Absence of Phase Transitions and Preservation of Gibbs Property Under Renormalization

Scientific Talk

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June 21, 2024

Preliminaries

Consider finite alphabet $\mathcal{A}$ and write $\Omega = \mathcal{A}^{\mathbb{Z}^d}$.

Definition (Interactions and Hamiltonians)

(1) **Interaction** is a collection of maps $\Phi = (\Phi_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$, where

$$\Phi_\Lambda(\omega) \;=\; \Phi_\Lambda(\omega(x) : x \in \Lambda), \quad \omega \in \Omega.$$

We say Φ is **uniformly absolutely convergent** (UAC), $\Phi \in \mathscr{B}^1(\Omega)$, if

$$\sup_{x \in \mathbb{Z}^d} \sum_{\Lambda \ni x} \|\Phi_\Lambda\|_\infty \;<\; \infty.$$

(2) For $\Phi \in \mathscr{B}^1(\Omega)$, we consider **Hamiltonians** $\mathcal{H} = (\mathcal{H}_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$,

$$\mathcal{H}_\Lambda(\omega) \;=\; \sum_{\Delta \cap \Lambda \neq \emptyset} \Phi_\Delta(\omega), \quad \omega \in \Omega.$$

Preliminaries

Definition (Specification and Gibbs measure)

(1) For $\Phi \in \mathscr{B}^1(\Omega)$, **specification** $\gamma = (\gamma_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$, is given so that $\gamma_\Lambda(\cdot|\cdot) : \mathcal{F}_\Lambda \times \Omega_{\Lambda^c} \to (0,1)$,

$$\gamma_\Lambda(\omega_\Lambda|\xi_{\Lambda^c}) \; = \; \frac{1}{\mathcal{Z}_\Lambda^\xi} \exp(-\mathcal{H}_\Lambda(\omega_\Lambda \xi_{\Lambda^c})),$$

where $\mathcal{Z}_\Lambda^\xi$ is the normalization constant.

(2) $\mu \in \mathcal{M}_1(\Omega)$ is a **Gibbs measure** on Ω consistent with Φ ($\mu \in \mathcal{G}_\Omega(\Phi)$) if for each $\Lambda \Subset \mathbb{Z}^d$,

$$\mu(\omega_\Lambda|\omega_{\Lambda^c}) \; = \; \gamma_\Lambda^\Phi(\omega_\Lambda|\omega_{\Lambda^c}) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

Preliminaries

Remark

(i) For $\Lambda \Subset \mathbb{Z}^d$ and ξ_{Λ^c} fixed, $\gamma_\Lambda(\cdot|\xi_{\Lambda^c})$ is a probability measure on $\Omega_\Lambda = \mathcal{A}^\Lambda$. This allows for construction of Gibbs measures via weak limits.

(ii) While $\mathcal{G}_\Omega(\Phi) \neq \emptyset$, we don't necessarily have that $|\mathcal{G}_\Omega(\Phi)| = 1$. How and when this happens is an important subject in statistical mechanics.

Characterization of Gibbsianity

Proposition

Let $\mu \in \mathcal{M}_1(\Omega)$. The following are equivalent:

(i) μ is Gibbs

(ii) μ has the following properties:

(a) **uniform non-nullness**: $\forall \Lambda \Subset \mathbb{Z}^d \; \exists \alpha_\Lambda, \beta_\Lambda \in (0,1)$ s.t.

$$\alpha_\Lambda \;\le\; \mu_\Lambda(\omega_\Lambda | \xi_{\Lambda^c}) \;\le\; \beta_\Lambda, \quad \forall \omega, \xi \in \Omega$$

(b) **quasilocality**: writing $\mathbb{B}_n = [-n, n]^d \cap \mathbb{Z}^d$, $\forall \Lambda \Subset \mathbb{Z}^d$,

$$\sup_{\omega} \sup_{\xi,\zeta} \left| \mu(\omega_\Lambda | \omega_{\mathbb{B}_n \setminus \Lambda} \xi_{\mathbb{B}_n^c \setminus \Lambda}) - \mu(\omega_\Lambda | \omega_{\mathbb{B}_n \setminus \Lambda} \zeta_{\mathbb{B}_n^c \setminus \Lambda}) \right| \;\to\; 0.$$

Renormalization group

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To be done later
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Dilemma: talk about renormalization or just fuzzy Gibbs measures?

