ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

ESTIMATION OF PARTIALLY OBSERVED GRAPH SIGNALS BY LEARNING SPECTRALLY MATCHED GRAPH DICTIONARIES

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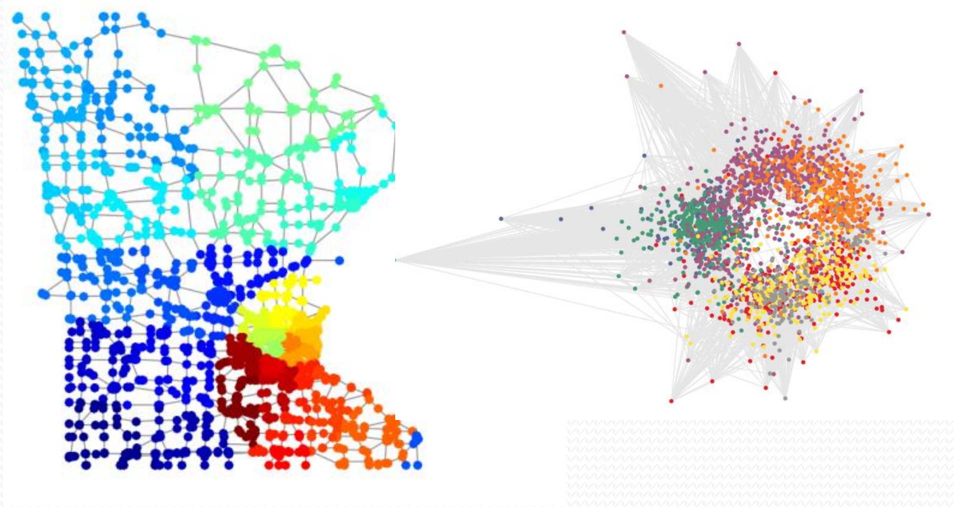
Room: D115

Graph Models

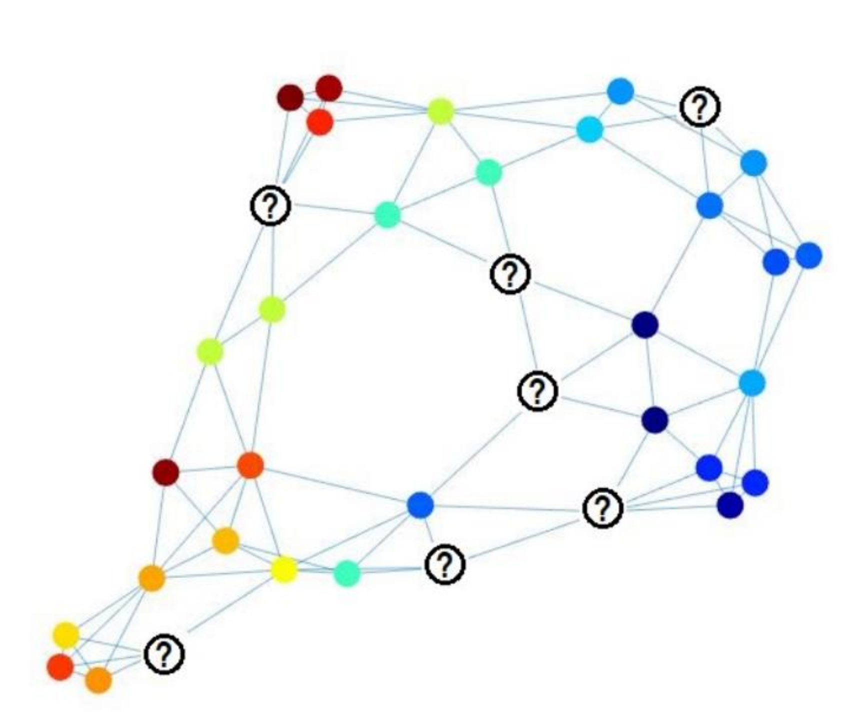
Social Network



Sensor Network

- Many modern applications involve data acquired on an irregular network topology.

Graph Models



Estimation/Inpainting of graph signals

OUTLINE

- Foundations of GSP
- Literature
- Proposed Method (SGKL)
  - Novelties
  - Signal Model, Objective function
  - Algorithm
- Experiments
  - Comparative
  - Performance and Sensitivity Analysis
- Conclusion

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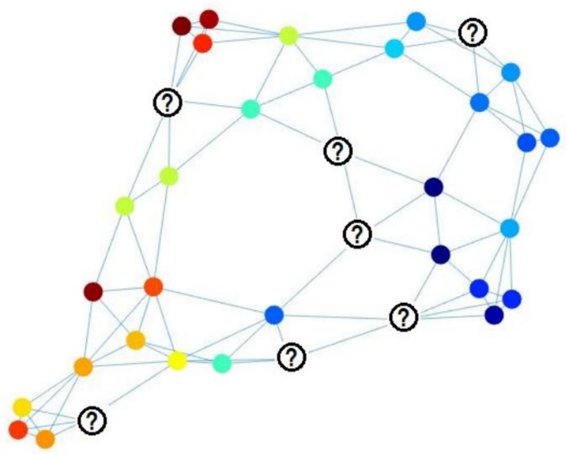
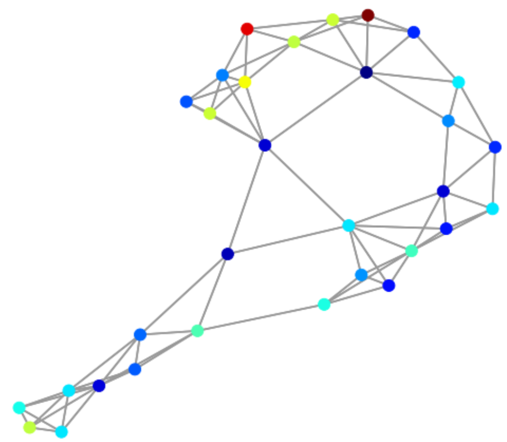
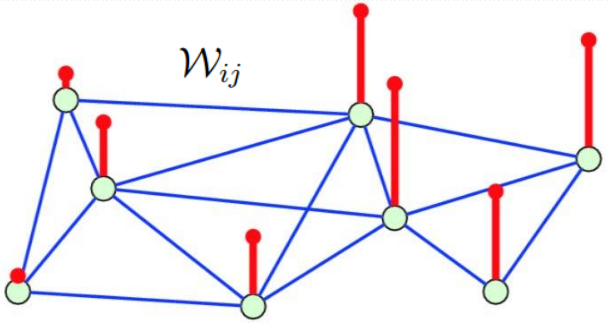
Introduction: Graph Signal Processing



