→ Effect of recombination on dispersal estimates (2 samples)

$$\vec{z}(0) = (1^T C^{-1} 1)^{-1} (1^T C^{-1} \vec{x})$$

$$= \frac{1^T C^{-1} \vec{x}}{c_{11}' + c_{12}' + c_{21}' + c_{22}'}$$

$$= \frac{(c_{11}' + c_{21}')x_1 + (c_{12}' + c_{22}')x_2}{c_{11}' + c_{12}' + c_{21}' + c_{22}'}$$

$$= \frac{\frac{1}{|C|}}{\frac{1}{|C|}} \frac{(c_{22} - c_{21})x_1 + (c_{11} - c_{12})x_2}{c_{11} - c_{12} - c_{21} + c_{22}}$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$\text{adj } C = \begin{bmatrix} c_{22} & -c_{12} \\ -c_{21} & c_{11} \end{bmatrix}$$

$$|C| = c_{11}c_{22} - c_{21}c_{22}$$

$$C^{-1} = \begin{bmatrix} c_{11}' & c_{12}' \\ c_{21}' & c_{22}' \end{bmatrix}$$

$$\sigma^2_{ML} = \frac{(\vec{x} - \vec{z}(0)\vec{1})^T C^{-1} (\vec{x} - \vec{z}(0)\vec{1})}{2}$$

But $$\vec{x} - \vec{z}(0)\vec{1} = \begin{bmatrix} (c_{11} - c_{12})(x_1 - x_2) \,/\, (\quad) \\ (c_{22} - c_{21})(x_2 - x_1) \,/\, (\quad) \end{bmatrix}$$

$$= \frac{x_2 - x_1}{c_{11} - c_{12} - c_{21} + c_{22}} \begin{bmatrix} c_{12} - c_{11} \\ c_{22} - c_{21} \end{bmatrix}$$

$$\sigma^2_{ML} = \frac{(\vec{x} - \vec{z}(0)\vec{1})^T}{2} \frac{x_2 - x_1}{c_{11} - c_{12} - c_{21} + c_{22}} \begin{bmatrix} c_{11}'(c_{12} - c_{11}) + c_{12}'(c_{22} - c_{21}) \\ c_{21}'(c_{12} - c_{11}) + c_{22}'(c_{22} - c_{21}) \end{bmatrix}$$

$$= \frac{(x_2 - x_1)}{2(c_{11} - c_{12} - c_{21} + c_{22})|C|} (\vec{x} - \vec{z}(0)\vec{1})^T \begin{bmatrix} c_{12}c_{21} - c_{22}c_{11} \\ c_{22}c_{11} - c_{12}c_{21} \end{bmatrix}$$

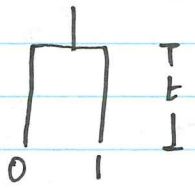
$$= \frac{x_2 - x_1}{2(c_{11} - c_{12} - c_{21} + c_{22})} \left(\vec{x} - \vec{x}(0)\vec{1}\right)^T \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{x_2 - x_1}{2(c_{11} - c_{12} - c_{21} + c_{22})} \circ \frac{(x_2 - x_1)}{(c_{11} - c_{12} - c_{21} + c_{22})} \circ (c_{11} - c_{12} - c_{21} + c_{22})$$

$$= \frac{(x_2 - x_1)^2}{2(c_{11} - c_{12} - c_{21} + c_{22})}$$

$$\therefore \boxed{\sigma_{ML}^2 = \frac{(x_2 - x_1)^2}{2(c_{11} - c_{12} + c_{21} + c_{22})}} \qquad \boxed{\vec{x}(0) = \frac{(c_{22} - c_{21})x_1 + (c_{11} - c_{12})x_2}{c_{11} - c_{12} - c_{21} + c_{22}}}$$

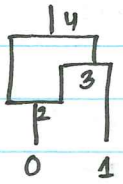
Case I: No Recombination




$$\because C = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$$

$$\therefore \vec{x}(0) = \frac{x_1 + x_2}{2} \qquad \sigma_{ML}^2 = \frac{(x_2 - x_1)^2}{4t}$$

Case II: Single Recombination Event




$$c_{00} = t_{20} + \frac{t_{42} t_{32}}{t_{42} + t_{32}} + \left(\frac{t_{42}}{t_{42} + t_{32}}\right)^2 t_{43}$$

$$c_{11} = t_{43} + t_{31} = t_{41}$$

$$c_{01} = \frac{t_{42}}{t_{42} + t_{32}} \circ t_{43}$$

$$c_{00} - c_{01} - c_{10} + c_{11} = t_{41} + t_{20} + \frac{t_{42} t_{32}}{t_{42} + t_{32}} + \left(\frac{t_{42}}{t_{42} + t_{32}}\right)^2 t_{43} - 2\,\frac{t_{42} t_{43}}{t_{42} + t_{32}}$$

$$= t_{41} + t_{20} + \frac{1}{(t_{42} + t_{32})^2}\left[t_{42} t_{32}(t_{42} + t_{32}) + t_{42}^2 (t_{42} - t_{32}) - 2 t_{42}(t_{42}^2 - t_{32}^2)\right]$$

$$= t_{41} + t_{20} + \frac{3t_{32}^2 - t_{42}^2}{(t_{32} + t_{42})^2} \cdot t_{42}$$

$$= t_{41} + t_{20} + \frac{3(t_{32}/t_{42})^2 - 1}{(t_{32}/t_{42} + 1)^2} \cdot t_{42}$$

$$= t_{41} + t_{42} + t_{20} + \left[\frac{2\left(\frac{t_{32}}{t_{42}}\right)^2 - 2\frac{t_{32}}{t_{42}} - 2}{\left(\frac{t_{32}}{t_{42}} + 1\right)^2}\right] t_{42}$$

$$= 2t + \left[\frac{3x^2 - 1}{(x+1)^2} - 1\right] t_{42} \cdot \cdot$$

$\because 0 < t_{32} < t_{42} \Rightarrow 0 < x < 1$

$$\Rightarrow \quad -1 < \frac{3x^2 - 1}{(x+1)^2} < \frac{1}{2}$$

$$\Rightarrow \quad -2 < \frac{3x^2 - 1}{(x+1)^2} < \frac{-1}{2}$$

$$\Rightarrow \quad 2t - 2t_{42} < \sigma_{ML}^2 < 2t - \frac{t_{42}}{2}$$

$$\Rightarrow \quad 2t_{20} < \sigma_{ML}^2 < \frac{2}{2}t - \frac{t_{20}}{2}$$

Compare No Recomb$^{n}$ (NR) to Single Recomb$^{n}$ Method 1 (SR1)

$$\Rightarrow \quad 1 - \frac{t_{42}}{t_{NR}} < \frac{\sigma^2_{ML}(SR1)}{\sigma^2_{ML}(NR)} < 1 - \frac{t_{42}}{4t_{NR}}$$

$$\Rightarrow \quad 1 - \frac{t_{SR1} - t_{RE}}{t_{NR}} < \frac{\sigma^2_{ML}(SR1)}{\sigma^2_{ML}(NR)} < 1 - \frac{t_{SR1} - t_{RE}}{t_{NR}}$$

$t_{SR1}$ = Time to GMRCA w/ single recomb event
$t_{NR}$ = Time to MRCA w/o recomb$^{n}$
$t_{RE}$ = Time to first recomb$^{n}$ event.