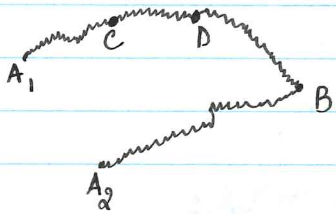
MATH ASIDE (contd...)

4.



$X_{CD} \sim ?$ $\quad \mathrm{Var}\, X_{CD} = ?$

$$X_{t_{A_1B}} \sim \mathcal{N}\left(\frac{\frac{a_1}{t_{A_1B}} + \frac{a_2}{t_{A_2B}}}{\frac{1}{t_{A_1B}}+\frac{1}{t_{A_2B}}}, \frac{1}{\frac{1}{t_{A_1B}}+\frac{1}{t_{A_2B}}}\right)$$

$$X_{CD} = X_{A_1D} - X_{A_1C}$$
$$= B_{t_{A_1D}} - \frac{t_{A_1D}}{t_{AB}}\left(B_{t_{A_1D}} - X_{t_{A_1D}}\right) - B_{t_{A_1C}} + \frac{t_{A_1C}}{t_{A_1B}}\left(B_{t_{A_1D}} - X_{t_{A_1D}}\right)$$

$$X_{CD} = B_{t_{A_1D}} - B_{t_{A_1C}} - \frac{t_{CD}}{t_{A_1B}}\left(B_{t_{A_1D}} - X_{t_{A_1D}}\right)$$

X_{CD} is normal since it is sum of multinormal RVs.

$$E[X_{CD}] = -\frac{t_{CD}}{t_{A_1B}}\left(a_1 - \frac{a_1 t_{A_2B} + a_2 t_{A_1B}}{t_{A_1B}+t_{A_2B}}\right)$$
$$= -\frac{t_{CD}}{t_{A_1B}}\left(\frac{a_1 t_{A_1B} - a_2 t_{A_1B}}{t_{A_1B}+t_{A_2B}}\right)$$
$$= \frac{t_{CD}\,(a_2-a_1)}{t_{A_1B}+t_{A_2B}}$$

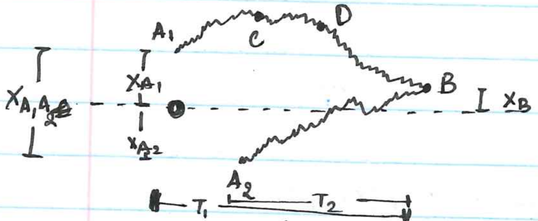
$$\mathrm{Var}\,X_{CD} = \mathrm{Var}\left(B_{t_{A_1D}} - B_{t_{A_2D}}\right) + \mathrm{Var}(\quad) - 2\,\mathrm{Cov}(\quad)$$

$$\cancel{\mathrm{Var}\,X_{CD}} = t_{CD} + \frac{t_{CD}^2}{t_{A_1B}^2}\left(t_{A_1B} + \frac{t_{A_1B}t_{A_2B}}{t_{A_1B}+t_{A_2B}}\right) - 2\frac{t_{CD}^2}{t_{A_1B}}$$

$$= t_{CD} + \frac{t_{CD}^2}{t_{A_1B}}\left(1 + \frac{t_{A_2B}}{t_{A_1B}+t_{A_2B}} - 2\right)$$

$$\boxed{\mathrm{Var}\,X_{CD} = \frac{t_{CD}\left(t_{A_1B}+t_{A_2B}-t_{CD}\right)}{t_{A_1B}+t_{A_2B}}}$$

(58) Suppose a_1 & a_2 are R.Vs i.e. $a_1 = X_{A_1}$ and $a_2 = X_{A_2}$.



$X_B = ?$ $\quad X_c = ?$ $\quad X_{CD} = ?$

$$X_B \sim \mathcal{N}\left(\frac{T_2 X_{A_1} + T_1 X_{A_2}}{T_1+T_2}, \frac{T_1T_2}{T_1+T_2}\right).$$

$$E[X_B] = E[E[X_B \mid X_{A_1}, X_{A_2}]]$$

$$= E\left[\frac{T_2 X_{A_1} + T_1 X_{A_2}}{T_1+T_2}\right] = \frac{T_2 E[X_{A_1}] + T_1 E[X_{A_2}]}{T_1+T_2}$$

$$\operatorname{Var} X_B = \operatorname{Var}(E[X_B \mid X_{A_1}, X_{A_2}]) + E[\operatorname{Var}(X_B \mid X_{A_1}, X_{A_2})]$$

$$= \operatorname{Var}\left(\frac{T_2 X_{A_1} + T_1 X_{A_2}}{T_1+T_2}\right) + \frac{T_1T_2}{T_1+T_2}$$

$$= \frac{T_2^2 \operatorname{Var} X_{A_1} + T_1^2 \operatorname{Var} X_{A_2}}{(T_1+T_2)^2} + \frac{T_1T_2}{T_1+T_2}$$