Fuzzy Gibbs measures

Consider $\Omega = \mathcal{A}^{\mathbb{Z}^d}$, with $\mathcal{A}$ finite, as before.

Let $\mathcal{B}$ be another alphabet, with $|\mathcal{B}| \leq |\mathcal{A}|$. Write $\Sigma = \mathcal{B}^{\mathbb{Z}^d}$.

We consider a surjection $\pi : \mathcal{A} \to \mathcal{B}$, which we call a **fuzzy map**.[1]

Definition

A fuzzy Gibbs measure ν on Σ is defined as

$$\nu \ = \ \mu \circ \pi^{-1},$$

where μ is some Gibbs measure on Ω.

Question: when is ν Gibbsian?

[1] π induces a surjection $\Omega \to \Sigma$ which we denote by the same letter

Hidden phase transitions

We can partition Ω w.r.t. π as follows:

pick $\sigma \in \Sigma$ and define

$$\Omega_\sigma \; = \; \pi^{-1}(\sigma);$$

we call sets $\{\Omega_\sigma : \sigma \in \Sigma\}$ **fibres**.

Definition (Hidden phase transition)

We say that a **hidden phase transition** occurs on Ω_σ if

$$|\mathcal{G}_{\Omega_\sigma}(\Phi)| \; > \; 1.$$

If $|\mathcal{G}_{\Omega_\sigma}(\Phi)| = 1$ for all $\sigma \in \Sigma$, we talk about **absence of hidden phase transitions**.

Hidden phase transitions

Proposition (Sufficient condition)

In the absence of hidden phase transitions, $\nu = \mu \circ \pi^{-1}$ is Gibbsian.

The following conjecture (stated informally here) was established by van Enter, Fernández and Sokal:

Conjecture (van Enter-Fernández-Sokal hypothesis, [vEFS93],[Ber20])

The fuzzy Gibbs measure is not Gibbsian *if and only if*

(i) $\exists \sigma \in \Sigma : |\mathcal{G}_{\Omega_\sigma}(\Phi)| > 1$, i.e., a hidden phase transition occurs, and

(ii) one can pick different phases of $\mathcal{G}_{\Omega_\sigma}(\Phi)$ by varying boundary conditions.

Construction of conditional measures

Goal: construct distribution of ω, conditional on $\pi(\omega) = \sigma$

Definition

Given $B \subseteq \Sigma$ measurable with $\nu(B) > 0$, define

$$\mu^B = \mu(\cdot|\pi^{-1}(B)).$$

One can consider a net of conditional measures μ^B (on Ω), indexed with pairs (V, B), where V is an open neighbourhood of σ and $B \subseteq V : \nu(B) > 0$.

Write $\overline{\mathfrak{M}}_\sigma$ for accumulation points of the above net, as open neighbourhoods (V) "approach" σ.

Tjur points

Definition

If $|\overline{\mathfrak{M}}_\sigma| = 1$ for a given $\sigma \in \Sigma$, denote by μ^σ the only member of $\overline{\mathfrak{M}}_\sigma$, the limit of the corresponding net. In this case, we say that σ is a **Tjur point**.

One can restate the previously presented conjecture as follows:

Conjecture (van Enter-Fernández-Sokal hypothesis, [Ber20])

The fuzzy Gibbs measure is Gibbsian *if and only if* $|\overline{\mathfrak{M}}_\sigma| = 1$ for all $\sigma \in \Sigma$, i.e., all points are Tjur.

Proposition (Berghout, Verbitskiy, [Ber20])

Direction $(\Leftarrow)$ holds.

Tjur points: sufficient condition revisited

Proposition

$\overline{\mathfrak{M}}_\sigma \neq 0$, each member is a probability measure supported on Ω_σ. If $\mu \in \mathcal{G}_\Omega(\Phi)$, then

$$\overline{\mathfrak{M}}_\sigma \subseteq \mathcal{G}_{\Omega_\sigma}(\Phi).$$

Corollary

Absence of phase transitions implies Gibbsianity of $\nu = \mu \circ \pi^{-1}$.

