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Leiden University

Absence of Phase Transitions and Preservation of Gibbs Property under Renormalization

Oskar Vavtar

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1 Introduction

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1.1 Preliminaries: interactions, Hamiltonians and Gibbs measures

The central aim of statistical mechanics is to provide a probabilistic description of microscopic physical phenomena. In particular one aims to study a collection of particles (often infinite), indexed in some lattice $\mathbb{L}$ and each taking values in some set $\mathcal{A}_x$, $x \in \mathbb{L}$, called the *(single-)spin space*, which we will assume to be finite.[1] Each manifestation of this macroscopic system corresponds to a (random) configuration in $\Omega = \prod_{x \in \mathbb{L}} \mathcal{A}_x$; often we will assume that the system is homogeneous, i.e., $\Omega = \mathcal{A}^{\mathbb{L}}$. In order to talk about randomness, we have to consider probability distributions on Ω; we will write $\mathcal{M}_1(\Omega)$ for the set of probability measures on Ω (or rather $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the corresponding σ-algebra, generated by the cylinder sets). One often calls those measures *random fields*.

There is however a particular class of probability measures that are of interest, called *Gibbs measures*, which we will start introducing formally in a moment, but not before giving some intuitive but *very* informal description. Generally, we assume that the particles interact in some sense, which contributes some (positive or negative) energy. If some configuration carries energy $\mathcal{H}(\omega)$, then the Gibbs measure (consistent with $\mathcal{H}$) would assign ω a probability proportional to $e^{-\mathcal{H}(\omega)}$. Intuitively, Gibbs measures award configurations with low energy and penalize configurations with high energy.

1.1.1 Formal definition, part 1: finite volume setting

In this section we assume $|\mathbb{L}| < \infty$; an often used example would be a finite graph $G = (V, E)$. In order to capture the interactions, one defines a collection of maps $\Phi = \{\Phi_\Lambda : \Lambda \subseteq \mathbb{L}\}$, called *interaction*, where for $\omega \in \Omega$, $\Phi_\Lambda(\omega)$, depends on ω only through its values in Λ, that is, $\Phi_\Lambda(\omega) = \Phi_\Lambda(\omega(x) : x \in \Lambda)$. The value $\Phi_\Lambda(\omega)$ represents the amount of energy that the interaction between particles in Λ contributes to the system.

Example 1.1. Often studied are the *nearest-neighbour* interactions. If $\mathbb{L}$ is a (not necessarily finite) graph $G = (V, E)$, then Φ_Λ is constant zero, unless $\Lambda = \{x, y\}$, where $x \sim y$.

The total energy of the systems is then obtained by summing up energies of all subsets:

$$\mathcal{H}(\omega) = \sum_{\Lambda \subseteq \mathbb{L}} \Phi_\Lambda(\omega), \quad \omega \in \Omega.$$

We call the function $\mathcal{H} = \mathcal{H}^\Phi$ a *Hamiltonian* consistent with Φ. As already suggested above, the Gibbs measure $\mu = \mu^\Phi$ is then defined so that

$$\mu(\omega) = \frac{1}{\mathcal{Z}} e^{-\mathcal{H}(\omega)}, \quad \omega \in \Omega,$$

[1] In general, one can take $\mathcal{A}$ to be infinite, even uncountable (e.g., $\mathbb{R}^d$).

where $\mathcal{Z} = \mathcal{Z}^{\Phi}$ is called the *partition function* and is given by $\mathcal{Z} = \sum_{\omega \in \Omega} e^{-\mathcal{H}(\omega)}$. In this finite volume setting, Gibbs measures also often called *Gibbs canonical ensembles.*

While from probabilistic point of view, this may be interesting enough model, it is rather disinteresting from the statistical mechanical point of view. In particular, there is no notion of phase transitions, i.e., coexistence of several Gibbs measure consistent with the same interaction, which is one of the main focuses of study in statistical mechanics. This does, however, change when we relax the assumption about finiteness of $\mathbb{L}$ and take it to be a countably infinite lattice, as we will see shortly.

1.1.2 Formal definition, part 2: infinite volume setting

In practice, one often wishes to consider $\mathbb{L}$ to be an infinite lattice. Unfortunately, the construction presented in the previous section doesn't translate to this setting, as the Hamiltonian would correspond to a infinite sum and hence the Gibbs measure wouldn't be well defined. We thus have to consider an alternative construction of Gibbs measures in infinite volumes. The most intuitive approach perhaps appears to be to construct them as (weak) limits of measures from the previous section, as finite volumes increase to $\mathbb{L}$ in a suitable sense. This indeed is a valid idea: one can construct infinite-volume Gibbs measures in this way and those measures form an import class. However, it is not the entire story. We will now proceed to define (repeating some already mentioned notions) a more general notion of infinite-volume Gibbs measures, which includes all such limit measures as well as other ones.

From now on we restrict ourselves to $\mathbb{L} = \mathbb{Z}^d$. In the rest of the thesis, we will only consider $\mathbb{L}$ to be either $\mathbb{Z}^d$ or some subset of it. Moreover, given $\omega \in \Omega$ and $\Lambda \subseteq \mathbb{Z}^d$, we write $\omega_\Lambda := \omega|_\Lambda$ as well as $\Omega_\Lambda := \prod_{x \in \Lambda} \mathcal{A}_x$, so that $\omega_\Lambda \in \Omega_\Lambda$, and $\mathcal{F}_\Lambda$ for the (corresponding) σ-algebra on Ω_Λ, generated by the cylinders on subsets of Λ.[2] Also, given another $\widetilde{\omega} \in \Omega$, we write

$$[\omega_\Lambda \widetilde{\omega}_\Lambda](x) \;:=\; \begin{cases} \omega(x): & x \in \Lambda, \\ \widetilde{\omega}(x): & x \in \Lambda^{\mathsf{c}}. \end{cases}$$

Definition 1.2.

(1) An *interaction* is a collection of maps $\Phi = \{\Phi_\Lambda : \Lambda \Subset \mathbb{Z}^d\}$, such that

$$\Phi_\Lambda(\omega) \;=\; \Phi_\Lambda(\omega_\Lambda), \quad \omega \in \Omega,$$

that is, Φ_Λ depends on ω only through its values in Λ. We say that Φ is *uniformly absolutely convergent* (write $\Phi \in \mathscr{B}^1(\Omega)$) if

$$\|\Phi\| \;:=\; \sup_{x \in \mathbb{Z}^d} \sum_{\substack{\Lambda \Subset \mathbb{Z}^d: \\ x \in \mathbb{Z}^d}} \|\Phi_\Lambda\|_\infty \;<\; \infty$$

and write $\mathscr{B}^1(\Omega)$ for the collection of all UAC interactions on Ω.

[2]The advantage of directly selecting ω_Λ from Ω_Λ, rather than ω from Ω and just restricting it, is that for $\Lambda \Subset \mathbb{Z}^d$ one has $|\Omega_\Lambda| < \infty$, which allows for finite sums.

(2) For $\Phi \in \mathscr{B}^1(\Omega)$, we define a corresponding collection of Hamiltonians $\mathcal{H}^\Phi = \{\mathcal{H}^\Phi_\Lambda : \Lambda \Subset \mathbb{Z}^d\}$, given by

$$\mathcal{H}^\Phi_\Lambda(\omega) \;=\; \sum_{\substack{\Delta \Subset \mathbb{Z}^d: \\ \Delta\cap\Lambda\neq\emptyset}} \Phi_\Lambda(\omega), \quad \omega \in \Omega.$$

Add definition of quasilocal Hamiltonian?

(3) For $\Phi \in \mathscr{B}^1(\Omega)$, we define a *specification* as a collection $\gamma^\Phi = \{\gamma^\Phi_\Lambda : \Lambda \Subset \mathbb{Z}^d\}$ via

$$\gamma^\Phi_\Lambda(\widetilde{\omega}_\Lambda|\omega_{\Lambda^c}) \;=\; \frac{1}{\mathcal{Z}^{\Phi,\omega}_\Lambda}\exp(-\mathcal{H}^\Phi_\Lambda(\widetilde{\omega}_\Lambda\omega_{\Lambda^c})), \quad \omega,\widetilde{\omega} \in \Omega,$$

where $\mathcal{Z}^{\Phi,\omega}_\Lambda = \sum_{\widetilde{\omega}_\Lambda\in\Omega_\Lambda}\exp(-\mathcal{H}^\Phi_\Lambda(\widetilde{\omega}_\Lambda\omega_{\Lambda^c}))$.

Remark 1.3. Formally, γ^Φ should be seen as a collection of probability kernels, where for each $\Lambda \Subset \mathbb{Z}^d$, $\gamma^\Phi_\Lambda : \mathcal{F}_\Lambda \times \Omega_{\Lambda^c} \to (0,1)$, so $\gamma^\Phi_\Lambda(\widetilde{\omega}_\Lambda|\omega_{\Lambda^c})$ should be actually read as $\gamma^\Phi_\Lambda(\{\widetilde{\omega}_\Lambda\}|\omega_{\Lambda^c})$. The probability of a general $A \in \mathcal{F}_\Lambda$ is now given by

$$\gamma^\Phi_\Lambda(A|\omega_{\Lambda^c}) \;=\; \sum_{\widetilde{\omega}_\Lambda\in A}\gamma^\Phi_\Lambda(\widetilde{\omega}_\Lambda|\omega_{\Lambda^c}).$$

The corresponding expectation is given for suitable $f : \Omega \to \mathbb{R}$ by

$$\gamma^\Phi_\Lambda[f|\omega_{\Lambda^c}] \;=\; \sum_{\widetilde{\omega}_\Lambda\in\Omega_\Lambda} f(\widetilde{\omega}_\Lambda\omega_{\Lambda^c})\gamma^\Phi_\Lambda(\widetilde{\omega}_\Lambda|\omega_{\Lambda^c}).$$

In further generality, we may call *a specification* any collection of probability kernels $\gamma = \{\gamma_\Lambda : \Lambda \Subset \mathbb{Z}^d\}$ such that for each $\Lambda \Subset \mathbb{Z}^d$

- and $A \in \mathcal{F}_\Lambda$, $\omega \mapsto \gamma(A|\omega)$ is $\mathcal{F}_{\Lambda^c}$-measurable,
- $\gamma_\Lambda(A|\cdot) = \mathbb{1}_A$ for each $A \in \mathcal{F}_\Lambda$,
- and $\Delta \subset \Lambda$, $\int \gamma_\Delta(A|\omega')\,\gamma_\Lambda(\mathrm{d}\omega'|\omega) = \gamma_\Lambda(A|\omega)$.

Finally, after introducing all this notions, we are able to formally define the infinite-volume Gibbs measures, using a very natural definition.

Definition 1.4. A probability measure $\mu \in \mathcal{M}_1(\Omega)$ is said to be a Gibbs measure consistent with $\Phi \in \mathscr{B}^1(\Omega)$, if it is consistent with γ^Φ, that is, if for each $\Lambda \Subset \mathbb{Z}^d$ we have

$$\mu(\omega_\Lambda|\omega_{\Lambda^c}) \;=\; \gamma^\Phi_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.^3$$

We write $\mathcal{G}_\Omega(\Phi)$ for the set of all Gibbs measure on Ω consistent with Φ. Alternatively, if our main reference is a specification γ or a collection of Hamiltonians $\mathcal{H}$, we may also write $\mathcal{G}_\Omega(\gamma)$ or $\mathcal{G}_\Omega(\mathcal{H})$, so in our notation $\mathcal{G}_\Omega(\Phi)$, $\mathcal{G}_\Omega(\mathcal{H}^\Phi)$ and $\mathcal{G}_\Omega(\gamma^\Phi)$ would refer to the same thing.

Remark 1.5.

(i) Equivalently, one could define μ to be Gibbs for γ if for each $\Lambda \Subset \mathbb{Z}^d$ and $f \in C(\Omega,\mathbb{R})$ one has

$$\int f\,\mathrm{d}\mu \;=\; \int \gamma_\Lambda[f|\omega_{\Lambda^c}]\,\mathrm{d}\mu(\omega).$$

[3] The notation $\mu(a_\Lambda|b_{\Lambda^c})$ should be read as $\mu(a \text{ on } \Lambda|b \text{ off } \Lambda)$.

(ii) For each $\Phi \in \mathscr{B}^1(\Omega)$, $\mathcal{G}_\Omega(\Phi) \neq \emptyset$.

Say something about infinite volume limits?

Example 1.6 (Ising model)**.** One of the most classical models in statistical mechanics is the Ising model, concerned with (ferro)magnetism. It is an example of a nearest-neighbour model with $\mathcal{A} = \{-1, +1\}$, where the Hamiltonian corresponding to a finite volume $\Lambda \Subset \mathbb{Z}^d$ is given by

$$\mathcal{H}_{\Lambda;\beta,h}(\omega) \;=\; -\beta \sum_{\substack{x\sim y:\\ x\in\Lambda \text{ or } y\in\Lambda}} \omega(x)\omega(y) + h\sum_{x\in\Lambda}\omega(x),$$

though one often writes "$2\mathbb{1}_{\{\omega(x)=\omega(y)\}} - 1$" instead of "$\omega(x)\omega(y)$", the latter ceasing to be suitable in case of choosing alternative $\mathcal{A}$ (such as $\{0,1\}$ for example). The parameters $\beta \geq 0$ and $h \in \mathbb{R}$ represent the *inverse temperature* and the *external field*, respectively. Thus, considering a boundary condition $\xi \in \Omega$, the probability of configuration agreeing with ω on Λ, conditional on it agreeing with ξ outside Λ^c, given by

$$\gamma_{\Lambda;\beta,h}(\omega_\Lambda|\xi_{\Lambda^c}) \;=\; \frac{1}{\mathcal{Z}^\xi_{\Lambda;\beta,h}} e^{-\mathcal{H}_{\Lambda;\beta,h}(\omega_\Lambda\xi_{\Lambda^c})},$$

where $\mathcal{Z}^\xi_{\Lambda;\beta,h} = \sum_{\widetilde{\omega}_\Lambda\in\Omega_\Lambda} \exp(-\mathcal{H}_{\Lambda;\beta,h}(\widetilde{\omega}_\Lambda\xi_{\Lambda^c}))$, only depends on the boundary condition through vertices that are adjacent to some vertex in Λ, i.e., are touching the boundary. This is in fact a general property of nearest-neighbour models.

Very important is the following characterization of Gibbs measures:

Proposition 1.7.

(i) Let $\Phi \in \mathscr{B}^1(\Omega)$. Then γ^Φ has the following two properties:

- *uniform non-nullness*: for every $\Lambda \Subset \mathbb{Z}^d$ there exist $\alpha_\Lambda, \beta_\Lambda \in (0,1)$ such that
$$\alpha_\Lambda \;\leq\; \gamma^\Phi_\Lambda(\widetilde{\omega}_\Lambda|\omega_{\Lambda^c}) \;\leq\; \beta_\Lambda, \quad \forall \widetilde{\omega}_\Lambda \in \Omega_\Lambda,\, \omega\in\Omega.$$
- *quasilocality*: writing $\mathbb{B}_n = [-n,n]^d \cap \mathbb{Z}^d$ for $n \in \mathbb{N}$, for every $\Lambda \Subset \mathbb{Z}^d$ one has
$$\sup_{\omega,\widetilde{\omega}\in\Omega}\sup_{\xi,\zeta\in\Omega} |\gamma^\Phi_\Lambda(\widetilde{\omega}_\Lambda|\omega_{\mathbb{B}_n\setminus\Lambda}\xi_{\mathbb{B}_n^c\setminus\Lambda}) - \gamma^\Phi_\Lambda(\widetilde{\omega}_\Lambda|\omega_{\mathbb{B}_n\setminus\Lambda}\zeta_{\mathbb{B}_n^c\setminus\Lambda})| \;\xrightarrow{n\to\infty}\; 0.$$

(ii) Let μ be a fully supported Borel probability measure on Ω. If there exists a specification (in the sense of [Remark]) which is uniformly non-null, quasilocal and with which μ is consistent (in the sense of [Definition]), then there exists some $\Phi \in \mathscr{B}^1(\Omega)$ so that $\mu \in \mathcal{G}_\Omega(\Phi)$.

