

Tarea 2 Probabilidad 2

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1. (X, Y) es un vector aleatorio con distribución hipergeométrica bivariada. Si su función de probabilidad está

dada por: $f(x, y) = \frac{\binom{N_1}{x} \binom{N_2}{y} \binom{N-N_1-N_2}{n-x-y}}{\binom{N}{n}}$; donde $x = 0, 1, 2, \dots, n; \quad y = 0, 1, 2, \dots, n; \quad x + y \leq n$
 $N_1 > 0; \quad N_2 > 0; \quad N_1 + N_2 \leq N$
 $0 < n < N$

- (a) Calcula $f_X(x)$ y $f_Y(y)$

$$f_X(x) = \sum_{y=0}^{n-x} \frac{\binom{N_1}{x} \binom{N_2}{y} \binom{N-N_1-N_2}{n-x-y}}{\binom{N}{n}} = \frac{\binom{N_1}{x}}{\binom{N}{n}} \sum_{y=0}^{n-x} \binom{N_2}{y} \binom{N-N_1-N_2}{n-x-y} = \frac{\binom{N_1}{x} \binom{N-N_1}{n-x}}{\binom{N}{n}} \text{ (Ver anexo)}$$

De forma análoga

$$f_Y(y) = \frac{\binom{N_2}{y} \binom{N-N_2}{n-y}}{\binom{N}{n}}$$

- (b) Calcula $Cov(X, Y)$ y $\rho(X, Y)$

2.

3. X y Y tienen función de densidad conjunta:

$$f(x, y) = \frac{1}{x^2 y^2} \mathbf{I}_{(1, \infty)}^{(x)} \mathbf{I}_{(1, \infty)}^{(y)}$$

- (a) Calcula la función de densidad conjunta de $U = XY$ y $V = \frac{X}{Y}$

$$UV = X^2 \Rightarrow X = \sqrt{UV}$$

$$\frac{U}{V} = Y^2 \Rightarrow Y = \sqrt{\frac{U}{V}}$$

$$\left| \frac{\partial(X, Y)}{\partial(U, V)} \right| = \left| \det \begin{pmatrix} \frac{\sqrt{UV}}{2\sqrt{U}} & \frac{1}{2\sqrt{UV}} \\ \frac{\sqrt{U}}{2\sqrt{V}} & -\frac{1}{2\sqrt{V^3}} \end{pmatrix} \right| = \frac{1}{4\sqrt{V}} \left| \det \begin{pmatrix} \sqrt{V} & \frac{1}{\sqrt{V}} \\ 1 & -\frac{1}{V} \end{pmatrix} \right| = \frac{1}{2V}$$

Así

$$f(u, v) = \frac{1}{2v} \frac{1}{(uv)^{\frac{u}{v}}} \mathbf{I}_{(1, \infty)}^{(\sqrt{uv})} \mathbf{I}_{(1, \infty)}^{(\frac{\sqrt{u}}{\sqrt{v}})} = \frac{1}{2u^2 v} \mathbf{I}_{(1, \infty)}^{(\sqrt{uv})} \mathbf{I}_{(1, \infty)}^{(\frac{\sqrt{u}}{\sqrt{v}})}$$

$$1 < uv < \infty \text{ y } 1 < \frac{u}{v} < \infty$$

$$\frac{1}{v} < u < \infty \text{ y } v < u < \infty$$

Por lo tanto

$$f(u, v) = \frac{1}{2u^2 v} (\mathbf{I}_{(v, \infty)}^{(u)} \mathbf{I}_{(1, \infty)}^{(v)} + \mathbf{I}_{(\frac{1}{v}, \infty)}^{(u)} \mathbf{I}_{(0, 1)}^{(v)}) = \frac{1}{2u^2 v} \mathbf{I}_{(\frac{1}{u}, u)}^{(v)} \mathbf{I}_{(1, \infty)}^{(u)}$$

- (b) Calcula $f_U(u)$ y $f_V(v)$

$$f_U(u) = \int_{\frac{1}{u}}^u \frac{1}{2u^2 v} dv \mathbf{I}_{(1, \infty)}^{(u)} = \frac{\ln u^2}{2u^2} \mathbf{I}_{(1, \infty)}^{(u)}$$

$$f_V(v) = \int_v^\infty \frac{1}{2u^2 v} du \mathbf{I}_{(1, \infty)}^{(v)} + \int_{\frac{1}{v}}^\infty \frac{1}{2u^2 v} du \mathbf{I}_{(0, 1)}^{(v)} = \frac{1}{2v^2} \mathbf{I}_{(1, \infty)}^{(v)} + \frac{1}{2} \mathbf{I}_{(0, 1)}^{(v)}$$

4. Sean X y Y independientes ambas con distribución uniforme en $(0, 1)$. Sea $U = X + Y$ y $V = X - Y$. Tenemos que $f_{X,Y}(x, y) = \mathbf{I}_{(0,1)}^{(x)} \mathbf{I}_{(0,1)}^{(y)}$

(a) Calcula $f_{U,V}(u, v)$

$$X = \frac{U+V}{2} \text{ y } Y = \frac{U-V}{2}$$

$$\left| \frac{\partial(X, Y)}{\partial(U, V)} \right| = \left| \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \right| = \frac{1}{2}$$

