Chapter 2

CONNECTEDNESS AND COMPACTNESS

The two principal topological properties studied in this chapter occupy a central position in both topology and analysis. The concepts of connectedness and compactness may be viewed as generalizations (in different ways) of two basic properties of intervals. Connectedness represents an extension of the idea that an interval is all in "one piece," while compactness may be construed as a generalization of the fact that a closed interval $[a,b]$ is both closed (as a subset of $\mathscr{E}^1$) and bounded.

A. CONNECTEDNESS: GENERAL RESULTS AND DEFINITIONS

The problem of deciding if a space X consists of just "one piece" is resolved by determining whether or not X may be broken up into disjoint open subsets.

(2.A.1) Definition. A pair (U,V) of nonempty open subsets of a space X is a *separation of* X if and only if $X = U \cup V$ and $U \cap V = \varnothing$.

(2.A.2) Remark. Suppose that X is a topological space and that A and B are disjoint nonempty closed subsets of X such that $X = A \cup B$. Then A and B effect a separation of X, since their complements are open. Thus, if X is separated by subsets A and B, then A and B are both open and closed.

(2.A.3) Definition. A topological space X is *connected* if and only if no separation of X exists. If a separation of X does exist, then X is *disconnected.* A subspace A of X is *connected* if and only if A with the relative topology is connected.

(2.A.4) Exercise. Show that a space X is connected if and only if X does not contain a proper nonempty subset that is both open and closed.

(2.A.5) Exercise. Suppose that A is a disconnected subspace of a space X. Show that there are closed subsets C_1 and C_2 of X such that $A \subset C_1 \cup C_2$, $A \cap C_1 \cap C_2 = \varnothing$, and both $A \cap C_1$ and $A \cap C_2$ are nonempty.

(2.A.6) Exercise. Show that if (U,V) is a separation of a space X and $C \subset X$ is connected, then either $C \subset U$ or $C \subset V$.

(2.A.7) Examples. (Proofs will follow from later results.)

Connected Spaces.

1. Intervals in $\mathscr{E}^1$
2. Disks (spaces homeomorphic with $\{(x,y) \in \mathscr{E}^2 \mid x^2 + y^2 \leq 1\}$)
3. $\mathscr{E}^n$
4. A sine curve
5. Any set with the indiscrete topology

Disconnected Spaces.

1. $(0,1) \cap (3,4)$ (with the relative topology)
2. $[0,1) \cap (1,4]$
3. Rational numbers (why?)
4. Any set with the discrete topology (and consisting of two or more points)

Although quite reasonable from a visual standpoint, the criterion that a space is connected if it consists of solely "one piece" is often difficult to apply in practice. For instance, while it seems apparent that $\mathscr{E}^2$ is connected, what can be said about $\mathscr{E}^2 \setminus \{(x,y) \mid x,y \in \mathbf{Q}\}$? Is this subspace still in "one piece" in spite of all the holes? In $\mathscr{E}^1$ no such complications arise, as we see from the following theorem.

(2.A.8) Theorem. A subspace B of $\mathscr{E}^1$ is connected if and only if B is a point or an interval.

Proof. Suppose that B is a subset of $\mathscr{E}^1$ that is neither an interval nor a point. Then there are points $a,b,c \in \mathscr{E}^1$ such that $a < b < c$, where $a,c \in B$ and $b \notin B$. Let $U = (-\infty,b) \cap B$ and $V = (b,\infty) \cap B$. Since U and V are clearly disjoint B-open subsets whose union is B, they form a separation of B, and consequently B is not connected.

To prove the converse, we suppose that B is not connected. We show that B cannot be an interval. It follows from (2.A.5) that there are closed subsets C and D of $\mathscr{E}^1$ such that $C \cap D \cap B = \varnothing$, $B \subset C \cap D$, and both $B \cap C$ and $B \cap D$ are nonempty. Let $c \in C \cap B$ and $d \in D \cap B$, and assume (relabel if necessary) that $c < d$. Let $s = \sup\{x \in B \mid x \in C \text{ and } c \leq x \leq d\}$. Note that $s \neq d$ (why?). Now let $t = \inf\{y \in B \mid y \in D \text{ and } s \leq y \leq d\}$. We consider two possibilities: $s = t$ and $s < t$. If $s = t$, then $s \in C \cap D$ (why?) and consequently s is not in B. If $s < t$, then $B \cap (s,t) = \varnothing$. In either case, it follows easily that B is not an interval.