1.1.3 Coexistence of several Gibbs measures

While the result in [Remark] (ii) tells us that for an UAC interaction, there's always a Gibbs measure, it tells us nothing else about the cardinality in $\mathcal{G}_\Omega(\Phi)$: in particular it doesn't guarantee that there's only one. In fact several Gibbs measures consistent with the same interaction may exist; in this case, we say that the system has a *phase transition*. Given a model with some parameter(s), an important question is for which parameter values there's only one Gibbs measure and for which parameter there's several.

We begin by introducing the most intuitive method of defining infinite volume Gibbs measure, which was already hinted on earlier, using the weak limits. Gibbs measures obtained in this way are often referred to as *thermodynamic limits (of finite volume Gibbs measures)*. Despite definition being straightforward, it requires us to first introduce some technical notions, which will be used throughout the thesis:

Definition 1.8.

(1) We say that $(\Lambda_n)_{n\in\mathbb{N}}$, where $\Lambda_n \Subset \mathbb{Z}^d$ for each $n \in \mathbb{N}$, is a *van Hove sequence*, if it satisfies the following conditions:

  (i) it is *increasing*: $\Lambda_n \subset \Lambda_{n+1}$ for each $n \in \mathbb{N}$,

  (ii) it *invades* $\mathbb{Z}^d$: $\bigcup_{n\geq 1} \Lambda_n = \mathbb{Z}^d$, and

  (iii) $\lim_{n\to\infty} \frac{|\partial\Lambda_n|}{|\Lambda_n|} = 0$. `Add definition of` $\partial\Lambda$`?`

  We denote van Hove convergence by $\Lambda_n \Uparrow \mathbb{Z}^d$.

(2) A function $f : \Omega \to \mathbb{R}$ is said to be *local* if there's $\Delta \Subset \mathbb{Z}^d$, such that for any choice of $\omega, \widetilde{\omega} \in \Omega$,

$$\omega(x) = \widetilde{\omega}(x)\ \forall x \in \Delta \implies f(\omega) = f(\widetilde{\omega}).$$

We sometimes write $\mathrm{supp}(f)$ for the minimal such Δ. We denote the collection of all local functions $f : \Omega \to \mathbb{R}$ by $\mathcal{L}(\Omega)$.

(c) Given (probability) measures $\mu, \mu_1, \mu_2, \ldots$ on Ω, we say that $(\mu_n)_{n\in\mathbb{N}}$ converges *weakly* to μ, writing $\mu_n \xrightarrow{w} \mu$, if

$$\int_\Omega f \,\mathrm{d}\mu_n \ \to\ \int_\Omega f \,\mathrm{d}\mu, \quad \forall f \in \mathcal{L}(\Omega).$$

The definitions then goes as follows:

Definition 1.9. We say that μ is a thermodynamic-limit Gibbs measure for $\Phi \in \mathscr{B}^1(\Omega)$, if there exists a boundary condition
$xi \in \Omega$, such that

$$\gamma^{\Phi}_{\Lambda_n}(\cdot|\xi_{\Lambda^c}) \ \xrightarrow{w}\ \mu \quad \text{as } n \to \infty,$$

for arbitrary van Hove sequence $(\Lambda_n)_{n\in\mathbb{N}}$.

Remark 1.10. The notation in the above definition is slightly dishonest, in terms of what γ^Φ_Λ is supposed to represent, though conceptually is "correct enough" to allow for an abuse of notation. While is supposed to be a probability measure on Ω, we have seen that given $\Lambda \Subset \mathbb{Z}^d$ and $\xi \in \Omega$ fixed, $\gamma^\Phi_\Lambda(\cdot|\xi_{\Lambda^c})$ is a probability measure on Ω_Λ, so one could argue that the weak limit in the above definition technically speaking does not make sense. The formal way to correct it would be to define, for each $\Lambda \Subset \mathbb{Z}^d$ and $\xi \in \Omega$ a measure $\mu^\xi_\Lambda = \mu^{\Phi,\xi}_\Lambda$ given by

$$\mu^\xi_\Lambda(\omega) \ =\ \mathbb{1}_{\{\omega_{\Lambda^c} = \xi_{\Lambda^c}\}} \gamma^\Phi_\Lambda(\omega_\Lambda|\xi_{\Lambda^c}), \quad \omega \in \Omega,$$

which is indeed a probability measure on Ω (while still in a sense "supported" on Ω_Λ, as $\mathrm{supp}(\mu^\xi_\Lambda) = \Omega_\Lambda \times \{\xi_{\Lambda^c}\}$), so the weak limit in the definition above indeed makes sense.

Example 1.11 (Ising: $+$ and $-$ limits)**.** Simple examples of thermodynamic limits are so-called $+$ and $-$ states. First, we define, for $\beta > 0$, $h \in \mathbb{R}$, $\Lambda \Subset \mathbb{Z}^d$ and $\xi \in \Omega$,

$$\mu^{\xi}_{\Lambda;\beta,h}(\omega) \;=\; \mathbb{1}_{\{\omega_{\Lambda^c}=\xi_{\Lambda^c}\}} \frac{1}{\mathcal{Z}^{\xi}_{\Lambda;\beta,h}} e^{-\mathcal{H}_{\Lambda;\beta,h}(\omega_\Lambda \xi_{\Lambda^c})}, \quad \omega \in \Omega.$$

Thermodynamic limits $\mu^+_{\beta,h}$ and $\mu^-_{\beta,h}$ are obtained as

$$\mu^+_{\mathbb{B}_n;\beta,h} \xrightarrow{w} \mu^+_{\beta,h} \quad \text{and} \quad \mu^-_{\mathbb{B}_n;\beta,h} \xrightarrow{w} \mu^-_{\beta,h},$$

where $+$ is constant $+1$ and $-$ is constant -1. An important question is whether those limits coincide, i.e., if $\mu^+_{\beta,h} = \mu^-_{\beta,h}$ as members of $\mathcal{M}_1(\Omega)$. For $h \neq 0$, the answer is *yes*. However, for $h = 0$ and $d \geq 2$, this is no longer the case. For each dimension $d \geq 2$, there exists a critical inverse temperature $\beta_c = \beta_c(d) \in (0, \infty)$ such that for $\beta < \beta_c$ we have $\mu^+_{\beta,0} = \mu^-_{\beta,0}$ (and in fact $|\mathcal{G}_\Omega(\Phi_{\beta,0})| = 1$) while for $\beta > \beta_c$ we have $\mu^+_{\beta,0} \neq \mu^-_{\beta,0}$ (so $|\mathcal{G}_\Omega(\Phi_{\beta,0})| > 1$).

Remark 1.12. Using the same procedure, we may also define non-Gibbs thermodynamic-limit measures, using a more general specificaiton γ (in the sense of [Remark]), not necessarily as configurations on vertices (but for example on edges, as will be seen later, in [Chapter 3]).

We now turn our attention to the general notion on Gibbs measures, as defined in [Definition], which are often referred to as the *DLR Gibbs measures*, named after Dobrushin, Lanford and Ruelle. Spaces of DLR Gibbs measures (from now on just Gibbs measures) do indeed carry some nice properties:

Proposition 1.13. Let $\Phi \in \mathscr{B}^1(\Omega)$.

(i) $\mathcal{G}_\Omega(\Phi)$ is convex.

(ii) Extremal measures of $\mathcal{G}_\Omega(\Phi)$ (denoted by $\mathrm{extr}(\mathcal{G}_\Omega(\Phi))$) are mutually singular.

(iii) For each $\mu \in \mathrm{extr}(\mathcal{G}_\Omega(\Phi))$,

$$\gamma^{\Phi}_{\mathbb{B}_n}(\cdot|\omega) \;\xrightarrow{w}\; \mu, \quad \text{for } \mu\text{-a.a. } \omega \in \Omega,$$

keeping in mind [Remark]. In other words, extremal μ can be obtained as a thermodynamic limit with some fixed boundary condition.

Example 1.14 (Ising: +/- vs. Dobrushin)**.** In the case of Ising model in the non-uniqueness regime (that is, $d \geq 2$, $h = 0$ and $\beta > \beta_c(d)$), the thermodynamic limits corresponding to boundary conditions $+$ and $-$ are both extremal Gibbs measures – in fact, they are the only ones:

$$\mathrm{extr}(\mathcal{G}_\Omega(\Phi_{\beta,h})) \;=\; \left\{\mu^+_{\beta,0}, \mu^-_{\beta,0}\right\}.$$

An interesting example of a non-extremal boundary condition is the *Dobrushin boundary condition*, which is given by

$$\mathrm{Dob}(x) \;=\; \begin{cases} +1: & x = (x_1, \ldots, x_d) \text{ with } x_d \geq 0, \\ -1: & \text{otherwise.} \end{cases}$$

While $\mu_{\beta,0}^{\mathrm{Dob}} \notin \mathrm{extr}(\mathcal{G}_\Omega(\Phi_{\beta,0}))$, it can still be obtained by a thermodynamical limit (and is a well-defined Gibbs measure), see [Velenik], Theorem 3.58. One is then tempted to ask, whether all members of $\mathcal{G}_\Omega(\Phi_{\beta,0})$ can be obtained as thermodynamical limit (as we have seen to be the case for the extremal ones). The answer is *no*, and it is precisely the Dobrushin boundary condition that helps us define a simple example. It is easy to see that considering a boundary condition $-$Dob, $\mu_{\beta,0}^{-\mathrm{Dob}} \in \mathcal{G}_\Omega(\Phi_{\beta,0})$. Defining

$$\widetilde{\mu} \;=\; \frac{1}{2}\mu_{\beta,0}^{\mathrm{Dob}} + \frac{1}{2}\mu_{\beta,0}^{-\mathrm{Dob}},$$

it immediately follows from [Proposition.(i)] that $\widetilde{\mu} \in \mathcal{G}_\Omega(\Phi_{\beta,0})$, however, for $d \geq 3$, $\widetilde{\mu}$ cannot possibly be obtained as a thermodynamical limit, as argued in [Velenik], Example 6.64.

Before concluding this preliminary section, we present an important result, which is very commonly used to verify uniqueness of a Gibbs measure with respect to the interaction:

Theorem 1.15 (Dobrushin's Uniqueness Condition)**.** Let $\Phi \in \mathscr{B}^1(\Omega)$ and define for $x, y \in \mathbb{Z}^d$,

$$c_{x,y}^{\Phi} \;:=\; \sup_{\substack{\omega,\widetilde{\omega}\in\Omega:\\ \omega\equiv\widetilde{\omega}\text{ off } y}} \|\gamma_x^{\Phi}(\cdot|\omega) - \gamma_x^{\Phi}(\cdot|\widetilde{\omega})\|_{\mathrm{TV}}.$$

If

$$\sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} c_{x,y}^{\Phi} \;<\; 1,$$

then $|\mathcal{G}_\Omega(\Phi)| = 1$.

In practice, the following criterion is useful in particular:

Corollary 1.16. $\Phi \in \mathscr{B}^1(\Omega)$ satisfies the Dobrushin's Uniqueness Condition as soon as

$$\sup_{x\in\mathbb{Z}^d} \sum_{\substack{\Lambda\Subset\mathbb{Z}^d:\\ x\in\Lambda}} (|\Lambda|-1) \sup_{\omega,\widetilde{\omega}} |\Phi_\Lambda(\omega) - \Phi_\Lambda(\widetilde{\omega})| \;<\; 2.$$

We will call interaction satisfying this *high-temperature interactions.*

In practice, one may sometimes approach a collection of Hamiltonians, that differ from each other purely by a single-site interaction. A very useful fact, which will be of great importance in a later chapter on spin-flip dynamics, is that as soon as one of them satisfies [Eq in Cor], so do all the other:

Lemma 1.17. Suppose that $\Phi \in \mathscr{B}^1(\Omega)$ satisfies Eq in Cor. Moreover, let $\Psi \in \mathscr{B}^1(\Omega)$ be a *single-site* interaction, i.e., Ψ_Λ is non-zero only for $\Lambda = \{x\}$, $x \in \mathbb{Z}^d$. Then, $\Phi^* := \Phi + \Psi$ satisfies [Eq in Cor] as well.

Proof. By triangle inequality,

$$\begin{aligned}
\sup_{x\in\mathbb{Z}^d} \sum_{\substack{\Lambda\Subset\mathbb{Z}^d:\\ x\in\Lambda}} (|\Lambda|-1) \sup_{\omega,\widetilde{\omega}} |\Phi^*_\Lambda(\omega) - \Phi^*_\Lambda(\widetilde{\omega})| \;&\leq\; \sup_{x\in\mathbb{Z}^d} \sum_{\substack{\Lambda\Subset\mathbb{Z}^d:\\ x\in\Lambda}} (|\Lambda|-1) \sup_{\omega,\widetilde{\omega}} |\Phi_\Lambda(\omega) - \Phi_\Lambda(\widetilde{\omega})| \\
&\quad+\; \sup_{x\in\mathbb{Z}^d} \sum_{\substack{\Lambda\Subset\mathbb{Z}^d:\\ x\in\Lambda}} (|\Lambda|-1) \sup_{\omega,\widetilde{\omega}} |\Psi_\Lambda(\omega) - \Psi_\Lambda(\widetilde{\omega})| \\
&<\; 2 \\
&\quad+\; \sup_{x\in\mathbb{Z}^d} \sum_{\substack{\Lambda\Subset\mathbb{Z}^d:\\ x\in\Lambda}} (|\Lambda|-1) \sup_{\omega,\widetilde{\omega}} |\Psi_\Lambda(\omega) - \Psi_\Lambda(\widetilde{\omega})|
\end{aligned}$$

Using that for any $x \in \mathbb{Z}^d$ and $\Lambda \Subset \mathbb{Z}^d$ containing x, $\Psi_\Lambda \equiv 0$ as soon as $\Lambda \neq \{x\}$, we have that

$$\sum_{\substack{\Lambda \Subset \mathbb{Z}^d: \\ x \in \Lambda}} (|\Lambda| - 1) \sup_{\omega, \widetilde{\omega}} |\Psi_\Lambda(\omega) - \Psi_\Lambda(\widetilde{\omega})| \;=\; (|\{x\}| - 1) \sup_{\omega, \widetilde{\omega} \in \Omega} |\Psi_x(\omega) - \Psi_x(\widetilde{\omega})|.$$

The right-hand side clearly equals zero, since $|\{x\}| = 1$, yielding the conclusion. □

1.2 Renormalisation group pathologies

We now direct our attention to the source of the questions that inspired the theory which is presented in this thesis. Those can be traced back to the seminal paper of Aernout van Enter, Roberto Fernández and Alan Sokal, carrying the title *Regularity Properties and Pathologies of Position-Space Renormalization-Group Transformations.*

Informally, the problem goes as follows. Consider (measurable) spaces Ω, Ω' of configurations on a (for now) finite volume Λ and suppose Ω is equipped with a probability measure μ. A renormalization-group (RG) map is given as a a probability kernel T from Ω to Ω'. T can be either a deterministic or a stochastic map.[4] This maps gives us a natural measure μ' on Ω', which is given by

$$\mu'(\omega') \;=\; (\mu T)(\omega') \;:=\; \sum_{\omega \in \Omega} \mu(\omega) T(\omega, \omega'), \quad \omega' \in \Omega'.$$

Note that if T is deterministic, given by a map $\widetilde{T} : \Omega \to \Omega'$, μT would correspond to the push-forward measure $\mu \circ \widetilde{T}^{-1}$. Thus, the RG map is without a problem defined between measures. In applications, however, an alternative route is usually taken. In particular, one defines the RG map as a map $\mathcal{R}$ between (suitable) Hamiltonians: given a Hamiltonian $\mathcal{H}$, with which μ is consistent ($\mu \propto e^{-\mathcal{H}}$), $\mathcal{R}$ assigns $\mathcal{H}$ the Hamiltonian $\mathcal{H}'$, which corresponds to $\mu' = \mu T$ ($\mu' \propto e^{-\mathcal{H}'}$). In particular, $\mathcal{H}'$ may be obtained via the formula

$$\mathcal{H}'(\omega') \;=\; (\mathcal{R}\mathcal{H})(\omega') \;=\; -\log \sum_{\omega \in \Omega} e^{-\mathcal{H}(\omega)} T(\omega, \omega)' + C,$$

with C a constant. The interplay between T and $\mathcal{R}$ is captured in the following diagram:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\mathcal{R}"] \arrow[d] & \mathcal{H}' \\ \mu \arrow[r, "T"'] & \mu' \arrow[u] \end{tikzcd}$$

This nice formalism, however, fails to translate when we instead take an infinite volume Λ, for example $\mathbb{Z}^d$. First clear problem is that there might be several Gibbs measures consistent with $\mathcal{H}$, implying that $\mathcal{H} \mapsto \mu$ is a multi-valued map. While μ' can still be defined rather simply,

$$\mu'(A') \;:=\; \int_\Omega T(\omega, A) \, \mathrm{d}\mu(\omega),$$

[4]In the deterministic case, $T(\cdot, \omega')$ corresponds to Dirac measure $\delta_{\omega'}$.

it yields a new problem, which has first demonstrated in [Isreal paper] and widely argued in [vEFS]: μ' might fail to be Gibbsian, that is, it might fail to be consistent with any suitable Hamiltonian, yielding that the map $\mu' \mapsto \mathcal{H}'$ might not be defined for all image measures μ'. This led authors of vEFS to reformulate the formalism, for which several results were proven and many new open questions posed.