If origin is at A_1 (or everything is measured relative to A_1)

$$\boxed{E[X_{A_1B}] = \frac{T_1 \, E[X_{A_1A_2}]}{T_1+T_2} \qquad \operatorname{Var} X_{A_1B} = \frac{T_1T_2}{T_1+T_2} + \frac{T_1^2 \operatorname{Var} X_{A_1A_2}}{(T_1+T_2)^2}}$$

Measuring relative to A_1 → Similarly, ~~$E[X_{A_1C}]$~~ X_{A_1C} = $B_{t_{A_1C}} - \frac{t_{A_1C}}{T_1}(B_{T_1} - X_{A_1B})$

$$E[X_{A_1C}] = \frac{t_{A_1C}}{T_1} E[X_{A_1B}] = \frac{t_{A_1C}}{T_1+T_2} E[X_{A_1A_2}]$$

$$\operatorname{Var}(X_{A_1C}) = \underbrace{\frac{t_{A_1C}(T_1+T_2-t_{A_1C})}{T_1+T_2}}_{E[\operatorname{Var}(X_{A_1C} \mid X_{A_1B})]} + \underbrace{\frac{t_{A_1C}^2}{T_1^2} \operatorname{Var} X_{A_1B}}_{\operatorname{Var}(E[X_{A_1C} \mid X_{A_1B}])}$$

$$= \frac{t_{A_1C}(T_1+T_2-t_{A_1C})}{T_1+T_2} + \frac{t_{A_1C}^2}{(T_1+T_2)^2} \operatorname{Var} X_{A_1A_2}$$

$$\boxed{E[X_{A_1C}] = \frac{t_{A_1C}}{T_1+T_2} E[X_{A_1A_2}] \qquad \operatorname{Var} X_{A_1C} = \frac{t_{A_1C}(T_1+T_2-t_{A_1C})}{T_1+T_2} + \frac{t_{A_1C}^2}{(T_1+T_2)^2} \operatorname{Var} X_{A_1A_2}}$$

$$X_{CD} \sim \mathcal{N}\left(\frac{t_{CD}\, X_{A_1A_2}}{T_1+T_2} , \frac{t_{CD}(T_1+T_2-t_{CD})}{T_1+T_2} \right).$$

$$E[X_{CD}] = \frac{t_{CD}}{T_1+T_2} E[X_{A_1A_2}]$$

$$\mathrm{Var}\, X_{CD} = E[\mathrm{Var}(X_{CD} \mid X_{A_1A_2})] + \mathrm{Var}(E[X_{CD} \mid X_{A_1A_2}])$$
$$= \frac{t_{CD}(T_1+T_2-t_{CD})}{T_1+T_2} + \frac{t_{CD}^2}{(T_1+T_2)^2} \mathrm{Var}\, X_{A_1A_2}.$$

$$\boxed{E[X_{CD}] = \frac{t_{CD}}{T_1+T_2} E[X_{A_1A_2}] + \frac{t_{CD}^2}{(T_1+T_2)^2} \mathrm{Var}\, X_{A_1A_2}}$$

$$\mathrm{Cov}(X_{CD}, X_{A_1A_2}) = ?$$

$$X_{CD} = B_{t_{A_1D}} - B_{t_{A_1C}} - \frac{t_{CD}}{T_1}\left(B_{T_1} - X_{A_1B} \right) \Rightarrow \mathrm{Cov}(X_{CD}, X_{A_1A_2}) = \frac{t_{CD}}{T_1} \mathrm{Cov}(X_{A_1B}, X_{A_1A_2})$$

∴ We need $\mathrm{Cov}(X_{A_1B}, X_{A_1A_2})$

$$X_{A_1B} \sim \mathcal{N}\left(\frac{T_1 X_{A_1A_2}}{T_1+T_2} , \frac{T_1T_2}{T_1+T_2} \right)$$

$$E[X_{A_1B} X_{A_1A_2}] = \cancel{\int_x \int_y p_{X_{A_1B}, X_{A_1A_2}}(x,y)\, dy\, dx}$$

