Домашнее задание 2 по БММО
«Матричные вычисления»

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Задача 1

Доказать тождество Вудбери: $(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1}$

$$
\begin{aligned}
I = (A+UCV)(A+UCV)^{-1} &= (A+UCV)(A^{-1} - A^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1}) = \\
&= I + UCVA^{-1} - (A+UCV)A^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1} = \\
&= I + UCVA^{-1} - (U+UCVA^{-1}U)(C^{-1}+VA^{-1}U)^{-1}VA^{-1} = \\
&= I + UCVA^{-1} - UC(C^{-1}+VA^{-1}U)(C^{-1}+VA^{-1}U)^{-1}VA^{-1} = \\
&= I + UCVA^{-1} - UCVA^{-1} = I
\end{aligned}
$$

Задача 2

$$
p(x|y) \propto p(y|x)p(x) \propto \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) - \frac{1}{2}(y-Ax)^T\Gamma^{-1}(y-Ax)\Big)
$$

$$
\begin{aligned}
&(x-\mu)^T\Sigma^{-1}(x-\mu) + (y-Ax)^T\Gamma^{-1}(y-Ax) = \\
&= x^T\Sigma^{-1}x - \mu^T\Sigma^{-1}x - x^T\Sigma^{-1}\mu + \mu^T\Sigma^{-1}\mu + y^T\Gamma^{-1}y - x^TA^T\Gamma^{-1}y - y^T\Gamma^{-1}Ax + x^TA^T\Gamma^{-1}Ax = \\
&= x^T(\Sigma^{-1} + A^T\Gamma^{-1}A)x - (\mu^T\Sigma^{-1} + y^T\Gamma^{-1}A)x - x^T(\Sigma^{-1}\mu + A^T\Gamma^{-1}y) + (\mu^T\Sigma^{-1}\mu + y^T\Gamma^{-1}y) = \\
&= x^T\Sigma'^{-1}x - \mu'^T\Sigma'^{-1}x - x^T\Sigma'^{-1}\mu' + \mu'^T\Sigma'^{-1}\mu' = (x-\mu')^T\Sigma'^{-1}(x-\mu')
\end{aligned}
$$

$$
\begin{gathered}
p(x|y) = \mathcal{N}(x|\mu', \Sigma'), \\
\mu' = (\Sigma^{-1} + A^T\Gamma^{-1}A)^{-1}(\Sigma^{-1}\mu + A^T\Gamma^{-1}y) \\
\Sigma' = (\Sigma^{-1} + A^T\Gamma^{-1}A)^{-1}
\end{gathered}
$$

Задача 3

$$
\begin{gathered}
\mathbb{E}(x-a)^T B(x-a) = \mathbb{E}[x^T Bx - x^T Ba - a^T Bx + a^T Ba] = \\
= \mathbb{E}(x^T Bx) - \mathbb{E}(x^T Ba) - \mathbb{E}(a^T Bx) + \mathbb{E}(a^T Ba) = \\
= tr(\mathbb{E}(x^T Bx) - \mathbb{E}(x^T)Ba - a^T B\mathbb{E}x + a^T Ba = \\
= \mathbb{E}(tr(x^T Bx)) - \mu^T Ba - a^T B\mu + a^T Ba = \\
= \Big\{ \mathbb{E}tr(x^T Bx) = \mathbb{E}tr(Bxx^T) = trB\mathbb{E}(xx^T) = trB(\Sigma + \mu\mu^T) = trB\Sigma + tr(\mu^T B\mu) = trB\Sigma + \mu^T B\mu \Big\} = \\
= tr(B\Sigma) + \mu^T B\mu - \mu^T Ba - a^T B\mu + a^T Ba = \\
= tr(B\Sigma) + (\mu - a)^T B(\mu - a)
\end{gathered}
$$

Задача 4

$$
\begin{gathered}
\frac{\partial}{\partial X}\det(X^{-1} + A) = \frac{\partial}{\partial X}\det A \det X^{-1} \det\left(X + A^{-1}\right) = \\
= \det A \frac{\partial}{\partial X}\det X^{-1} \det\left(X + A^{-1}\right) = \\
= \det A \Big[\det\left(X + A^{-1}\right)\frac{\partial}{\partial X}\frac{1}{\det X} + \frac{1}{\det X}\frac{\partial}{\partial X}\det\left(X + A^{-1}\right) \Big] = \\
= \det A \Big[\det\left(X + A^{-1}\right)\frac{-\det\left(X\right)X^{-T}}{(\det X)^2} + \frac{1}{\det X}\det\left(X + A^{-1}\right)(X + A^{-1})^{-T} \Big] = \\
= \det A \frac{\det\left(X + A^{-1}\right)}{\det X}[(X + A^{-1})^{-T} - X^{-T}] = \\
= \{\text{т. Вудбери} : (X + A^{-1})^{-T} = X^{-T} - X^{-T}(A^T + X^{-T})^{-1}X^{-T}\} = \\
= -\det A \frac{\det\left(X + A^{-1}\right)}{\det X} X^{-T}(A^T + X^{-T})^{-1}X^{-T}
\end{gathered}
$$

Задача 5

$$\frac{\partial}{\partial X} tr(AX^{-T}BXC) = \{trA^T = trA\} =$$
$$= \frac{\partial}{\partial X} tr(C^T X^T B^T X^{-1} A^T) =$$
$$= \{\text{Matrix Calculus – Notes on the Derivative of a Trace (Johannes Traa), стр.3}\} =$$
$$= \frac{\partial}{\partial X} tr(C^T X^T D) + \frac{\partial}{\partial X} tr(E X^{-1} A^T) = \left\{ \begin{array}{l} D = B^T X^{-1} A^T \\ E = C^T X^T B^T \end{array} \right\} =$$
$$= \left\{ \begin{array}{l} \frac{\partial}{\partial X} tr(AX^T B) = BA \\ \frac{\partial}{\partial X} tr(AX^{-1}B) = -(X^{-1}BAX^{-1})^T \end{array} \right\} = DC^T - (X^{-1}A^T E X^{-1})^T =$$
$$= B^T X^{-1} A^T C^T - (X^{-1} A^T C^T X^T B^T X^{-1})^T =$$
$$= B^T X^{-1} A^T C^T - X^{-T} B X C A X^{-T}$$