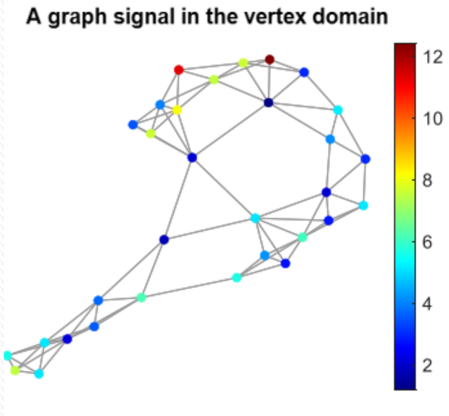
$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$$

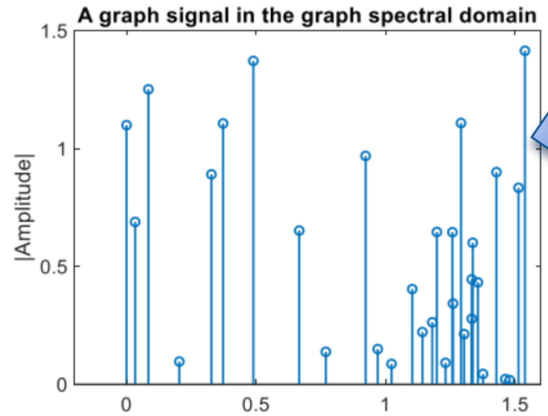
$f : \mathcal{V} \rightarrow \mathbb{R}$: A graph signal, $f \in \mathbb{R}^N$





Introduction: Graph Signal Processing





$$D_{ii} = \sum_j \mathcal{W}_{ij}$$

$$\mathcal{L}_{\mathcal{D}} := \mathcal{D} - \mathcal{W}.$$

$$\mathcal{L} := \mathcal{I} - \mathcal{D}^{-1/2}\mathcal{W}\mathcal{D}^{-1/2}$$

$$\mathcal{L}\, u_k = \lambda_k u_k$$

Classical Signal Processing

$$\hat{f}(\xi) := \langle f, e^{2\pi j \xi t} \rangle = \int_{\mathbb{R}} f(t) e^{-2\pi j \xi t} dt.$$

Graph Signal Processing

$$\hat{f}\,(\lambda_\ell) := \langle \mathbf{f}, \mathbf{u}_\ell \rangle = \sum_{i=1}^{N} f(i) u_\ell^*(i).$$

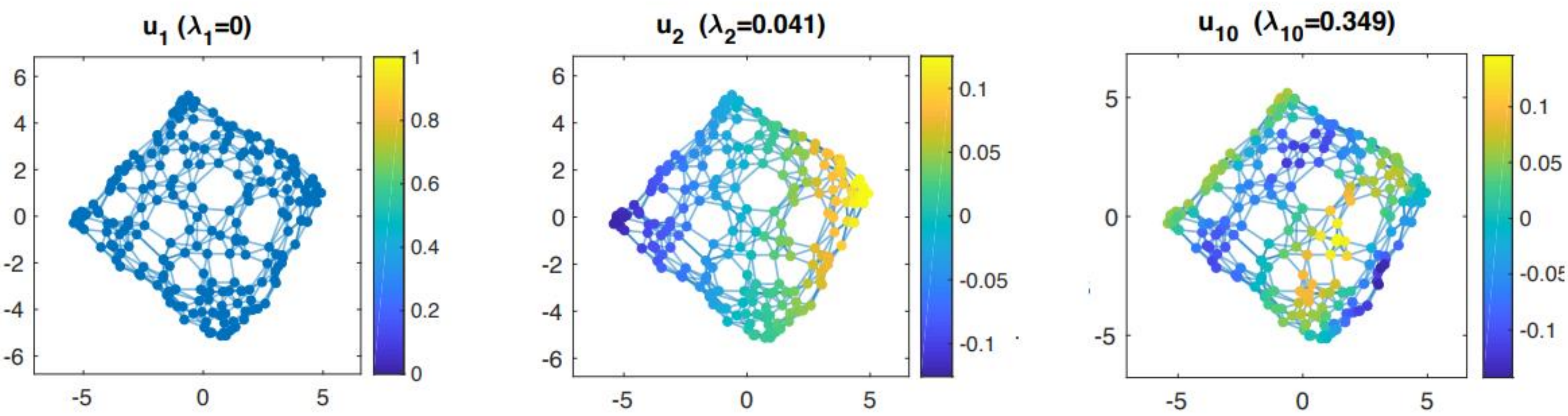
Introduction: Graph Signal Processing

Classical Signal Processing

$$\hat{f}(\xi) := \langle f, e^{2\pi j \xi t} \rangle = \int_{\mathbb{R}} f(t) e^{-2\pi j \xi t} dt.$$

Graph Signal Processing

$$\hat{f}(\lambda_\ell) := \langle \mathbf{f}, \mathbf{u}_\ell \rangle = \sum_{i=1}^{N} f(i) u_\ell^*(i).$$



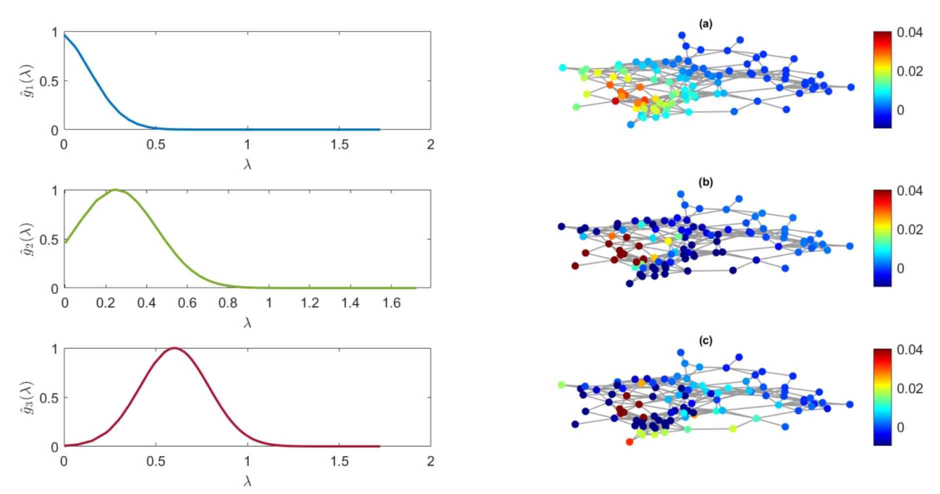
Introduction: Graph Signal Processing

Graph signal processing: Irregular topologies

Define:

- Inverse GFT
- Translation & Localization
- Modulation
- Filtering
- Convolution
- …

in Spectral Domain



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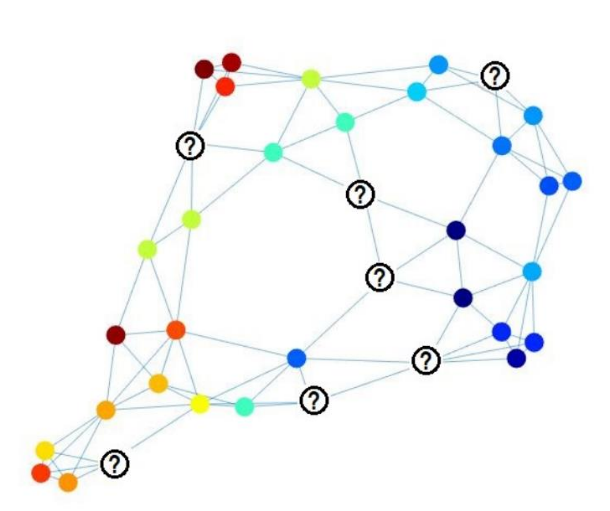
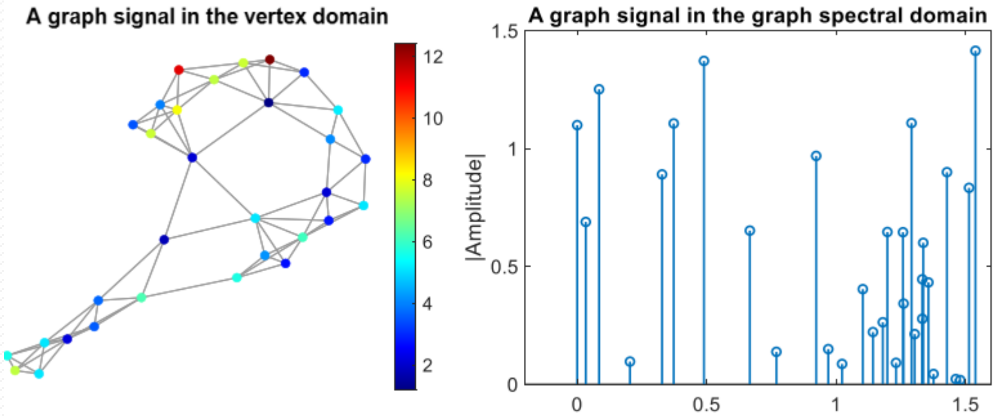


Literature

- Inference of graph signals based on multi-scale dictionary learning techniques (GEMS) (Yankelevsky, 2019)
- Tikhonov regularization (Graph Based Tik.) (Zhou, 2004)
- Total Variation Minimization (TVMin) (Jung, 2019)
- Interpolation of non-smooth graph signals (LSEM) (Mazarguil, 2022)
- Iterative graph signal reconstruction methods (O-PGIR) (Brugnoli, 2020)
- Learning spectrally concentrated kernels method (SCGDL) (Turhan, 2021)
- Graph neural networks and deep algorithm unrolling based methods

Literature

- Smoothness,
- Bandlimitedness,



Wind-Speed Measurement, Molene Dataset

- Lack of capacity to generalize

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Proposed Method

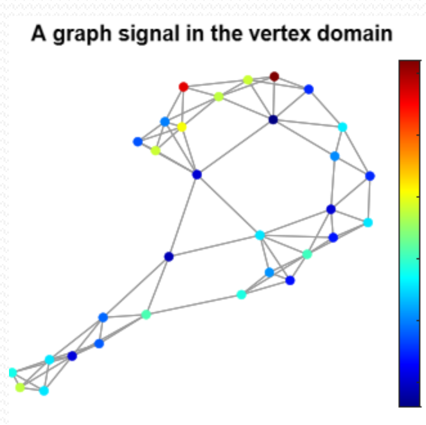
- Estimation of partially observed graph signals

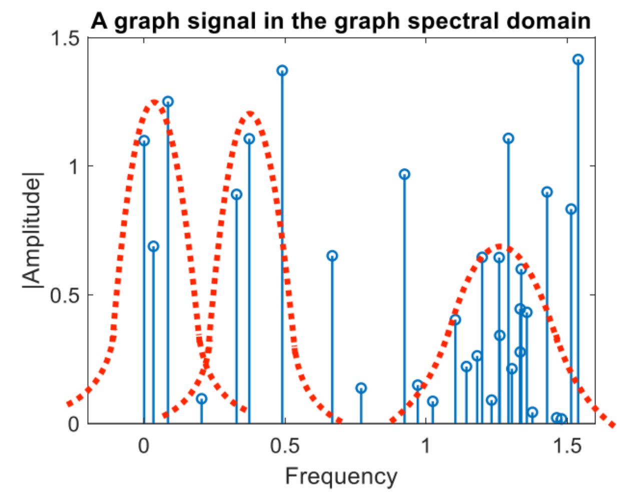
•Narrowband Spectral Graph Kernels
•Spectral Graph Dictionaries

