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Chapter 3

METRIC SPACES

In order to consolidate and extend the ideas of the previous two chapters, we now restrict our attention to metric spaces. A variety of metric spaces are considered; our study begins with those that are compact.

A. COMPACTNESS IN A METRIC SETTING

Sequences are especially useful and important in the theory of metric spaces. For example, we have already seen that in a metric context, the continuity of a function is completely determined by the behavior of convergent sequences (1.F.9). Subsequences are the key to characterizing compactness in metric spaces.

(3.A.1) Definition. A point x in a topological space X is a *cluster point* of a sequence $\{x_i\}$ if and only if for each neighborhood U of x and for each $i \in \mathbf{Z}^+$, there is an integer $k > i$ such that $x_k \in U$.

(3.A.2) Exercise. Suppose that X is a metric space and that x is a cluster point of a sequence $\{x_i\}$ in X. Show that $\{x_i\}$ contains a subsequence converging to x.

(3.A.3) Exercise. Find a topological space X, a sequence $\{x_i\}$ in X, and a cluster point x of $\{x_i\}$ that is not a limit point of any subsequence of $\{x_i\}$. [Hint: cf. problem 2 of section A.]

In metric spaces the notions of compactness, sequential compactness, and countable compactness coincide. Before proving this, we introduce one additional concept that appears frequently in analysis.

(3.A.4) ***Definition.*** A metric space (X,d) is *totally bounded* if and only if for each $\varepsilon > 0$, there is a positive integer N_ε and a finite subset $\{x_1, x_2, \ldots, x_{N_\varepsilon}\}$ of X such that $X = \bigcup_{i=1}^{N_\varepsilon} S_\varepsilon(x_i)$.

(3.A.5) ***Remark.*** Total boundedness is a metric property rather than a topological property. For instance, although the open interval $(0,1)$ with the usual metric is totally bounded, it is homeomorphic to $\mathscr{E}^1$, which is not totally bounded. Nevertheless, if certain conditions are added to a metric space, then total boundedness becomes of topological interest. This is illustrated in the following theorem, which establishes a key property of sequentially compact metric spaces.

(3.A.6) ***Theorem.*** If (X,d) is a sequentially compact metric space, then (X,d) is totally bounded.

Proof. Let $\varepsilon > 0$ be given. Pick an arbitrary point $x_1 \in X$. If $X = S_\varepsilon(x_1)$, we are done; if $X \neq S_\varepsilon(x_1)$, choose a point $x_2 \in X \setminus S_\varepsilon(x_1)$. Again, if $X = S_\varepsilon(x_1) \cup S_\varepsilon(x_2)$ we stop; otherwise select a point $x_3 \in X \setminus (S_\varepsilon(x_1) \cup S_\varepsilon(x_2))$. Continuing in this fashion, we will eventually encounter an integer N_ε such that $X = \bigcup_{i=1}^{N_\varepsilon} S(x_i)$, for, if not, we would have constructed a sequence that fails to have a convergent subsequence.

(3.A.7) ***Corollary.*** Sequentially compact metric spaces have a countable basis.

Proof. For each $n \in \mathbf{Z}^+$, there is a family of neighborhoods $S_{1/n}(x_1), \ldots, S_{1/n}(x_{k_n})$ that cover X. The collection of all these neighborhoods is clearly a countable basis for X.

(3.A.8) ***Theorem.*** Suppose that (X,d) is a metric space. Then (X,d) is countably compact if and only if (X,d) is sequentially compact.

Proof. We have already seen that sequential compactness implies countable compactness (2.I.9). Hence, suppose that there is a metric space (X,d) that is countably compact but not sequentially compact. Then there is a

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sequence $\{x_i\} \subset X$ that has no convergent subsequence. We will construct a countable cover of X that has no finite (in fact no proper) subcover. First observe that (3.A.2) implies that the sequence $\{x_i\}$ does not have a cluster point. Let $A = \{x_i \mid i \in \mathrm{N}\}$. Clearly A is infinite (otherwise $\{x_i\}$ would have a cluster point). For each $x \in X$, there is an open neighborhood $\hat{U}_x$ of x such that $\hat{U}_x \cap A$ is finite. Since points are closed, there is an open neighborhood U_x of x such that $U_x \cap A = \varnothing$ if $x \notin A$, and $U_x \cap A = \{x\}$ if $x \in A$. Then if $U_0 = \{U_x \mid x \in X \setminus A\}$, we have that $\{\{U_a\} \mid a \in A\} \cup \{U_0\}$ is a countably infinite cover of X which has no proper subcover.

(3.A.9) ***Definition.*** If A is a subset of a metric space (X,d), then the *diameter of A*, denoted by *diam A*, is defined to be $\sup\{d(x,y) \mid x,y \in A\}$.

(3.A.10) ***Theorem.*** Suppose that $\mathscr{U}$ is an open cover of a sequentially compact metric space X. Then there is a $\lambda > 0$ such that for each $x \in X$, there is a set $U \in \mathscr{U}$ with $S_\lambda(x) \subset U$. (Equivalently, there is a $\lambda > 0$ such that whenever diam $A < \lambda$, then there is a $U \in \mathscr{U}$ that contains A.) The number λ is called a *Lebesgue number* for the cover $\mathscr{U}$.

Proof. If no such λ exists, then for each n, there is an $x_n \in X$ such that $S_{1/n}(x_n)$ fails to be in any member of $\mathscr{U}$. Since X is sequentially compact, there is a cluster point x of the sequence $\{x_n\}$ (why?). Since $\mathscr{U}$ is a cover of X, we have that $x \in U$ for some $U \in \mathscr{U}$. Let $r = d(x, X \setminus U) > 0$ and choose m large enough so that $d(x_m, x) < r/2$ and $1/m < r/4$. Then $S_{1/m}(x_m)$ is contained in U, which contradicts the way that x_m was chosen.

The following is our principal theorem.

(3.A.11) ***Theorem.*** Suppose that X is a metric space. Then X is compact if and only if X is sequentially compact.