This thesis in particular is concerned with a established conjecture, stated in the following chapter, regarding the sufficient condition for Gibbsianity of μ', in a certain deterministic setting, which was already heavily explored in [Berghout].

To conclude this section, we provide two examples of RG transformations, one deterministic (decimation) and one stochastic (majority vote).

Example 1.18 (Decimation of Ising model)**.** Given $\Omega = \Omega' = \{-1,+1\}^{\mathbb{Z}^d}$, consider an integer $b \geq 2$. The decimation map $T = T_b : \Omega \to \Omega'$ is given so that for $\omega' = T(\omega)$,

$$\omega'(x) \;=\; \omega(bx), \quad x \in \mathbb{Z}^d,$$

where $b(x_1, \ldots, x_d) = (bx_1, \ldots, bx_d)$. More generally one could also consider a homomorphism $R : \mathbb{Z}^d \to \mathbb{Z}^d$ with $\det R \neq 0$ and define $T = T_R$ so that $\omega'(x) = \omega(R(x))$. Pathologies of the renormalized Gibbs measure(s) of this model were already pointed out by Israel in [paper] and rigorously treated in by van Enter, Fernández and Sokal in [vEFS]. In particular, for $d = 2$ and $h = 0$, it was verified in [vEFS] that for $b = 2$ and any $\beta > \frac{1}{2}\cosh^{-1}(1+\sqrt{2}) \approx 1.73\beta_c$, transformed Gibbs measure is not Gibbsian. On the other hand, it was demonstrated in [paper] by Haller and Kennedy, that for the same choice of d, h and b, taking $\beta < 1.36\beta_c$ results in Gibbsian transformed measure. It was also verified in [vEFS], see Theorem 4.3, that for $h = 0$ and any choice of $d \geq 2$, $b \geq 2$, there's always some large β_* so that choosing $\beta \geq \beta_*$ results in a non-Gibbsian renormalized measure.

Example 1.19 (Majority vote on Ising model)**.** Again taking $\Omega = \Omega' = \{-1,+1\}^{\mathbb{Z}^d}$, we consider an integer $b \geq 1$. We now let Λ_* be a fixed finite subset of $\mathbb{Z}^d$ (also referred to as *block*) and write $\Lambda_x = \Lambda_* + bx$ for the translations of Λ_* by bx, that is, $\Lambda_x = \{bx + y : y \in \Lambda_*\}$. The majority vote map (or rather kernel) $T = T_{b,\Lambda_*}$ is given so that for each $x \in \mathbb{Z}^d$,

$$\mathrm{Law}(\omega'(x)) \;=\; \begin{cases} \delta_{+1} : & \sum_{y\in\Lambda_x} \omega(y) > 0, \\ \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1} : & \sum_{y\in\Lambda_x} \omega(y) = 0, \\ \delta_{-1} : & \sum_{y\in\Lambda_x} \omega(y) < 0, \end{cases}$$

that is $\omega'(x)$ equals $+1$ (resp., -1) if more sites in Λ_x "vote" for $+1$ than for -1 (resp., more sites "vote" for -1 than for $+1$), and is decided by a fair coin flip in case of a draw. It was conjectured in vEFS, that for $d = 2$ and $b \geq 3$ odd[5] (with Λ_* a $b \times b$ box), one can always find β_* sufficiently large, so that choosing $\beta \geq \beta_*$ results in a non-Gibbsian renormalized measure. This was in fact verified for the case $b = 7$, see Theorem 4.5. Any progress since then?

Maybe a subsection on formalism from Chapter 3 in [vEFS]?

[5] Notice that for b odd, a draw cannot occur, so T becomes a deterministic transformation.

2 Fuzzy Gibbs formalism

In this chapter, we introduce a type of deterministic renormalization transformations, with which the rest of the thesis is concerned – the *fuzzy maps*. A sentence or two more. This chapter is mostly based on the Chapter 5 in [Berghout's thesis].

2.1 Set-up

In the remaining of this thesis, we consider two configuration spaces, $\Omega = \mathcal{A}^{\mathbb{Z}^d}$ and $\Sigma = \mathcal{B}^{\mathbb{Z}^d}$, where $\mathcal{A}$ and $\mathcal{B}$ are finite alphabets, such that $|\mathcal{A}| \geq |\mathcal{B}|$. Spaces Ω and Σ are endowed with the corresponding cylinder-generated σ-algebras $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. We consider a surjective map $\pi : \mathcal{A} \to \mathcal{B}$, which automatically induces a surjective map $\pi^* : \Omega \to \Sigma$ and a collection of surjective maps $\pi^*_V : \Omega_V \to \Sigma_V$ for $V \subset \mathbb{Z}^d$ (where $\Omega_V = \mathcal{A}^V$ and $\Sigma_V = \mathcal{B}^V$, endowed with $\mathcal{F}_V$ and $\widetilde{\mathcal{F}}_V$). We abuse notation and write π to refer to all of them, as it should be always clear on what space we apply it. We call π a *fuzzy map* or a *(single-block) factor map*.

As indicated in the previous chapter, we are interested in the distributions of the renormalized system, in particular, in the distributions that arise:

Definition 2.1. Suppose μ is a Gibbs measure on Ω, that is, $\mu \in \mathcal{G}_\Omega(\Phi)$ for some $\Phi \in \mathscr{B}^1(\Omega)$. We define a measure ν on Σ by

$$\nu \;:=\; \mu \circ \pi^{-1},$$

which we refer to as a *fuzzy Gibbs measure*.

It was argued in the previous chapter, that we cannot immediately assume that ν is Gibbsian. The doubt is justified, as indeed, in general, fuzzy Gibbs measures aren't Gibbsian. Referring to the equivalent conditions for Gibbsianity in [Proposition], one can in fact show that the uniform non-nullness is always inherited by the fuzzy Gibbs measure, so non-Gibbsianity is always a consequence of non-quasilocality of conditional probabilities.

In many cases, the problem can be sourced back to a rather intuitive reason, which we refer to a the *hidden phase transitions*. Due to the assumptions on $\mathcal{A}$, $\mathcal{B}$ and π, many different configurations in Ω can be mapped to the same configuration in Σ. This may or may not prove problematic, depending on the properties of the configurations sharing the image. This motivates us to introduce the notions of *fibres*: given an image configuration $\sigma \in \Sigma$, one defines a fibre $\Omega_\sigma \subseteq \Omega$ as

$$\Omega_\sigma \;:=\; \pi^{-1}(\sigma).$$

Given an underlying interaction Φ (w.r.t. which μ is Gibbsian), one can also consider the system on Ω_σ equipped with Φ – in particular, one can ask question about $\mathcal{G}_{\Omega_\sigma}(\Phi)$, the Gibbs measures on Ω_σ w.r.t. Φ.

Definition 2.2. We say that $\pi : \Omega \to \Sigma$ *admits no hidden phase transitions*, if

$$|\mathcal{G}_{\Omega_\sigma}(\Phi)| \;=\; 1, \quad \forall \sigma \in \Sigma.$$

We will see later, that absence of hidden phase transition is indeed the desirable scenario, as it ensures Gibbsianity of the fuzzy Gibbs meausure. This fact proves very handy, as fibres are usually not difficult to describe. The focus of the last two chapters of this thesis will in fact be providing alternative proofs of Gibbsianity using this approach. On the other hand, presence of phase transitions can cause trouble, as Gibbsianity of the fuzzy Gibbs measure is concerned. They do not, however, guarantee anything: it will be demonstrated in the following example, that there are cases where hidden phase transitions are present and yet the fuzzy Gibbs measure remains Gibbsian.

Example 2.3 ([Berghout], pp. 138)**.** Consider (appropriate) space Ω_1, Ω_2 and consider

$$\Omega = \Omega_1 \times \Omega_2, \quad \Sigma = \Omega_1.$$

A natural $\pi : \Omega \to \Sigma$ to consider is a projection onto the first coordinate, i.e., $\pi(\omega_1, \omega_2) = \omega_1$. Let now $\Phi_1 \in \mathscr{B}^1(\Omega_1)$ be arbitrary and assume $\Phi_2 \in \mathscr{B}^1(\Omega_2)$ is such that $|\mathcal{G}_{\Omega_2}(\Phi_2)| > 1$. Letting $\Phi := \Phi_1 + \Phi_2$, one has $\Phi \in \mathscr{B}^1(\Omega)$ and given any $\mu \in \mathcal{G}_\Omega(\Phi)$, we have $\mu = \mu_1 \otimes \mu_2$, where $\mu_1 \in \mathcal{G}_{\Omega_1}(\Phi_1)$ and $\mu_2 \in \mathcal{G}_{\Omega_2}(\Phi_2)$. Given arbitrary $\sigma \in \Sigma = \Omega_1$, one has $\Omega_\sigma = \{\sigma\} \times \Omega_2$ and hence $\mathcal{G}_{\Omega_\sigma}(\Phi) = \{\delta_\sigma\} \otimes \mathcal{G}_{\Omega_2}(\Phi_2)$, which is clearly not a singleton, so hidden phase transitions do indeed occur on all fibres. However, $\nu = \mu \circ \pi^{-1} = \mu_1$, which is indeed Gibbsian.

Thus, the notion of hidden phase transitions does not solve the problem of sufficient and necessary conditions for Gibbsianity of a fuzzy Gibbs measure by itself. In fact, the problem is still open, though there is a conjecture due to van Enter, Fernández and Sokal [vEFS], also referred to as the *van Enter-Fernández-Sokal hypothesis*, which predicts that Gibbsianity of the fuzzy Gibbs measure is equivalent to the existence of a certain continuous measure disintegration, which we will introduce in the next section.[6] One of the directions (disintegration implying Gibbsianity) has already been verified, by Berghout and Verbitskiy [source]. The other direction, believed to be a more difficult problem, still remains unsolved. While this thesis does not tackle the problem directly, the proofs given to verify Gibbsianity of a fuzzy Gibbs measure also verify that the examples do not contradict this direction of the hypothesis. A better explanation of why this is the case will follow towards the end of the chapter, once we introduce a sufficient theoretical framework.

Example or two?

Section based on 5.2.2 in [Berghout]?

2.2 Conditional measures, part 1: continuous measure disintegrations

Some blahblah?

Definition 2.4. A family of measures $\{\mu_\sigma : \sigma \in \Sigma\} \subset \mathcal{M}_1(\Omega)$ is called a *family of conditional measures for μ on fibres* if

(i) $\mu_\sigma(\Omega_\sigma) = 1$ for each $\sigma \in \Omega$, i.e., each conditional measure is supported its corresponding fibre,

[6] This is not the original formulation from [vEFS], but a correction due to Berghout and Verbitskiy [source].

(ii) for any choice of $f \in \mathrm{L}^1(\mu)$, the map $y \mapsto \int_{\Omega_\sigma} f \, \mathrm{d}\mu_\sigma$ is measurable and

$$\int_\Omega f \, \mathrm{d}\mu \;=\; \int_\Sigma \Big[\int_{\Omega_\sigma} f \, \mathrm{d}\mu_\sigma \Big] \mathrm{d}\nu(\sigma).$$

Remark 2.5. Given a continuous surjection $\pi : \Omega \to \Sigma$, such disintegration exists for any Borel (probability) measure on Ω.

`Mention the` $\int_{\Omega_\sigma} f \, \mathrm{d}\mu_\sigma = \mathbb{E}_\mu[f|\pi^{-1}\widetilde{\mathcal{F}}]$ `interpretation?`

As previously mentioned, the conjecture requires an additional continuity assumption:

Definition 2.6. We say that the family $\{\mu_\sigma : \sigma \in \Sigma\}$ defined above is a *continuous measure disintegration* if the map $\sigma \mapsto \mu_\sigma$ is weakly continuous, that is, if for each $f \in C(\Omega, \mathbb{R})$, the map $y \mapsto \int_{\Omega_\sigma} f \, \mathrm{d}\mu_\sigma$ is continuous on Σ.

The above notion is precisely the one, which is conjecture to be the equivalent condition for Gibbsianity of the fuzzy Gibbs measure:

Conjecture 2.7. Given $\mu \in \mathcal{G}_\Omega(\Phi)$, $\Phi \in \mathscr{B}^1(\Omega)$, and a fuzzy map $\pi : \Omega \to \Sigma$, $\nu = \mu \circ \pi^{-1}$ is Gibbsian if and only μ admits a continuous measure disintegration w.r.t. ν.

The already proven direction, due to Berghout and Verbitskiy, is the following:

Theorem 2.8 (`[Berghout's thesis]`)**.** Let $\mu \in \mathcal{G}_\Omega(\Phi)$ for some $\Phi \in \mathscr{B}^1(\Omega)$, $\pi : \Omega \to \Sigma$ a fuzzy map and suppose that μ admits a continuous measure disintegration $\{\mu_\sigma : \sigma \in \Sigma\}$ on fibres $\{\Omega_\sigma : \sigma \in \Sigma\}$ w.r.t. $\nu = \mu \circ \pi^{-1}$. Then ν is Gibbsian, i.e., there exists $\Psi \in \mathscr{B}^1(\Sigma)$ such that $\nu \in \mathcal{G}_\Sigma(\Psi)$.

Being an alternative sufficient condition for Gibbsianity of the fuzzy Gibbs measure, along with absence of hidden phase transition, one is tempted to ask how the two notions are related. Sadly, given the non-constructive nature of the above result, difficulties arise with exploring the properties of conditional measures given by the disintegration.

In the next section, we will define an alternative – constructive – approach to defining conditional measures, which turns out to be equivalent to the one presented in this section.