Dictionary

$$Y^{m}_{\text{estimated}} = D^{m} X^{m}$$

Sparse Codes

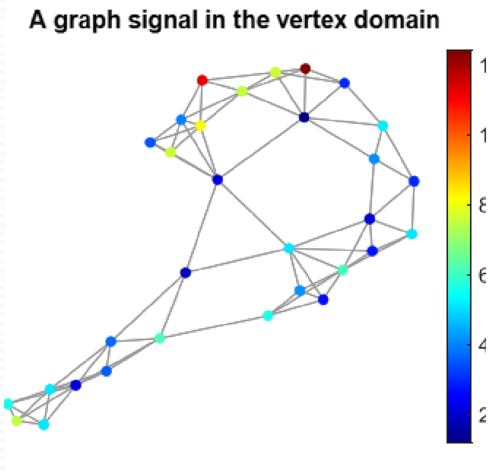


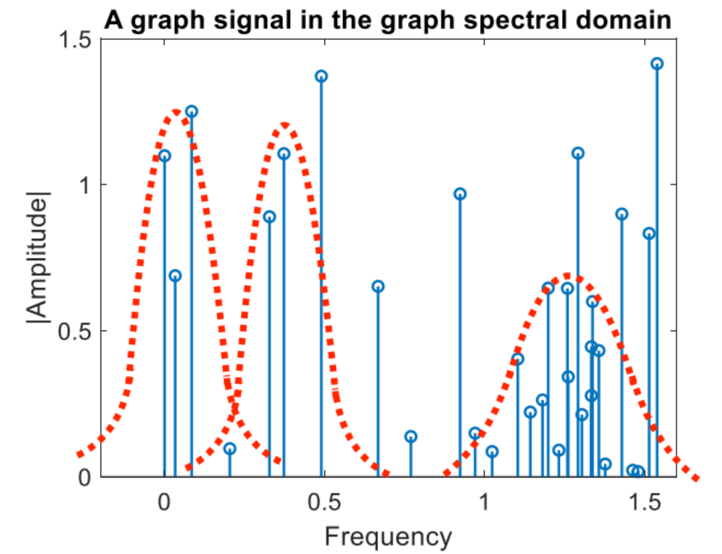


Novelties

- Explicitly model the spectral components of the data and fits them to the parameters of narrowband spectral graph kernels

- Data have missing entries while learning where missing entries of each signal in the dataset differ from each other

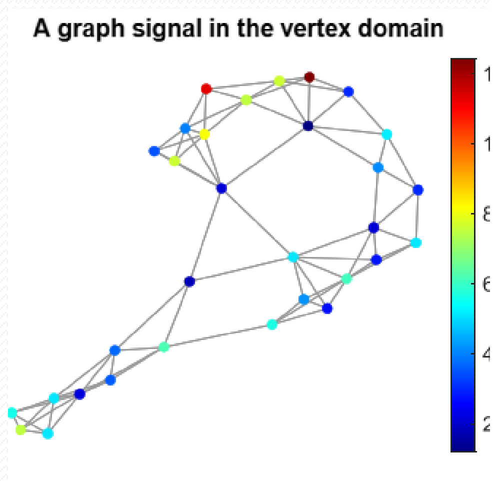


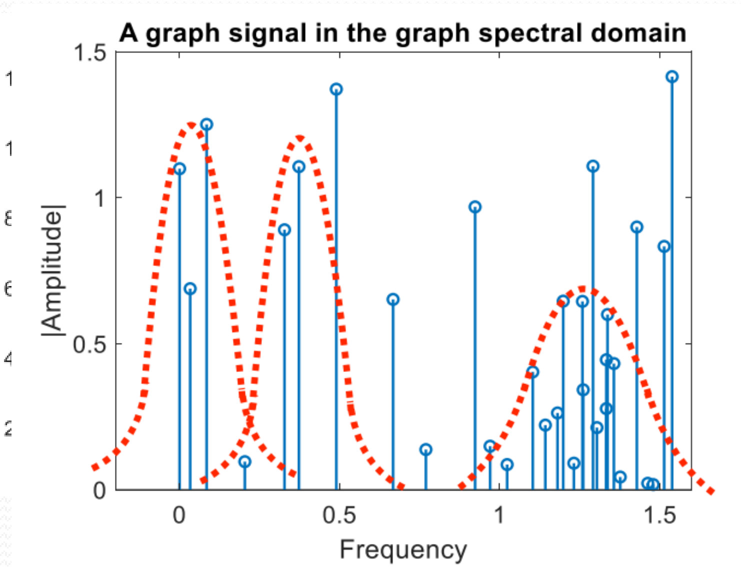


Novelties

- The capacity to generalize the model to any given graph by learning the graph dictionaries solely in the spectral domain

- Ability to jointly learn graph dictionaries and their sparse codings for multiple graphs and data sets.





Signal Model and Notation

$$\hat{g}_j(\lambda) = \exp\left(-\frac{\|\lambda-\mu_j\|^2}{s_j^2}\right)$$ → Gaussian Spectral Kernels

$$D_j^m = U^m \hat{g}_j(\Lambda^m)(U^m)^T \in \mathbb{R}^{N^m \times N^m}$$ → Spectral Sub-Dictionaries

$$D^m = \begin{bmatrix} D_1^m & D_2^m & \cdots & D_J^m \end{bmatrix} \in \mathbb{R}^{N^m \times JN^m}$$ → Spectral Dictionaries

$$y_i^m = D^m x_i^m + w_i^m$$

Proposed Method

Two Step Minimization. Not Jointly Convex.

$$
\begin{aligned}
\min_{\{X^m\},\psi} \; & \overbrace{\sum_{j=1}^{J} (\mu_j)^2 + \eta_s \sum_{j=1}^{J} (s_j - s_0)^2}^{\text{Spectral Kernel Parameters}} + \eta_x \sum_{m=1}^{M} \|X^m\|_1 \\
& + \eta_w \sum_{m=1}^{M} \sum_{i=1}^{K^m} \left\| S^{m,i} y_i^m - S^{m,i} D^m x_i^m \right\|^2 \\
& + \underbrace{\eta_y \sum_{m=1}^{M} \operatorname{tr}((X^m)^T (D^m)^T L^m D^m X^m)}_{\text{Smoothly Varying Reconstructed Signal}} + \eta_c \sum_{m=1}^{M} \operatorname{tr}((X^m) \widetilde{L}^m (X^m)^T)
\end{aligned}
$$

Sparse Representation over Graph Dictionary

Coherency with partial observations

Similar Reconstructed Signals with Similar Dictionary Atoms

Proposed Method

Two Step Minimization. Not Jointly Convex.

