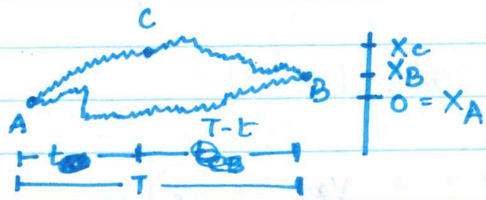
MATH ASIDE

(1)



Var X_c = ?

$X_c = B_t^{(1)} \mid B_T^{(1)} = B_T^{(2)}$ where $B_t^{(i)}$ $i \in \{1,2\}$ are 2 independent Brownian motions starting at 0 (= A)

$$X_T = B_T^{(1)} \mid B_T^{(1)} = B_T^{(2)}$$

$\Rightarrow$ Let $p_X(x)$ be the probability density of X.

Then,

$$p_{X_T}(x) = \frac{p_{B_T^{(1)}}(x)\, p_{B_T^{(2)}}(x)}{\int_y p_{B_T^{(1)}}(y)\, p_{B_T^{(2)}}(y)\, dy} = \frac{\frac{1}{2\pi T} e^{-x^2/T}}{\frac{1}{2\pi T} \sqrt{2\pi T/2}} = \frac{1}{\sqrt{2\pi T/2}} e^{-x^2/2\, T/2}$$

$\Rightarrow X_T \sim \mathcal{N}(0, T/2)$.

$$\text{Denominator} = \int_y p_{B_T^{(1)}}(y)\, p_{B_T^{(2)}}(y)\, dy$$

$$= \int_y \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} * \frac{1}{\sqrt{2\pi T}} e^{-y^2/2T}\, dy$$

$$= \frac{1}{2\pi T} \int_y e^{-y^2/T}\, dy$$

$$= \frac{1}{2\pi T} * \sqrt{2\pi T/2}$$

$X_c = B_t - \frac{t}{T}(B_T - X_T)$ where B_t is a brownian motion independent of ~~B~~ X_T

$$\operatorname{Var} X_c = E[\operatorname{Var}(X_c|X_T)] + \operatorname{Var}(E[X_c|X_T])$$

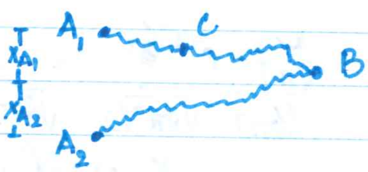
$$= E\left[t + \frac{t^2}{T^2} * T - 2\frac{t^2}{T}\right] + \operatorname{Var}\left(\frac{t}{T}X_T\right)$$

$$= t - \frac{\cancel{2}t^2}{T} + \frac{t^2}{2T}$$

$$= \frac{t(2T-t)}{2T}$$

$$\boxed{\text{Summary: } X_T \sim \mathcal{N}(0, T/2) \ \& \ \operatorname{Var} X_c = \frac{t(2T-t)}{2T}}$$

(2)



$\operatorname{Var} X_c = ?$ $\operatorname{Var} X_B = ?$

$$p_{X_B}(x) = \frac{p_{B_T^{(1)}}(x)\, p_{B_T^{(2)}}(x)}{\int_y p_{B_T^{(1)}}(y)\, p_{B_T^{(2)}}(y)\, dy} = \frac{\dfrac{e^{-\left(x - \frac{(a_1+a_2)}{2}\right)^2/T} e^{-\left(\frac{a_1-a_2}{2}\right)^2/T}}{2\pi T}}{\dfrac{\sqrt{2\pi T/2}}{2\pi T} e^{-(a_1-a_2/2)^2/T}}$$

$$\Rightarrow \boxed{X_B \sim \mathcal{N}\left(\frac{a_1+a_2}{2}, \frac{T}{2}\right)}$$

$$\text{Denominator} = \int_y \frac{e^{-(y-a_1)^2/2T}}{\sqrt{2\pi T}} \frac{e^{-(y-a_2)^2/2T}}{\sqrt{2\pi T}} dy$$

$$= \int_y \frac{1}{2\pi T} e^{-2(y^2 - y(a_1+a_2) + (a_1^2+a_2^2)/2)/2T} dy$$

$$= \frac{1}{2\pi T} \int_y e^{-\left(y - \frac{(a_1+a_2)}{2}\right)^2/T} e^{-\left(\frac{a_1-a_2}{2}\right)^2/T} dy$$

$$= \sqrt{2\pi T/2} \cdot \frac{e^{-\left(\frac{a_1-a_2}{2}\right)^2/T}}{2\pi T}$$

where $X_c = B_t - \frac{t}{T}(B_T - X_T)$

B_t is BM starting at a_1

$$\Rightarrow \mathrm{Var}\, X_c = E[\mathrm{Var}(X_c | X_T)] + \mathrm{Var}(E[X_c | X_T])$$