Proof. If X is compact, then clearly X is countably compact. Hence by (3.A.8) we have that X is sequentially compact.

Conversely, suppose that X is sequentially compact. Let $\mathscr{U}$ be an open cover of X. By (3.A.10), there is a $\lambda > 0$ such that if $x \in X$, then $S_\lambda(x) \subset U$ for some $U \in \mathscr{U}$. Furthermore, by (3.A.6), there is a finite subset $\{x_1, \ldots, x_n\}$ of X such that $X = \bigcup_{i=1}^{n} S_\lambda(x_i)$. To complete the proof, simply choose for each $i = 1, 2, \ldots, n$, a set $U_i \in \mathscr{U}$ with $S_\lambda(x_i) \subset U_i$. Then $\{U_i \mid i = 1, 2, \ldots, n\}$ is a finite subcover.

The following corollary has many applications in analysis.

(3.A.12) Corollary (Bolzano-Weierstrass Theorem). Every bounded sequence of points $\{x_i\}$ in $\mathscr{E}^n$ contains a convergent subsequence.

Proof. Let A be the set whose members are precisely the elements of the sequence. Then $\bar{A}$ is certainly closed and bounded, and hence compact. The corollary now follows immediately from the theorem.

(3.A.13) Corollary. Suppose that (X,d) is a compact metric space and that $\mathscr{U}$ is an open cover of X. Then there is a Lebesgue number λ for $\mathscr{U}$.

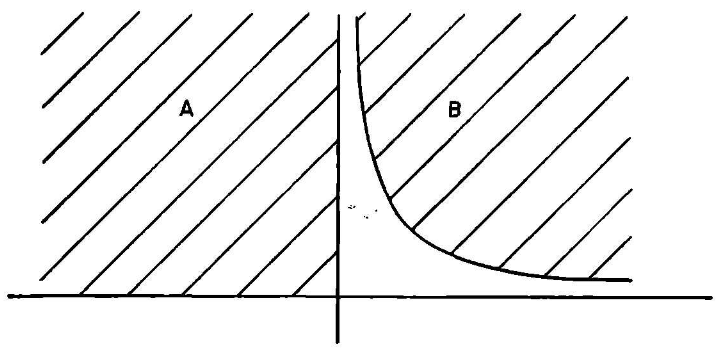
Theorem (3.A.6) can be applied to prove the following basic result.

(3.1.14) Theorem. Suppose that A and B are disjoint subsets of a metric space (X,d). If A is compact and B is closed, then $d(A,B) > 0$. Furthermore, if $d(A,B) = r$, then there is an $a \in A$ such that $d(a,B) = r$.

Proof. Suppose to the contrary that $d(A,B) = 0$. Then for each positive integer n, there are points $a_n \in A$ and $b_n \in B$ such that $d(a_n,b_n) < 1/n$. By the previous theorem, there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to some point $a \in A$. But then the corresponding subsequence $\{b_{n_k}\}$ must also converge to a, contradicting the fact that B is closed. Therefore, $d(A,B) > 0$.

To prove the second part of the theorem, suppose that $d(A,B) = r$ and let $d_B : A \to \mathscr{E}^1$ be defined by $d_B(a) = d(a,B)$. Then d_B is continuous (1.F.12), and since A is compact, we have that $d_B(A)$ is compact and hence closed in $\mathscr{E}^1$. Therefore, $r \in d_B(A)$, and consequently there is an $a \in A$ such that $d_B(a) = d(a,B) = r$.

The following example in $\mathscr{E}^2$ shows that the compactness of A is a necessary condition in (3.A.14).



(3.A.15) Definition. A function $f : (X,d) \rightarrow (X',d')$ is *uniformly continuous* if and only if for each $\varepsilon > 0$, there is a $\delta > 0$ such that if $d(x,y) < \delta$, then $d'(f(x),f(y)) < \varepsilon$.

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Note that δ depends only on ε and not on the individual points x and y. Obviously, uniform continuity implies continuity, but the converse does not hold.

(3.A.16) Exercise. Find an example of a homeomorphism between two metric spaces that is not uniformly continuous.

(3.A.17) Theorem. Suppose that (X,d) and (X',d) are metric spaces. If $f : X \rightarrow X'$ is continuous and X is compact, then f is uniformly continuous.

Proof. Let $\varepsilon > 0$ be given. For each $x \in X$, there is a $\delta_x > 0$, such that if $d(x,y) < \delta_x$, then $d(f(x),f(y)) < \varepsilon/2$. Cover X with sets of the form $S_{\delta_x}(x)$, and let λ be a Lebesgue number of this cover. Now suppose w and z are points of X such that $d(w,z) < \lambda$. Then both w and z belong to some member of the cover, say $S_{\delta_x}(x)$. If follows that $d(f(w),f(z)) < d(f(w),f(x)) + d(f(x),f(z)) < \varepsilon/2 + \varepsilon/2$.

Strangely enough, from a strictly topological point of view, all continuous functions between metric spaces are in a sense uniformly continuous. That is, a continuous function f mapping a metric space (X,d) into a metric space (Y,d') can be converted into a uniformly continuous function by switching to an appropriate equivalent metric on X. Thus, uniform continuity of a given function is a metric property rather than a topological one.

(3.A.18) Theorem. (Levine [1960]). Suppose that (X,d) and (Y,d') are metric spaces and that $f : X \rightarrow Y$ is a continuous function. Then there is a metric d^* on X, equivalent to d, that makes f uniformly continuous.

Proof. Define $d^*(a,b) = d(a,b) + d'(f(a),f(b))$. To see that d^* is a metric, we need only check that $d^*(a,c) \leq d^*(a,b) + d^*(b,c)$ for $a,b,c \in X$. But this is immediate, since $d^*(a,b) + d^*(b,c) = d(a,b) + d'(f(a),f(b)) + d(b,c) + d'(f(b),f(c)) \geq d(a,c) + d'(f(a),f(c)) = d^*(a,c)$.

