Absence of Phase Transitions and Preservation of Gibbs Property Under Renormalization

Scientific Talk

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Preliminaries

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(1) **Interaction** is a collection of maps $\Phi = (\Phi_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$, where

$$\Phi_\Lambda(\omega) \;=\; \Phi_\Lambda(\omega(x) : x \in \Lambda), \quad \omega \in \Omega.$$

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We say Φ is **uniformly absolutely convergent** (UAC), $\Phi \in \mathscr{B}^1(\Omega)$, if

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$$\sup_{x \in \mathbb{Z}^d} \sum_{\Lambda \ni x} \|\Phi_\Lambda\|_\infty \; < \; \infty.$$

(2) For $\Phi \in \mathscr{B}^1(\Omega)$, we consider **Hamiltonians** $\mathcal{H} = (\mathcal{H}_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$,

$$\mathcal{H}_\Lambda(\omega) \; = \; \sum_{\Delta \cap \Lambda \neq \emptyset} \Phi_\Delta(\omega), \quad \omega \in \Omega.$$

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$$\mu(\omega_\Lambda|\omega_{\Lambda^c}) \;=\; \gamma_\Lambda^\Phi(\omega_\Lambda|\omega_{\Lambda^c}) \quad \text{for } \mu\text{-a.a. } \omega \in \Omega.$$

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Remark

(i) For $\Lambda \Subset \mathbb{Z}^d$ and ξ_{Λ^c} fixed, $\gamma_\Lambda(\cdot|\xi_{\Lambda^c})$ is a probability measure on $\Omega_\Lambda = \mathcal{A}^\Lambda$. This allows for construction of Gibbs measures via weak limits.

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(ii) While $\mathcal{G}_\Omega(\Phi) \neq \emptyset$, we don't necessarily have that $|\mathcal{G}_\Omega(\Phi)| = 1$. How and when this happens is an important subject in statistical mechanics.

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(a) **uniform non-nullness**: $\forall \Lambda \Subset \mathbb{Z}^d \; \exists \alpha_\Lambda, \beta_\Lambda \in (0,1)$ s.t.

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   (b) **quasilocality**: writing $\mathbb{B}_n = [-n, n]^d \cap \mathbb{Z}^d$, $\forall \Lambda \Subset \mathbb{Z}^d$,

$$\sup_{\omega} \sup_{\xi,\zeta} \left| \mu(\omega_\Lambda | \omega_{\mathbb{B}_n \setminus \Lambda} \xi_{\mathbb{B}_n^c \setminus \Lambda}) - \mu(\omega_\Lambda | \omega_{\mathbb{B}_n \setminus \Lambda} \zeta_{\mathbb{B}_n^c \setminus \Lambda}) \right| \;\to\; 0.$$

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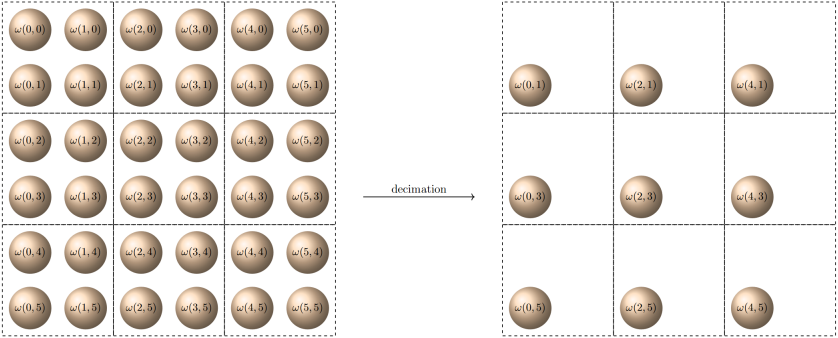
Example: decimation



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Question: If $\omega \sim \mu$, with μ Gibbs (consistent with $\mathcal{H}$), what about the law of ω'.

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In finite volume:

$$\begin{tikzcd} \mathcal{H} \arrow[r, mapsto, "\mathcal{R}"] \arrow[d, mapsto] & \mathcal{H}' \\ \mu \arrow[r, mapsto, "T"] & \mu' \arrow[u, mapsto] \end{tikzcd}$$

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Problem: In finite volume, μ' might not be Gibbsian at all, so we cannot speak of $\mathcal{H}'$,[1] $\mathcal{R}$ doesn't make sense.

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Fuzzy Gibbs measures

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Question: when is ν Gibbsian?

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$$|\mathcal{G}_{\Omega_\sigma}(\Phi)| \;>\; 1.$$

If $|\mathcal{G}_{\Omega_\sigma}(\Phi)| = 1$ for all $\sigma \in \Sigma$, we talk about **absence of hidden phase transitions**.

