Домашнее задание 3 по БММО
«Вариационный вывод»

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Содержание

1 Формулы пересчёта для компонент вариационного приближения

1.1 $q_T(T)$

$$
\begin{aligned}
\log q_T(T) &= \mathbb{E}_{q_Z(Z)} \log p(X, T, Z \mid \omega, \mu, \Sigma_k, \nu) + const = \\
&= \mathbb{E}_{q_Z(Z)} \sum_{n=1}^{N} \sum_{k=1}^{K} t_{nk} \left[\log \omega_k + \log \mathcal{N}(x_n \mid \mu_k, \frac{\Sigma_k}{z_n}) + \log \mathcal{G}(z_n \mid \frac{\nu}{2}, \frac{\nu}{2}) \right] + const = \\
&= \sum_{n=1}^{N} \mathbb{E}_{q_Z(Z)} \sum_{k=1}^{K} t_{nk} \left[\log \omega_k + \log \mathcal{N}(x_n \mid \mu_k, \frac{\Sigma_k}{z_n}) + \log \mathcal{G}(z_n \mid \frac{\nu}{2}, \frac{\nu}{2}) \right] + const = \\
&= \sum_{n=1}^{N} \sum_{k=1}^{K} t_{nk} \left[\log \omega_k + \log \mathcal{N}(x_n \mid \mu_k, \frac{\Sigma_k}{\mathbb{E}_{q_Z(Z)} [z_n]}) \right] + const
\end{aligned}
$$

Тогда:

$$q_T(T) = \prod_{n=1}^{N}\prod_{k=1}^{K}\left[\frac{\omega_k\mathcal{N}(x_n \mid \mu_k, \frac{\Sigma_k}{\mathbb{E}_{q_Z(Z)}[z_n]})}{\sum_{l=1}^{K}\omega_l\mathcal{N}(x_n \mid \mu_l, \frac{\Sigma_l}{\mathbb{E}_{q_Z(Z)}[z_n]})}\right]^{t_{nk}} \tag{1}$$

1.2 $q_Z(Z)$

$$\begin{aligned}
\log q_{z_n}(z_n) = \mathbb{E}_{q_T(T)}\sum_{k=1}^{K} t_{nk}\left[\log\omega_k + \log\mathcal{N}(x_n \mid \mu_k, \frac{\Sigma_k}{z_n}) + \log\mathcal{G}(z_n \mid \frac{\nu}{2}, \frac{\nu}{2})\right] + const = \\
= \left\{\text{Учитывая, что } \sum_j t_{nj} = 1\text{, можем отбросить слагаемые, которые не зависят } k\right\} = \\
= \sum_{k=1}^{K}\mathbb{E}_{q_T(T)}[t_{nk}]\left[-\frac{1}{2}\log\det\frac{\Sigma_k}{z_n} - \frac{1}{2}(x_n-\mu_k)\Sigma_k^{-1}(x_n-\mu_k)\cdot z_n + (\frac{\nu}{2}-1)\cdot\log z_n - \frac{\nu}{2}\cdot z_n\right] + const = \\
= \sum_{k=1}^{K}\mathbb{E}_{q_T(T)}[t_{nk}]\left[\left(\frac{D}{2}+\frac{\nu}{2}-1\right)\cdot\log z_n - \left(\frac{1}{2}(x_n-\mu_k)\Sigma_k^{-1}(x_n-\mu_k)+\frac{\nu}{2}\right)\cdot z_n\right] + const = \\
= \left(\frac{D}{2}+\frac{\nu}{2}\right)\cdot\log z_n + \sum_{k=1}^{K}\left[-\left(\frac{\mathbb{E}_{q_T(T)}[t_{nk}]}{2}(x_n-\mu_k)^T\Sigma_k^{-1}(x_n-\mu_k)+\frac{\nu}{2}\right)\right]\cdot z_n + const
\end{aligned}$$

$$q_Z(Z) = \prod_{n=1}^{N} q_{z_n}(z_n) = \prod_{n=1}^{N}\mathcal{G}\left(z_n \middle| \frac{D}{2}+\frac{\nu}{2}, \frac{\sum_{k=1}^{K}\left[\mathbb{E}_{q_T(T)}[t_{nk}](x_n-\mu_k)^T\Sigma_k^{-1}(x_n-\mu_k)+\nu\right]}{2}\right) \tag{2}$$