It follows from (1.G.5) and (1.F.9) that d and d^* will be equivalent provided it is shown that a sequence $\{x_n\}$ converges to a point x with respect to the d metric if and only if it converges to x with respect to the d^* metric. Suppose that the sequence $\{x_n\}$ converges to x with respect to the metric d. Let $\varepsilon > 0$ be given. There is an integer N_1, such that $d(x_n,x) < \varepsilon/2$ whenever $n > N_1$. Since f is continuous, the sequence $\{f(x_n)\}$ converges to $f(x)$. Hence there is an integer N_2 such that $d'(f(x_n),f(x)) < \varepsilon/2$ whenever $n > N_2$. Thus, if $n > \max(N_1, N_2)$ we have that $x_n \in S_\varepsilon^{d*}(x)$. Conversely, if $\{x_n\}$ converges to x with respect to the metric d^*, then $\{x_n\}$ converges to x with respect to the d metric, since $d(x_n,x) \leq d^*(x_n,x)$ for each n.

To see that $f : (X,d^*) \to (Y,d')$ is uniformly continuous, let $\varepsilon > 0$ be given and set $\delta = \varepsilon$. Then if $d^*(a,b) < \delta$, we have that $d(a,b) + d'(f(a),f(b)) < \delta = \varepsilon$, which of course implies that $d'(f(a),f(b)) < \varepsilon$.

B. COMPLETE METRIC SPACES

Although compactness is convenient when dealing with convergence, frequently, less stringent conditions are sufficient to achieve adequate control over sequences in metric spaces. Complete metric spaces possess enough structure to enable one to establish many important theorems that have wide applications in both topology and analysis. For instance, a key fixed point result (problem B.2), that holds in complete metric spaces can be used to prove theorems involving the existence and uniqueness of solutions to differential equations.

(3.B.1) Definition. Suppose that (X,d) is a metric space. A sequence $\{x_i\}$ in X is a *Cauchy sequence* if and only if for each $\varepsilon > 0$ there is an integer N such that $d(x_m,x_n) < \varepsilon$ whenever $m,n > N$. The metric space (X,d) is *complete* if and only if for each Cauchy sequence $\{x_i\}$ in X, there is a point $x \in X$ such that $\{x_i\}$ converges to x.

(3.B.2) Exercise. Give an example of a noncomplete metric space.

It is easy to show that if $\{x_i\}$ is a Cauchy sequence for which some subsequence converges to a point x, then the entire sequence must also converge to x. Thus it follows from (3.A.11) that compact metric spaces are complete. However, there is no dearth of noncompact, complete metric spaces. For instance, $\mathscr{E}^1$ is complete, as is shown in the next theorem.

(3.B.3) Theorem. The space $(\mathscr{E}^1,d)$ where d is the usual metric ($d(x,y) = |x - y|$) is complete.

Proof. Let $\{x_n\}$ be a Cauchy sequence in $\mathscr{E}^1$. Consider $A = \{x_n \mid n \in \mathbf{Z}^+\}$. Since A is bounded (why?), $\bar{A}$ is compact (2.G.12). Now apply (3.A.11) to obtain a convergent subsequence of $\{x_n\}$. The remarks preceeding the theorem now yield that $\{x_n\}$ also converges.

We observe next that completeness is not a topological property, i.e., there are homeomorphic metric spaces one of which is complete, and the other is not. For instance, in (1.G.6), it was observed that $\mathscr{E}^1$ and the open interval $(-1,1)$ are homeomorphic; however, $(-1,1)$ is not complete. This rather distressing situation (after all, this text purports to study topological invariants) may be remedied with the introduction of the concept of topological completeness.

(3.B.4) Definition. A metric space (X,d) is *topologically complete* if and only if there is an equivalent metric d' for X such that (X,d') is complete.

Of course, a complete metric space is automatically topologically complete. A natural question to ask is whether or not all metric spaces are topologically complete. Although the answer is no [see (10.E.5)], it will be shown in Chapter 10 that any metric space may be embedded isometrically as a dense subset of complete metric space. [If (X,d) and (Y,d') are metric spaces, then a function $f: X \to Y$ is an *isometry* if and only if $d(x_1,x_2) = d'(f(x_1),f(x_2))$, for each $x_1,x_2 \in X$.]

(3.B.5) Exercise. Show that a closed subset of a complete metric space is complete.

(3.B.6) Exercise. Show that a complete subspace of a metric space is closed.

We prove later that any locally compact separable metric space is topologically complete (10.E.3). In fact, it can be shown that every locally compact metric space is topologically complete (Dugundji [1966]).

(3.B.7) Exercise. Find a locally compact metric space that is not complete.

An important property of complete metric spaces is given in the following theorem.

(3.B.8) Theorem. Suppose that $A_1 \supset A_2 \supset A_3 \supset \cdots$ is a decreasing sequence of nonempty closed subsets of a complete metric space (X,d) and that the limit of the diameters of the A_i is 0. Then $\bigcap_{i=1}^{\infty} A_i = \{x\}$ for some $x \in X$.

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Proof. For each positive integer i, select a point $x_i \in A_i$. Since the diameters of the A_i's converge to 0 and the A_i's are nested, it is clear that $\{x_i\}$ is a Cauchy sequence and hence converges to a point $x \in X$. We show that

$\bigcap_{i=1}^{\infty} A_i = \{x\}$. If $x \notin \bigcap_{i=1}^{\infty} A_i$, let A_j be the set with smallest index that fails to contain x. By (3.A.14), $d(x,A_j) = r > 0$. However, this is impossible, since it implies that $d(x,A_i) \geq r$ for $i > j$, which prevents the sequence from converging to x. Therefore, we have that $x \in \bigcap_{i=1}^{\infty} A_i$. That x is the only point in the intersection follows readily from the observation that if y is any other point in X, then $d(x,y) = s > 0$; however, for i sufficiently large, diam $A_i < s$.

(3.B.9) Corollary. Suppose that (X,d) is a complete metric space and let $D_1, D_2, \ldots$ be a sequence of open dense subsets of X. Then $\bigcap_{i=1}^{\infty} D_i$ is dense in X.