⇒ This is correct but unnecessary

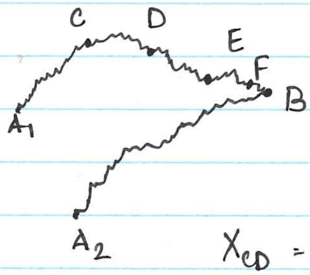
$$\cancel{= \int_x \int_y xy\, p_{A_1A_2}(y) \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu(X_{A_1A_2}))^2/2\sigma^2}\, dx\, dy}$$
$$\cancel{= \int_y y\, \mu(y)\, p_{A_1A_2}(y)\, dy = E[X_{A_1A_2}\, \mu(X_{A_1A_2})]}$$
$$\cancel{= E\left[\frac{T_1}{T_1+T_2} X_{A_1A_2}^2 \right]}$$

$$E[X_{A_1B} X_{A_1A_2}] = E\left[E[X_{A_1B} X_{A_1A_2} \mid X_{A_1A_2}] \right] = E\left[X_{A_1A_2} E[X_{A_1B} \mid X_{A_1A_2}] \right]$$
$$= \frac{T_1}{T_1+T_2} E[X_{A_1A_2}^2]$$

$$\mathrm{Cov}(X_{A_1B}, X_{A_1A_2}) = E[X_{A_1B} X_{A_1A_2}] - E[X_{A_1B}]E[X_{A_1A_2}]$$
$$= \frac{T_1}{T_1+T_2} \mathrm{Var}\, X_{A_1A_2}$$

$$\therefore \boxed{\mathrm{Cov}(X_{A_1B}, X_{A_1A_2}) = \frac{T_1}{T_1+T_2} \mathrm{Var}\, X_{A_1A_2}}$$

(6.) $\mathrm{Cov}(X_{CD}, X_{EF}) = ?$ Fixed A_1 A_2.



$\cancel{\mathrm{Cov}\ X_{CD} \sim B_{t_{CD}} - t_{CD}}$

$X_{A_1B} \sim \mathcal{N}\left(\frac{t_{A_1B}(a_2-a_1)}{t_{A_1B}+t_{A_2B}}, \frac{t_{A_1B}t_{A_2B}}{t_{A_1B}+t_{A_2B}}\right)$

$$X_{CD} = B_{t_{A_1D}} - B_{t_{A_1C}} - \frac{t_{CD}}{t_{A_1B}}\left(\alpha B_{t_{A_1B}} - X_{A_1B}\right)$$

$$X_{EF} = B_{t_{A_1F}} - B_{t_{A_1E}} - \frac{t_{EF}}{t_{A_1B}}\left(B_{t_{A_1B}} - X_{A_1B}\right)$$

$$\mathrm{Cov}(X_{CD}, X_{EF}) = t_{A_1D} - t_{A_1D} - t_{A_1C} + t_{A_1C} - \frac{t_{EF}}{t_{A_1B}} \cdot t_{CD} - \frac{t_{CD}}{t_{A_1B}} t_{EF}$$

$$+ \frac{t_{CD}t_{EF}}{t_{A_1B}^2} \cdot \left(t_{A_1B} + \frac{t_{A_1B}t_{A_2B}}{t_{A_1B}+t_{A_2B}}\right).$$

$$= -\cancel{\frac{t_{CD}t_{EF}}{t_{A_1B}}} \frac{t_{CD}t_{EF}}{t_{A_1B}}\left(-2+1+\frac{t_{A_2B}}{t_{A_1B}+t_{A_2B}}\right).$$

$$= -\frac{t_{CD}t_{EF}}{t_{A_1B}+t_{A_2B}}$$

$$\therefore \boxed{\mathrm{Cov}(X_{CD}, X_{EF}) = -\frac{t_{CD}t_{EF}}{t_{A_1B}+t_{A_2B}}}$$

(7.) Same as (6) but A_1 & A_2 are not fixed.

$$\therefore X_{A_2B} \sim \mathcal{N}\left(\frac{t_{A_1B}X_{A_1A_2}}{t_{A_1B}+t_{A_2B}}, \frac{t_{A_1B}t_{A_2B}}{t_{A_1B}+t_{A_2B}}\right)$$

$$\mathrm{Cov}(X_{CD}, X_{EF}) = -\frac{t_{CD}t_{EF}}{t_{A_1B}+t_{A_2B}} + \frac{t_{CD}t_{EF}}{t_{A_1B}} \cdot \frac{t_{A_1B}^2}{(t_{A_1B}+t_{A_2B})^2} \cdot \mathrm{Var}\, X_{A_1A_2}$$

$$\boxed{\mathrm{Cov}(X_{CD}, X_{EF}) = -\frac{t_{CD}t_{EF}}{t_{A_1B}+t_{A_2B}} + \frac{t_{CD}t_{EF}}{(t_{A_1B}+t_{A_2B})^2}\mathrm{Var}\, X_{A_1A_2}}$$