- Fix D, Optimize X
  - ADMM
- Fix X, Optimize D
  - Gradient Descent

$$\min_{\{X^m\}} \eta_x \sum_{m=1}^{M} \|X^m\|_1 + \eta_w \sum_{m=1}^{M} \sum_{i=1}^{K^m} \left\|S^{m,i} y_i^m - S^{m,i} D^m x_i^m\right\|^2$$
$$+ \eta_y \sum_{m=1}^{M} ((X^m)^T (D^m)^T \mathcal{L}^m D^m X^m)$$
$$+ \eta_c \sum_{m=1}^{M} ((X^m) \widetilde{\mathcal{L}}^m (X^m)^T).$$

$$\min_{\psi} f(\psi) = \min_{\psi} \sum_{j=1}^{J} (\mu_j)^2 + \eta_s \sum_{j=1}^{J} (s_j - s_0)^2$$
$$+ \eta_w \sum_{m=1}^{M} \sum_{i=1}^{K^m} \left\|S^{m,i} y_i^m - S^{m,i} D^m x_i^m\right\|^2$$
$$+ \eta_y \sum_{m=1}^{M} ((X^m)^T (D^m)^T \mathcal{L}^m D^m X^m).$$

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Experiments: Baseline Methods

- Spectral Graph Wavelet Dictionaries (tight frame, Meyer, Mexican hat, and ab-spline) (Hammond, 2011)
- Inference of graph signals based on multi-scale dictionary learning (GEMS) (Yankelevsky, 2019)
- Tikhonov regularization (Graph Based Tik.) (Zhou, 2004)
- Total Variation Minimization (TVMin) (Jung, 2019)
- Interpolation of non-smooth graph signals (LSEM) (Mazarguil, 2022)
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- Learning spectrally concentrated kernels method (SCGDL) (Turhan, 2021)

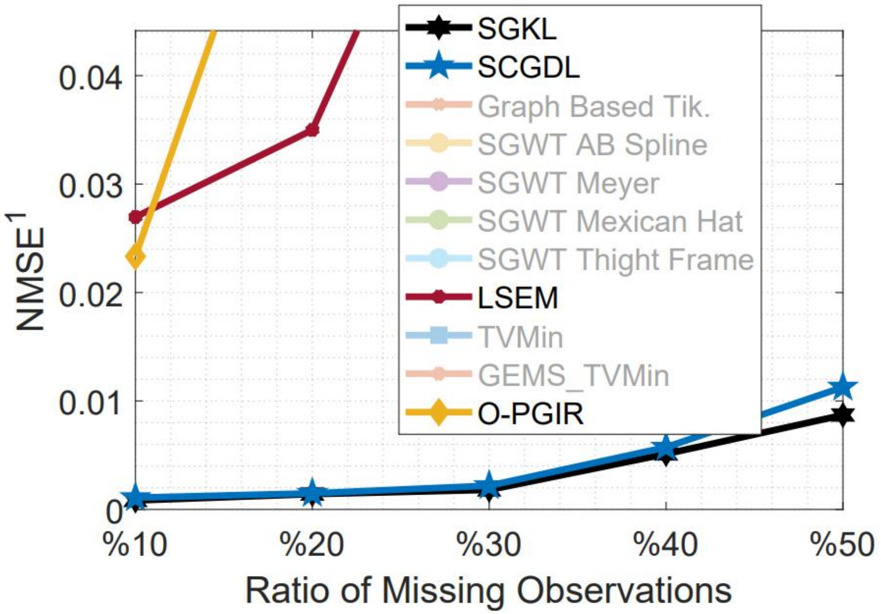
Experiments: Data Set

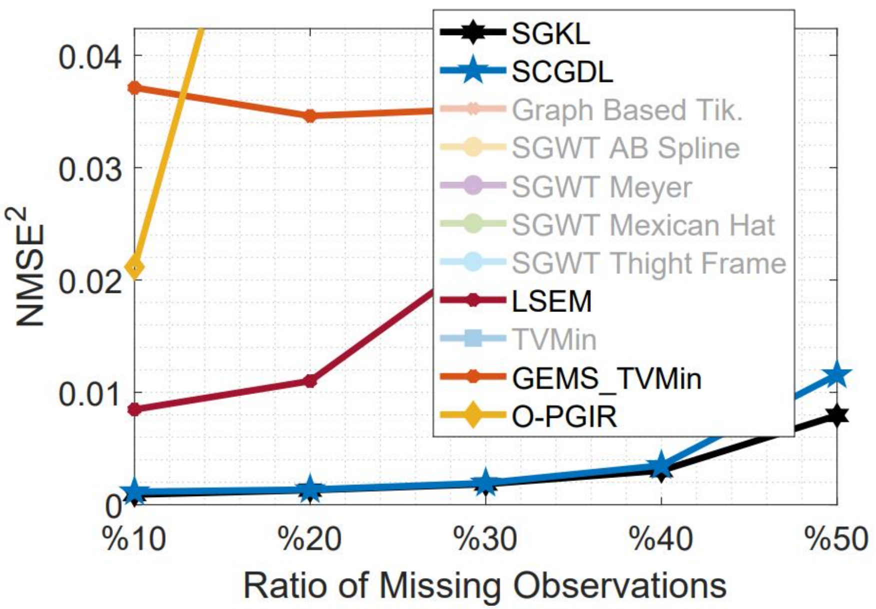
Synthetic Data Set:

- G^1 and G^2 10-NN graphs with 100 Nodes.
- J=4 Spectral Kernels
- K^1 = 200 and K^2 = 400 Signals

Experiments: Results

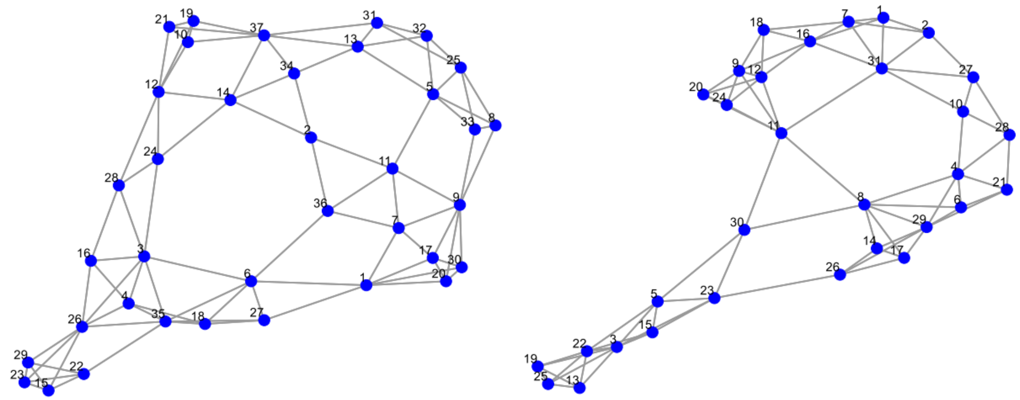
$$NMSE^m = \left\| Y_u^m - \tilde{Y}_u^m \right\|^2 / \| Y_u^m \|^2$$





Experiments: Data Set

Molene Data Set:
Released by the French national meteorological service which consists of temperature and wind speed measurements taken at different locations in the Brittany region of France.