Proof. Let U be an open set in X. It suffices to show that $U \cap (\bigcap_{i=1}^{\infty} D_i) \neq \varnothing$. Since D_1 is open and dense, $D_1 \cap U$ must be nonempty and open. Hence, there is a positive number $\varepsilon_1 < 1$ and a point $x_1 \in X$ such that $\overline{S_{\varepsilon_1}(x_1)} \subset D_1 \cap U$. The open set $S_{\varepsilon_1}(x_1)$ has nonempty intersection with D_2, and consequently there is a point x_2 and a positive number $\varepsilon_2 < 1/2$ such that $\overline{S_{\varepsilon_2}(x_2)} \subset D_2 \cap S_{\varepsilon_1}(x_1)$. Similarly, $S_{\varepsilon_2}(x_2) \cap D_3$ is open and nonempty, and therefore there is an $\varepsilon_3 < 1/3$, etc. Repeated application of this procedure leads to a decreasing sequence of closed sets with diameters converging to 0. By the previous theorem, there is a point x common to each $\overline{S_{\varepsilon_i}(x_i)}$. Hence, x is contained in both U and $\bigcap_{i=1}^{\infty} D_i$, which completes the proof.

Spaces other than complete metric spaces may also have the property described in the foregoing corollary; such spaces are known as Baire spaces.

(3.B.10) Definition. A topological space X is a *Baire space* if and only if the intersection of any countable family of open dense sets in X is dense.

(3.B.11) Exercise. Show that if X is a locally compact topological T_2 space then X is a Baire space. [Hint: Use an argument similar to that given in proof of (3.B.9).]

An important property of Baire spaces is given in the following theorem.

(3.B.12) Theorem. Suppose that X is a Baire space and $C_1, C_2, \ldots$ is a countable closed cover of X. Then at least one of the C_i's contains a nonempty open (in X) set.

Proof. This theorem follows easily from the De Morgan rules. Since $X = \bigcup_{i=1}^{\infty} C_i$, it must be the case that $\bigcap_{i=1}^{\infty} (X \setminus C_i)$ is empty. However, $X \setminus C_i$ is open for each i. Hence, since X is a Baire space, at least one of the sets, say $X \setminus C_j$, is not dense, which implies that C_j contains an open set.

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(3.B.13) Definition. A subset A of a space X is *nowhere dense* in X if and only if $\bar{A}$ contains no open subsets of X, i.e., int $\bar{A} = \varnothing$.

(3.B.14) Definition. A subset A of a space X is a *set of the first category* in X if and only if A can be written as a countable union of sets nowhere dense in X. The subset A is of the *second category* if and only if A is not of the first category.

This use of category here should not be confused with the category used in the context of functors and categories (see Chapter 12).

(3.B.15) Examples.

1. The rationals are of the first category in $\mathscr{E}^1$.

2. The irrationals are of the second category in $\mathscr{E}^1$. If the irrationals could be written as a countable union of nowhere dense sets, then with the addition of the individual rational points as nowhere dense subsets, $\mathscr{E}^1$ itself could be represented as a countable union of nowhere dense subsets and would be of the first category (in itself). This contradicts the following classical theorem.

(3.B.16) Theorem (Baire Category Theorem). Any complete metric space is of the second category in itself.

Proof. Suppose that X is a complete metric space and that X is of the first category. Then X can be written as a countable union of nowhere dense sets $A_1, A_2, \cdots$. Since each A_i is nowhere dense, $X \setminus \bar{A}_i$ is both open and dense. From the completeness of X, we have by (3.B.9) that $\bigcap_{i=1}^{\infty} (X \setminus \bar{A}_i)$ is nonempty. Since $\bigcap_{i=1}^{\infty} (X \setminus \bar{A}_i) = X \setminus (\bigcup_{i=1}^{\infty} \bar{A}_i) = \varnothing$, it follow that X must be of the second category.

(3.B.17) Definition. A point x of a space X is *isolated* if and only if x has a neighborhood that contains no other point of X, i.e., $\{x\}$ is open in X.

(3.B.18) Corollary. Every countable complete metric space has an isolated point.

Note that every point of a discrete topological space is isolated. At the other end of the spectrum are the perfect spaces.

(3.B.19) ***Definition.*** A closed subset A of a space X is *perfect* if and only if each point of A is an accumulation point of A.

The reader should note that a set A is perfect if and only if $A = A'$.

(3.B.20) ***Exercise.*** Suppose that X and Y are topological spaces. Show that if X is perfect, then $X \times Y$ is perfect.

As an application of the Baire category theorem, we consider in detail an interesting example given by Knaster and Kuratowski [1921]. This space is of importance in dimension theory, since it is a one dimensional set that is not totally disconnected. It also is an ingenious example of a subspace of the plane possessing an explosion point.

Before constructing the example, let us first investigate one of the most unusual sets in all of mathematics, the Cantor set. To define the Cantor set, we begin by forming the following subsets of the closed interval $[0,1]$: $A_1 = [0,1/3] \cup [2/3,1]$; $A_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$; $A_3 = [0,1/27] \cup [1/9,4/27] \cup [2/9,7/27] \cup [8/27,1/3] \cup [2/3,19/27] \cup [20/27,7/9] \cup [8/9,25/27] \cup [26/27,1]$; etc. In general, A_{i+1} is obtained from A_i by removing the middle third from each component of A_i. The Cantor set K is defined by $K = \bigcap_{i=1}^{\infty} A_i$. Clearly, end points of the intervals in the various A_i's belong to K. It is equally obvious from the construction that for every point $x \in K$, there are points of K distinct from x that are arbitrarily close to x. Thus, K possesses no isolated points. Since K is closed (actually, K is compact) it is complete (3.B.5). Hence, we have by (3.B.18) that K is uncountable, and thus K consists of points other than the end points already noted.

We would be derelict if we failed to mention that the Cantor set was actually discovered independently by H. J. S. Smith in 1875 (Hawkins, [1970]). Nevertheless, "*Smith set*" fails to have the ring of a "*Cantor set*."

