Tarea 2 Probabilidad 2

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1. (X,Y) es un vector aleatorio con distribución hipergeométrica bivariada. Si su función de probabilidad está dada por: $f(x,y)=\dfrac{\binom{N_1}{x}\binom{N_2}{y}\binom{N-N_1-N_2}{n-x-y}}{\binom{N}{n}}$; donde $x=0,1,2,...,n;\quad y=0,1,2,...,n;\quad x+y\le n$; $N1>0;\quad N2>0;\quad N1+N2\le N$; $0<n<N$

   (a) Calcula $f_X(x)$ y $f_Y(y)$

   $f_X(x)=\sum\limits_{y=0}^{n-x}\dfrac{\binom{N_1}{x}\binom{N_2}{y}\binom{N-N_1-N_2}{n-x-y}}{\binom{N}{n}}=\dfrac{\binom{N_1}{x}}{\binom{N}{n}}\sum\limits_{y=0}^{n-x}\binom{N_2}{y}\binom{N-N_1-N_2}{n-x-y}=\dfrac{\binom{N_1}{x}\binom{N-N_1}{n-x}}{\binom{N}{n}}$(Ver anexo)

   De forma análoga

   $f_Y(y)=\dfrac{\binom{N_2}{y}\binom{N-N_2}{n-y}}{\binom{N}{n}}$

   (b) Calcula $Cov(X,Y)$ y $\rho(X,Y)$

2.

3. X y Y tienen función de densidad conjunta:

$$f(x,y)=\frac{1}{x^2y^2}\mathbf{I}_{(1,\infty)}^{(x)}\mathbf{I}_{(1,\infty)}^{(y)}$$

   (a) Calcula la función de densidad conjunta de $U=XY$ y $V=\dfrac{X}{Y}$

   $UV=X^2\Rightarrow X=\sqrt{UV}$

   $\dfrac{U}{V}=Y^2\Rightarrow Y=\sqrt{\dfrac{U}{V}}$

   $\left|\dfrac{\partial(X,Y)}{\partial(U,V)}\right|=\left|\det\begin{pmatrix}\dfrac{\sqrt{V}}{2\sqrt{U}} & \dfrac{1}{2\sqrt{UV}}\\ \dfrac{\sqrt{U}}{2\sqrt{V}} & -\dfrac{\sqrt{U}}{2\sqrt{V^3}}\end{pmatrix}\right|=\dfrac{1}{4\sqrt{V}}\left|\det\begin{pmatrix}\sqrt{V} & \dfrac{1}{\sqrt{V}}\\ 1 & -\dfrac{1}{V}\end{pmatrix}\right|=\dfrac{1}{2V}$

   Así

   $f(u,v)=\dfrac{1}{2v}\dfrac{1}{(uv)(\frac{u}{v})}\mathbf{I}_{(1,\infty)}^{(\sqrt{uv})}\mathbf{I}_{(1,\infty)}^{(\frac{\sqrt{u}}{\sqrt{v}})}=\dfrac{1}{2u^2v}\mathbf{I}_{(1,\infty)}^{(\sqrt{uv})}\mathbf{I}_{(1,\infty)}^{(\frac{\sqrt{u}}{\sqrt{v}})}$

   $1<uv<\infty$ y $1<\dfrac{u}{v}<\infty$

   $\dfrac{1}{v}<u<\infty$ y $v<u<\infty$

   Por lo tanto

   $f(u,v)=\dfrac{1}{2u^2v}(\mathbf{I}_{(v,\infty)}^{(u)}\mathbf{I}_{(1,\infty)}^{(v)}+\mathbf{I}_{(\frac{1}{v},\infty)}^{(u)}\mathbf{I}_{(0,1)}^{(v)})=\dfrac{1}{2u^2v}\mathbf{I}_{(\frac{1}{u},u)}^{(v)}\mathbf{I}_{(1,\infty)}^{(u)}$

   (b) Calcula $f_U(u)$ y $f_V(v)$

$$f_U(u)=\int_{\frac{1}{u}}^{u}\frac{1}{2u^2v}dv\mathbf{I}_{(1,\infty)}^{(u)}=\frac{\ln u^2}{2u^2}\mathbf{I}_{(1,\infty)}^{(u)}$$

$$f_V(v)=\int_{v}^{\infty}\frac{1}{2u^2v}du\mathbf{I}_{(1,\infty)}^{(v)}+\int_{\frac{1}{v}}^{\infty}\frac{1}{2u^2v}du\mathbf{I}_{(0,1)}^{(v)}=\frac{1}{2v^2}\mathbf{I}_{(1,\infty)}^{(v)}+\frac{1}{2}\mathbf{I}_{(0,1)}^{(v)}$$

4. Sean X y Y independientes ambas con distribución uniforme en $(0,1)$. Sea $U = X+Y$ y $V = X-Y$. Tenemos que $f_{X,Y}(x,y) = 1\mathbf{I}_{(0,1)}^{(x)}\mathbf{I}_{(0,1)}^{(y)}$