2 Формулы пересчета параметров $\omega_k, \mu_k, \Sigma_k$ на М-шаге

$$\begin{aligned}
&\mathbb{E}_{q(T,Z)}\log p(X,T,Z \mid \omega,\mu,\Sigma_k,\nu) = \\
&= \mathbb{E}_{q_Z(Z)q_T(T)}\sum_{n=1}^{N}\sum_{k=1}^{K} t_{nk}\left[\log\omega_k + \log\mathcal{N}(x_n \mid \mu_k, \frac{\Sigma_k}{z_n}) + \log\mathcal{G}(z_n \mid \frac{\nu}{2}, \frac{\nu}{2})\right] = \\
&= \sum_{n=1}^{N}\sum_{k=1}^{K}\mathbb{E}_{q_T(T)}[t_{nk}]\left[\log\omega_k + \mathbb{E}_{q_Z(Z)}\left(\log\mathcal{N}(x_n \mid \mu_k, \frac{\Sigma_k}{z_n}) + \log\mathcal{G}(z_n \mid \frac{\nu}{2}, \frac{\nu}{2})\right)\right] = \\
&= \{\text{Последнее слагаемое не зависит от } \omega_k, \mu_k, \Sigma_k\} = \\
&= \sum_{n=1}^{N}\sum_{k=1}^{K}\mathbb{E}_{q_T(T)}[t_{nk}]\left[\log\omega_k - \frac{1}{2}\log\det\frac{\Sigma_k}{z_n} + \mathbb{E}_{q_Z(Z)}\left(-\frac{1}{2}(x_n-\mu_k)\Sigma_k^{-1}(x_n-\mu_k)\cdot z_n\right)\right] = \\
&= \sum_{n=1}^{N}\sum_{k=1}^{K}\mathbb{E}_{q_T(T)}[t_{nk}]\left[\log\omega_k - \frac{1}{2}\log\det\Sigma_k + \mathbb{E}_{q_Z(Z)}[z_n]\left(-\frac{1}{2}(x_n-\mu_k)^T\Sigma_k^{-1}(x_n-\mu_k)\right)\right]
\end{aligned}$$

2.1 ω_k

Известно, что $\sum_{k=1}^{K} \omega_k = 1$. Тогда используя метод Лагранжа:

$$\frac{\partial \left[\mathbb{E}_{q_Z(Z) q_T(T)} \log p(X, T, Z \mid \omega, \mu, \Sigma_k, \nu) + \lambda \left(1 - \sum_{k=1}^{K} \omega_k\right)\right]}{\partial \omega_k} =$$

$$= \frac{\partial \left[\sum_{n=1}^{N} \sum_{k=1}^{K} \mathbb{E}_{q_T(T)} \left[t_{nk}\right] \log \omega_k\right]}{\partial \omega_k} - \lambda$$

$$= \sum_{n=1}^{N} \frac{\mathbb{E}_{q_T(T)} \left[t_{nk}\right]}{\omega_k} - \lambda = 0$$

$$\omega_k = \frac{\sum_{n=1}^{N} \mathbb{E}_{q_T(T)} \left[t_{nk}\right]}{\lambda} = \frac{\sum_{n=1}^{N} \mathbb{E}_{q_T(T)} \left[t_{nk}\right]}{\sum_{n=1}^{N} \sum_{k=1}^{K} \mathbb{E}_{q_T(T)} \left[t_{nk}\right]} = \frac{\sum_{n=1}^{N} \mathbb{E}_{q_T(T)} \left[t_{nk}\right]}{N} \tag{3}$$

2.2 μ_k

$$\frac{\partial \left[\sum_{n=1}^{N} \sum_{k=1}^{K} \mathbb{E}_{q_T(T)} \left[t_{nk}\right] \mathbb{E}_{q_Z(Z)} \left[z_n\right] \left(-\frac{1}{2}(x_n - \mu_k)^T \Sigma_k^{-1} (x_n - \mu_k)\right)\right]}{\partial \mu_k} =$$

$$= \sum_{n=1}^{N} \mathbb{E}_{q_T(T)} \left[t_{nk}\right] \mathbb{E}_{q_Z(Z)} \left[z_n\right] \frac{\partial \left[x_n^T \Sigma_k^{-1} \mu_k - \frac{1}{2} \mu_k^T \Sigma^{-1} \mu_k\right]}{\partial \mu_k} =$$

$$= \sum_{n=1}^{N} \mathbb{E}_{q_T(T)} \left[t_{nk}\right] \mathbb{E}_{q_Z(Z)} \left[z_n\right] \left[\Sigma_k^{-1} x_n - \Sigma_k^{-1} \mu_k\right] = 0$$

$$\mu_k = \frac{\sum_{n=1}^{N} \mathbb{E}_{q_T(T)} \left[t_{nk}\right] \mathbb{E}_{q_Z(Z)} \left[z_n\right] x_n}{\sum_{n=1}^{N} \mathbb{E}_{q_T(T)} \left[t_{nk}\right] \mathbb{E}_{q_Z(Z)} \left[z_n\right]} \tag{4}$$

2.3 Σ_k

$$\frac{\partial \left[\sum_{n=1}^{N} \sum_{k=1}^{K} \mathbb{E}_{q_T(T)} \left[t_{nk}\right] \left((-\frac{1}{2} \log \det \Sigma_k - \mathbb{E}_{q_Z(Z)} \left[z_n\right] \frac{1}{2} (x_n - \mu_k)^T \Sigma_k^{-1} (x_n - \mu_k)\right)\right]}{\partial \Sigma_k^{-1}} =$$

$$= \left\{ \frac{\partial \log \det \Sigma_k^{-1}}{\Sigma_k^{-1}} = \Sigma_k \right\} =$$

$$= \sum_{n=1}^{N} \mathbb{E}_{q_T(T)} \left[t_{nk}\right] \left[\frac{1}{2} \Sigma_k - \mathbb{E}_{q_Z(Z)} \left[z_n\right] \frac{1}{2} (x_n - \mu_k)(x_n - \mu_k)^T\right] = 0$$

$$\Sigma_k = \frac{\sum_{n=1}^{N} \mathbb{E}_{q_T(T)} \left[t_{nk}\right] \mathbb{E}_{q_Z(Z)} \left[z_n\right] (x_n - \mu_k)(x_n - \mu_k)^T}{\sum_{n=1}^{N} \mathbb{E}_{q_T(T)} \left[t_{nk}\right]} \tag{5}$$