Remark

By demonstrating the absence of phase transitions, we not only obtain Gibbsianity of the fuzzy Gibbs measure, but also verify that the example doesn't contradict the unproven direction of the van Enter-Fernández-Sokal hypothesis.

Classical Potts model

Write E^d for the (nearest-neighbour) edge set of $\mathbb{Z}^d$ and

$$\mathsf{E}_\Lambda = \left\{ \langle x, y \rangle \in \mathsf{E}^d : x, y \in \Lambda \right\}, \quad \partial\mathsf{E}_\Lambda = \left\{ \langle x, y \rangle \in \mathsf{E}^d : x \in \Lambda, y \notin \Lambda \right\}.$$

Definition (Interaction of Potts model)

The interaction of q-state Potts model $\Phi_{\beta,q}$ is given by

$$\Phi_{\Lambda;\beta,q}(\omega) \;=\; \begin{cases} 2\mathbb{1}_{\{\omega(x)\neq\omega(y)\}} - 1, & \Lambda = \{x, y\} : x \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

Hamiltonians are thus given by

$$\mathcal{H}_{\Lambda;\beta,q}(\omega) \;=\; \sum_{\langle x,y\rangle \in \mathsf{E}_\Lambda \cup \partial\mathsf{E}_\Lambda} (2\mathbb{1}_{\{\omega(x)\neq\omega(y)\}} - 1).$$

Classical Potts model: phase transition

Write $\Omega = \{1, \ldots, q\}^{\mathbb{Z}^d}$.

Theorem

For each $q \geq 2$ and $d \geq 2$, there exists $\beta_c(d, q) \in (0, \infty)$, such that

(i) for $\beta < \beta_c(d, q)$, $|\mathcal{G}_\Omega(\Phi_{\beta,q})| = 1$,

(ii) for $\beta > \beta_c(d, q)$, $\mathcal{G}_\Omega(\Phi_{\beta,q})$ contains q distinct mutually singular measures .

Mutually singular measures in (ii) are precisely measures $\mu_{\beta,q}^{\mathbb{Z}^d,1}, \ldots, \mu_{\beta,q}^{\mathbb{Z}^d,q}$, corresponding to constant boundary conditions $1, \ldots, \mathsf{q}$.

Fuzzy Potts model

Let $1 < s < q$ and $\mathrm{r} = (r_1, \ldots, r_s)$, such that $r_1 + \ldots + r_s = q$.

Definition

Fuzzy Potts map $\pi_{\mathrm{r}} : \{1, \ldots, q\} \to \{1, \ldots, s\}$ is given by

$$\pi_{\mathrm{r}}(a) = \begin{cases} 1: & 1 \leq a \leq r_1, \\ 2: & r_1 + 1 < a \leq r_1 + r_2, \\ \ldots & \\ n: & r_1 + \ldots + r_{n-1} < a \leq r_1 + \ldots r_n, \\ \ldots & \\ s: & r_1 + \ldots + r_{s-1} < a \leq q. \end{cases}$$

Fuzzy Gibbs measure corresponding to $\mu_{\beta,q}^{\mathbb{Z}^d,\xi}$ is given by

$$\nu_{\beta,q}^{\mathbb{Z}^d,\xi} = \mu_{\beta,q}^{\mathbb{Z}^d,\xi} \circ \pi_{\mathrm{r}}^{-1}.$$

Fuzzy Potts model: Gibbsianity

Write $r^* = \min(\{r_1, \ldots, r_s\} \cap \mathbb{N}_{\geq 2})$

Theorem (Häggström, [Häg03])

Let $d \geq 2$, $q \geq 3$ and $\xi \in \{\emptyset, 1, \ldots, \mathrm{q}\}$; consider fuzzy Potts measure $\mu_{\beta,q}^{\mathbb{Z}^d,\xi}$.