(2.A.9) Exercise. Fill in the details of the foregoing proof.

(2.A.10) Theorem. Suppose that $\{A_\alpha \mid \alpha \in \Lambda\}$ is a collection of connected subspaces of a topological space X, and that for each $\alpha,\beta \in \Lambda$, $A_\alpha \cap A_\beta \neq \varnothing$. Then $\bigcup \{A_\alpha \mid \alpha \in \Lambda\}$ is connected.

Proof. Let $Y = \bigcup \{A_\alpha \mid \alpha \in \Lambda\}$ and suppose that Y is not connected. Then there are disjoint, nonempty, Y-open sets U and V whose union is Y. Since $U \neq \varnothing$, there is an $\alpha \in \Lambda$ such that $A_\alpha \cap U \neq \varnothing$, and similarly there is a $\beta \in \Lambda$ such that $A_\beta \cap V \neq \varnothing$. By (2.A.6), either $A_\alpha \subset U$ or $A_\alpha \subset V$, and since $A_\alpha \cap U \neq \varnothing$, it follows that $A_\alpha \subset U$. Similarly, we have that $A_\beta \subset V$. However, this is impossible, since $A_\alpha \cap A_\beta \neq \varnothing$.

(2.A.11) Exercise.

1. Show that $\mathscr{E}^1$ is connected.
2. Suppose that $\{A_\alpha \mid \alpha \in \Lambda\}$ is a collection of connected subspaces of a space X, and that $B \subset X$ is also connected. Show that if $B \cap A_\alpha \neq \varnothing$ for each $\alpha \in \Lambda$, then $B \cup \{A_\alpha \mid \alpha \in \Lambda\}$ is connected.

The following theorem, quite useful in spite of its trivial proof, gives a convenient characterization of connectedness. For the remainder of this chapter, the discrete space consisting of just two points, $\{0,1\}$, will be denoted by $\mathscr{S}$.

(2.A.12) Theorem. A space X is connected if and only if there is no continuous function $f : X \to \mathscr{S}$ which is onto.

Proof. If $f : X \to \mathscr{S}$ is onto, then the sets $f^{-1}(0)$ and $f^{-1}(1)$ form a separation of X. On the other hand, if X is not connected, then there is a separation (U,V) of X. Let $f : X \to \mathscr{S}$ be defined by $f(x) = 0$ if $x \in U$ and $f(x) = 1$ if $x \in V$. Then f is clearly continuous and maps X onto $\mathscr{S}$.

(2.A.13) Theorem. If A is a connected subspace of a topological space X and $A \subset B \subset \bar{A}$, then B is connected.

Proof. We apply (2.A.12). Suppose that $f : B \to \mathscr{S}$ is continuous. We must show that f fails to be onto. Since $f_{|A}$ is continuous and A is connected, we may assume $f(a) = 0$ for each $a \in A$. Suppose that for some $b \in B, f(b) = 1$. Then $f^{-1}(1)$ is a B-open set that contains b but misses A. This, however, is impossible, since b is an accumulation point of A.

Connectedness is preserved by continuous functions.

(2.A.14) Theorem. If f is a continuous function from a connected space X onto a space Y, then Y is connected.

Proof. If Y is not connected, then by (2.A.12) there is a continuous map g from Y onto $\mathscr{S}$. Hence, gf is a continuous function from X onto $\mathscr{S}$, which contradicts the fact that X is connected.

We have two immediate corollaries, the second of which should be familiar from elementary calculus.

(2.A.15) Corollary. Connectedness is a topological invariant.

(2.A.16) Corollary (The Intermediate Value Theorem). Let $f : \mathscr{E}^1 \to \mathscr{E}^1$ be a continuous function. Suppose that $a,b \in f(\mathscr{E}^1)$ and that $a < b$. Then if z is any number such that $a < z < b$, there is at least one point $c \in \mathscr{E}^1$ such that $f(c) = z$.

Proof. Suppose that no such point c exists. Let $U = \{x \in \mathscr{E}^1 \mid x < z\}$ and $V = \{x \in \mathscr{E}^1 \mid x > z\}$. U and V form a separation of $f(\mathscr{E}^1)$, which contradicts (2.A.11) and (2.A.14).

One should observe that the domain of f in the foregoing corollary can be replaced by any connected space and the corresponding proposition will still hold.