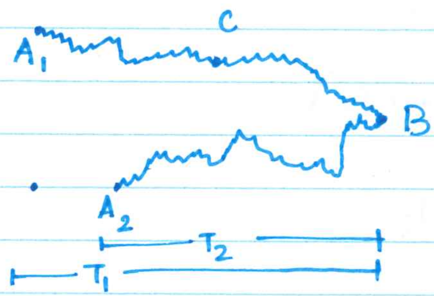
$$= t - \frac{t^2}{T} + \mathrm{Var}\left(a_1 - \frac{t}{T} a_1 + \frac{t}{T} X_T\right)$$

$$= t - \frac{t^2}{T} + \frac{t^2}{T^2} \mathrm{var}\, X_T$$

$$\mathrm{Var}\, X_c = \frac{t(2T-t)}{2T}$$

$$\boxed{\text{Summary}: \; X_T \sim \mathcal{N}\left(\frac{a_1 + a_2}{2}, \frac{T}{2}\right) \;\&\; \mathrm{Var}\, X_c = \frac{t(2T-t)}{2T}}$$

3.)



$X_B \sim$? var X_c ?

$$p_{X_B}(x) = \frac{p_{B^{(1)}_{T_1}}(x)\; p_{B^{(2)}_{T_2}}(x)}{\int_y p_{B_{T_1}}(y)\; p_{B_{T_2}}(y)\, dy} \quad \Rightarrow X_B \sim \mathcal{N}\left(\frac{\frac{a_1}{T_1} + \frac{a_2}{T_2}}{\frac{1}{T_1} + \frac{1}{T_2}}, \frac{1}{\frac{1}{T_1} + \frac{1}{T_2}}\right).$$

$$\text{Denominator} = \frac{1}{2\pi\sqrt{T_1 T_2}} \int_y \exp\left[-\frac{1}{2}\left(\frac{1}{T_1} + \frac{1}{T_2}\right)\left(y^2 - 2\frac{\frac{a_1}{T_1} + \frac{a_2}{T_2}}{\frac{1}{T_1} + \frac{1}{T_2}} y + \frac{\frac{a_1^2}{T_1} + \frac{a_2^2}{T_2}}{\frac{1}{T_1} + \frac{1}{T_2}}\right)\right] dy$$

$$= \frac{1}{2\pi\sqrt{T_1 T_2}} \int \mathrm{Exp}\left[\frac{-1}{2}\left(\frac{1}{T_1} + \frac{1}{T_2}\right)\left(y - \frac{\frac{a_1}{T_1} + \frac{a_2}{T_2}}{\frac{1}{T_1} + \frac{1}{T_2}}\right)^2\right] dy\, \mathrm{Exp}\left[\frac{-1}{2}(\quad)(\quad)\right]$$

$$= \frac{1}{2\pi\sqrt{T_1 T_2}} * \sqrt{2\pi\left(\frac{1}{T_1} + \frac{1}{T_2}\right)}\; \mathrm{Exp}\left[\frac{-1}{2}(\quad)(\quad)\right]$$

$$\operatorname{Var} X_c = E[\operatorname{Var}(X_c \mid X_T)] + \operatorname{Var}(E[X_c \mid X_T]) \qquad X_c = B_t - \frac{t}{T_1}(B_{T_1} - X_B)$$

$$= \cancel{\frac{t}{T_1}}\, t - \frac{t^2}{T_1} + \frac{t^2}{T_1^2}\left(\frac{1}{\frac{1}{T_1}+\frac{1}{T_2}}\right)$$

$$= t - \frac{t^2}{T_1} + \frac{t^2}{T_1^2}\left(\frac{T_1 T_2}{T_1 + T_2}\right)$$

$$= t - \frac{t^2}{T_1} + \frac{t^2 T_2}{T_1 (T_1 + T_2)}$$

$$= \frac{t(T_1 - t)}{T_1} + \frac{t^2}{T_1}\frac{T_2}{T_1 + T_2}$$

$$= \frac{t\left((T_1 - t)(T_1 + T_2) + t T_2\right)}{T_1 (T_1 + T_2)}$$

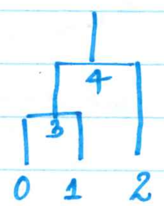
$$= \frac{t\left((T_1 - t)T_1 + T_1 T_2\right)}{T_1 (T_1 + T_2)}$$