(i) For each $\beta < \beta_c(d, r^*)$, $\nu_{\beta,q}^{\mathbb{Z}^d,\xi}$ is a Gibbs measure.

(ii) For each $\beta > \frac{1}{2}\log\frac{1+(r^*-1)p_c(d)}{1-p_c(d)}$, $\nu_{\beta,q}^{\mathbb{Z}^d,\xi}$ is *not* a Gibbs measure.[a]

[a] $p_c(d)$ = critical probability for Bernoulli percolation on $\mathbb{Z}^d$

Goal: provide an alternative proof of (i), using absence of hidden phase transitions.

Idea of alternative proof

Want to show: for each $\sigma \in \{1, \dots, s\}^{\mathbb{Z}^d}$, $|\mathcal{G}_{\Omega_\sigma}(\Phi_{\beta,q})| = 1$.

Notice:

$$\Omega_\sigma = \prod_{x \in \mathbb{Z}^d} \pi^{-1}(\sigma(x));$$

write, for $j = 1, \dots, s$,

$$\mathsf{A}_j = \pi^{-1}(j) = \{r_1 + \ldots + r_{j-1} + 1, \dots, r_1 + \ldots + r_j\}$$

and

$$U_j = \left\{x \in \mathbb{Z}^d : \sigma(x) = j\right\}.$$

Then,

$$\Omega_\sigma = \prod_{x \in \mathbb{Z}^d} \begin{cases} \mathsf{A}_1, & x \in U_1, \\ \ldots & \\ \mathsf{A}_s, & x \in U_s \end{cases} =: \bigotimes_{j=1}^{s} \mathsf{A}_j^{U_j}.$$

Idea of alternative proof

It is enough to show that:

(i) If β is such that

$$|\mathcal{G}_{\mathsf{A}_j^{\mathbb{Z}^d}}(\Phi_{\beta,|\mathsf{A}_j|})| = 1, \quad \forall j = 1, \ldots, s,$$

then

$$|\mathcal{G}_{\mathsf{A}_j^{U_j}}(\Phi_{\beta,|\mathsf{A}_j|})| = 1, \quad \forall j = 1, \ldots, s.$$

(ii) If

$$|\mathcal{G}_{\mathsf{A}_j^{U_j}}(\Phi_{\beta,|\mathsf{A}_j|})| = 1, \quad \forall j = 1, \ldots, s,$$

then

$$|\mathcal{G}_{\otimes_j \mathsf{A}_j^{U_j}}(\Phi_{\beta,q})| = 1.$$

Idea of alternative proof

Clear: enough to show above for $s = 2$, induction takes care of the rest.
Thus sufficient to prove:

Proposition (Part I)

Let $U \subset \mathbb{Z}^d$ and $q \in \mathbb{N}_{\geq 2}$. For $\beta < \beta_c(d, q)$,

$$|\mathcal{G}_{\{1,\ldots,q\}^U}(\Phi_{\beta,q})| = 1.$$

Proposition (Part II)

Let $\mathbb{Z}^d = U \sqcup V$ and $\mathsf{A} \cap \mathsf{B} = \emptyset$. If β is such that

$$|\mathcal{G}_{\mathsf{A}^U}(\Phi_{\beta,|\mathsf{A}|})| = |\mathcal{G}_{\mathsf{B}^V}(\Phi_{\beta,|\mathsf{B}|})| = 1,$$

then

$$|\mathcal{G}_{\mathsf{A}^U \otimes \mathsf{B}^V}(\Phi_{\beta,|\mathsf{A}|+|\mathsf{B}|})| = 1.$$

Spin-flip dynamics: general model

Idea: Pick initial configuration $\omega_0 \in \{-1, +1\}^{\mathbb{Z}^d}$ according to some Gibbs measure and randomly flip spins as time runs.

Question: Having obtained $(\omega_t)_{t\geq 0}$, when is $\mathrm{Law}(\omega_t)$ Gibbsian?

Spin-flip dynamics: general model

Let $\Omega_0 = \{-1, +1\}^{\mathbb{Z}^d}$.

Pick $\mu \in \mathcal{G}_{\Omega_0}$ and draw $\omega_0 \sim \mu$.

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