(3.B.21) ***Exercise.*** Show that the Cantor set is totally disconnected.

(3.B.22) ***Exercise.*** Show that the Cantor set is homeomorphic to the set of points in $\mathscr{E}^1$ of the form $\sum_{i=1}^{\infty} k_i/3^i$, where k_i is either 0 or 2.

Let us now construct the Knaster-Kuratowski example to which we previously alluded. Consider the Cantor set K in $\mathbf{I}$ as a subspace of the x axis in $\mathscr{E}^2$ and set $p = (1/2,1/2)$. For each $c \in K$, let $L(c)$ be the line segment con-

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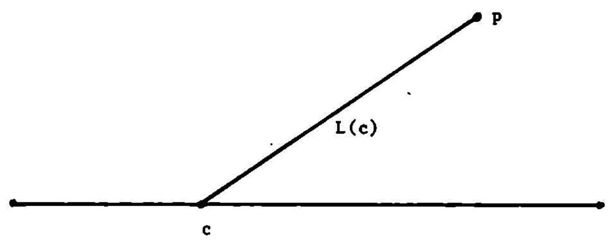
necting p and c. Let E be the subset of K that consists of the end points of the deleted intervals in the construction of K, and let $F = K \setminus E$. Define sets E', F', X_E, and X_F as follows: $E' = \bigcup \{L(c) \mid c \in E\}$; $F' = \bigcup \{L(c) \mid c \in F\}$; $X_E = \{(x,y) \in E' \mid y \text{ is rational}\}$; $X_F = \{(x,y) \in F' \mid y \text{ is irrational}\}$. Finally, set $X = X_E \cup X_F$.

We now show that X is connected, even though intuitively X seems too sievelike to be in "one piece."



If X is not connected, then there are open sets U and V in $\mathscr{E}^2$ such that $X \subset U \cup V$ and $X \cap U \cap V = \varnothing$. We assume that $p \in U$. For each $t \in (0,1/2)$, let $K_t = \{c \in K \mid L(c) \text{ contains a boundary point } (x_c,t) \text{ of } U\}$. We claim that for each t, K_t is nowhere dense in K. Suppose not; then there are points $c_1,c_2 \in K$ such that $\varnothing \neq (c_1,c_2) \cap K \subset K_t$. Let $e \in E \cap (c_1,c_2)$. Then the boundary point (x_e,t) of U has an irrational y coordinate, i.e., t is irrational. If t were rational then (x_e,t) would be in X and hence in either U or V, both alternatives being impossible. Now suppose that $f \in F \cap (c_1,c_2)$; we conclude in a similar way that t is irrational. Hence, K_t must be nowhere dense in K.

Let $K_0 = \{c \in K \mid L(c) \text{ has no boundary points on } U\}$. Since $p \in U$, we have that for any $c \in K_0$, $L(c)$ is contained in U, and consequently, $\overline{K}_0 \neq K$ (otherwise, $V \cap X$ would be empty). Thus, there are points c_1 and c_2 in K such that $(c_1,c_2) \cap K \cap \overline{K}_0 = \varnothing$. Therefore, one may assume that $K_0 = \phi$, since otherwise the space X could be replaced with a "new" X that sits over a part of the interval (c_1,c_2) (the "new" X is a subset of X that is homeomorphic to X). With this minor point out of the way, we observe that $K = (\bigcup \{K_t \mid t \text{ is rational}\}) \cup E$, (why?), or in other words, K is a countable union of nowhere dense sets, which contradicts (3.B.16). Hence, X must be connected.

It is easy to establish that the removal of point p totally disconnects X. Each component of $X \setminus \{p\}$ must be contained in $L(c)$ for some c, since any subset of X that contains points in two $L(c)$'s can be separated by a line running between the $L(c)$'s and through the missing point p. But clearly the points of X lying in each $L(c)$ are totally disconnected, because either points with rational second coordinates or points with irrational second coordinates are missing. Thus, components consist of individual points, or, more graphically, X has been completely shattered by the loss of p.

As another vivid illustration of the category theorem in action, we prove the following rather delicate and unexpected result.

(3.B.23) Theorem. Suppose that X is a connected, locally connected, complete metric space. Then X cannot be written as a countable union of proper disjoint closed subsets.

Proof. Suppose to the contrary that $X = \bigcup_{i=1}^{\infty} C_i$, where the C_i's are closed and disjoint. Let $D = X \setminus (\bigcup_{i=1}^{\infty} \text{int } C_i) = \bigcup_{i=1}^{\infty} \text{Fr } C_i$. Then D is a closed subset of X, and, hence, by (3.B.5) and (3.B.16), D is second category (in itself).

Claim: If U is an open set in X such that $U \cap \text{Fr } C_i \neq \varnothing$, then $U \cap (D \setminus \text{Fr } C_i) \neq \varnothing$.

To establish the claim, first observe that by hypothesis it may be assumed that U is connected. Since U intersects $\text{Fr } C_i$, we have that $U \cap (X \setminus C_i)$ is nonempty, and hence, there is a positive integer $j \neq i$ such that $U \cap C_j \neq \varnothing$. If U fails to intersect $\text{Fr } C_j$, then $C_j \cap U = (\text{int } C_j) \cap U$, which implies that $U \cap C_j$ is both open and closed (in U), contradicting the connectedness of U. Therefore, $U \cap \text{Fr } C_j \neq \varnothing$, and the claim is proven.

Since K is second category, not all of the $\text{Fr } C_i$ can be nowhere dense (in K). Furthermore, by the connectedness of X, $\text{Fr } C_i \neq \varnothing$ whenever $C_i \neq \varnothing$. Thus, there is an open set V in X and an integer j such that $\varnothing \neq V \cap K \subset \overline{\text{Fr } D_j} = \text{Fr } C_j$. Therefore, we have that $V \cap \text{Fr } C_j \neq \varnothing$ and $V \cap (D \setminus \text{Fr } C_j) = \varnothing$, which contradicts the claim.