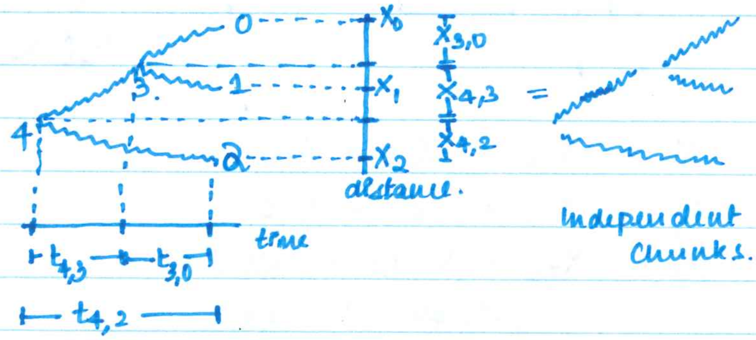
$$= \frac{t(T_1 + T_2 - t)}{\cancel{T_1}(T_1 + T_2)}.$$

$$\boxed{\begin{array}{l} \text{Summary:} \quad X_B \sim \mathcal{N}\left(\dfrac{\frac{a_1}{T_1} + \frac{a_2}{T_2}}{\frac{1}{T_1} + \frac{1}{T_2}}, \dfrac{1}{\frac{1}{T_1} + \frac{1}{T_2}}\right) \\[2ex] \qquad\qquad \operatorname{Var} X_c = \dfrac{t(T_1 + T_2 - t)}{\cancel{T_1}(T_1 + T_2)} \end{array}}$$

ANCESTRAL RECOMBINATION GRAPHS (ARGs)

~~Non spatial model~~ (No recombination)





$X_0 = X_{4,3} + X_{3,0}$ $\quad X_1 = X_{4,3} + X_{3,1}$ $\quad X_2 = X_{4,2}$

But $X_{4,3} = B^{(4,3)}_{t_{4,3}}$ $\quad X_{3,0} = B^{(3,0)}_{t_{3,0}}$ $\quad X_{3,1} = B^{(3,1)}_{t_{3,1}}$ where B^k_t are indep. brownian motions

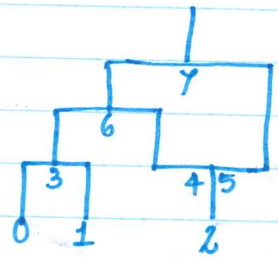
→ $\mathrm{Cov}(X_0, X_1) = \mathrm{Var}\, X_{4,3} = t_{4,3}$

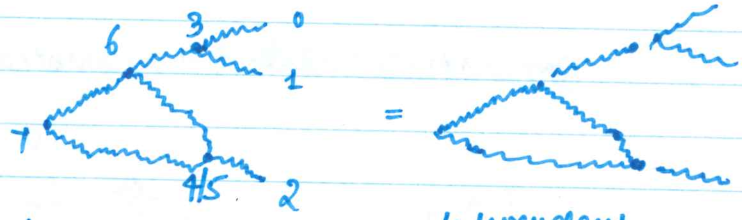
→ $\mathrm{Cov}(X_0, X_2) = 0 = \mathrm{Cov}(X_1, X_2)$

→ $\mathrm{Var}(X_0) = t_{4,3} + t_{3,0} = t_{4,2} = \mathrm{Var}\, X_1 = \mathrm{Var}\, X_2$

$$\therefore \quad C = \begin{bmatrix} t_{4,2} & t_{4,3} & 0 \\ t_{4,3} & t_{4,2} & 0 \\ 0 & 0 & t_{4,2} \end{bmatrix}$$

Single Recombination





Independent chunks.

$$X_0 = X_{7,6} + X_{6,3} + X_{3,0}$$

$$X_1 = X_{7,6} + X_{6,3} + X_{3,1}$$

$$X_2 = X_{7,6} + X_{6,4/5} + X_{4/5,2}$$

$$= X_{7,4/5} + X_{4/5,2}$$

$$\mathrm{Cov}(X_0, X_1) = \mathrm{Var}\, X_{7,6} + t_{6,3} \cancel{+ t_{3,0}}$$

$$= \frac{t_{7,6}(t_{6,4/5} + t_{7,4/5})}{\cancel{t_{7,6}+} 2t_{7,4/5}} + t_{6,3} \cancel{+ t_{3,0}}$$

$$\mathrm{Cov}(X_0, X_2) = \cancel{\mathrm{Var}\, X_{7,6} + 2\,\mathrm{Cov}(X_{7,6}, X_{6,4/5})}$$

$$= \mathrm{Cov}(X_{7,6}, X_{7,4/5}) = \frac{t_{7,6}}{2}$$

$$\mathrm{Var}\, X_0 = \mathrm{Var}\, X_{7,6} + \mathrm{Var}\, X_{6,3} + \mathrm{Var}\, X_{3,0}$$