B. SLIGHTLY DEEPER RESULTS CONCERNING CONNECTEDNESS

Corollary (2.A.16) of Theorem (2.A.14) proves to be a key tool in establishing certain basic theorems in calculus. However, rather than pursue these results, we give an illustration of how this corollary may be employed in a non-calculus setting.

An interesting problem that has kept many topologists and analysts occupied concerns the study of fixed point properties. A space X is said to have the *fixed point property* if and only if each continuous function mapping X into itself leaves some point alone, i.e., for each continuous map $f : X \to X$, there is an $x \in X$ such that $f(x) = x$. Theorems involving the fixed point property are not only aesthetically pleasing in themselves, but also have significant applications to other parts of mathematics. Our first fixed point result is a special case of the Brouwer fixed point theorem, which states that the space obtained by taking the product of $\mathbf{I} = [0,1]$ with itself a finite number of times has the fixed point property.

(2.B.1) Theorem. The interval $\mathbf{I}$ has the fixed point property.

Proof. Suppose that $f : \mathbf{I} \to \mathbf{I}$ is continuous. We will find a point $c \in \mathbf{I}$ such that $f(c) = c$. If $f(0) = 0$ or $f(1) = 1$, we are finished. Hence, let us assume that $0 < f(0)$ and $f(1) < 1$. Define a continuous function g from $\mathbf{I}$ into $\mathscr{E}^1$ by setting $g(x) = x - f(x)$. Note that $g(0) = -f(0) < 0$ and $g(1) = 1 - f(1) < 0$. From the intermediate value theorem it follows that there is a point $c \in \mathbf{I}$ such that $g(c) = 0$. However, this implies that $c - f(c) = 0$, and hence we have that $f(c) = c$.

Given spaces $X_1, X_2, \ldots, X_n$ all having a common topological property P, it is natural to inquire if the product space $\prod_{i=1}^{n} X_i$ also enjoys P. This proves to be the case if the property in question is connectedness.

(2.B.2) Theorem. If $X_1, X_2, \ldots, X_n$ are connected topological spaces, then $Y = \prod_{i=1}^{n} X_i$ is also connected.

Proof. We prove the theorem for $n = 2$, and a trivial inductive argument will yield the more general result. If Y is not connected, then by (2.A.12) there is a map f from Y onto $\mathscr{S}$. Let $a = (a_1,a_2)$ and $b = (b_1,b_2)$ be points in Y with $f(a) = 0$ and $f(b) = 1$. Define functions $f_1 : X_1 \to \mathscr{S}$ and $f_2 : X_2 \to \mathscr{S}$, by setting $f_1(x_1) = f(x_1,b_2)$ for each $x_1 \in X_1$, and $f_2(x_2) = f(a_1,x_2)$ for each $x_2 \in X_2$. It is easily seen that f_1 and f_2 are continuous, and since X_1 and X_2 are connected, we conclude that f_1 and f_2 must be constant maps. Since $f_1(b_1) = f(b_1,b_2) = 1$, it follows that f_1 must be identically 1. Since $f_2(a_2) = f(a_1,a_2) = 0$, it follows that f_2 is identically 0. On the other hand we have that $f_1(a_1) = f_2(b_2) = f(a_1,b_2)$, and thus we may conclude that Y is connected.

(2.B.3) Corollary. The space $\mathscr{E}^n$ is connected.

(2.B.4) Exercise. Prove the converse of (2.B.2).

(2.B.5) Theorem. If B is a subset of $\mathscr{E}^n$ $(n > 1)$ and B contains a countable number of points, then $\mathscr{E}^n \setminus B$ is connected.

Proof. Let $c \in \mathscr{E}^n \setminus B$ and select a line L_c in $\mathscr{E}^n \setminus B$ containing c. (Why does such a line exist?) For each point $x \in \mathscr{E}^n \setminus B$, there is a line L_x in $\mathscr{E}^n \setminus B$ that contains x and intersects L_c. Thus, $\mathscr{E}^n \setminus B$ may be considered as the union of L_c and the L_x's, and hence by (2.A.11), $\mathscr{E}^n \setminus B$ is connected.