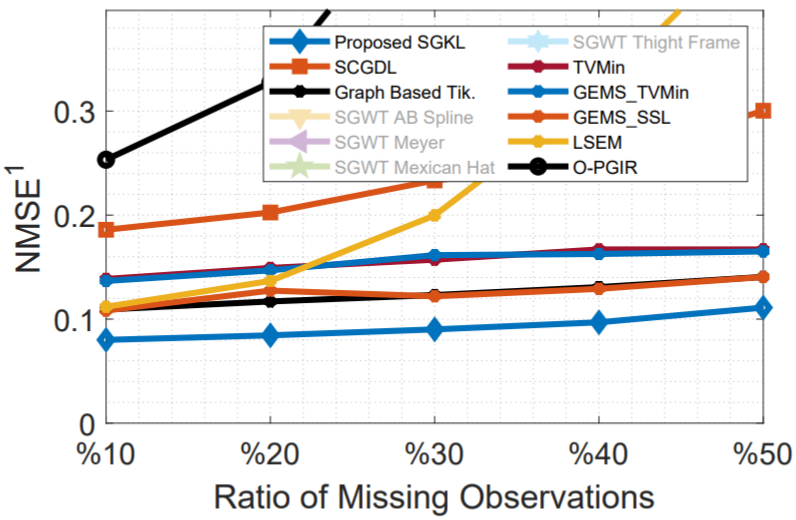


(a) Temperature (b) Wind-Speed

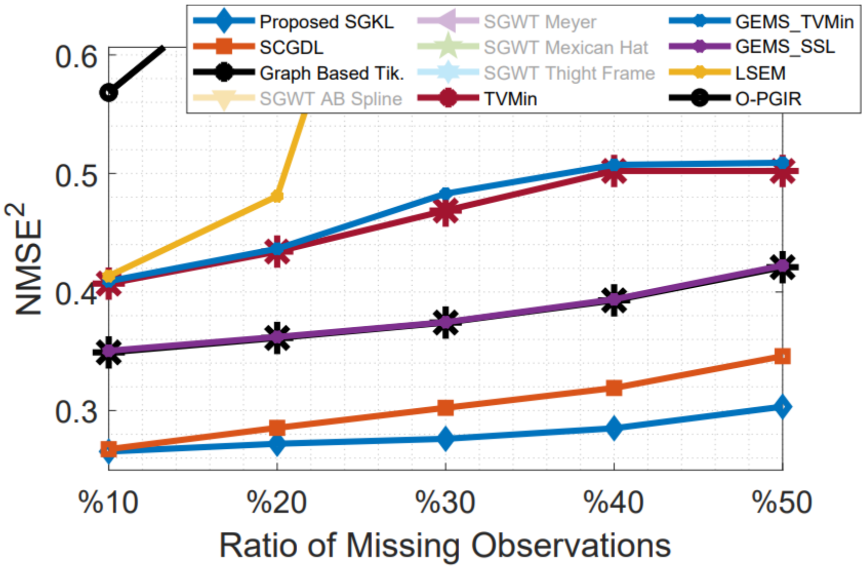
$N^1 = 37$ $N^2 = 31$, 5-NN
$K^1 = K^2 = 744$

Experiments: Results

$$NMSE^m = \left\| Y_u^m - \tilde{Y}_u^m \right\|^2 / \| Y_u^m \|^2$$



(a) G1-Temperature



(b) G2-Wind-Speed

Experiments: Data Set

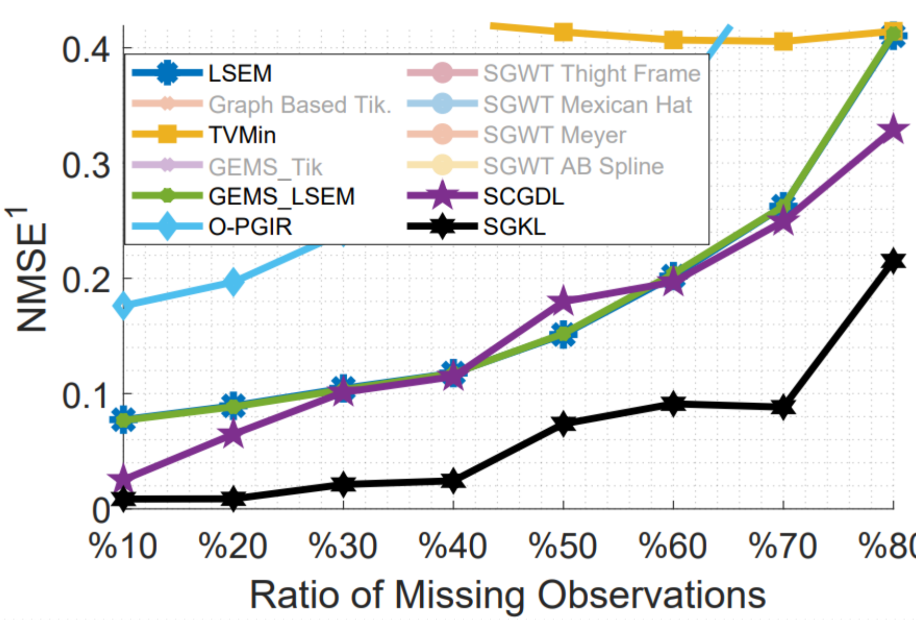
Covid Data Set:
a publicly available online dataset which contains the number of daily new cases per country