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(i) $\exists \sigma \in \Sigma : |\mathcal{G}_{\Omega_\sigma}(\Phi)| > 1$, i.e., a hidden phase transition occurs, and

(ii) one can pick different phases of $\mathcal{G}_{\Omega_\sigma}(\Phi)$ by varying boundary conditions.

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Write $\overline{\mathfrak{M}}_\sigma$ for accumulation points of the above net, as open neighbourhoods (V) "approach" σ.

Tjur points

Definition

If $|\overline{\mathfrak{M}}_\sigma| = 1$ for a given $\sigma \in \Sigma$, denote by μ^σ the only member of $\overline{\mathfrak{M}}_\sigma$, the limit of the corresponding net. In this case, we say that σ is a **Tjur point**.

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Proposition (Berghout, Verbitskiy, [Ber20])

Direction $(\Leftarrow)$ holds.

Tjur points: sufficient condition revisited

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$\overline{\mathfrak{M}}_\sigma \neq 0$, each member is a probability measure supported on Ω_σ. If $\mu \in \mathcal{G}_\Omega(\Phi)$, then

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Remark

By demonstrating the absence of phase transitions, we not only obtain Gibbsianity of the fuzzy Gibbs measure, but also verify that the example doesn't contradict the unproven direction of the van Enter-Fernández-Sokal hypothesis.

Classical Potts model

Write E^d for the (nearest-neighbour) edge set of $\mathbb{Z}^d$ and

$$\mathsf{E}_\Lambda = \left\{ \langle x, y \rangle \in \mathsf{E}^d : x, y \in \Lambda \right\}, \quad \partial \mathsf{E}_\Lambda = \left\{ \langle x, y \rangle \in \mathsf{E}^d : x \in \Lambda, y \notin \Lambda \right\}.$$

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Definition (Interaction of Potts model)

The interaction of q-state Potts model $\Phi_{\beta,q}$ is given by

$$\Phi_{\Lambda;\beta,q}(\omega) \;=\; \begin{cases} 2\mathbb{1}_{\{\omega(x)\neq\omega(y)\}} - 1, & \Lambda = \{x, y\} : x \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

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Hamiltonians are thus given by

$$\mathcal{H}_{\Lambda;\beta,q}(\omega) \;=\; \beta \sum_{\langle x,y \rangle \in \mathsf{E}_\Lambda \cup \partial \mathsf{E}_\Lambda} (2\mathbb{1}_{\{\omega(x)\neq\omega(y)\}} - 1).$$

Classical Potts model: phase transition

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Theorem

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Mutually singular measures in (ii) are precisely measures $\mu_{\beta,q}^{\mathbb{Z}^d,1}, \ldots, \mu_{\beta,q}^{\mathbb{Z}^d,q}$, corresponding to constant boundary conditions $1, \ldots, \mathsf{q}$.

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Fuzzy Potts map $\pi_{\mathrm{r}} : \{1, \ldots, q\} \to \{1, \ldots, s\}$ is given by

$$\pi_{\mathrm{r}}(a) = \begin{cases} 1: & 1 \leq a \leq r_1, \\ 2: & r_1 + 1 < a \leq r_1 + r_2, \\ \ldots & \\ n: & r_1 + \ldots + r_{n-1} < a \leq r_1 + \ldots r_n, \\ \ldots & \\ s: & r_1 + \ldots + r_{s-1} < a \leq q. \end{cases}$$

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Fuzzy Gibbs measure corresponding to $\mu_{\beta,q}^{\mathbb{Z}^d,\xi}$ is given by

$$\nu_{\beta,q}^{\mathbb{Z}^d,\xi} = \mu_{\beta,q}^{\mathbb{Z}^d,\xi} \circ \pi_{\mathrm{r}}^{-1}.$$

Fuzzy Potts model: Gibbsianity

Write $r^* = \min(\{r_1, \ldots, r_s\} \cap \mathbb{N}_{\geq 2})$

Theorem (Häggström, [Häg03])

Let $d \geq 2$, $q \geq 3$ and $\xi \in \{\emptyset, 1, \ldots, \mathsf{q}\}$; consider fuzzy Potts measure $\mu_{\beta,q}^{\mathbb{Z}^d,\xi}$.

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(ii) For each $\beta > \frac{1}{2} \log \frac{1+(r^*-1)p_c(d)}{1-p_c(d)}$, $\nu_{\beta,q}^{\mathbb{Z}^d,\xi}$ is *not* a Gibbs measure.[a]

[a] $p_c(d)$ = critical probability for Bernoulli percolation on $\mathbb{Z}^d$

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Goal: provide an alternative proof of (i), using absence of hidden phase transitions.