Así

$$f_{U,V}(u, v) = \frac{1}{2} \mathbf{I}_{(0,1)}^{(\frac{u+v}{2})} \mathbf{I}_{(0,1)}^{(\frac{u-v}{2})}$$

$$0 < u + v < 2 \text{ y } 0 < u - v < 2$$

Entonces

$(0 < u < 2)$ y $(-u < v < 2 - u \text{ y } u - 2 < v < u)$ o bien
 $(-1 < v < 1)$ y $(-v < u < 2 - v \text{ y } v < u < 2 + v)$

Por lo tanto

$$f_{U,V}(u, v) = \frac{1}{2} (\mathbf{I}_{(-v, 2+v)}^{(u)} \mathbf{I}_{(-1,0)}^{(v)} + \mathbf{I}_{(v, 2-v)}^{(u)} \mathbf{I}_{(0,1)}^{(v)}) = \frac{1}{2} \mathbf{I}_{(-u, u)}^{(v)} \mathbf{I}_{(0,1)}^{(u)} + \mathbf{I}_{(u-2, 2-u)}^{(v)} \mathbf{I}_{(1,2)}^{(u)}$$

(b) Demuestra que $Cov(U, V) = 0$, pero no son independientes.

$$E(U, V) = \frac{1}{2} \left(\int_{-1}^0 \int_{-v}^{2+v} uv du dv + \int_0^1 \int_v^{2-v} uv du dv \right) = \frac{1}{2} \left(\int_{-1}^0 4(v + v^2) dv + \int_0^1 4(v - v^2) dv \right) = 0$$

$$E(U) = \frac{1}{2} \left(\int_{-1}^0 \int_{-v}^{2+v} u du dv + \int_0^1 \int_v^{2-v} u du dv \right) = \frac{1}{2} \left(\int_{-1}^0 4(1 + v) dv + \int_0^1 4(1 - v) dv \right) = 2$$

$$E(V) = \frac{1}{2} \left(\int_{-1}^0 \int_{-v}^{2+v} v du dv + \int_0^1 \int_v^{2-v} v du dv \right) = \int_{-1}^0 v + v^2 dv + \int_0^1 v - v^2 dv = 0$$

Por lo tanto

$$Cov(U, V) = E(U, V) - E(U)E(V) = 0$$

Y claramente U y V no son independientes.

5.

6.

7.

8.

9. Sea $U \sim U(0, 2\pi)$ y $Z \sim exp(1)$ tal que U y Z son independientes. Sea $X = \sqrt{2Z} \cos U$ y $Y = \sqrt{2Z} \sin U$. Demuestra que X y Y son variables aleatorias independientes con distribución normal estándar.

$$f_{U,Z}(u, z) = \frac{1}{2\pi} e^{-z} \mathbf{I}_{(0,2\pi)}^{(u)} \mathbf{I}_{(0,\infty)}^{(z)}$$

$$X = \sqrt{2Z} \cos U \text{ y } Y = \sqrt{2Z} \sin U$$

$$\frac{X^2 + Y^2}{4} = Z \text{ y } \arctan\left(\frac{Y}{X}\right) = U$$

$$\left| \frac{\partial(U, Z)}{\partial(X, Y)} \right| = \left| \det \begin{pmatrix} \frac{-X^2 Y}{X^2(X^2 + Y^2)} & \frac{1}{2} X \\ \frac{X^2}{X(X^2 + Y^2)} & \frac{1}{2} Y \end{pmatrix} \right| = \frac{1}{2}$$

$$f_{X,Y}(x, y) = \frac{1}{4\pi} e^{-\frac{x^2 + y^2}{4}} \mathbf{I}_{(0,2\pi)}^{(\arctan \frac{y}{x})} \mathbf{I}_{(0,\infty)}^{(\frac{x^2 + y^2}{4})} = \frac{1}{4\pi} e^{-\frac{x^2 + y^2}{4}} \mathbf{I}_{(-\infty, \infty)}^{(x)} \mathbf{I}_{(-\infty, \infty)}^{(y)}$$

Así

$$f_X(x) = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}} \mathbf{I}_{(-\infty, \infty)}^{(x)} \Rightarrow X \sim N(0, 2)$$

$$f_Y(y) = \frac{1}{2\sqrt{\pi}} e^{-\frac{y^2}{4}} \mathbf{I}_{(-\infty, \infty)}^{(y)} \Rightarrow Y \sim N(0, 2)$$

Ambas son normal standar.

1 Anexo

Sabemos que:

$$(1+x)^n = \sum_{i=0}^n \binom{n}{i} x^i$$

Además

$$\left(\sum_{i=0}^n a_i x^i\right) \left(\sum_{j=0}^m b_j x^j\right) = \sum_{r=0}^{n+m} \left(\sum_{i=0}^r a_i b_{r-i}\right) x^r$$

Así

$$(1+x)^{n+m} = (1+x)^n (1+x)^m = \left(\sum_{i=0}^n \binom{n}{i} x^i\right) \left(\sum_{j=0}^m \binom{m}{j} x^j\right) = \sum_{r=0}^{n+m} \left(\sum_{i=0}^r \binom{n}{i} \binom{m}{r-i}\right) x^r = \sum_{r=0}^{n+m} \binom{n+m}{r} x^r$$

Por lo tanto

$$\sum_{i=0}^r \binom{n}{i} \binom{m}{r-i} = \binom{n+m}{r}$$