(a) Europe (b) US

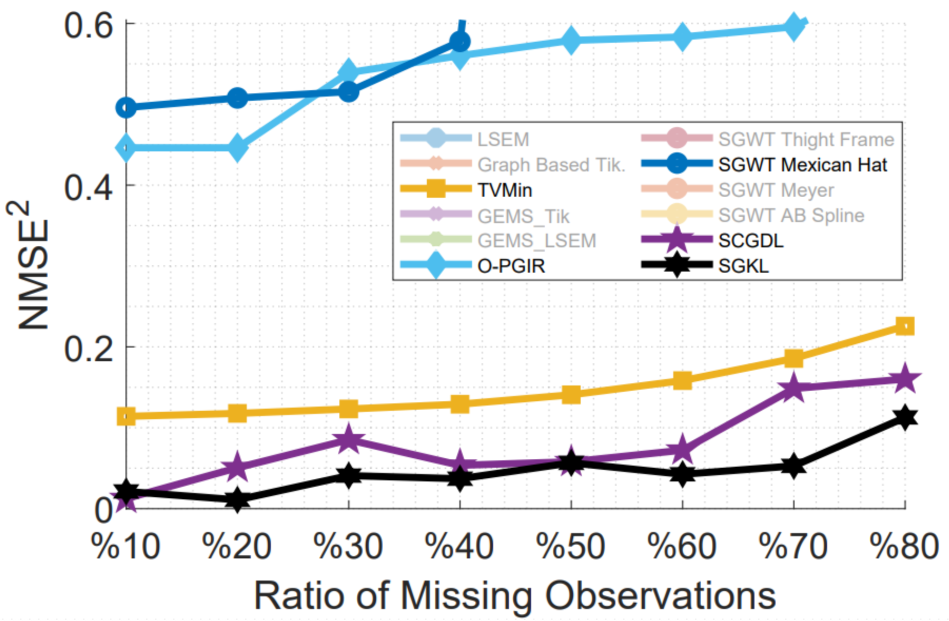
$N^1 = 37$ $N^2 = 50$, 4-NN, 3-NN
$K^1 = K^2 = 483$

Experiments: Results

$$NMSE^m = \left\|Y_u^m - \tilde{Y}_u^m\right\|^2 / \|Y_u^m\|^2$$

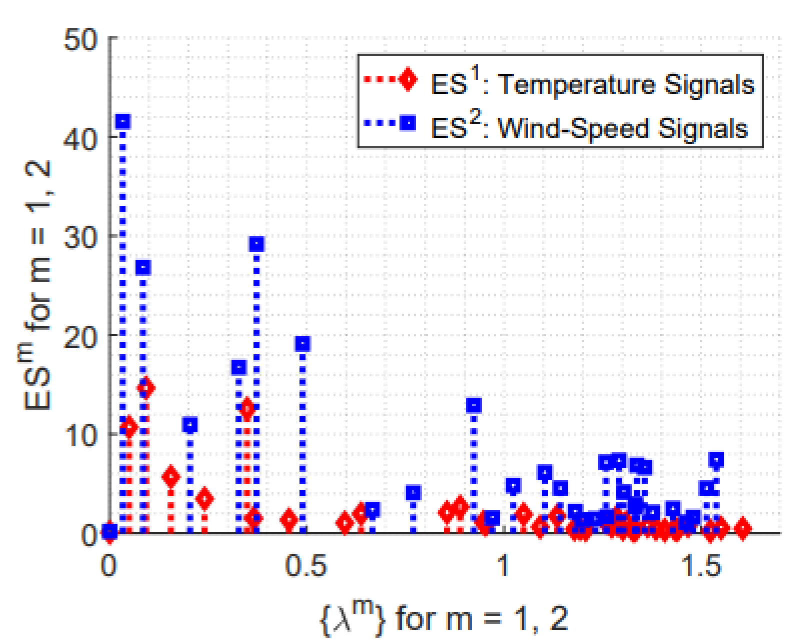


(a) G[1]-Europe

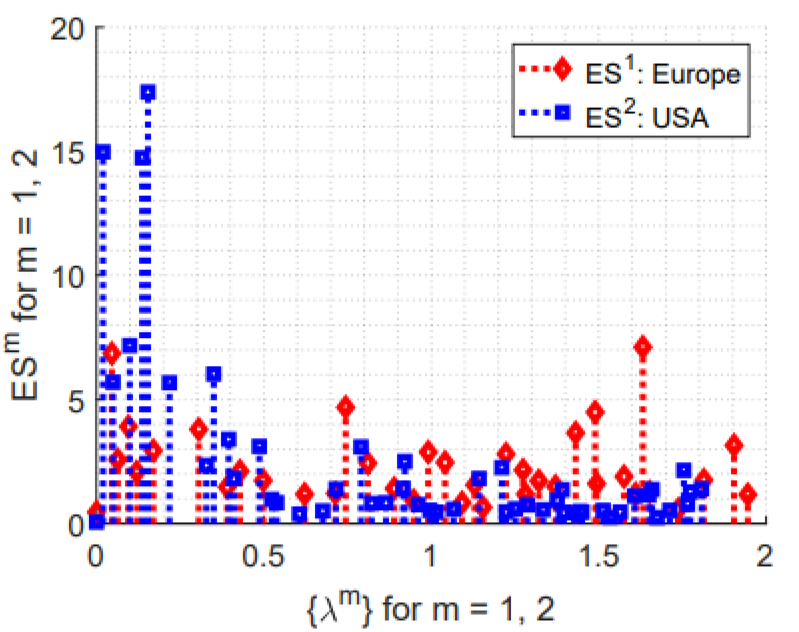


(b) G[2]-US

Experiments: Energy Spectrum of Datasets



Molene



Covid

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Experiments: Performance and Sensitivity Analysis

Effect of:

- Noise Level
- Learning different numbers of spectral kernels
- Ground-truth sparsity
- Learning
    - Jointly on multiple graphs or
    - Independently on each single graph
- Sensitivity analysis of the hyperparameters
    - η_ω
    - η_y
    - η_c
    - η_x
    - η_s

Experiments: Noise Level

$$NMSE^m = \left\| Y_u^m - \tilde{Y}_u^m \right\|^2 / \|Y_u^m\|^2$$

SNR, dB	$\mathcal{G}^1$	$\mathcal{G}^2$
-8	1.61	1.49
-6	1.38	1.31
-5	1.02	1.02
-3	0.61	0.59
-1	0.43	0.42
6	0.13	0.12
9	0.01	0.01
15	0.003	0.004
21	$4{\cdot}10^{-4}$	$5{\cdot}10^{-4}$
35	$4{\cdot}10^{-4}$	$4{\cdot}10^{-4}$

NMSE vs SNR for $\mathcal{G}^1$ and $\mathcal{G}^2$

$$NMSE^m = \left\| Y_u^m - \tilde{Y}_u^m \right\|^2 / \| Y_u^m \|^2$$

Experiments: Number of Kernels

J	$\mathcal{G}^1$	$\mathcal{G}^2$
1	0.0027	0.0016
2	$4.38 \cdot 10^{-4}$	$3.71 \cdot 10^{-4}$
3	$3.36 \cdot 10^{-4}$	$2.53 \cdot 10^{-4}$
4	$4.06 \cdot 10^{-4}$	$3.6 \cdot 10^{-4}$
5	0.0039	0.0051

