega(y) - h \sum_{x\in\Lambda} \omega(x),$$

with $\beta > 0$ inverse temperature, $h \in \mathbb{R}$ external magnetic field.

Theorem ([vEFdHR02])

Let $\mu_{\beta,h}$ be as above for $\beta \gg \beta_c(d)$.

(i) There exists $t_0 = t_0(\beta, h)$, so that $\mu_{\beta,h} S(t)$ is Gibbsian for $t \leq t_0$.

(ii) Moreover, if $h > 0$ then there exists $t_1 = t_1(\beta, h)$, so that $\mu_{\beta,h} S(t)$ is Gibbsian for $t \geq t_1$.

High-temperature dynamics as a fuzzy model

We notice that since each $(\omega_t(x))_{t\geq 0}$ evolves independently, being flipped at exponential rate $c > 0$,

$$[\mu S(t)](\omega_t(x) = 1) = \mu(\omega_0(x) = 1)\mathbb{P}\binom{\text{even flips}}{\text{until } t} + \mu(\omega_0(x) = -1)\mathbb{P}\binom{\text{odd flips}}{\text{until } t},$$

where probabilities of even/odd flips are given by

$$\frac{1 \pm e^{-2tc}}{2}.$$

Writing $\kappa(t) = \frac{1}{2}(1 - e^{-2tc})$, we obtain

$$[\mu S(t)](\omega_t(x) = 1) \; = \; \mu(\omega_0(x) = 1)(1 - \kappa(t)) + \mu(\omega_0(x) = -1)\kappa(t).$$

High-temperature dynamics as a fuzzy model

Fix $\kappa \in [0,1]$ and let $X \sim \mu$.

Define Ω_*-valued r.v. Y^κ, so that $(Y^\kappa(x))_{x\in\mathbb{Z}^d}$ are i.i.d. with

$$Y^\kappa(x) \;=\; \begin{cases} 1, & \text{with probability } 1-\kappa, \\ -1, & \text{with probability } \kappa, \end{cases}$$

and denote its law by ρ^κ.

Define now $Z^\kappa = X \cdot Y^\kappa$ and denote its law by ν^κ.

Clearly

$$\nu^\kappa(Z^\kappa(x) = 1) \;=\; \mu(X(x) = 1)(1-\kappa) + \mu(X(x) = -1)\kappa.$$

High-temperature dynamics as a fuzzy model

Define

$$\pi : \Omega \to \Omega_*$$
$$(\omega_{\mathsf{c}}, \omega_{\mathsf{d}}) \mapsto \omega_{\mathsf{c}} \cdot \omega_{\mathsf{d}}$$

It is clear that

$$\nu^\kappa \; = \; (\mu \otimes \rho^\kappa) \circ \pi^{-1}.$$

Both μ, ρ^κ are Gibbsian $\Rightarrow \mu \otimes \rho^\kappa$ is Gibbsian $\Rightarrow \nu^\kappa$ is fuzzy Gibbs.

Moreover, choosing $\kappa(t)$ (as defined before), we obtain precisely

$$\nu^{\kappa(t)} \; = \; \mu S(t).$$

Alternative proof strategy

To verify Gibbsianity of $\mu S(t) = \nu^{\kappa(t)}$, it is enough to show that for each $\sigma \in \Omega_*$,

$$|\mathcal{G}_{\Omega_\sigma}(\mathcal{H}^{\kappa(t)})| \ = \ 1,$$

where $\mathcal{H}^{\kappa(t)}$ is a Hamiltonian with which $\mu \otimes \rho^{\kappa(t)}$ is consistent.

One notices that in fact

$$\Omega_\sigma \ = \ \left\{\left(\omega, \frac{\sigma}{\omega}\right) : \omega \in \Omega_*\right\} \ = \ \{(\omega, \omega\sigma) : \omega \in \Omega_*\}\,.$$

Thus sufficient to show that for each $\sigma \in \Omega_*$,

$$|\mathcal{G}_{\Omega_*}(\mathcal{H}^{\kappa(t)}(\cdot, \cdot\sigma))| \ = \ 1.$$

Alternative proof strategy

Main trick: Given any fixed $\sigma \in \Omega_*$,

$$\mathcal{H}^{\kappa(t)}(\cdot, \cdot\sigma) - \hat{\mathcal{H}}^{(t)}(\cdot, \sigma)$$

corresponds to a single-site interaction $\Rightarrow$ one is high-temperature iff the other is.

S. Berghout. *Gibbs Processes and Applications*. Ph.D. thesis. Leiden University, 2020.

A.C.D. van Enter, R. Fernández, F. den Hollander, F. Redig. *Possible Loss and Recovery of Gibbsianness During the Stochastic Evolution of Gibbs Measures*. Commun. Math. Phys. 226 (2002), 101-130.

A.C.D. van Enter, R. Fernández, A.D. Sokal. *Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory*. J. Statist. Phys. 72 (1993), no. 5-6, 879-1167.

O. Häggström. *Is the fuzzy Potts model Gibbsian?* Ann. I. H. Poincaré 39 (2003), no. 5, 891-917.