2.3 Conditional measures, part 2: Tjur points

In this section we present a constructive approach to defining conditional measure, which has an enormous advantage (over the previous one) of being rather intuitive. Considering some finite volume system, say on $\mathcal{A}^\Lambda$ with $\Lambda \Subset \mathbb{Z}^d$, one could simply define a conditional measure w.r.t. configuration $\sigma \in \mathcal{B}^\Lambda$ by

$$\mu^\sigma \;=\; \mu(\cdot|\pi^{-1}(\sigma)).$$

This approach, however, fails in the infinite volume setting, as we would possibly be conditioning on sets of measure zero, so one has to invent a workaround. The most logical approach appears to be considering conditional measures on "smaller-and-smaller" sets (of positive measure) containing σ, hoping to obtain some sort of "limit".

When trying to obtain such a "limit", one should be cautious, as there's many ways to "approach" a given configuration with sets containing it. We thus have to use the notion of nets. Recalling that nets are in general not indexed in $\mathbb{N}$ but in (upwards) directed sets, we need to define a directed set of "sets" containing, and in some sense approaching, σ. The reason for apostrophes in the last sentence is the fact, that instead of sets, we will consider pairs of sets, one open (for constructing the net) and one measurable (for conditioning). It is of course clear, that each $\sigma \in \Sigma$ will be assigned its own directed set.

Indeed, fix $\sigma \in \Sigma$ and define

$$D_\sigma \; := \; \Big\{(V, B) : V \text{ open neighbourhood of } \sigma,\, \widetilde{\mathcal{F}} \ni B \subseteq V,\, \nu(B) > 0\Big\}.$$

Following the idea of the previously mentioned conditional probability, one can define, for each pair $(V, B) \in D_\sigma$, a conditional measure μ^B on Ω, given by

$$\mu^B \; = \; \mu(\cdot|\pi^{-1}(B)).$$

Equipping D_σ with a partial order $\preceq$, given by

$$(V, B) \preceq (V', B') \iff V' \subseteq V,$$

to be interpreted as "(V', B') *is closer to* σ *than* (V, B)", defines an upwards directed set $(D_\sigma, \preceq)$, which turns the collection $\big\{\mu^B : (B, V) \in D_\sigma\big\}$ into a net, so we are able to talk about limits and related notions.

To be precise, the limits we are talking about are the weak limits: we say that limit μ^σ of $\big\{\mu^B : (B, V) \in D_\sigma\big\} \subset \mathcal{M}_1(\Omega)$ exists, if for any choice of $\varepsilon > 0$ and $f \in C(\Omega, \mathbb{R})$, there exists an open neighbourhood V of σ, such that for any $B \subseteq V$: $\nu(B) > 0$ one has

$$\Big|\int f \,\mathrm{d}\mu^\sigma - \int f \,\mathrm{d}\mu^B\Big| \; < \; \varepsilon.$$

Since Ω is compact, we automatically have that, if it indeed exists, μ^σ is a probability measure.

Definition 2.9. If the limit μ^σ of $\big\{\mu^B : (B, V) \in D_\sigma\big\}$ exists, we call it the *conditional distribution of* ω *given* $\pi(\omega) = \sigma$ and call σ a *Tjur point*. We write Σ_0 for the set of all Tjur points.

We mentioned before, that the conditional measures constructed in this section will be equivalent to ones defined in the previous section. This will be seen in the next two results. The first one tells us, that conditional measures μ^σ, provided "enough" of them exist, define a continuous measure disintegration:

Theorem 2.10 ([some Tjur paper], Theorem 4.1 and 5.1)**.** Suppose that the limit measure μ^σ exists for ν-almost all $\sigma \in \Sigma$, i.e., $\nu(\Sigma_0) = 1$. Then, for each $f \in \mathrm{L}^1(\Omega, \mathcal{F}, \mu)$,

(i) f is μ^σ-integrable, i.e., $f \in \mathrm{L}^1(\Omega, \mathcal{F}, \mu^\sigma)$,

(ii) $\sigma \mapsto \int_{\Omega_\sigma} f, \mathrm{d}\mu^\sigma$ is ν-integrable, i.e., belongs $\mathrm{L}^1(\Sigma, \widetilde{F}, \nu)$,

(iii) and finally,

$$\int_\Omega f \, \mathrm{d}\mu \;=\; \int_\Sigma \left[\int_\Omega f \, \mathrm{d}\mu^\sigma \right] \mathrm{d}\nu(\sigma).$$

Moreover, the map $\sigma \mapsto \mu^\sigma$ is (weakly) continuous on Σ_0.

The second results tells us, that this is in fact the same continuous measure disintegration as in the previous section.

Theorem 2.11 ([some Tjur paper], Theorem 7.1)**.** Let $\pi : \Omega \to \Sigma$ be a continuous, surjective map and μ a Radon measure on Ω. Then the following are equivalent:

(i) the family of measures $\{\mu_\sigma : \sigma \in \Sigma\}$ from [previous section] constitutes a continuous measure disintegration,

(ii) conditional distributions μ^σ are defined for all $\sigma \in \Sigma$, i.e., $\Sigma_0 = \Sigma$, and

$$\mu^\sigma \;=\; \mu_\sigma, \quad \forall \sigma \in \Sigma.$$

The results listed above are viable for a rather general Ω (and Σ). The specific spaces we consider, allow for a nice additional results, which allows us to drop the notion of nets and obtain limit purely through sequences. Very important again is the notion of cylinders. In particular we will be interested in cylinders on boxes $\mathbb{B}_n$: writing

$$C_n^\sigma \;:=\; [\sigma_{\mathbb{B}_n}] \;=\; \{\widetilde{\sigma} \in \Sigma : \sigma_{\mathbb{B}_n} = \widetilde{\sigma}_{\mathbb{B}_n}\},$$

we obtain a van Hove sequence $(C_n^\sigma)_{n\in\mathbb{N}}$, which proves to be sufficient to verify convergence on, assuming it is uniform:

Theorem 2.12 ([Berghout's thesis], Theorem 5.17)**.** Let $\Omega = \mathcal{A}^{\mathbb{Z}^d}$, $\Sigma = \mathcal{B}^{\mathbb{Z}^d}$, $\pi : \Omega \to \Sigma$ a fuzzy map, $\mu \in \mathcal{G}_\Omega(\Phi)$ for some $\Phi \in \mathscr{B}^1(\Omega)$ and $\nu = \mu \circ \pi^{-1}$. Then, the following are equivalent:

(i) $\Sigma_0 = \Sigma$, i.e., μ^σ exists for all $\sigma \in \Sigma$,

(ii) for all $\sigma \in \Sigma$, $\mu^{C_n^\sigma} \xrightarrow{w} \mu^\sigma$ uniformly in σ as $n \to \infty$.

If the family $\{\mu^\sigma : \sigma \in \Sigma\}$ is continuous, convergence of (ii) need not be uniform?

The following result, which is a direct consequence of the last two results, finally gives the conclusion, that the two notions of conditional measures are equivalent:

Theorem 2.13. The measure $\mu \in \mathcal{M}_1(\Omega)$ admits a continuous measure disintegration $\{\mu_\sigma : \sigma \in \Sigma\}$ under fuzzy map $\pi : \Omega \to \Sigma$ if and only if conditional measures $(\mu^{C_n^\sigma})_{n\in\mathbb{N}}$ convergence uniformly in σ as $n \to \infty$.

2.4 Conditional measures, part 3: hidden phase transitions

At the end of [Section 2.2], we raised the question of how (existence of) continuous measure disintegration might relate to hidden phase transitions. Given the equivalence established in [Section 2.3], we are now better equipped to explore this question.

It was pointed out that in some situations, hidden phase transitions are present, yet the fuzzy Gibbs measure remains Gibbsian, however, absence of hidden phase transition immediately gives the Gibbsianity. This is precisely what will be addressed in this section.

First we want to further explore the nets $\{\mu^B : (B,V) \in D_\sigma\}$, $\sigma \in \Sigma$, beyond the case of convergence, as several *accumulation points* may exist. Recall that a measure $\widetilde{\mu}$ is an accumulation point of $\{\mu^B : (B,V) \in D_\sigma\}$ if for arbitrary $f \in C(\Omega,\mathbb{R})$, $\varepsilon > 0$ and $V \ni \sigma$ open, one can always find a ($\widetilde{\mathcal{F}}$-)measurable set $B \subseteq V$ such that

$$\Big|\int f \,\mathrm{d}\mu^B - \int f \,\mathrm{d}\widetilde{\mu}\Big| \;<\; \varepsilon.$$

We will denote the set of all accumulation points of $\{\mu^B : (B,V) \in D_\sigma\}$ by $\overline{\mathfrak{M}}_\sigma$. It turns out that not only do accumulation points always exist (due to compactness of Ω), they are in our setting in fact always Gibbs measures:

Theorem 2.14 ([Berghout's thesis])**.** For each $\sigma \in \Sigma$, the following is true:

(i) $\overline{\mathfrak{M}}_\sigma \neq \emptyset$ and for each $\widetilde{\mu} \in \overline{\mathfrak{M}}_\sigma$, $\lambda_\sigma(\Omega_\sigma) = 1$.

(ii) If the original measure μ belongs to $\mathcal{G}_\Omega(\Phi)$, for some $\Phi \in \mathscr{B}^1(\Omega)$, then $\overline{\mathfrak{M}}_\sigma \subseteq \mathcal{G}_{\Omega_\sigma}(\Phi)$.

This yields two important corollaries. Firstly, having $|\overline{\mathfrak{M}}_\sigma| = 1$ for some $\sigma \in \Sigma$ is equivalent to $\sigma \in \Sigma_0$, so by the prior demonstration of equivalence of the two notions of conditional measure [result] (what about uniformity?) along with [Theorem] gives the following sufficient condition for Gibbsianity of the fuzzy Gibbs measure:

Corollary 2.15. Suppose $\nu = \mu \circ \pi^{-1}$ is a fuzzy Gibbs measure for some Gibbs measure μ under the fuzzy map π and assume that

$$|\overline{\mathfrak{M}}_\sigma| \;=\; 1, \quad \forall \sigma \in \Sigma.$$

Then ν is Gibbsian.

Given the [equivalence], this may be seen precisely as verification of one of the directions of the van Enter-Fernández-Sokal hypothesis, presented in [Conjecture], which may also be reformulated in a more accessible way:

Conjecture 2.16. In the setting of Conjecture,

$$\nu = \mu \circ \pi^{-1} \text{ is Gibbsian} \quad \Longleftrightarrow \quad |\overline{\mathfrak{M}}_\sigma| = 1 \, \forall \sigma \in \Sigma.$$

Moreover, given the property [Theorem.(ii)], we can establish that absence of hidden phase transitions indeed implies Gibbsianity of the fuzzy Gibbs measure:

Corollary 2.17. In the setting of [previous corollary], assume that

$$|\mathcal{G}_{\Omega_\sigma}(\Phi)| \;=\; 1, \quad \forall \sigma \in \Sigma,$$

i.e., hidden phase transitions do not occur. Then ν is Gibbsian.

2.5 Implications on what follows

The rest of the thesis will be concerned with two fuzzy Gibbs models, where certain sufficient conditions for Gibbsianity of the fuzzy Gibbs measure are known. In particular, the goal is to obtain alternative proofs of Gibbsianity of the fuzzy Gibbs measure, given the same conditions, using [Corollary]. While an alternative proof is a nice enough acquisition, the exact method we are to use provides a further information as van Enter-Fernández-Sokal hypothesis is concerned. Given the fact that absence of phase transitions implies existence of a continuous measure disintegration, such a proof additionally verifies, that the known example indeed does not contradict the yet unconfirmed direction of the hypothesis [Conjecture].

`Maybe a diagriam demonstrating what is argued above?`

The first example, explored in [Chapter 3] is concerned with the fuzzy Potts model, for which a certain sufficient condition was found by Häggström in [reference]. The alternative proof we give is completely independent, as the original proof in the paper is based on the characterisation of Gibbsianity presented in [Proposition.(i)], i.e., on verifying the quasilocality.

The second example, treated in [Chapter 4] is concerned with the random spin-flip dynamics model, explored in [vE,F,dH,R paper]. While the paper is not at all concerned with fuzzy Gibbs formalism – and neither can the general model presented be seen as a fuzzy Gibbs model – we demonstrate that a special case of the model, for which in particular some interesting results were give, can indeed be easily seen as a fuzzy Gibbs model. Morevoer, we demonstrate that the proof given in the paper are based on a very similar idea, allowing to give a proof using [Corollary] only by introducing minor modification to the original proof.

3 (Non-)Gibbsianity of fuzzy Potts model

This chapter aims to introduce the fuzzy Potts model and provide an alternative, independent proof of Häggström's result [reference] about conditions for the Gibbsianity of the involed fuzzy Gibbs measure, using the results due to Berghout and Verbitskiy [reference]. Moreover, it introduces the notion of random cluster representations, a powerful tool in the theory of Potts model, which is used in the proof.

3.1 Classical Potts model

In this introductory section of the chapter, we define the classical Potts model, with respect to which the fuzzy Potts model will be later defined. We also state the celebrated result the existence of phase transition with respect to the inverse temperature parameter.

The Potts model is a natural generalization of (ferromagnetic) Ising model with zero external field ($h = 0$), where instead of two possible values ($+1$ and -1, as in the context of the Ising model), the spins may assume $q \geq 2$ different values, taking $\mathcal{A} = \{1, \dots, q\}$ to be the single-spin space, and consequently $\Omega = \{1, \dots, q\}^{\mathbb{Z}^d}$ to be the configuration space. As in the case of the Ising model, Potts model is a nearest-neighbour model, so the interaction Φ is defined for $\Lambda = \{x, y\}$, with $x \sim y$ (i.e., $|x - y| = 1$) and is given by

$$\Phi_{x\sim y}(\omega) \;=\; 2\mathbb{1}_{\{\omega(x)\neq\omega(y)\}} - 1 \;=\; \begin{cases} 1: & \omega(x) \neq \omega(y), \\ -1: & \omega(x) = \omega(y). \end{cases}$$

One can check that this is (as interactions go) equivalent to the construction of the Ising model in [Example]: clearly, taking $q = 2$ and identifying $\{0, 1\}$ with $\{-1, +1\}$, one obtains the zero external field Ising model. The above interaction yields the following finite volume Hamiltonian expression: given a finite volume $\Lambda \Subset \mathbb{Z}^d$ and boundary condition $\xi \in \Omega$, we have

$$\mathcal{H}_{\Lambda;\beta}(\omega_\Lambda \xi_{\Lambda^c}) \;=\; \beta \sum_{(x,y)\in\widetilde{\Lambda}} (2\mathbb{1}_{\{\omega(x)\neq\omega(y)\}} - 1) + \beta \sum_{(x,y)\in\partial\widetilde{\Lambda}} (2\mathbb{1}_{\{\omega(x)\neq\xi(y)\}} - 1), \quad \omega \in \Omega,$$

where $\widetilde{\Lambda} := \{(x, y) : x \sim y,\, x, y \in \Lambda\}$ and $\partial\widetilde{\Lambda} := \{(x, y) : x \sim y,\, x \in \Lambda, y \in \Lambda^c\}$. We might, however, take a simpler equivalent interaction, which we justify shortly, in a remark. In the spirit of thermodynamical formalism, the system of Hamiltonians $(\mathcal{H}_\Lambda)_{\Lambda\Subset\mathbb{Z}^d}$ admits a specification $\gamma = (\gamma_\Lambda)_{\Lambda\Subset\mathbb{Z}^d}$, where for each $\Lambda \Subset \mathbb{Z}^d$ we have

$$\begin{aligned} &\gamma_\Lambda : \mathcal{F}_\Lambda \times \Omega_{\Lambda^c} \to (0, 1) \\ &(\{\omega_\Lambda\}, \xi_{\Lambda^c}) \mapsto \gamma_\Lambda(\omega_\Lambda|\xi_{\Lambda^c}) = \frac{1}{\mathcal{Z}^\xi_{\Lambda;\beta,q}} \exp(-\mathcal{H}_{\Lambda;\beta}(\omega_\Lambda\xi_{\Lambda^c})), \end{aligned}$$

where $\mathcal{Z}^\xi_{\Lambda;\beta,q} = \sum_{\widetilde{\omega}_\Lambda\in\Omega_\Lambda} \exp(-\mathcal{H}_\Lambda(\widetilde{\omega}_\Lambda\xi_{\Lambda^c}))$.