NMSE vs J for $\mathcal{G}^1$ and $\mathcal{G}^2$

$$NMSE^m = \left\|Y_u^m - \tilde{Y}_u^m\right\|^2 / \|Y_u^m\|^2$$

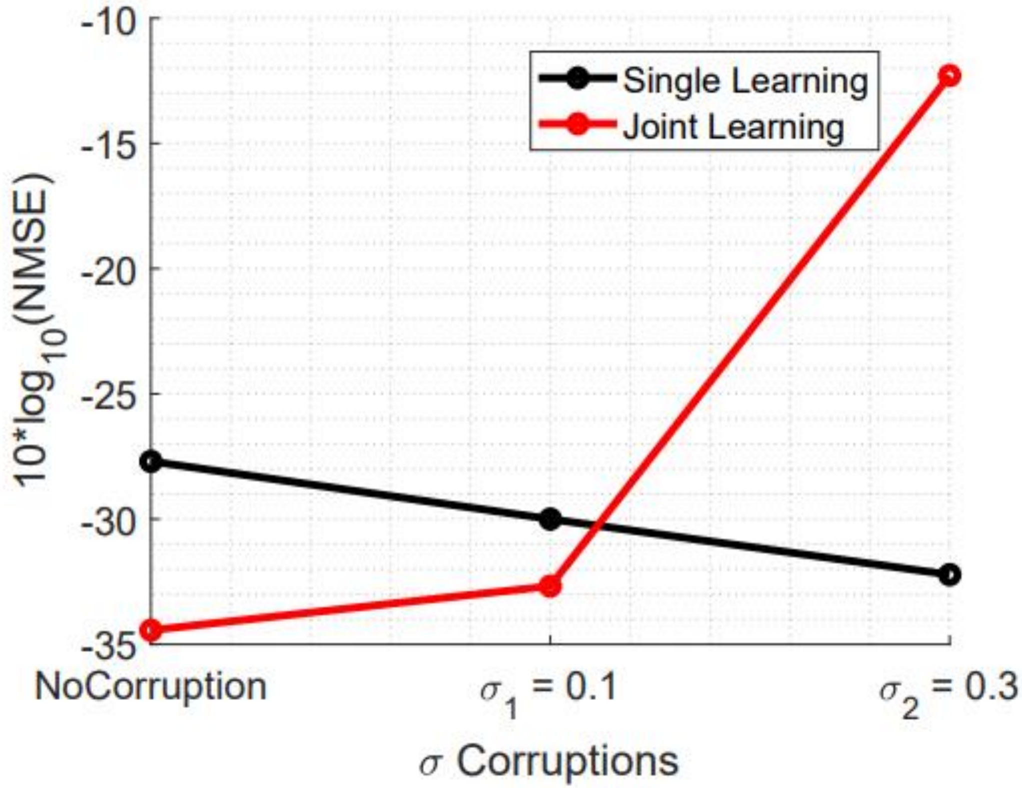
Experiments: Ground-truth sparsity

$T_{0,GT}$	$\mathcal{G}^1$	$\mathcal{G}^2$
10	$62 \cdot 10^{-4}$	$58 \cdot 10^{-4}$
20	$17 \cdot 10^{-4}$	$18 \cdot 10^{-4}$
30	$18 \cdot 10^{-4}$	$22 \cdot 10^{-4}$
40	$4.06 \cdot 10^{-4}$	$3.6 \cdot 10^{-4}$
50	$4 \cdot 10^{-4}$	$5 \cdot 10^{-4}$
60	$14 \cdot 10^{-4}$	$15 \cdot 10^{-4}$
70	$43 \cdot 10^{-4}$	$47 \cdot 10^{-4}$
80	$25 \cdot 10^{-4}$	$27 \cdot 10^{-4}$
90	$33 \cdot 10^{-4}$	$29 \cdot 10^{-4}$
100	$25 \cdot 10^{-4}$	$25 \cdot 10^{-4}$

NMSE vs $T_{0,GT}$ for $\mathcal{G}^1$ and $\mathcal{G}^2$

$$NMSE^m = \left\| Y_u^m - \tilde{Y}_u^m \right\|^2 / \|Y_u^m\|^2$$

Experiments: Joint vs Independent Learning



Single vs joint learning of graph dictionaries in the case of mismatches in the ground-truth kernels.

Experiments: η_y

$$NMSE^m = \left\|Y_u^m - \tilde{Y}_u^m\right\|^2 / \|Y_u^m\|^2$$

η_y	$\mathcal{G}^1$	$\mathcal{G}^2$
10^0	0.0042	0.0052
10^1	0.0091	0.010
10^2	$4.06 \cdot 10^{-4}$	$3.6 \cdot 10^{-4}$
10^3	0.0034	0.041
10^4	0.013	0.014
10^5	0.022	0.022

NMSE vs η_y for $\mathcal{G}^1$ and $\mathcal{G}^2$

$$\eta_y \sum_{m=1}^{M} ((X^m)^T (D^m)^T \mathcal{L}^m D^m X^m)$$

Experiments: η_c

$$NMSE^m = \left\|Y_u^m - \tilde{Y}_u^m\right\|^2 / \|Y_u^m\|^2$$

η_c	$\mathcal{G}^1$	$\mathcal{G}^2$
10^0	0.01	0.01
10^1	0.006	0.006
10^2	0.002	0.002
10^3	$4.06 \cdot 10^{-4}$	$3.6 \cdot 10^{-4}$
10^4	0.003	0.002
10^5	0.003	0.003

NMSE vs η_c for $\mathcal{G}^1$ and $\mathcal{G}^2$

$$\eta_c \sum_{m=1}^{M} ((X^m)\tilde{\mathcal{L}}^m(X^m)^T)$$

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Conclusion

- We propose a graph signal model based on representations using narrowband graph kernel prototypes.
- The initially unknown observations of the signals are then estimated based on this learnt model.
- Experiments on synthetic and real graph signal sets show that the proposed method provides promising signal estimation performance compared to baseline solutions.

Conclusion: Future Remarks

- Exploring more challenging scenarios with missing entries on the graph in regional, grid, or patterned forms.

- Developing approximate solutions for faster computation and enhanced scalability.

Thank You