   (a) Calcula $f_{U,V}(u,v)$
   $X = \dfrac{U+V}{2}$ y $Y = \dfrac{U-V}{2}$
   $\Big|\dfrac{\partial(X,Y)}{\partial(U,V)}\Big| = \Big|\det(\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{array})\Big| = \dfrac{1}{2}$
   Así
   $$f_{U,V}(u,v) = \frac{1}{2}\mathbf{I}_{(0,1)}^{(\frac{u+v}{2})}\mathbf{I}_{(0,1)}^{(\frac{u-v}{2})}$$
   $0 < u+v < 2$ y $0 < u-v < 2$
   Entonces
   $(0 < u < 2)$ y $(-u < v < 2-u$ y $u-2 < v < u)$ o bien
   $(-1 < v < 1)$ y $(-v < u < 2-v$ y $v < u < 2+v)$
   Por lo tanto
   $$f_{U,V}(u,v) = \frac{1}{2}(\mathbf{I}_{(-v,2+v)}^{(u)}\mathbf{I}_{(-1,0)}^{(v)} + \mathbf{I}_{(v,2-v)}^{(u)}\mathbf{I}_{(0,1)}^{(v)}) = \frac{1}{2}\mathbf{I}_{(-u,u)}^{(v)}\mathbf{I}_{(0,1)}^{(u)} + \mathbf{I}_{(u-2,2-u)}^{(v)}\mathbf{I}_{(1,2)}^{(u)})$$

   (b) Demuestra que $Cov(U,V) = 0$, pero no son independientes.
   $$E(U,V) = \frac{1}{2}(\int_{-1}^{0}\int_{-v}^{2+v} uvdudv + \int_{0}^{1}\int_{v}^{2-v} uvdudv) = \frac{1}{2}(\int_{-1}^{0} 4(v+v^2)dv + \int_{0}^{1} 4(v-v^2) = 0$$
   $$E(U) = \frac{1}{2}(\int_{-1}^{0}\int_{-v}^{2+v} ududv + \int_{0}^{1}\int_{v}^{2-v} ududv) = \frac{1}{2}(\int_{-1}^{0} 4(1+v)dv + \int_{0}^{1} 4(1-v)) = 2$$
   $$E(V) = \frac{1}{2}(\int_{-1}^{0}\int_{-v}^{2+v} vdudv + \int_{0}^{1}\int_{v}^{2-v} vdudv) = \int_{-1}^{0} v+v^2dv + \int_{0}^{1} v-v^2dv = 0$$
   Por lo tanto
   $$Cov(U,V) = E(U,V) - E(U)E(V) = 0$$
   Y claramente U y V no son independientes.

5.

6.

7.

8.

9. Sea $U \sim U(0,2\pi)$ y $Z \sim exp(1)$ tal que U y Z son independientes. Sea $X = \sqrt{2Z}\cos U$ y $Y = \sqrt{2Z}\sin U$. Demuestra que X y Y son variables aleatorias independientes con distribución normal estándar.
   $$f_{U,Z}(u,z) = \frac{1}{2\pi}e^{-z}\mathbf{I}_{(0,2\pi)}^{(u)}\mathbf{I}_{(0,\infty)}^{(z)}$$
   $X = \sqrt{2Z}cosU$ y $Y = \sqrt{2Z}sinU$
   $\dfrac{X^2+Y^2}{2} = Z$ y $\arctan(\dfrac{Y}{X}) = U$
   $\Big|\dfrac{\partial(U,Z)}{\partial(X,Y)}\Big| = \Big|\det(\begin{array}{cc} \dfrac{-X^2Y}{X^2(X^2+Y^2)} & X \\ \dfrac{X^2}{X(X^2+Y^2} & Y \end{array})\Big| = 1$
   $$f_{X,Y}(x,y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{4}}\mathbf{I}_{(0,2\pi)}^{(\arctan\frac{y}{x})}\mathbf{I}_{(0,\infty)}^{(\frac{x^2+y^2}{4})} = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{4}}\mathbf{I}_{(-\infty,\infty)}^{(x)}\mathbf{I}_{(-\infty,\infty)}^{(y)}$$

Así

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathbf{I}_{(-\infty,\infty)}^{(x)} \Rightarrow X \sim N(0,1)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \mathbf{I}_{(-\infty,\infty)}^{(y)} \Rightarrow Y \sim N(0,1)$$

Ambas son normal standar.

1 Anexo

Sabemos que:

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

Además

$$(\sum_{i=0}^{n} a_i x^i)(\sum_{j=0}^{m} b_j x^j) = \sum_{r=0}^{n+m} (\sum_{i=0}^{r} a_i b_{r-i}) x^r$$

Así

$$(1+x)^{n+m} = (1+x)^n (1+x)^m = (\sum_{i=0}^{n} \binom{n}{i} x^i)(\sum_{j=0}^{m} \binom{m}{j} x^j) = \sum_{r=0}^{n+m} (\sum_{i=0}^{r} \binom{n}{i} \binom{m}{r-i}) x^r = \sum_{r=0}^{n+m} \binom{n+m}{r} x^r$$

Por lo tanto

$$\sum_{i=0}^{r} \binom{n}{i} \binom{m}{r-i} = \binom{n+m}{r}$$