Remark 3.1. (1) Notice that

$$\mathcal{H}_{\Lambda;\beta}(\omega_\Lambda\xi_{\Lambda^c}) \;=\; 2\beta \sum_{(x,y)\in\widetilde{\Lambda}} \mathbb{1}_{\{\omega(x)\neq\omega(y)\}} + 2\beta \sum_{(x,y)\in\partial\widetilde{\Lambda}} \mathbb{1}_{\{\omega(x)\neq\xi(y)\}} - \beta(|\widetilde{\Lambda}| + |\partial\widetilde{\Lambda}|),$$

where $C_{\beta,\Lambda} := \beta(|\widetilde{\Lambda}| + |\partial\widetilde{\Lambda}|)$ is a constant independent of ω, ξ, giving us

$$\begin{aligned}
\gamma_\Lambda(\omega_\Lambda|\xi_{\Lambda^c}) &= \frac{\exp(-\mathcal{H}_{\Lambda;\beta}(\omega_\Lambda\xi_{\Lambda^c}))}{\sum_{\widetilde{\omega}_\Lambda\in\Omega_\Lambda}\exp(-\mathcal{H}_{\Lambda;\beta}(\widetilde{\omega}_\Lambda\xi_{\Lambda^c}))} \\
&= \frac{\exp\Big(-2\beta\Big[\sum_{(x,y)\in\widetilde{\Lambda}}\mathbb{1}_{\{\omega(x)\neq\omega(y)\}} + \sum_{(x,y)\in\partial\widetilde{\Lambda}}\mathbb{1}_{\{\omega(x)\neq\xi(y)\}}\Big]\Big)\exp(C_{\beta,\Lambda})}{\exp(C_{\beta,\Lambda})\sum_{\widetilde{\omega}_\Lambda\in\Omega_\Lambda}\exp\Big(-2\beta\Big[\sum_{(x,y)\in\widetilde{\Lambda}}\mathbb{1}_{\{\widetilde{\omega}(x)\neq\widetilde{\omega}(y)\}} + \sum_{(x,y)\in\partial\widetilde{\Lambda}}\mathbb{1}_{\{\widetilde{\omega}(x)\neq\xi(y)\}}\Big]\Big)} \\
&= \frac{\exp\Big(-2\beta\Big[\sum_{(x,y)\in\widetilde{\Lambda}}\mathbb{1}_{\{\omega(x)\neq\omega(y)\}} + \sum_{(x,y)\in\partial\widetilde{\Lambda}}\mathbb{1}_{\{\omega(x)\neq\xi(y)\}}\Big]\Big)}{\sum_{\widetilde{\omega}_\Lambda\in\Omega_\Lambda}\exp\Big(-2\beta\Big[\sum_{(x,y)\in\widetilde{\Lambda}}\mathbb{1}_{\{\widetilde{\omega}(x)\neq\widetilde{\omega}(y)\}} + \sum_{(x,y)\in\partial\widetilde{\Lambda}}\mathbb{1}_{\{\widetilde{\omega}(x)\neq\xi(y)\}}\Big]\Big)},
\end{aligned}$$

so the same specification would be obtained by selecting an alternative interaction and system of Hamiltonians

$$\begin{aligned}
\Phi'_{x\sim y} &= 2\mathbb{1}_{\{\omega(x)\neq\omega(y)\}}, \\
\mathcal{H}'_{\Lambda;\beta}(\omega_\Lambda\xi_{\Lambda^c}) &= 2\beta\sum_{(x,y)\in\widetilde{\Lambda}}\mathbb{1}_{\{\omega(x)\neq\omega(y)\}} + 2\beta\sum_{(x,y)\in\partial\widetilde{\Lambda}}\mathbb{1}_{\{\omega(x)\neq\sigma(y)\}}.
\end{aligned}$$

For the sake of simplicity, from now on, we use will use Φ' and $(\mathcal{H}'_{\Lambda;\beta})_{\Lambda\Subset\mathbb{Z}^d}$ in our computations instead, and simply denote them by Φ and $(\mathcal{H}_{\Lambda;\beta})_{\Lambda\Subset\mathbb{Z}^d}$.

(2) The interaction we initially used, is often also written in the literature as

$$\Phi_{x\sim y}(\omega) = 1 - 2\mathbb{1}_{\{\omega(x)=\omega(y)\}}.$$

Had we used this version, we would in point (1) obtain $\Phi'_{x\sim y}(\omega) = -2\mathbb{1}_{\{\omega(x)=\omega(y)\}}$ instead.

(3) There are several other interactions that equivalently define the Potts model, such as $\Phi''_{x\sim y}(\omega) = \mathbb{1}_{\{\omega(x)\neq\omega(y)\}}$ and $\Phi'''_{x\sim y}(\omega) = -\mathbb{1}_{\{\omega(x)=\omega(y)\}}$ for example.

In the spirit of thermodynamical formalism, the (infinite volume) Gibbs measure for q-state Potts model on Ω with inverse temperature β is any probability measure on Ω, whose finite-volume conditional probabilities agree with the specification γ, that is, a measure $\mu \in \mathcal{M}_1(\Omega)$ such that for each $\Lambda \Subset \mathbb{Z}^d$ and $\omega \in \Omega$, one has

$$\mu(\omega \text{ on } \Lambda|\xi \text{ off } \Lambda) = \gamma_\Lambda(\omega_\Lambda|\xi_{\Lambda^c}),$$

for μ-a.e. $\xi \in \Omega$.

Remark 3.2. Notice that for any fixed boundary condition $\xi \in \Omega$, the map $\gamma_\Lambda(\cdot|\xi_{\Lambda^c})$ is a probability measure on Ω_Λ, which we can denote as $\widetilde{\mu}^\xi_{\Lambda;\beta,q}$. We can, in this case, restate $\mu \in \mathcal{M}_1(\Omega)$ to be Gibbs measure for q-state Potts model with inverse temperature β if for each $\Lambda \Subset \mathbb{Z}^d$ and μ-a.e. $\xi \in \Omega$,

$$\mu(\cdot \text{ on } \Lambda|\xi \text{ off } \Lambda) = \widetilde{\mu}^\xi_{\Lambda;\beta,q}.$$

Moreover, in the spirit of [Remark], we can for each $\widetilde{\mu}^\xi_{\Lambda;\beta,q}$ define an analogue probability measure on Ω, given by

$$\mu^\xi_{\Lambda;\beta,q}(\omega) = \mathbb{1}_{\{\omega_{\Lambda^c}=\xi_{\Lambda^c}\}}\widetilde{\mu}^\xi_{\Lambda;\beta,q}(\omega_\Lambda), \quad \omega \in \Omega,$$

which one uses to obtain thermodynamical limits.

The Potts model enjoys a similar phase transition result as the Ising model, as is stated in the following theorem.

Theorem 3.3 (By whom?ACCN?)**.** For q-state Potts model on $\mathbb{Z}^d$ with $d \geq 2$, there exists a critical inverse temprature $\beta_c = \beta_c(d, q) \in (0, \infty)$ such that

(i) for each $\beta < \beta_c$ there is a unique Gibbs measure, and

(ii) for each $\beta > \beta_c$, there exist q mutually singular Gibbs measures.

It is worth mentioning that the mutually singular Gibbs measures in the $\beta > \beta_c$ regime are precisely the measures $\mu^1_{\beta,q}, \ldots, \mu^q_{\beta,q}$, where for $i \in \{1, \ldots, q\}$, $\mu^i_{\beta,q}$ is the thermodynamical limit

$$\mu^i_{\beta,q} \;=\; w\text{-}\lim_{\Lambda \Uparrow \mathbb{Z}^d} \mu^{\mathsf{i}}_{\Lambda;\beta,q},$$

where $\mathsf{i} = \{i\}^{\mathbb{Z}^d}$ is the constant boundary condition. In other words, $\mathrm{extr}(\mathcal{G}_\Omega(\Phi_{\beta,q})) = \left\{\mu^1_{\beta,q}, \ldots, \mu^q_{\beta,q}\right\}$

3.2 Fuzzy Potts model

We now turn our attention to the fuzzy Potts model, which we define and state the celebrated result about its (non-)Gibbsianity, due to Häggström [reference]. Moreover, we explain the strategy and structure of our alternative proof of (part of) the said result.

Consider the Potts model with spin space $\mathcal{A} = \{1, \ldots, q\}$, $q \geq 2$, on lattice $\mathbb{L}$, say $\mathbb{L} = \mathbb{Z}^d$, which defines a model on $\Omega = \{1, \ldots, q\}^{\mathbb{L}}$. The *fuzzy Potts model* is defined by considering some integer $1 < s < q$, so that the spin space is $\mathcal{B} = \{1, \ldots, s\}$ and the whole model defined on $\Sigma = \{1, \ldots, s\}^{\mathbb{L}}$. Moreover, we consider a vector $\mathrm{r} = (r_1, \ldots, r_s) \in \mathbb{N}^s$, such that $r_1 + \ldots + r_s = q$ and define a fuzzy transformation $\pi_{\mathrm{r}} : \{1, \ldots, q\} \to \{1, \ldots, s\}$ by putting

$$\pi_{\mathrm{r}}(a) \;:=\; \begin{cases} 1: & 1 \leq a \leq r_1, \\ 2: & r_1 + 1 \leq a \leq r_1 + r_2 \\ \ldots & \\ n: & \sum_{i=1}^{n-1} r_i + 1 \leq a \leq \sum_{i=1}^{n} r_i, \\ \ldots & \\ s: & \sum_{i=1}^{s-1} r_i + 1 \leq a \leq q, \end{cases}$$

i.e., $\pi_a = n$ iff $a \in (\sum_{i=1}^{n-1} r_i, \sum_{i=1}^{n} r_i] \cap \mathbb{N}$, $n \in \{1, \ldots, s\}$. In other words, the entire fuzzy map $\pi = \pi_{\mathrm{r}}$ is encoded by a single s-vector r.

Fixing $q \geq 2$, $\beta \geq 0$ and writing $\mu^{\mathbb{Z}^d,\xi}_{\beta,q}$ for the Gibbs measure of the Potts model on $\{1, \ldots, q\}^{\mathbb{Z}^d}$ for boundary condition $\xi \in \{\emptyset, 1, \ldots, q\}$ with inverse temperature β, the fuzzy transformation π_{r} induces the fuzzy Gibbs measure

$$\nu^{\mathbb{Z}^d,\xi}_{\beta,q,\mathrm{r}} \;:=\; \mu^{\mathbb{Z}^d,\xi}_{\beta,q} \circ \pi_{\mathrm{r}}^{-1}.$$

Of great interest is the potential Gibbsianity of such measure. `Something about the Häggström's result blahblahblah.` Recall that for $q \geq 2$ and $d \geq 2$, there exists $\beta_c(d, q) \in (0, \infty)$ such that for each $\beta < \beta_c(d, q)$, $\mu^{\mathbb{Z}^d,\emptyset}_{\beta,q} = \mu^{\mathbb{Z}^d,1}_{\beta,q} = \ldots = \mu^{\mathbb{Z}^d,q}_{\beta,q}$, i.e., there is a unique Gibbs measure of the Potts model on $\{1,\dots,q\}^{\mathbb{Z}^d}$ with inverse temperature β, while for each $\beta > \beta_c(d,q)$ there are q mutually singular Gibbs measures [free b.c. distrinc but not mut. singular?].

Theorem 3.4 (Häggström, 2003, [reference]). Let $d \geq 2$, $q \geq 3$, $\xi \in \{\emptyset, 1, \dots, q\}$ and $\mathrm{r} = (r_1, \dots, r_s)$ with $1 < s < q$, $r_1 + \ldots + r_s = q$, and write $r^* = \min_{1 \leq i \leq s} r_i$. Consider a fuzzy Gibbs measure $\nu^{\mathbb{Z}^d,\xi}_{q,\beta,\mathrm{r}} = \mu^{\mathbb{Z}^d,\xi}_{q,\beta} \circ \pi_{\mathrm{r}}^{-1}$.

(i) For each $\beta < \beta_c(d, r^*)$, the measure $\nu^{\mathbb{Z}^d,\xi}_{\beta,q,\mathrm{r}}$ is a Gibbs measure.

(ii) For any $\beta > \frac{1}{2} \log \frac{1+(r^*-1)p_c(d)}{1-p_c(d)}$, $\nu^{\mathbb{Z}^d,\xi}_{\beta,q,\mathrm{r}}$ is not a Gibbs measure. Here $p_c(d)$ denotes the critical probability for Bernoulli percolation, see [Appendix].

Remark 3.5. Given $q', q'' \geq 2$ such that $q' \leq q''$, one has $\beta_c(d,q') \leq \beta_c(d,q'')$. The requiring that β be smaller than the critical inverse temperature corresponding to the minimal r_i guarantees that $\beta < \beta_c(d,r_i)$ for all $i = 1,\dots,s$. This indeed yields uniqueness of Gibbs measure for Potts model on $\{1,\dots,s_i\}^{\mathbb{Z}^d}$ (with this choice of β) for all $i = 1,\dots,s$.

In light of the theory from the previous chapter, we are particularly interested in part (i) of the theorem, the Gibbs regime. The van Enter-Fernández-Sokal hypothesis, would suggest that, since for $\beta < \beta_c(d,r^*)$ the Gibbs property is preserved, each $\mu^{\mathbb{Z}^d,\xi}_{\beta,q}$ should admit a continuous disintegration in terms of $\nu^{\mathbb{Z}^d,\xi}_{\beta,q,\mathrm{r}}$. Moreover, proving the latter would – applying the result of [theorem reference] – constitute an alternative and independent proof of [Theorem].(i).

Given [Theorem reference], it is sufficient to verify (for a fixed $\beta < \beta_c(d,r^*)$), that for each $\sigma \in \Sigma = \{1,\dots,s\}^{\mathbb{Z}^d}$, there is a unique Gibbs measure for q-Potts model with inverse temperature beta on the fibre $\Omega_\sigma = \pi_{\mathrm{r}}^{-1}(\sigma)$, i.e., that

$$|\mathcal{G}_{\Omega_\sigma}(\Phi_{\beta,q})| \;=\; 1, \quad \forall \sigma \in \Sigma.$$

Luckily, one can express the fibres in a rather nice way, allowing for an easier procedure. Given $\sigma \in \Sigma$, we simply have

$$\Omega_\sigma \;=\; \pi_{\mathrm{r}}^{-1} \;=\; \prod_{i \in \mathbb{Z}^d} \pi_{\mathrm{r}}^{-1}(\sigma_i).$$

Writing $\mathcal{A}_j := \pi_{\mathrm{r}}^{-1}(j)$ and $U_j := \left\{ i \in \mathbb{Z}^d : \pi_{\mathrm{r}}(\sigma_i) = j \right\}$, $j = 1,\dots,s$, we could also write

$$\Omega_\sigma \;=\; \bigotimes_{j=1}^{s} \mathcal{A}_j^{U_j} \;:=\; \prod_{i \in \mathbb{Z}^d} \begin{cases} \mathcal{A}_1 : & i \in U_1, \\ \cdots & \\ \mathcal{A}_s : & i \in U_s. \end{cases}$$

One way of proving the uniqueness of q-Potts Gibbs measure on such Ω_σ for an appropriate inverse temperature β, is via the following two steps (so far stated informally):

(1) Given the assumption on β, we know that for each $j = 1,\dots,s$, there is a unique Gibbs measure for Potts model on $\mathcal{A}_j^{\mathbb{Z}^d}$ with inverse temperature β. We wish to show that this implies also uniqueness of Gibbs measure for Potts model on $\mathcal{A}_j^{U_j}$, given the same inverse temperature.

