Proba 2 Tarea 3

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1. Los montos de reclamaciones por daño a casa-habitación a causa de huracán son variables aleatorias independientes con la misma función de densidad
$$f(x) = \frac{3}{x^4}\mathbf{I}_{(1,\infty)}^{(x)}$$
donde x es el monto de reclamación en miles de dólares. Supóngase que 3 reclamaciones fueron hechas.¿Cuál es el monto esperado de la reclamación por mayor monto de las tres?
$$X_{(3)} = 2F(x)^2 f(x) = 3(1-\frac{1}{x^3})^2\frac{3}{x^4} \Rightarrow E[X_{(3)}] = \int_1^\infty (1-\frac{1}{x^3})^2\frac{9}{x^3}dx = 9\int_1^\infty \frac{1}{x^3}-\frac{2}{x^6}+\frac{1}{x^9}dx = \frac{81}{40} = 2.025$$

2. Sea X_1, ... , X_n una muestra aleatoria de una distribución uniforme en [0, 1]. Demuestra que $X_{(i)} \sim Beta(i, n-i+1)$
$$f_{X_{(i)}}(x_i) = \frac{n!}{(i-1)!(n-i)!}F^{i-1}(x)[1-F(x)]^{n-i}f(x) = \frac{n!}{(i-1)!(n-i)!}(x)^{i-1}(1-x)^{(n-i+1)-1}(1) = Beta(i, n-i+1)$$

4. Sea X_1, X_2 muestra aleatoria de la distribución $N(\mu, \sigma^2)$. Demuestra que $E[X_{(1)}] = \mu - \dfrac{\sigma}{\sqrt{\pi}}$

   $X_{(1)} = \min\{X_1, X_2\} \Rightarrow f_{X_{(1)}} = 2[1-F(x)]f(x)$
$$\Rightarrow E[X_{(1)}] = 2E[X] - 2\int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(t-\mu)^2}{2\sigma^2}}dtdx = 2\mu - \frac{1}{\pi\sigma^2}\int_{-\infty}^{\infty}xe^{-\frac{(x-\mu)^2}{2\sigma^2}}\int_{-\infty}^{x}e^{-\frac{(t-\mu)^2}{2\sigma^2}}dtdx$$
   Resolvamos por partes

   $u = \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt$; $dv = xe^{-\frac{(x-\mu)^2}{2\sigma^2}}$

   $du = e^{-\frac{(x-\mu)^2}{2\sigma^2}}$; $v = \mu\int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt - \sigma^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
$$= 2\mu - \frac{1}{\pi\sigma^2}((\int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt)(\mu\int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt - \sigma^2e^{-\frac{(x-\mu)^2}{2\sigma^2}}))\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(\mu\int_{-\infty}^{x}e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt - \sigma^2e^{-\frac{(x-\mu)^2}{2\sigma^2}})(e^{-\frac{(x-\mu)^2}{2\sigma^2}})dx$$
$$= 2\mu - \frac{1}{\pi\sigma^2}(\mu(\int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt)^2\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(\mu\int_{-\infty}^{x}e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt - \sigma^2e^{-\frac{(x-\mu)^2}{2\sigma^2}})(e^{-\frac{(x-\mu)^2}{2\sigma^2}})dx$$
$$= 2\mu - \frac{1}{\pi\sigma^2}(\mu(\int_{-\infty}^{\infty} e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt)^2 - \int_{-\infty}^{\infty}(\mu\int_{-\infty}^{x}e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt - \sigma^2e^{-\frac{(x-\mu)^2}{2\sigma^2}})(e^{-\frac{(x-\mu)^2}{2\sigma^2}})dx$$
$$= 2\mu - \frac{1}{\pi\sigma^2}(2\mu\pi\sigma^2 - \int_{-\infty}^{\infty}(\mu\int_{-\infty}^{x}e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt - \sigma^2e^{-\frac{(x-\mu)^2}{2\sigma^2}})(e^{-\frac{(x-\mu)^2}{2\sigma^2}})dx$$
$$= \frac{1}{\pi\sigma^2}\int_{-\infty}^{\infty}\mu e^{-\frac{(x-\mu)^2}{2\sigma^2}}\int_{-\infty}^{x}e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt - \sigma^2e^{-\frac{(x-\mu)^2}{2(\frac{\sigma}{\sqrt{2}})^2}}dx = \frac{1}{\pi\sigma^2}(\frac{\mu}{2}[\int_{-\infty}^{x}e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt]^2\Big|_{-\infty}^{\infty} - \sigma^2\int_{-\infty}^{\infty}e^{-\frac{(x-\mu)^2}{2(\frac{\sigma}{\sqrt{2}})^2}}dx)$$
$$= \frac{1}{\pi\sigma^2}(\mu\pi\sigma^2 - \sigma^2\sqrt{2\pi}\frac{\sigma}{\sqrt{2}}) = \mu - \frac{\sigma}{\sqrt{\pi}}$$

6. Sea $X \sim F(m, n)$.Demuestra que $Y = \dfrac{\frac{m}{n}X}{1+\frac{m}{n}X}$ tiene distribución Beta. ¿Cuáles son sus parámetros?
$$Y = \frac{\frac{m}{n}X}{1+\frac{m}{n}X} \Rightarrow X = \frac{nY}{m(1-Y)} \Rightarrow |J| = \frac{n}{m(1-Y)^2}$$

$$f_X(x) = \frac{\Gamma(\frac{m}{2}, \frac{n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}(\frac{mx}{mx+n})^{\frac{m}{2}}(1-\frac{mx}{mx+n})^{\frac{n}{2}}x^{-1}$$

$$f_Y(y) = \frac{n}{m(1-y)^2}\frac{\Gamma(\frac{m}{2}, \frac{n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}(y)^{\frac{m}{2}}(1-y)^{\frac{n}{2}}\frac{m(1-y)}{ny} = \frac{\Gamma(\frac{m}{2}, \frac{n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}(y)^{\frac{m}{2}-1}(1-y)^{\frac{n}{2}-1} = Beta(\frac{m}{2}, \frac{n}{2})$$

8. Sea $X \sim t_{(n)}$. Demuestra que $Y = \dfrac{1}{1+\frac{X^2}{n}}$ tiene distribución Beta

$$\Rightarrow X = \sqrt{\frac{n(1-y)}{y}} \Rightarrow \left|\frac{dx}{dy}\right| = \frac{\sqrt{n}}{2\sqrt{1-y}y\sqrt{y}}$$

$$f_X(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})}(1+\frac{x^2}{n})^{-\frac{n+1}{2}} \Rightarrow f_Y(y) = \frac{\sqrt{n}}{2\sqrt{1-y}y\sqrt{y}}\frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})}(\frac{1}{y})^{-\frac{n+1}{2}} = \frac{\Gamma(\frac{n+1}{2})}{2\sqrt{\pi}\Gamma(\frac{n}{2})}(y)^{\frac{n}{2}-1}(1-y)^{\frac{1}{2}-1} = Beta(\frac{n}{2}, \frac{1}{2})$$