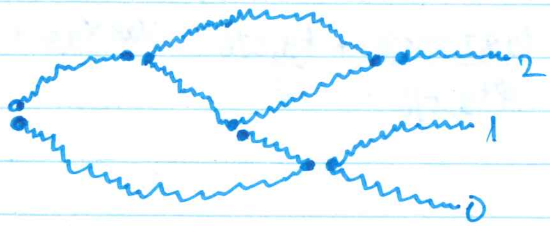
$$= \frac{t_{7,6}(t_{6,4/5} + t_{7,4/5})}{2t_{7,45}} + t_{6,3} + t_{3,0} \quad = \mathrm{Var}\, X_1$$

$$\mathrm{Var}\, X_2 = \mathrm{Var}\, X_{7,4/5} + \mathrm{Var}\, X_{4/5,2}$$

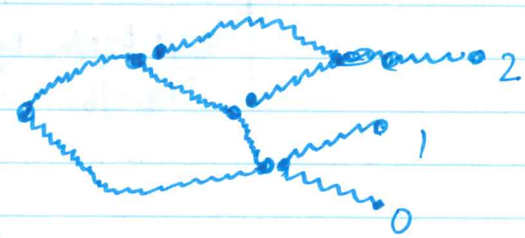
$$= \frac{t_{7,4/5}}{2} + t_{4/5,2}$$

$$C = \begin{bmatrix} \mathrm{Var}\, X_{7,6} + t_{6,3} + t_{3,0} & \mathrm{Var}\, X_{7,6} + t_{6,3} & t_{7,6}/2 \\ \mathrm{Var}\, X_{7,6} + t_{6,3} & \mathrm{Var}\, X_{7,6} + t_{6,3} + t_{3,1} & t_{7,6}/2 \\ \frac{t_{7,6}}{2} & \frac{t_{7,6}}{2} & \frac{t_{7,4/5}}{2} + t_{4/5,2} \end{bmatrix}$$

Joint Recombination



OR



Independent crunkes

opt^n 1 Opt^n 2

Opt 1. $X_0 = X_{9,5/6} + X_{5/6,0} = X_{9,8} + X_{8,7} + X_{7,5/6} + X_{5/6,0}$

$X_1 = X_{9,5/6} + X_{5/6,1} = \quad '' \quad + X_{5/6,1}$

$X_2 = X_{9,8} + X_{8,3/4} + X_{3/4,2} = X_{9,8} + X_{8,7} + X_{7,3/4} + X_{3/4,2}$

Opt 2. $\mathrm{Cov}(X_0, X_1) = \frac{t_{9,5/6}}{2}$

$$\mathrm{Cov}(X_0, X_2) = \mathrm{Var}(X_{9,8,7}) + \mathrm{Cov}(X_{9,8,7}, X_{7,5/6})$$

$$= \frac{t_{9,8,7}(2t_{9,5/6} - t_{9,8,7})}{2t_{9,5/6}} + \mathrm{cov}(X_{9,5/6}, X_{9,8,7}) + \mathrm{Var}(X_{9,5/6})$$

$$= \frac{t_{9,8,7}(2t_{9,5/6} - t_{9,8,7})}{2t_{9,5/6}} - \frac{t_{9,8,7}}{2} + \frac{t_{9,5/6}}{2} \quad \Big| \quad \frac{t_{9,8,7}\, t_{7,5/6}}{2t_{9,5/6}} + \frac{t_{9,5/6}}{2} \quad ''$$

$$= \frac{t_{9,8,7}}{2}\left(2 - \frac{t_{9,8,7}}{t_{9,5/6}} - 1\right) + \frac{t_{9,5/6}}{2} = \frac{t_{9,8,7}}{2}\left(1 - \frac{t_{9,8,7}}{t_{9,5/6}}\right) + \frac{t_{9,5/6}}{2}$$

$$\mathrm{Var}(X_0) = \frac{t_{9,5/6}}{2} + t_{5/6,0} = \mathrm{Var}(X_1)$$

$$\mathrm{Var}(X_2) = \frac{t_{9,8}(2t_{9,5/6} - t_{9,8})}{2t_{9,5/6}} + \frac{t_{8,3/4}\, t_{7,3/4}}{t_{8,3/4} + t_{7,3/4}} + t_{3/4,2}$$

$$C = \begin{bmatrix} \frac{t_{9,5/6}}{2} + t_{5/6,0} & \frac{t_{9,5/6}}{2} & \frac{t_{9,8,7}\, t_{7,5/6}}{2t_{9,5/6}} + \frac{t_{9,5/6}}{2} \\ \frac{t_{9,5/6}}{2} & \frac{t_{9,5/6}}{2} + t_{5/6,0} & '' \\ \frac{t_{9,8,7}\, t_{7,5/6}}{2t_{9,5/6}} + \frac{t_{9,5/6}}{2} & \frac{t_{9,8,7}\, t_{7,5/6}}{2t_{9,5/6}} + \frac{t_{9,5/6}}{2} & \mathrm{Var}\, X_{9,8} + \mathrm{Var}\, X_{8,3/4} + t_{3/4,2} \end{bmatrix}$$