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Notice:

$$\Omega_\sigma \; = \; \prod_{x\in\mathbb{Z}^d} \pi^{-1}(\sigma(x));$$

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Notice:

$$\Omega_\sigma = \prod_{x \in \mathbb{Z}^d} \pi^{-1}(\sigma(x));$$

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Proposition (Part I)

Let $U \subset \mathbb{Z}^d$ and $q \in \mathbb{N}_{\geq 2}$. For $\beta < \beta_c(d, q)$,

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Proposition (Part I)

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Proposition (Part II)

Let $\mathbb{Z}^d = U \sqcup V$ and $\mathsf{A} \cap \mathsf{B} = \emptyset$. If β is such that

$$|\mathcal{G}_{\mathsf{A}^U}(\Phi_{\beta,|\mathsf{A}|})| = |\mathcal{G}_{\mathsf{B}^V}(\Phi_{\beta,|\mathsf{B}|})| = 1,$$

then

$$|\mathcal{G}_{\mathsf{A}^U \otimes \mathsf{B}^V}(\Phi_{\beta,|\mathsf{A}|+|\mathsf{B}|})| = 1.$$

Spin-flip dynamics: general model

Idea: Pick initial configuration $\omega_0 \in \{-1, +1\}^{\mathbb{Z}^d}$ according to some Gibbs measure and randomly flip spins as time runs.

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Question: Having obtained $(\omega_t)_{t\geq 0}$, when is $\mathrm{Law}(\omega_t)$ Gibbsian?

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This induces a semigroup $(S(t))_{t\geq 0}$ acting on $\mathcal{M}_1(\Omega_*)$, so that

$$\mu S(t) \; = \; \mathrm{Law}(\omega_t).$$

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$$|\mathcal{G}_{\Omega_*}(\hat{\mathcal{H}}^{(t)}(\cdot, \eta))| \;=\; 1, \quad \forall \eta \in \Omega_*.$$

Two theorems

We will say that $\Phi \in \mathscr{B}^1(\Phi)$ (or its associated Hamiltonian) is *high-temperature* if

$$\sup_{x \in \mathbb{Z}^d} \sum_{\Lambda \ni x} (|\Lambda| - 1) \sup_{\omega, \tilde{\omega}} |\Phi_\Lambda(\omega) - \Phi_\Lambda(\tilde{\omega})| \; < \; 2.$$

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Theorem ([vEFdHR02])

Assuming dynamics as above and μ either high-temperature or infinite-temperature (i.i.d.), then $\mu S(t)$ is Gibbsian for all $t \geq 0$.

Two theorems

Write $\mu_{\beta,h}$ for some Gibbs measure for Ising model, i.e., consistent with

$$\mathcal{H}_\Lambda(\omega) \;=\; -\beta \sum_{\langle x,y\rangle \in \mathsf{E}_\Lambda \cup \partial \mathsf{E}_\Lambda} \omega(x)\omega(y) - h \sum_{x\in\Lambda} \omega(x),$$

with $\beta > 0$ inverse temperature, $h \in \mathbb{R}$ external magnetic field.

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(i) There exists $t_0 = t_0(\beta, h)$, so that $\mu_{\beta,h}S(t)$ is Gibbsian for $t \leq t_0$.

(ii) Moreover, if $h > 0$ then there exists $t_1 = t_1(\beta, h)$, so that $\mu_{\beta,h}S(t)$ is Gibbsian for $t \geq t_1$.

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Writing $\kappa(t) = \frac{1}{2}(1 - e^{-2tc})$, we obtain

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Clearly

$$\nu^\kappa(Z^\kappa(x) = 1) = \mu(X(x) = 1)(1-\kappa) + \mu(X(x) = -1)\kappa.$$

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$$\begin{aligned} \pi : \Omega &\to \Omega_* \\ (\omega_{\mathsf{c}}, \omega_{\mathsf{d}}) &\mapsto \omega_{\mathsf{c}} \cdot \omega_{\mathsf{d}} \end{aligned}$$

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Both μ, ρ^κ are Gibbsian $\Rightarrow$ $\mu \otimes \rho^\kappa$ is Gibbsian $\Rightarrow$ ν^κ is fuzzy Gibbs.

Moreover, choosing $\kappa(t)$ (as defined before), we obtain precisely

$$\nu^{\kappa(t)} \ = \ \mu S(t).$$

Alternative proof strategy

To verify Gibbsianity of $\mu S(t) = \nu^{\kappa(t)}$, it is enough to show that for each $\sigma \in \Omega_*$,

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Thus sufficient to show that for each $\sigma \in \Omega_*$,

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Main trick: Given any fixed $\sigma \in \Omega_*$,

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Main trick: Given any fixed $\sigma \in \Omega_*$,

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corresponds to a single-site interaction $\Rightarrow$ one is high-temperature iff the other is.

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