C. CONVEX AND HAUSDORFF METRICS

A fairly strong condition that may be imposed on metric spaces is convexity. We do not deal extensively with this concept, although it should be realized that convexity serves as a backdrop for a significant amount of work in mathematics.

(3.C.1) Definition. A subset C of $\mathscr{E}^n$ is *convex* if and only if each pair of points a and b in C may be connected by a straight line segment that lies

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entirely in C, i.e., if a and b are points of C, then the set $\{ta + (1 - t)b \mid 0 \le t \le 1\}$, is a subset of C, where $a = (a_1, a_2, \ldots, a_n)$ and $ta = (ta_1, ta_2, \ldots, ta_n)$.

The *standard n-ball*, $\mathscr{B}^n$, in $\mathscr{E}^n$ is defined to be the set of points in $\mathscr{E}^n$ whose distance from the origin is less than or equal to 1. It should be obvious that $\mathscr{B}^n$ is convex.

The following theorem will prove useful in later chapters.

(3.C.2) ***Theorem.*** Suppose that A is a compact convex subset of $\mathscr{E}^n$ with nonempty interior. Then A is homeomorphic to $\mathscr{B}^n$.

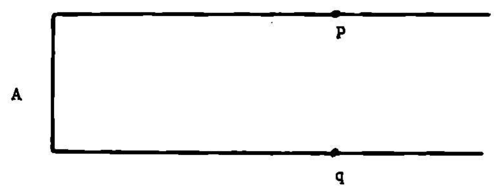
Proof. Let z be an interior point of A, and choose ε small enough so that the set $B_\varepsilon(z) = \{w \in \mathscr{E}^n \mid d(w,z) \le \varepsilon\}$ is contained in the interior of A. For each $x \in A \setminus \{z\}$, let $L(x)$ be the ray starting at z that passes through x. We assume for the moment that $L(x) \cap \operatorname{Fr} A$ consists of a single point x'. Define $H(x)$ to be the unique point on $L(x)$ with the property that $d(x,z)/d(z,x') = d(z,H(x))/\varepsilon$. A somewhat tedious argument (and hence left to the reader) can be employed to show that h is a homeomorphism between A and $B_\varepsilon(z)$. Since $B_\varepsilon(z)$ is homeomorphic to $\mathscr{B}^n$, it follows that A and $\mathscr{B}^n$ are homeomorphic. It remains to show that $L(x) \cap \operatorname{Fr} A$ consists of a single point for each $x \in A \setminus \{z\}$.

Suppose that x' and x'' are distinct points in $L(x) \cap \operatorname{Fr} A$ and that the order of the points on $L(x)$ is z, x', x''. Since $x' \in \operatorname{Fr} A$, there is a point w outside of A near to x' (and not on $L(x)$), such that a line segment starting at x'' and ending inside of $B_\varepsilon(z)$ passes through w. This contradicts the convexity of A; hence, no such x'' exists.

The definition of convexity may be extended to arbitrary metric spaces as follows.

(3.C.3) ***Definition.*** Suppose that (X,d) is a metric space and that $x,y \in X$. A point $m \in X$ is a *midpoint of x and y* if and only if $d(x,m) = d(m,y) = \frac{1}{2}\, d(x,y)$. The metric space X is *convex* if and only if each pair of points has at least one midpoint. The metric space X is *strongly convex* if and only if each pair has a unique midpoint. The metric space X is *without ramifications* if and only if no midpoint of x and y is also a midpoint of x' and y, where $x \ne x'$.

Consider the following subset A of the plane.



It should be apparent that the usual metric for $\mathscr{E}^2$, restricted to A, is not convex. Nevertheless, little ingenuity is required to discover a convex metric for A—simply define the distance between two points p and q to be the (usual) distance traversed when one moves along A from p to q.

Obvious query: for what spaces do convex metrics exist?

A remarkable result due to Moise [1949] and Bing [1949] ensures the existence of a convex metric for all compact, connected, locally connected metric spaces. The reader now has sufficient background to be able to follow the arguments given in those papers.

The convex metric constructed in the Bing and Moise papers need not be strongly convex. For example, the reader should be able to convince himself that $\mathscr{S}^1$ admits no strongly convex metric.

Strongly convex metric spaces without ramifications are less abundant. In fact, Rolfson [1970] has recently shown that any compact 3-dimensional strongly convex metric space without ramifications is homeomorphic to $\mathscr{B}^3 = \{(x_1,x_2,x_3) \in \mathscr{E}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$. Higher dimensional analogs of this result are unknown.

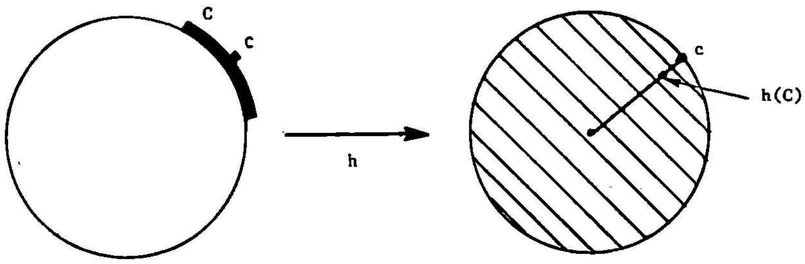
We now introduce an entirely different type of metric, one that is defined on closed subsets of a given metric space. Earlier, we saw how one might define the distance between two subsets of a metric space (X,d) by setting $d(A,B) = \inf\{d(x,y) \mid x \in A \text{ and } y \in B\}$. This distance is quite useful in many contexts, but it has the disadvantage that distinct but intersecting sets always have distance 0; hence, this particular distance function does not lead to a metric for families of subsets of X. One way of creating such a metric is described as follows.

(3.C.4) Definition. Suppose that (X,d) is a metric space with finite diameter and $\mathscr{H}$ is a collection of nonempty closed subsets of X. For $C,D \in \mathscr{H}$ let $d_C(D) = \sup\{d(x,C) \mid x \in D\}$. Then $\rho : \mathscr{H} \times \mathscr{H} \to [0,\infty)$ defined by $\rho(A,B) = \max\{d_A(B),d_B(A)\}$ is the *Hausdorff metric* for $\mathscr{H}$, and $(\mathscr{H},\rho)$ is called a *hyperspace* of X.