We have stated before that a central problem in topology is that of classifying spaces up to homeomorphism. This involves finding a method for deciding whether any arbitrary pair of topological spaces are homeomorphic. In this generality the problem cannot be solved, as mathematical logicians have shown (Markov, [1960]). However, if one reduces the problem to that of trying to determine whether a particular pair of topological spaces are homeomorphic, positive results are sometimes obtainable. As indicated in Chapter 1, a common way of attacking this problem is to find a topological invariant that is shared by one but not by both of the two spaces in question. Theorem (2.A.14) shows that connectedness is a topological (actually even a continuous) invariant. The reader is asked to show in the next exercises that the fixed point property is another such topological invariant.

(2.B.6) Exercises.

1. Show that the fixed point property is a topological invariant, i.e., show that if $h : X \to Y$ is a homeomorphism and X has the fixed point property, then so does Y.
2. Show that [0,1] and $\mathscr{S}^1$ are not homeomorphic.

The topological invariant, connectedness, may be exploited to show that $\mathscr{E}^1$ is homeomorphically distinct from $\mathscr{E}^2$.

(2.B.7) Theorem. The space $\mathscr{E}^1$ is not homeomorphic to $\mathscr{E}^2$.

Proof. Suppose that a homeomorphism $h : \mathscr{E}^2 \to \mathscr{E}^1$ exists. Let $c \in \mathscr{E}^2$ and note that by (2.B.5), $\mathscr{E}^2 \setminus \{c\}$ is connected. However, $h(\mathscr{E}^2 \setminus \{c\}) = \mathscr{E}^1 \setminus \{h(c)\}$ and $\mathscr{E}^1 \setminus \{h(c)\}$ is not connected, which contradicts (2.A.14).

(2.B.8) Exercise. Show that (0,1) does not have the fixed point property, and hence (0,1) is not homeomorphic with [0,1].

(2.B.9) Exercise. Using the topological invariant, connectedness, give another proof that (0,1) and [0,1] are not homeomorphic.

We conclude this section with a rather strange result, which will be used frequently in succeeding chapters.

(2.B.10) Theorem. Suppose that X is connected and that A is a connected subset of X. Suppose further that $X \setminus A = U \cup V$. where U and V are nonempty disjoint open (in $X \setminus A$) subsets of $X \setminus A$. Then $A \cup U$ is connected.

Proof. If $A \cup U$ is not connected, then $A \cup U = C_1 \cup C_2$, where C_1 and C_2 are nonempty, disjoint open and closed subsets of $A \cup U$ Since A is connected, either $A \subset C_1$ or $A \subset C_2$. If $A \subset C_1$, then we have that $C_2 \subset U$. Therefore, C_2 is open and closed in U. This, however, is impossible, since U is open and closed in $X \setminus A$, and hence it follows from (1.D.4) that C_2 is both open and closed in $(X \setminus A) \cup (A \cup U) = X$, which contradicts the connectedness of X. (Obviously, (1.D.4) also holds for closed subsets.)

A similar argument holds if $A \subset C_2$.

C. PATH CONNECTEDNESS

In this section, we introduce a somewhat different concept of connectedness—one that will find repeated application in later chapters. A layman's notion of connectedness would probably center on the idea that a space X is connected if one can move from one point in X to another without ever leaving X. This concept is given mathematical import by the following two definitions.

(2.C.1) Definition. Suppose that X is a topological space. A *path* in X is a continuous function $f : \mathbf{I} \to X$. If $f(0) = a$ and $f(1) = b$, then f is said to be a *path from a to b*; a is the *initial point* of f, and b is the *end* or *terminal point*.

(2.C.2) Definition. A space X is *path connected* if and only if each pair of distinct points in X can be joined by a path, i.e., for each $x,y \in X$ there is a continuous map $f : \mathbf{I} \to X$ such that $f(0) = x$ and $f(1) = y$.

It should be obvious to the reader that $\mathscr{E}^n$ is path connected for every $n \in \mathbf{Z}^+$.

(2.C.3) Theorem. Suppose that a, b, and c are points in a space X and that there are paths from a to b and from b to c. Then there is a path from a to c.