(2) Given the uniqueness of Gibbs measure for Potts model on $\mathcal{A}_j^{U_j}$ with inverse temperature β for all $j = 1, \dots, s$, we need to show that this implies the uniqueness of Gibbs measure for Potts model on $\bigotimes_{j=1}^{s} \mathcal{A}_j^{U_j}$, given the same inverse temperature.

It is clear that it is enough to verify the above steps for $s = 2$, as the argument for larger s follows immediately by induction. To be precise about what we wish to prove, we formulate two propositions that are to be verified, in order to obtain conclusion.

Proposition 3.6 (Part 1)**.** Let $\beta < \beta_c(d, q)$ (i.e., there's unique Gibbs measure for Potts model on $\{1, \dots, q\}^{\mathbb{Z}^d}$) and let $U \subset \mathbb{Z}^d$. Then, there's a unique Gibbs measure for Potts model on $\{1, \dots, q\}^U$ with inverse temprature β.[7]

Proposition 3.7 (Part 2)**.** Suppose $U, V \subset \mathbb{Z}^d$ form a partition of $\mathbb{Z}^d$ and let A, B be disjoint alphabets. Suppose that for $\beta \in (0, \infty)$, Potts model with inverse temperature β admits a unique Gibbs measure on both A^U and B^V (i.e., $|\mathcal{G}_{\mathsf{A}^U}(\Phi_{\beta,|\mathsf{A}|}| = |\mathcal{G}_{\mathsf{B}^V}(\Phi_{\beta,|\mathsf{B}|})| = 1)$. Then the same is the case for the Potts model with inverse temperature β on $\mathsf{A}^U \otimes \mathsf{B}^V$, i.e.,

$$|\mathcal{G}_{\mathsf{A}^U \otimes \mathsf{B}^V}(\Phi_{\beta,|\mathsf{A}|+|\mathsf{B}|})| \;=\; 1.$$

Before proving the proposition, we need to introduce some more tools, which are regulars in the mathematical statistical mechanics of lattice systems.

3.3 Preliminaries, part 1: stochastic domination

In this and the next section, we will introduce some tools, which will be used to prove [Proposition reference]. The section will cover the basics on stochastic domination, a well-established tool in mathematical statistical mechanics. In the section that follows, we introduce the theory of random cluster representations, which are important in study of the Potts model in particular.

Assume firstly that $\mathcal{A} \subset \mathbb{R}$ is linearly ordered; in our case, we are generally assuming $\mathcal{A}$ to be finite, though this theory can be extended to closed subsets of $\mathbb{R}$. Then linear order of $\mathcal{A}$ induces a natural (coordinatewise) partial order on $\Omega = \mathcal{A}^{\mathbb{L}}$: given two configurations $\xi, \xi' \in \Omega$, we write

$$\xi \preceq \xi' \iff \xi(i) \le \xi'(i), \; \forall i \in \mathbb{L}.$$

This allows us to introduce a notion of increasing functions:

Definition 3.8 (Increasing functions and events)**.**

(1) We say that a function $f : \Omega \to \mathbb{R}$ is *increasing* (or *non-decreasing*), if for each $\xi, \xi' \in \Omega$, $\xi \preceq \xi'$ implies $f(\xi) \le f(\xi')$.

(2) Equipped with the previous notion, we say that an event A (or a simply $A \subset \Omega$ measurable) is *increasing* if $\mathbb{1}_A$ is an increasing function.

Example 3.9. A very simple example of an increasing event is $\{x \leftrightarrow \infty\}$, the event that $x \in \mathbb{L}$ belongs to an infinite cluster, see [Appendix]. Indeed, if x belongs to some infinite cluster C, and we obtain a cluster C' by opening some additional edges of the percolation configuration, we get that $C \subseteq C'$ and hence $x \in C'$.

[7] The subset U is understood to be equipped with an edge set $\{(x, y) : x \sim y,\, x, y \in U\}$

This allows us to define a certain partial order on $\mathcal{M}_1(\Omega)$, the set of probability measures on Ω. `Some comment about motivation for defining stochastic domination/ordering on measures?`

Definition 3.10 (Stochastic domination)**.** Let μ and μ' be probability measures on Ω. We say that μ is *stochastically dominated* by μ', writing $\mu \preceq_{\mathcal{D}} \mu'$, if

$$\int f \, \mathrm{d}\mu \;\le\; \int f \, \mathrm{d}\mu', \quad \forall f \in \mathrm{BM}(\Omega, \mathbb{R}).$$

Remark 3.11. An important fact to note is, that given $\mu \preceq_{\mathcal{D}} \mu'$ and in increasing event A (such that $\{x \leftrightarrow \infty\}$ for examples, one has

$$\mu(A) \;\le\; \mu'(A).$$

A reverse inequality holds for decreasing event.

`Should add something, but it really depends on how the final proof of Part 1 will look like.`

3.4 Preliminaries, part 2: random cluster representations

We now present the *random cluster representations*, first introduced by Fortuin and Kesteleyn (after whom it is also known as the *Fortuin-Kesteleyn mode*, or simply *FK model*), which is an indispensable tool for studying phase transition in Ising and Potts model. First, we will introduce the theory for the case of finite graphs, which will give the much needed intuition, and continue to extend it to the more useful infinite lattice setting.

3.4.1 Random cluster on a finite graph

Throughout the section, we consider a finite graph $G = (V, E)$. The *random cluster measure*, defined shortly, is a percolation measure, i.e., a probability measure defined on $\{0,1\}^E$. Given $\eta \in \{0,1\}^E$, we write $G_\eta = (V, \eta)$ for the subgraph of G given by

$$G_\eta \;=\; \{V, \{e \in E : \eta(e) = 1\}\}.$$

Moreover, we define the quantity $k(\eta)$, which represents the number of connected components in G_η (including the isolated vertices).

Definition 3.12. The *random cluster measure* $\phi^G_{p,q}$ for G, with parameters $p \in [0,1]$ and $q > 0$, is the probability measure on $\{0,1\}^E$, given by

$$\phi^G_{p,q}(\eta) \;=\; \frac{1}{\mathcal{Z}^G_{p,q}} \left\{ \prod_{e \in E} p^{\eta(e)} (1-p)^{1-\eta(e)} \right\} q^{k(\eta)}, \quad \eta \in \{0,1\}^E,$$

where $\mathcal{Z}^G_{p,q}$ is the normalizing constant.

Clearly, $q = 1$ yields Bernoulli percolation measure, $\phi^G_{p,1} = \psi^G_p$, see `[Appendix]`; this is the only case where $\phi^G_{p,q}$ can be seen as a product measure (outside $p = 1$ and $p = 0$).

It turns out, that for the case of $q \in \{2, 3, \ldots\}$, there is an intimate relation with the Potts model. Writing $\mathcal{H}^G_\beta$ for the Hamiltonian of Potts model on $\{1, \ldots, q\}^V$ (in the sense of the finite volume setting, see [Section 1.1.1], we write

$$\mu^G_{\beta,q}(\omega) \;=\; \frac{1}{\mathcal{Z}^G_{\beta,q}} \exp(-\mathcal{H}^G_\beta(\omega)), \quad \omega \in \{1, \ldots, q\}^V,$$

for the finite volume Gibbs measure for Potts model on $\{1, \ldots, q\}^V$ with inverse temperature β. It turns out that $\mu^G_{\beta,q}$ and $\phi^G_{p,q}$ can be very fruitfully related via a certain coupling, introduced by Swendsen and Wang in [reference] and further explored by Edwards and Sokal in [reference].

Definition 3.13. Fix $q \in \{2, 3, \ldots\}$ and $p \in [0, 1]$. Let now $\mathsf{P}^G_{p,q}$ be a probability measure on $\{1, \ldots, q\}^V \times \{0, 1\}^E$, which corresponds to picking a random element according to the following procedure:

(1) To each $x \in V$, assign $X(x) \sim \mathrm{Unif}(1, \ldots, q)$, and to each $e \in E$, $Y(e) \sim \mathrm{Ber}(p)$, independently for all vertices and edges.

(2) Condition on the event that for any choice of $x \sim y$, $X(x) \neq X(y)$ imples $Y((x, y)) = 0$, i.e., vertices assuming different spin values are not connected by open edges.

In fact, one has

$$\mathsf{P}^G_{p,q}(\omega, \eta) \;\propto\; \prod_{e=(x,y)\in E} p^{\eta(e)}(1-p)^{1-\eta(e)} \mathbb{1}_{\{(\omega(x)-\omega(y))\eta(e)=0\}},$$

where the event $\{(\omega(x) - \omega(y))\eta(e) = 0\}$ enforces precisely the conditioning described in (2). We call $\mathsf{P}^G_{p,q}$ the *Edwards-Sokal measure.*

Theorem 3.14 ([GHM], Theorem 6.2)**.** Taking $\beta = \frac{1}{2}\log(1 - p)$, $\mathsf{P}^G_{p,q}$ constitutes a coupling of $\mu^G_{\beta,q}$ and $\phi^G_{p,q}$.

The usefulness of this coupling can be seen in the following corollary:

Corollary 3.15. One can very simply “obtain” $\mu^G_{\beta,q}$ from $\phi^G_{p,q}$, and the other way around.

(i) Let $p = 1 - e^{-2\beta}$. Suppose we pick a $\{1, \ldots, q\}^V$-values random configuration X as follows:

  (1) Pick a random edge configuration $Y \sim \phi^G_{p,q}$.

  (2) For each connected component C of $G_Y = (V, Y)$, pick $Z_C \sim \mathrm{Unif}(1, \ldots, q)$ (for each C independently) and assign to each vertex in C the value assumed by Z_C.

  Then, $X \sim \mu^G_{\beta,q}$.

(i) Let again $p = 1 - e^{-2\beta}$. Suppose now we pick a $\{0, 1\}^E$-valued random edge configuration Y according to the following procedure:

  (1) Pick a random configuration $X \sim \mu^G_{\beta,q}$.

  (2) For each $e = (x, y) \in E$ independently, assign

$$Y(e) \;\sim\; \begin{cases} \mathrm{Ber}(p): & X(x) = X(y), \\ \delta_0: & X(x) \neq X(y). \end{cases}$$

  Then, $Y \sim \phi^G_{p,q}$.

3.4.2 Infinite volume random cluster and phase transitions

To be written after the proof of Part 1 if finalized. Should include:

(i) General (DLR-type) definition of infinite volume random cluster measure.

(ii) Expression of $\phi(e$ open$|\eta)$.

(iii) Definition of relevant infinite volume limits, with some properties.

(iv) Statement of the theorem relating uniqueness of Potts Gibbs measure and $\phi^1_{p,q}(x \leftrightarrow \infty) = 0$.

3.5 Proof of Part 1

3.6 Proof of Part 2

4 (Non-)Gibbsianity of random spin-flip dynamics

In this chapter, we introduce the random spin-flip dynamics model, which was explored by van Enter, Fernández, den Hollander and Redig in [reference], and proceed to demonstrate, that a special case of this model may be seen as a fuzzy Gibbs model. We present some results about conditions for Gibbsianity and demonstrate that in the fuzzy Gibbs case, the proofs are analogue to our approach, which in turn verifies, that the example does not contradict the van Enter-Fernández-Sokal hypothesis.

4.1 The general model

We proceed to introduce the model, as defined in [reference], with some minor changes in notation. In principle, we are interested in a (probability) measure valued stochastic process $(\nu_t)_{t\geq 0}$, starting from a Gibbs measure. In particular the question the original paper explores is how the conditions on the starting measure and on process-inducing dynamics influence the (non-)Gibbsianity of ν_t, $t > 0$.

Define first $\Omega_0 := \{-1, +1\}^{\mathbb{Z}^d}$ to be the configuration space. We first consider dynamics on configurations, which is governed by spin-flip rates $\mathbb{Z}^d \times \Omega_0 \ni (x, \omega) \mapsto c(x, \omega)$, which satisfy the following conditions:

(i) Finite range: for any $x \in \mathbb{Z}^d$, the map $c_x := (\omega \mapsto c(x, \omega))$ is a local function of ω, with $\operatorname{diam}(\mathcal{D}_{c_x}) \leq R < \infty$.

(ii) Translation invariance: for any $x \in \mathbb{Z}^d$, $\tau_x c_0 = c_x$.

(iii) Strict positivity: for all $x \in \mathbb{Z}^d$ and $\omega \in \Omega$, $c(x, \omega) > 0$.

Remark 4.1. It is a direct consequence of assuming (i)-(iii), that there's $m, M \in (0, \infty)$, such that

$$0 \;<\; m \;\leq\; c(x,\omega) \;\leq\; M \;<\; \infty, \quad \forall x \in \mathbb{Z}^d, \omega \in \Omega.$$

The idea behind the dynamics is as follows: one starts with a (possibly random) configuration ω_0 at time 0 and flips (random) spins at random times, which are being governed by $(c_x)_{x\in\mathbb{Z}^d}$. That is, site $x \in \mathbb{Z}^d$ is being flipped at (exponential?) rate c_x, which in general depends on the current configuration, through its value at x and a certain neighbourhood of x. Note that unless $\operatorname{diam}(\mathcal{D}_{c_x}) = 0$ for each $x \in \mathbb{Z}^d$ (i.e., $c(x, \omega) = c(x, \omega(x))$), the flips of individual sites are not independent.[8]

The goal now is to describe a semigroup $(S(t))_{t\geq 0}$ that will describe the dynamics induced by $(c_x)_{x\in\mathbb{Z}^d}$. This motivates us to define a generator L on the space of local functions; writing ω^x for the configuration that agrees with ω off x and with $-\omega$ on x, we can define

$$Lf \;=\; \sum_{x\in\mathbb{Z}^d} c_x(f(\omega^x) - f(\omega)), \quad f \in \mathcal{L}.$$

It is known [reference?] that the closure of L on $\mathcal{C}(\Omega_0)$ constitutes a generator of a unique Feller process $(\omega_t)_{t\geq 0}$, with $\mathbb{P}_\omega$ the corresponding path-measure, conditional

[8] This scenario, however, will be the one we will focus on later on.

on $\omega_0 = \omega$. Furthermore, we obtain the associated semigroup $(S(t))_{t\geq 0}$, which is given by

$$S(t) \;=\; \exp(tL), \quad t \geq 0.$$

It is precisely this semigroup, that in turn also gives us dynamics on measures. To be precise, $(S(t))_{t\geq 0}$ acts on probability measures via

$$\int_{\Omega_0} [S(t)f](\omega)\,\mathrm{d}\nu(\omega) \;=\; \int_{\Omega_0} f(\omega)\,\mathrm{d}[\nu S(t)](\omega), \quad f \in \mathcal{L},\, \nu \in \mathcal{M}_1(\Omega),\, t \geq 0.$$

In a more explicit language, the measure $\nu S(t)$ assigns a measurable set $A \subset \Omega_0$ a value $[\nu S(t)](A) = \int S(t)\mathbb{1}_A\,\mathrm{d}\nu$.