3 Функционал $L(q, \omega_k, \mu_k, \Sigma_k)$

$$L(q, \omega_k, \mu_k, \Sigma_k) = \mathbb{E}_{q(T,Z)}\left[\log \frac{p(X, T, Z \mid \omega_k, \mu_k, \Sigma_k, \nu)}{q(T, Z)}\right] =$$
$$= \mathbb{E}_{q(T,Z)}\left[\log p(X, T, Z \mid \omega_k, \mu_k, \Sigma_k, \nu)\right] - \mathbb{E}_{q(T,Z)}\left[\log q(T, Z)\right] =$$
$$= \sum_{n=1}^{N}\sum_{k=1}^{K} \mathbb{E}_{q_T(T)}\left[t_{nk}\right]\left[\log \omega_k + \mathbb{E}_{q_Z(Z)}\left(\log \mathcal{N}(x_n \mid \mu_k, \frac{\Sigma_k}{z_n}) + \log \mathcal{G}(z_n \mid \frac{\nu}{2}, \frac{\nu}{2})\right)\right] - \mathbb{E}_{q(T,Z)}\left[\log q_T(T) + \log_Z(Z)\right] =$$
$$= \sum_{n=1}^{N}\sum_{k=1}^{K} \mathbb{E}_{q_T(T)}\left[t_{nk}\right]\left[\log \omega_k - \frac{D}{2}\log 2\pi - \frac{1}{2}\log\det\Sigma_k + \frac{\nu}{2}\log\frac{\nu}{2} - \log\Gamma\left(\frac{\nu}{2}\right) + \mathbb{E}_{q_Z(Z)}\Big[\frac{D}{2}\log z_n - \right.$$
$$\left. - \frac{1}{2}(x_n - \mu_k)^T\Sigma_k^{-1}(x_n - \mu_k)z_n + \left(\frac{\nu}{2} - 1\right)\log z_n - \frac{\nu}{2}z_n\Big]\right] - \mathbb{E}_{q_T(T)}\left[\log q_T(T)\right] - \mathbb{E}_{q_Z(Z)}\left[\log q_Z(Z)\right] =$$
$$= \sum_{n=1}^{N}\sum_{k=1}^{K} \mathbb{E}_{q_T(T)}\left[t_{nk}\right]\left[\log \omega_k - \frac{D}{2}\log 2\pi - \frac{1}{2}\log\det\Sigma_k + \frac{\nu}{2}\log\frac{\nu}{2} - \log\Gamma\left(\frac{\nu}{2}\right) + \right.$$
$$\left. + \mathbb{E}_{q_Z(Z)}\left[z_n\right]\left[-\frac{1}{2}(x_n - \mu_k)^T\Sigma_k^{-1}(x_n - \mu_k) - \frac{\nu}{2}\right] + \mathbb{E}_{q_Z(Z)}\left[\log z_n\right]\left[\frac{D}{2} + (\frac{\nu}{2} - 1)\right]\right] -$$
$$- \mathbb{E}_{q_T(T)}\log q_T(T) - \mathbb{E}_{q_Z(Z)}\log q_Z(Z)$$

4 Формулы для статистик распределений

$$\mathbb{E}_{q_T(T)}\left[t_{nk}\right] = q(t_{nk} = 0)\cdot 0 + q(t_{nk} = 1)\cdot 1 = q(t_{nk} = 1) = \frac{\omega_k \mathcal{N}(x_n \mid \mu_k, \frac{\Sigma_k}{\mathbb{E}_{q_Z(Z)}[z_n]})}{\sum_{l=1}^{K}\omega_l \mathcal{N}(x_n \mid \mu_l, \frac{\Sigma_l}{\mathbb{E}_{q_Z(Z)}[z_n]})}$$

Учитывая 2:

$$\mathbb{E}_{q_Z(Z)}\left[z_n\right] = \frac{D + \nu}{\sum_{k=1}^{K}\left[\mathbb{E}_{q_T(T)}\left[t_{nk}\right](x_n - \mu_k)^T\Sigma_k^{-1}(x_n - \mu_k) + \nu\right]}$$

$$\mathbb{E}_{q_Z(Z)}\left[\log z_n\right] = \psi\left(\frac{D + \nu}{2}\right) - \log\frac{\sum_{k=1}^{K}\left[\mathbb{E}_{q_T(T)}\left[t_{nk}\right](x_n - \mu_k)^T\Sigma_k^{-1}(x_n - \mu_k) + \nu\right]}{2}$$