(3.C.5) Exercise. Show that ρ is a metric.

Note that for points $x,y \in X$, we have that $\rho(\{x\},\{y\}) = d(x,y)$, and hence X is isometrically embedded in $(\mathscr{H},\rho)$. Of considerable interest is the problem of determining the actual nature of $(\mathscr{H},\rho)$ for a given collection of closed subsets of a metric space (X,d). Let us examine what happens in a relatively simple case.

Suppose that X is the unit circle $\mathscr{S}^1$ with the usual topology, and let $\mathscr{H}$ be the collection of all closed connected subsets of X. We show that the hyperspace $(\mathscr{H},\rho)$ is homeomorphic to the unit ball $\mathscr{B}^2 = \{(x,y) \in \mathscr{E}^2 \mid x^2 + y^2 \leq 1\}$. A homeomorphism h between the two spaces is given as follows. Define $h(\mathscr{S}^1)$ to be the origin (0,0). If C is a proper closed connected subset of $\mathscr{S}^1$, let c be its midpoint and let $r =$ (length $C)/2\pi$. Then h maps C onto the point of $\mathscr{B}^2$ lying on the radius passing through c, and at a distance of $1 - r$ from the origin.



There is no problem in verifying that h is a homeomorphism.

Other examples of this sort are given in the problem set. Schori and West [1972] have recently resolved a long-standing problem involving hyperspaces by showing that if $X = \mathbf{I}$ and $\mathscr{H}$ is the collection of all closed subsets of X, then the resulting hyperspace is the Hilbert cube. (The Hilbert cube is the space obtained by taking the product of $\mathbf{I}$ with itself a countable number of times and will be discussed in Chapter 7.) Amazingly, Schori went on to show in subsequent papers that the hyperspace associated with any compact, connected, locally connected metric space is also the Hilbert cube.

PROBLEMS

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Section A

1. Show that a metric space is compact if and only if every infinite subset has an accumulation point (cf. chapter 2, problem I-13).

2. Suppose that X is a first countable space, $\{x_i\}$ is a sequence in X, and x is a cluster point of $\{x_i\}$. Show that x is a limit point of a subsequence of $\{x_i\}$.
3.* A subset $A \subset (X,d)$ is *metrically isolated* if and only if $d(A,B) > 0$ for each closed set B contained in $X \setminus A$. Show that if A is metrically isolated, then A is closed and has compact boundary.
4. Suppose that (X,d) and (Y,d') are metric spaces. Show that a function $f : X \to Y$ is uniformly continuous if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in X, $\lim_{n\to\infty} d(x_n,y_n) = 0$ implies that $\lim_{n\to\infty} d'(f(x_n),f(y_n)) = 0$.
5.* Let (X,d) be a compact metric space and suppose that $f : X \to X$ is an isometry. Show that f is onto.
6. Suppose that X is a metric space with the property that whenever $f : X \to X$ is an isometry, then f is onto. Is X compact?
7.* Suppose that (X,d) is a metric space with the following two properties:

   (i) for each $x \in X$ and $r > 0$, $\{y \mid d(x,y) \leq r\}$ is compact;
   (ii) for each $x,y \in X$, there is an isometry $\psi : X \to X$ such that $\psi(x) = y$.

   Show that every isometry from X into X is onto.
8. Find an example to show that if (X,d) and (Y,d') are metric spaces and $f : X \to Y$ is continuous, there need not be an equivalent metric for Y under which f is uniformly continuous.
9. Suppose that A and B are disjoint compact subsets of a metric space (X,d). Show that there is an $a \in A$ and $b \in B$ such that $d(A,B) = d(a,b)$.
10.* Suppose that (X,d) and (Y,d') are metric spaces. Show that a function $f : X \to Y$ is uniformly continuous if and only if whenever A and B are nonempty subsets and $d(A,B) = 0$, then $d'(f(A),f(B)) = 0$.
11. Suppose that (X,d) and (Y,d') are metric spaces. If $h : X \to Y$ is a uniformly continuous homeomorphism, must h^{-1} be uniformly continuous?
12. A metrizable space $(X,\mathscr{U})$ is *topologically totally bounded* if and only if there is a metric d for X such that the topology induced by d coincides with $\mathscr{U}$ and (X,d) is totally bounded. Show that a space $(X,\mathscr{U})$ is topologically totally bounded if and only if $(X,\mathscr{U})$ is separable.
13. Determine which of the following are uniformly continuous:

   a) $f(x) = x^2$
   b) $f(x) = \sqrt{x}$
   c) $f(x) = e^x$

14. Is the sum (product) of two uniformly continuous functions uniformly continuous?

Section B

1. Show that in (3.B.8), $\bigcap_{i=1}^{\infty} A_i$ may be empty if the limit of the diameters of the A_i is not 0.
2. Let $r \in (0,1)$ and suppose that (X,d) is a complete metric space, S is a closed subset of X, and $f : S \to S$ is a map with the property that $d(f(x),f(y)) \le rd(x,y)$ for all $x,y \in S$. (Such a map is called a *contractive map*.) Show that f has a unique fixed point.
3. Suppose that d is a metric for a space X with the property that every closed set of finite diameter is compact. Show that (X,d) is a complete metric space.
4. Suppose that X is a metric space with the property that if S is any nonempty closed subset of X and $f : S \to S$ is any contractive map (see problem 2 above), then f has a fixed point. Show that X is complete. [Hint: Let $\{x_n\}$ be a nonconvergent Cauchy sequence and for each $x \in X$, let $L(x) = \inf\{d(x,x_n) \mid x_n \neq x\}$. Note that $L(x) > 0$. Choose r such that $0 < r < 1$. Define $\sigma : \mathrm{N} \to \mathrm{N}$ inductively by setting $\sigma(0) = 0$ and defining $\sigma(n)$ to be an integer $> \sigma(n-1)$ such that $d(x_i,x_j) \le rL(x_{\sigma(n-1)})$ for all integers $i,j \ge \sigma(n)$. Let $S = \{x_{\sigma(n)} \mid n = 0, 1, 2, \ldots\}$ and let $f : S \to S$ be defined by $f(x_{\sigma(n)}) = x_{\sigma(n+1)}$].
5. Show that any subsequence of a Cauchy sequence is Cauchy.
6.* Show that a metric space is compact if and only if it is complete in every equivalent metric.
7.* Show that a metric space is compact if and only if it is bounded in every equivalent metric.
8. Show that every subset of a first category set is first category.
9. Prove or disprove: every uncountable space is second category (in itself).
10. Show that if A is a nowhere dense subset of a topological space X, then $X \setminus A$ is dense in X.
11. Suppose that X and Y are metric spaces and that $A \subset X$ and $B \subset Y$. Show that if $A \times B$ is nowhere dense in $X \times Y$, then either A is nowhere dense in X or B is nowhere dense in Y.
12. Suppose that X is a metric space and that A is a countable subset of X. Show that A is of the first category in X if and only if A has no isolated point in X.
13. Prove or disprove: every open subset of a second category space is of the second category.
14. Show that the countable union of first category sets is of the first category.
15. Show that every countably infinite complete metric space contains an infinite number of isolated points,

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16. The Cantor set may be used as follows to describe a map $f : \mathbf{I} \to \mathscr{E}^1$ that is open but not continuous. On each component C_α of the complement of the Cantor set in I, let $f_\alpha : C_\alpha \to \mathscr{E}^1$ be any strictly increasing continuous function. On the Cantor set itself, define f to be identically 0. Show that f is open but not continuous.
17. Show that if U is an open subset of a space X, then Fr U is nowhere dense.
18. Prove the following 'converse' of (3.B.8). Suppose that (X,d) is a metric space and that the intersection of each decreasing sequence of nonempty closed balls whose radii converge to zero is nonempty. Then (X,d) is complete.
19.* Another subset of the plane possessing an explosion point may be constructed as follows. As before, start with the "cone" over the Cantor set. Let $\mathscr{S}$ denote the family of all half-closed segments each connecting a point of the Cantor set with $p = (1/2,1/2)$ but excluding p, and let $\phi : [0,\Omega] \to \mathscr{S}$ be 1–1 and onto. Let $\mathscr{A}$ be the family of all compact subsets of $\mathscr{E}^2$ that intersect an uncountable number of members of $\mathscr{S}$. Let $\psi : [0,\Omega) \to \mathscr{A}$ be 1–1 and onto (why is this possible?). For each $\alpha \in [0,\Omega)$, let $\phi(\alpha) = S_\alpha$ and $\psi(\alpha) = A_\alpha$. Let α_0 be the smallest element of $[0,\Omega)$ such that $A_0 \cap S_{\alpha_0} \neq \varnothing$, and select a point c_{α_0} in this intersection. Let α_1 be the smallest element in $[0,\Omega)$ (and $\neq \alpha_0$) such that $A_1 \cap S_{\alpha_1} \neq \varnothing$. Select a point c_{α_1} in the intersection. Continuing in this fashion, show that for each $\beta \in [0,\Omega]$, exactly one point c_β is selected in S_β. Let $X = \{p\} \cup \{c_\beta \mid \beta \in [0,\Omega)\}$. Show that X is connected, but $X \setminus \{p\}$ is totally disconnected.
20.* Show that an uncountable second countable space can be written as the union of a perfect set and a countable set and that these two sets may be chosen to be disjoint.
21. Show that a subset A of a complete metric space is countably compact if and only if A is closed and totally bounded.
22.* Show that the Knaster-Kuratowski example has the fixed point property (the fixed point need not be p.)
23. Show that if A is a subset of a separable metric space, then A has at most a countable number of isolated points.

Section C

1. Show that the conclusion of (3.C.2) need not hold if A has empty interior.
2. Suppose that $p,q,r \in (X,d)$. Then q is said to be *between* p and r if and only if $d(p,r) = d(p,q) + d(q,r)$. A subset $A \subset (X,d)$ is *linear* if and only if there is an isometry (into) $\Phi : A \to \mathscr{E}^1$.

(i) Suppose that $p,q,r \in X$. Show that $\{p,q,r\}$ is linear if and only if one of the points p, q, or r is between the other two.

(ii) Give an example of a metric space consisting of four points that is not linear but for which every proper subset is linear.

3. An arc α contained in (X,d) is a *segment* if and only if Im α is linear. Suppose that (X,d) is a strongly convex metric space without ramifications. Let pq and pr be segments in X such that $pq \cap pr \setminus \{p\} \neq \varnothing$. Show that either $pq \subset pr$ or $pr \subset pq$. Here pq denotes the segment with endpoints p and q.
4. If (X,d) is strongly convex and $p,q,r \in X$, show that q is between p and r if and only if $q \in pr$.
5. If $U_1, U_2, \ldots, U_n$ are open subsets of a topological space X, let $\langle U_1, \ldots, U_n \rangle = \{A \subset X \mid \text{such that } A \subset \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \varnothing \text{ for each } i\}$. Show that sets of this form determine a basis for a topology for $V = \{A \subset X \mid A \text{ is closed and } A \neq \varnothing\}$. This topology is called the *Vietoris topology*.
6.* Show that in compact metric spaces, the Vietoris topology and the topology generated by the Hausdorff metric coincide.
7. Let $X = \mathbf{I}$, and let $\mathscr{H}$ be the family of all subsets of $\mathbf{I}$ that consist of one or two points. Show that $\mathscr{H}$ with the Hausdorff metric is homeomorphic to a (solid) triangle.
8.* Let $X = \mathbf{I}$ and let $\mathscr{H}$ be the family of all subsets of $\mathbf{I}$ that consist of one, two, or three points. Show that $\mathscr{H}$ with the Hausdorff metric is homeomorphic to a tetrahedron.