The interpretation turn out to be very intuitive. Letting $\omega_0 \in \nu$ be the initial configuration and running the spin-flip dynamics the measure, then

$$\nu S(t) \;=\; \mathrm{Law}(\omega_t), \quad \forall t \geq 0.$$

Of central interest is the situation where $\omega_0 \sim \mu$, where μ is a Gibbs measure, that is, $\mu \in \mathcal{G}_{\Omega_0}(\Phi)$ for some translation invariant, finite range good interaction Φ. It is also worth noting, that the dynamics induced by $(c_x)_{x\in\mathbb{Z}^d}$ admits (was this assumption or fact) a L-invariant[9] Borel probability measure ρ on Ω_0 which is *reversible*, that is,

$$\int_{\Omega_0} (Lf)g\,\mathrm{d}\rho \;=\; \int_{\Omega_0} f(Lg)\,\mathrm{d}\rho, \quad \forall f, g \in \mathcal{L}.$$

Writing ρ^x for the law of ω^x where $\omega \sim \rho$, the reversibility of ρ for L-induced spin flip dynamics is equivalent to

$$\frac{\mathrm{d}\rho^x}{\mathrm{d}\rho} \;=\; \frac{c(x,\omega)}{c(x,\omega^x)}, \quad \forall x \in \mathbb{Z}^d,\, \sigma \in \Omega_0,$$

which implies that there exists a continuous version of Radon-Nikodým derivative $\frac{\mathrm{d}\rho^x}{\mathrm{d}\rho}$. It then follows from Proposition 2.2. in `[reference]` that there exists a good interaction Ψ_ρ such that $\rho \in \mathcal{G}_{\Omega_0}(\Psi_\rho)$. Since we assumed $(c_x)_{x\in\mathbb{Z}^d}$ to have finite range and be translation invariant, we can choose Ψ_ρ to be of finite range and translation invariant as well.

The most general result given in the paper is the following:

Theorem 4.2. Let $\mu \in \mathcal{G}_{\Omega_0}(\Phi)$ be the law of the original configuration and $\rho \in \Phi(\Psi)$ the reversible measure associated with $(c_x)_{x\in\mathbb{Z}^d}$-dynamics, with Φ, Ψ finite range and translation invariant. Then there exists $t_0 = t_0(\mu, \rho) > 0$, such that $\mu S(t)$ is Gibbsian for each $0 \leq t \leq t_0$.

`A brief comment.`

4.2 Infinite-temperature dynamics

A particularly nice setting, which my much nicer computational prospects is one of "infinite-temperature" dynamics, which means simply that spin flips of sites are

[9] $\int Lf\,\mathrm{d}\rho = 0$ for each $f \in \mathcal{L}$.

independent of each other, i.e., the path measure $\mathbb{P}_\omega$ may be expressed as a product measure,

$$\mathbb{P}_\omega \;=\; \bigotimes_{x\in\mathbb{Z}^d} \mathbb{P}_{\omega(x)}.$$

This regime corresponds to c_x depending on configuration only through its values at site x, that is, $\operatorname{diam}(\mathcal{D}_{c_x}) = 0$ for each $x \in \mathbb{Z}^d$. We will see why this model is computationally much simpler in the next section.

In this section, we will present some results from [reference] regarding the Gibbsianity of $(\mu S(t))_{t\geq 0}$ as well as present their proofs, which we will in the next section argue to be almost analogous to our approach.

Before that, we present a result that is the central tool used in those proofs. If $\mu \in \mathcal{G}_{\Omega_0}(\Phi)$ is the law of the initial configuration, write $\mathcal{H} = (\mathcal{H}_\Lambda)_{\Lambda\Subset\mathbb{Z}^d}$ for the Hamiltonians consistent with Φ. In all the proofs that will be presented, one argues by considering the distributions $\widehat{\mu}_t$ of random variables of form (ω_0, ω_t), $t \geq 0$. Formally, $\widehat{\mu}_t$ enjoys the following correspondence with μ and $S(t)$:

$$\int_{\Omega_0^2} f(\sigma)g(\eta)\,\mathrm{d}\widehat{\mu}_t(\sigma,\eta) \;=\; \int_{\Omega_0} f(\omega)[S(t)g](\omega)\,\mathrm{d}\mu(\omega), \quad f, g \in \mathcal{L}.$$

Technically speaking, due to its definition as a joint distribution, $\widehat{\mu}_t$ has “better chances” of being Gibbsian than $\mu S(t)$. (Elaboration? Any implication?) Assuming that it is indeed Gibbsian and writing $(p_t^x)_{t\geq 0}$ for the transition kernel associated with the dynamics at site x, the Hamiltonians associated with $\widehat{\mu}_t$ may be expressed as

$$\mathcal{H}_\Lambda^{(t)}(\sigma_\Lambda, \eta_\Lambda) \;=\; \mathcal{H}_\Lambda(\sigma_\Lambda) - \sum_{x\in\Lambda} \log p_t^x(\sigma(x), \eta(x)), \quad \sigma, \eta \in \Omega_0.$$

Another assumption that authors make in this section is, that the generators of local spin-sites $(L_x)_{x\in\mathbb{Z}^d}$ are independent of x (i.e., are all the same), and are of form

$$L_x \;=\; \frac{1}{2}\begin{pmatrix} -1+\varepsilon & 1-\varepsilon \\ 1+\varepsilon & -1-\varepsilon \end{pmatrix}, \quad 0 \leq \varepsilon < 1,$$

which indeed yields that rates $(p_t^x)_{x\in\mathbb{Z}^d}$ are also independent of x, hence all the same.

The following result is the first part of the Proposition 3.7. in [reference]:

Proposition 4.3. Assume that $\widehat{\mu}_t$ is indeed Gibbsian. If for each fixed $\eta \in \Omega_0$, there is a unique Gibbs measure associated with the Hamiltonians $(\mathcal{H}_\Lambda^{(t)}(\cdot,\eta))_{\Lambda\Subset\mathbb{Z}^d}$ (which one could denote by $|\mathcal{G}_{\Omega_0}(\mathcal{H}^{(t)}(\cdot,\eta))| = 1$), then $\nu S(t)$ is Gibbsian.

One can quickly notice a similarity with Berghout's resut.

Background about Dobrushin's uniqueness condition/high temperature Gibbs measures, if not done before.

The first result of interest is rather general, as it assumes nothing about the Gibbs measure of the initial configuration, other than its uniqueness w.r.t. the associated interaction:

Theorem 4.4. Let μ be either an infinite- or high-temprature Gibbs measure, that is, $\mu \in \mathcal{G}_{\Omega_0}(\Phi)$, where Φ is such that $|\mathcal{G}_{\Omega_0}(\Phi)| = 1$. Then, $\mu S(t)$ is Gibbsian for each $t \geq 0$.

Before stating the proof, we remark that $\mathcal{H}^t_\Lambda$ may be rewritten as

$$\mathcal{H}^{(t)}_\Lambda(\sigma_\Lambda, \eta_\Lambda) = \mathcal{H}_\Lambda(\sigma_\Lambda) - \sum_{x\in\Lambda} h^x_1(t)\sigma(x) - \sum_{x\in\Lambda} h^x_2(t)\eta(x) - \sum_{x\in\Lambda} h^x_{1,2}(t)\sigma(x)\eta(x), \quad \sigma,\eta \in \Omega_0,$$

where

$$\begin{aligned} h^x_1(t) &= \frac{1}{4}\log\frac{p^x_t(+1,+1)p^x_t(+1,-1)}{p^x_t(-1,+1)p^x_t(-1,-1)}, \\ h^x_2(t) &= \frac{1}{4}\log\frac{p^x_t(+1,+1)p^x_t(-1,+1)}{p^x_t(+1,-1)p^x_t(-1,-1)} \\ h^x_{1,2}(t) &= \frac{1}{4}\log\frac{p^x_t(+1,+1)p^x_t(-1,-1)}{p^x_t(+1,-1)p^x_t(-1,+1)}. \end{aligned}$$

The assumption of independence of p^x_t from x, allows us to simply write $h_1, h_2, h_{1,2}$. In fact, the precise expression by which L_x is given also allows us to write those values out nicely, as can be seen in Equation (5.9) in [reference], but is for our purposes unimportant.

Proof. In this case, we know the joint distribution $\widehat{\mu}_t$ of (ω_0, ω_t) to be Gibbsian and consistent with the Hamiltonians $\mathcal{H}^{(t)}$, which can – due to assumptions of independence of L_x from x – be written out as

$$\mathcal{H}^{(t)}_\Lambda(\sigma_\Lambda, \eta_\Lambda) = \mathcal{H}_\Lambda(\sigma_\Lambda) - \sum_{x\in\Lambda}\left(h_1(t) + h_{1,2}(t)\eta(x)\right)\sigma(x) - h_2(t)\sum_{x\in\Lambda}\eta(x).$$

We wish to show, that for each fixed $\eta \in \Omega_0$, $|\mathcal{G}_{\Omega_0}(\mathcal{H}^{(t)}(\cdot,\eta))| = 1$, as the conclusion then follows from [reference of Proposition]. Indeed, fix arbitrary $\sigma \in \Omega_0$; we immediately notice, that the last term on the RHS is constant in σ, so we can consider an alternative collection of Hamiltonians $\widetilde{\mathcal{H}}^{(t)}(\cdot;\eta)$, given by

$$\widetilde{\mathcal{H}}^{(t)}_\Lambda(\sigma_\Lambda;\eta) = \mathcal{H}_\Lambda(\sigma_\Lambda) - \sum_{x\in\Lambda}\left(h_1(t) + h_{1,2}(t)\eta(x)\right)\sigma(x),$$

which induces the same specification as $\mathcal{H}^{(t)}(\cdot,\eta)$, and hence

$$\mathcal{G}_{\Omega_0}(\widetilde{\mathcal{H}}^{(t)}(\cdot;\eta)) = \mathcal{G}_{\Omega_0}(\mathcal{H}^{(t)}(\cdot,\eta)).$$

Since $\widetilde{\mathcal{H}}^{(t)}(\cdot;\eta)$ differs from $\mathcal{H}$ only in the single site interaction, by [Lemma], $\widetilde{\mathcal{H}}^{(t)}(\cdot;\eta)$ satisfies [Dobrushin] if and only if $\mathcal{H}$ satisfies [Dobrushin]. $\mathcal{H}$ indeed satisfies it by assumption, from which it follows that $|\mathcal{G}_{\Omega_0}(\widetilde{\mathcal{H}}^{(t)}(\cdot;\eta))| = 1$ and hence

$$|\mathcal{G}_{\Omega_0}(\mathcal{H}^{(t)}(\cdot,\eta))| = 1.$$

As this works for any choice of η, [reference of Proposition] allows us to conclude. $\square$

Another example the authors consider is one of the Ising model in the non-uniqueness regime, in particular of $\beta \gg \beta_c$. Here they make an additional assumption on the dynamics, which is that $\varepsilon = 0$ in the definition of L_x, which translates on neither $+1$ nor -1 being favoured, and results in $h_1 \equiv 0$, $h_2 \equiv 0$ and $h_{1,2}(t) =: h_t = -\frac{1}{2}\log\tanh(\frac{t}{2})$.

Theorem 4.5. Let $\mu_{\beta,h}$ denote some Gibbs measure of Ising model on Ω_0 with $\beta \gg \beta_c$.

(1) There exists a $t_0 = t_0(\beta, h)$, such that $\nu_{\beta,h}S(t)$ is Gibbsian for all $0 \leq t \leq t_0$.

(2) Moreover, if $h > 0$, then there exists a $t_1 = t_1(h)$, such that $\mu_{\beta,h}S(t)$ is Gibbsian for all $t \geq t_1$.

Proof. Recalling that the Hamiltonian of the Ising model is of form

$$\mathcal{H}(\sigma) = -\beta \sum_{(x,y)} \sigma(x)\sigma(y) - h \sum_x \sigma(x),$$

slightly ignoring the finite volume and boundary condition business, we obtain that $\mathcal{H}^{(t)}$ is of form

$$\mathcal{H}^{(t)}(\sigma,\eta) \;=\; -\beta \sum_{(x,y)} \sigma(x)\sigma(y) - h\sum_x \sigma(x) - h_t \sum_x \sigma(x)\eta(x).$$

(1) We want to exploit the fact that for small t, the "dynamical field" h_t is large and – intuitively speaking – "forces" σ in "direction" of η. In this spirit, we wish to rewrite the joint Hamiltonian as

$$\begin{aligned}\mathcal{H}^{(t)}(\sigma,\eta) \;&=\; \sqrt{h_t}\left(-\frac{\beta}{\sqrt{h_t}} \sum_{(x,y)} \sigma(x)\sigma(y) - \frac{h}{\sqrt{h_t}} \sum_x \sigma(x) - \sqrt{h_t}\sum_x \sigma(x)\eta(x)\right) \\ &=: \; \sqrt{h_t}\widehat{\mathcal{H}}^{(t)}(\sigma,\eta).\end{aligned}$$

It is known from `reference` that for fixed $\eta \in \Omega_0$ and for $0 \leq t \leq t_0'$ small enough, η is the unique ground state of $\widehat{\mathcal{H}}^{(t)}(\cdot,\eta)$, so for any $\lambda \geq \lambda_0$ large enough, $\lambda\widehat{\mathcal{H}}^{(t)}$ satisfies `[Dobrushin]`. It follows that for $0 \leq t \leq t_0$ such that $\sqrt{h_t} \geq \lambda_0$, $|\mathcal{G}_{\Omega_0}(\mathcal{H}^{(t)}(\cdot,\eta))|$ for all $\eta \in \Omega_0$, allowing us to conclude.

(2) We want to exploit the fact that h_t is small for large field and is hence overpowered by h; this time around, we write

$$\begin{aligned}\mathcal{H}^{(t)}(\sigma,\eta) \;&=\; \beta\left(-\sqrt{\beta} \sum_{(x,y)} \sigma(x)\sigma(y) - \frac{h}{\beta} \sum_x \sigma(x) - \frac{h_t}{\sqrt{\beta}}\sum_x \sigma(x)\eta(x)\right) \\ &=: \; \sqrt{\beta}\overline{\mathcal{H}}^{(t)}(\sigma,\eta).\end{aligned}$$

For fixed $\eta \in \Omega_0$ and $t > t_1(h)$ large enough, $\frac{h}{|h|}$ is the unique ground state of $\overline{\mathcal{H}}^{(t)}(\cdot,\eta)$. Thus, again by `[reference]`, for β large enough, $\sqrt{\beta}\overline{\mathcal{H}}^{(t)}(\cdot,\eta)$ satisfies `[Dobrushin]`. Why is this enough? Is "large enough" independent of η? Or is there a uniform bound?

□

Remark 4.6. The same result can be obtained for $\varepsilon > 0$, though in that case t_i's also depend on the proportion $\frac{1-\varepsilon}{1+\varepsilon}$.

4.3 Alternative approach: dynamics as a fuzzy map

In the previous section, we have remarked on the similarity between [Proposition] and [Berghout's result]. The aim of this section is firstly to argue that one can view the dynamics described in the previous section as a fuzzy model, and secondly to demonstrate that the proofs of those same results are very similar, despite the alternative approach.

Recall again that we assumed that the dynamics are i.i.d.[10], with no bias towards either $+1$ or -1 spins ($\varepsilon = 0$). Considering the spin flip rates $c(x, \omega)$, the first assumption removes the dependence on the first coordinate and restricts the dependence in the second coordinate to only $\omega(x)$, while the second assumptions fully removes the dependence on the second coordinate. This yields that the dynamics are given by a collection of independent Poisson clocks with rate $c > 0$.

It is thus rather simple to describe probabilities of a certain value being assumed at a specific spin at any given time. Indeed, on has that, for $x \in \mathbb{Z}^d$ and $t > 0$,

$$[\mu S(t)](\omega_t(x) = 1) \;=\; \mu(\omega_0(x) = 1)\mathbb{P}(\text{even flips of } x) + \mu(\omega_0(x) = -1)\mathbb{P}(\text{odd flips of } x),$$

where $\mathbb{P}(\text{odd flips of } x)$ of course corresponds to the probability of there being even number of arrivals of Poisson process with intensity c until time t. Famously, the latter carries the probability of $\frac{1}{2}(1 - e^{-2tc})$, yielding

$$[\mu S(t)](\omega_t(x) = 1) \;=\; \mu(\omega_0(x) = 1)\frac{1 + e^{-2tc}}{2} + \mu(\omega_0(x) = -1)\frac{1 - e^{-2tc}}{2}.$$

The assumptions of this particular model allow us to consider the alternative to considering the joint distribution of (ω_0, ω_t), as was done before. In particular, we will rather focus on the joint distribution of ω_0 and "dynamics" (that is, whether there was odd or even number of spins at any site) until time t. The advantage of this approach is, that while ω_t does indeed depend on ω_0 (and their joint distribution is not necessarily Gibbsian), the information about the dynamics up until t *is* – due to no bias via $\varepsilon = 0$ – independent of ω_0. The joint distribution is thus a product measure and hence automatically Gibbsian on Ω_0^2. Moreover, it is also easy to see that such construction allows us to see each $\mu S(t)$ as a fuzzy Gibbs measure, allowing us to use tools from [second chapter].

We first fix some value $\kappa \in [0, 1]$ and define an Ω_0-valued random variable $X \sim \mu$. Moreover, define another Ω_0-valued random variable, Y^κ, whose law ρ^κ is given as

$$\rho^\kappa \;=\; \left[(1-\kappa)\delta_{+1} - \kappa\delta_{\{-1\}}\right]^{\otimes \mathbb{Z}^d},$$

that is $(Y^\kappa_x)_{x \in \mathbb{Z}^d}$ are i.i.d. with

$$Y^\kappa_x \;=\; \begin{cases} 1, & \text{with probability } 1 - \kappa, \\ -1, & \text{with probability } \kappa. \end{cases}$$

At last we define a random variable $Z^\kappa := X \cdot Y^\kappa$, and denote its law by ν^κ. Clearly, due to independence of X and Y^κ, one has $(Z^\kappa_i)_{x \in \mathbb{Z}^d}$ i.i.d. with

$$Z^\kappa_x \;=\; \begin{cases} X_i, & \text{with probability } 1 - \kappa, \\ -X_i, & \text{with probability } \kappa. \end{cases}$$

[10] This is concluded by combining the assumption that $\mathbb{P}_\omega = \bigotimes_{x \in \mathbb{Z}^d} \mathbb{P}_{\omega(x)}$ and the assumption that $(L_x)_{x \in \mathbb{Z}^d}$ are independent of x.

It follows that arbitrary $x \in \mathbb{Z}^d$, one has

$$\begin{aligned}\nu^\kappa(Z_x^\kappa = 1) &= \mu(X_x = 1)\rho^\kappa(Y_x^\kappa = 1) + \mu(X_x = -1)\rho^\kappa(Y_x^\kappa = -1)\\ &= \mu(X_x = 1)(1-\kappa) + \mu(X_x = -1)\kappa.\end{aligned}$$

It is clear, that if define

$$\kappa(t) = \frac{1 - e^{-2ct}}{2}, \quad t \geq 0,$$

we obtain

$$\mu S(t) = \nu^{\kappa(t)}, \quad t \geq 0.$$

One the other hand, defining the map

$$\begin{aligned}\pi : \Omega_0 \times \Omega_0 &\to \Omega_0\\ (\omega^{\mathsf{c}}, \omega^{\mathsf{d}}) &\mapsto \omega^{\mathsf{c}}\omega^{\mathsf{d}},\end{aligned}$$

we can clearly see that

$$\nu^\kappa = (\mu \otimes \rho^\kappa) \circ \pi^{-1}.$$

Clearly, such map is a surjection from $\Omega = \Omega_0^2$ to $\Sigma = \Omega_0$, so $\mu \otimes \rho^\kappa$ being Gibbsian, it yields that ν^κ is indeed a fuzzy Gibbs measure, so the same holds true for $\mu S(t)$, $t \geq 0$.

This allows us to utilize the fibre approach on measures $\mu S(t)$. In this case, the particular fibres are very simple: for each $\sigma \in \Sigma$ and $\omega \in \Omega_0$, there a unique $\omega' \in \Omega_0$, such that $\omega\omega' = \sigma$. In fact, one precisely has that

$$\omega' = \left(\frac{\sigma(x)}{\omega(x)}\right)_{x \in \mathbb{Z}^d}.$$

Since individual sites can only assume values $+1$ and -1, we can utilize the fact that for $\alpha, \beta \in \{-1, +1\}$ one has $\frac{\alpha}{\beta} = \alpha\beta$, to write

$$\Omega_\sigma = \{(\omega, \omega\sigma) : \omega \in \Omega_0\}.$$ [11]

First we demonstrate, how we can prove `[Theorem for unique` μ`]`, using the fibre approach. Before starting, we remark that since $\mu \otimes \rho^\kappa$ is a product measure, its Hamiltonians $\mathcal{H}^\kappa = (\mathcal{H}_\Lambda^\kappa)_{\Lambda \Subset \mathbb{Z}^d}$ take the from

$$\mathcal{H}^\kappa(\omega^{\mathsf{c}}, \omega^{\mathsf{d}}) = \mathcal{H}(\omega^{\mathsf{c}}) - \sum_x \log \rho^\kappa(\omega^{\mathsf{d}}(x)),$$

where $\rho^\kappa(\omega^{\mathsf{d}}(x))$ equals $1 - \kappa$ is $\omega^{\mathsf{d}}(x) = +1$ and κ if $\omega^{\mathsf{d}}(x) = -1$. In light of `[Berghout's result]` and the fact that $\{\kappa(t) : t \geq 0\} \subset [0, 1]$, we only need to prove the following:

Proposition 4.7. For each $\kappa \in [0, 1]$, we have $|\mathcal{G}_{\Omega_\sigma}(\mathcal{H}^\kappa)| = 1$ for each $\sigma \in \Sigma$.

[11] Here, there should be no confusion between $\Omega_0 = \{-1, +1\}^{\mathbb{Z}^d}$ and $\Omega_{\mathbf{0}} = \pi^{-1}(\mathbf{0})$, where $\mathbf{0}$ is the configuration with all sites equal to 0.

Proof. Fixing some $\sigma \in \Sigma$, if one is to restrict possible values of $(\omega^{\mathsf{c}}, \omega^{\mathsf{d}})$ to Ω_σ, we are left to consider $\mathcal{H}^\kappa(\omega, \omega\sigma)$, $\omega \in \Omega_0$, where

$$\begin{aligned}\mathcal{H}^\kappa(\omega, \omega\sigma) &= \mathcal{H}(\omega) - \sum_{x:\sigma(x)=1} \log \rho^\kappa(\omega(x)) - \sum_{x:\sigma(x)=-1} \log \rho^\kappa(-\omega(x)) \\ &= \mathcal{H}(\omega) - \sum_{x:\sigma(x)=1} \log \rho^\kappa(\omega(x)) - \sum_{x:\sigma(x)=-1} \log(1 - \rho^\kappa(\omega(x)))\,.\end{aligned}$$

Similar as in the proof in the previous section, $(\omega \mapsto \mathcal{H}^\kappa(\omega, \omega\sigma))$ differs from $\mathcal{H}$ only by a single-site interaction, yielding that one admits a unique Gibbs measure on Ω_0 iff the other does. By assumption, $|\mathcal{G}_{\Omega_0}(\mathcal{H})| = 1$, so $|\mathcal{G}_{\Omega_0}(\mathcal{H}^\kappa(\cdot, \cdot\sigma))| = 1$, regardless of σ. Given that $\Omega_\sigma = \{(\omega, \omega\sigma) : \omega \in \Omega\}$, this is equivalent to $|\mathcal{G}_{\Omega_\sigma}(\mathcal{H}^\kappa)| = 1$ for all $\sigma \in \Sigma$. □

In light of their proof of `Ising theorem`, in particular the fact that they argued via `[Eq in Cor]`, the sensible only way to obtain – for sure – the same t_0 and t_1, it to verify that we have Gibbsianity as soon as they have it. Due to the fact that they argued via `[Eq in Cor]`, it is sufficient to verify the following:

Proposition 4.8. Let $t \geq 0$ and $\sigma \in \Sigma$ be arbitrary. Then $\mathcal{H}^{(t)}(\cdot, \sigma)$ and $\mathcal{H}^{\kappa(t)}(\cdot, \cdot\sigma)$ differ only by a single site interaction.

Proof. In this particular case, one has

$$\mathcal{H}^{\kappa(t)}(\omega, \omega\sigma) = -\beta \sum_{(x,y)} \sigma(x)\sigma(y) - h \sum_x \sigma(x) - \sum_x \log \rho^{\kappa(t)}(\omega(x)\sigma(x)), \omega \in \Omega.$$

It follows that

$$\mathcal{H}^{\kappa(t)}(\omega, \omega\sigma) - \mathcal{H}^{(t)}(\omega, \sigma) = h_t \sum_x \sigma(x)\eta(x) - \sum_x \log \rho^{\kappa(t)}(\omega(x)\sigma(x)),$$

which indeed corresponds to a single-site interaction. □

This tells us that as soon as $|\mathcal{G}_{\Omega_0}(\mathcal{H}^{(t)}(\cdot, \sigma)| = 1$, one also has $|\mathcal{G}^{\kappa(t)}_{\Omega_0}(\omega, \omega\sigma)| = |\mathcal{G}_{\Omega_\sigma}(\mathcal{H}^{\kappa(t)})| = 1$, so `[Berghout's result]` concludes the argument.

A Technical appendix

This appendix introduces some technical notions that need to be understood, but whose definitions would disturb the flow of the main text, if included there.

A.1 Regular conditional probability

Blahblahblah

Consider an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and measurable space $(E, \mathcal{E})$. Recall that a *probabilty kernel* from $(E, \mathcal{E})$ to $(\Omega, \mathcal{F})$ is a map $\kappa : E \times \mathcal{F} \mapsto [0, 1]$, such that

(i) for each $x \in E$, $\kappa(x, \boldsymbol{\cdot})$ is a probability measure on $(\Omega, \mathcal{F})$,

(ii) for each $A \in \mathcal{F}$, $\kappa(\boldsymbol{\cdot}, A)$ is a measurable map $(E \to [0, 1])$.

Definition A.1. Let $X : \Omega \to E$ be a measurable function (random variable). A regular conditional probability w.r.t. X is a transition kernel $\kappa : E \times \mathcal{F} \to [0, 1]$, such that

$$\mathbb{P}(A \cap X^{-1}(B)) \;=\; \int_B \kappa(x, A) \,\mathrm{d}[\mathbb{P} \circ T^{-1}](x), \quad \forall A \in \mathcal{F},\ B \in \mathcal{E}.^{12}$$

Given fixed $x \in E$ and $A \in \mathcal{F}$, one usually writes $\mathbb{P}(A|X = x)$ or $\mathbb{P}(A|x)$ instead of $\kappa(x, A)$.

In our case, we consider (usually) $\Omega = \mathcal{A}^{\mathbb{Z}^d}$ with some $\mu \in \mathcal{M}_1(\Omega)$. We then simply take $E = \Omega$ and $X = \mathrm{id}$, i.e., $X(\omega) = \omega$ for each congifuration $\omega \in \Omega$. In the definition of DLR Gibbs measure, we write the following:

$$\mu(\omega \text{ on } \Lambda|\xi \text{ off } \Lambda), \quad \omega, \xi \in \Omega,\ \Lambda \Subset \mathbb{Z}^d.$$

By event $\{\omega \text{ on } \Lambda\}$, we refer to $\{\widetilde{\omega} \in \Omega : \widetilde{\omega}_\Lambda = \omega_\Lambda\}$, so we could consider $\mu(\omega \text{ on } \Lambda|\boldsymbol{\cdot})$ as above, so we'd have (using that $X = \mathrm{id}$),

$$\mu(\{\omega \text{ on } \Lambda\} \cap B) \;=\; \int_B \mu(\omega \text{ on } \Lambda|\xi) \,\mathrm{d}\mu(\xi).$$

However $\{\xi \text{ off } \Lambda\}$ is a set of configurations, rather than a single configuration as desired. In particular $\{\xi \text{ off } \Lambda\} = \{\widetilde{\xi} \in \Omega : \widetilde{\xi}_{\Lambda^c} = \xi_{\Lambda^c}\}$; however, since Λ is finite, $\{\xi \text{ off } \Lambda\}$ is finite too, so one has, now formally (as it is clear what we mean by $\mu(\omega \text{ on } \Lambda|\xi)$),

$$\mu(\omega \text{ on } \Lambda|\xi \text{ off } \Lambda) \;=\; \sum_{\substack{\widetilde{\xi} \in \Omega: \\ \widetilde{\xi}_{\Lambda^c} = \xi_{\Lambda^c}}} \mu(\omega \text{ on } \Lambda|\widetilde{\xi}).$$

A.2 Coupling

Blahblahblah

We consider two (possibly equal) probability spaces, $(\Omega', \mathcal{F}', \mathbb{P}')$ and $(\Omega'', \mathcal{F}'', \mathbb{P}'')$.

Definition A.2. A coupling of $\mathbb{P}'$ and $\mathbb{P}''$ is any probability measure $\widehat{\mathbb{P}}$ on product space $(\Omega' \times \Omega'', \mathcal{F}' \otimes \mathcal{F}'')$ such that

$$\begin{aligned} \widehat{\mathbb{P}}((\omega', \omega'') : \omega' \in A) \;&=\; \mathbb{P}'(A), \quad \forall A \in \mathcal{F}', \\ \widehat{\mathbb{P}}((\omega', \omega'' : \omega'' \in B) \;&=\; \mathbb{P}''(B), \quad \forall B \in \mathcal{F}''. \end{aligned}$$

Mention alternative notion of coupling of random variables.

[12] Instead of $\mathbb{P}(A \cap X^{-1}(B))$, one could also write $\mathbb{P}(A \cap \{X \in B)\}$ or $\mathbb{P}(A \text{ and } X \in B)$.

A.3 Bernoulli percolation

Blahblahblah

Consider first a finite graph $G = (V, E)$.

Definition A.3. Let $p \in [0,1]$ be the probability parameter. *Bernoulli percolation* is a probability measure $\psi_p = \psi_p^G$ on $\{0,1\}^E$, where each configuration η (called a *(site) percolation configuration*) is assigned probability

$$\psi_p(\eta) \;=\; \prod_{e\in E} p^{\eta(e)}(1-p)^{1-\eta(e)}.$$

This can indeed be extended to an infinite graph $G = (V, E)$, where $\psi_p = \mathrm{Ber}(p)^{\otimes E}$.[13] The configuration η can be understood as a subgraph $(V, \{e : \eta(e) = 1\})$ of G, which we will denote shorter by $G_\eta = (V, \eta)$.

We are interested in $G = (\mathbb{Z}^d, \mathbb{E}^d)$ situation. Writing $\{x \leftrightarrow \infty\}$ for the event that $x \in \mathbb{Z}^d$ belongs to an infinite cluster of $(\mathbb{Z}^d, \eta)$, one might as what is the probability of such event – due to translation invariance, this probability should be same for any choice of x. Writing

$$\theta(p) \;=\; \psi_p(0 \leftrightarrow \infty),$$

we define the critical probability

$$p_c \;:=\; \inf\left\{p \in [0,1] : \theta(p) > 0\right\}.$$

It should be noted that p_c does in fact depend on dimension d. For $d = 1$, we clearly have $\theta(p) = 0$ (unless $p = 1$), and hence $p_c(1) = 1$.

Theorem A.4 ([D-C], Theorem 1.1)**.** For $d \geq 2$, $p_c(d) \in (0,1)$. Some comment that it is a 0-1 thing?

[13] It is clear that the finite graph version obeys this as well.

References

To be typeset...