Proof. Let $f : \mathbf{I} \to X$ and $g : \mathbf{I} \to X$ be paths such that $f(0) = a$, $f(1) = b$, $g(0) = b$, and $g(1) = c$. Define $h : \mathbf{I} \to X$ by setting

$$h(x) = \begin{cases} f(2t) & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ g(2t - 1) & \text{if} \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

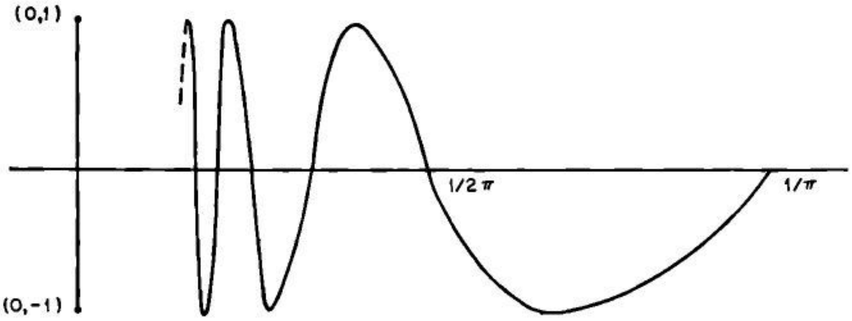
Then by the map gluing theorem (1.F.6), (1.F.3), and (1.F.1), h is continuous. Clearly, $h(0) = a$ and $h(1) = c$.

(2.C.4) Exercise. Suppose that $\{A_\alpha \mid \alpha \in \Lambda\}$ is a collection of path connected subsets of a space X such that $\bigcap \{A_\alpha \mid \alpha \in \Lambda\} \neq \varnothing$. Show that $\bigcup \{A_\alpha \mid \alpha \in \Lambda\}$ is path connected.

(2.C.5) Exercise. Show that every path connected space is connected.

Although path connected spaces are connected, there do exist connected spaces that are not path connected, as we see in the following example.

(2.C.6) Example. Let $Y = \{(x, \sin 1/x) \in \mathscr{E}^2 \mid 0 < x < 1/\pi\}$ and let $Z = \{(x,y) \in \mathscr{E}^2 \mid x = 0 \text{ and } -1 \le y \le 1\}$. The space $X = Y \cup Z$ with the relative topology inherited from $\mathscr{E}^2$ is frequently referred to as the *topologist's sine curve.*



It follows immediately from (2.A.13) that X is connected. However, X is not path connected. To see this, we show that no path can stretch from the point $(1/\pi,0)$ to the point $(0,0)$. Suppose to the contrary that there is a path $f : \mathbf{I} \to X$ with $f(0) = (1/\pi,0)$ and $f(1) = (0,0)$. Since $f(\mathbf{I})$ is connected, every point on the sine curve (for $0 < x < 1/\pi$) must be included in the range of f. Thus we may select a sequence of points in $\mathbf{I}$, $x_1 < x_2 < x_3 < \cdots$, such that the sequence $\{x_i\}$ converges to 1 and the second coordinate of $f(x_i)$ is 1 if i is odd and is -1 if i is even. This, however, is absurd, since as $\{x_i\}$ converges to 1, the sequence $\{f(x_i)\}$ attempts to simultaneously approach both $(0,-1)$ and $(0,1)$ which is impossible by (1.F.9) and (1.F.8).

(2.C.7) Definition. A topological space X is *locally path connected* if and only if each point $x \in X$ has a neighborhood base of path connected open sets, i.e., if $x \in U$, where U is open, then there is a path connected open subset V of X such that $x \in V \subset U$.

(2.C.8) Theorem. If X is a connected, locally path connected topological space, then X is path connected.

Proof. Let $x \in X$, and consider the set $A = \{z \in X \mid \text{there is a path from } x \text{ to } z\}$. We show that A is nonempty, open, and closed. Then, since X is connected, this will imply that $A = X$ (2.A.4). It is clear that A is nonempty, since $x \in A$. To show that A is open, suppose that $z \in A$. Since X is locally path connected, there is an open set V containing z such that every two points in V can be joined by a path lying in V. Hence, by (2.C.3), we have that $V \subset A$ and thus A must be open. To see that A is closed, we show that $X \setminus A$ is open. Suppose that $z \in X \setminus A$ and let V be a neighborhood of z that is path connected. If $V \cap A \neq \varnothing$, then we may join z to x by a path (2.C.3), which contradicts the fact that $z \in X \setminus A$. Hence, $V \cap A = \varnothing$, and $X \setminus A$ is open.

(2.C.9) Corollary. Connected open subsets of $\mathscr{E}^n$ are path connected.

The concept of a path is perhaps less intuitive than it originally appears. For example, we have noted earlier that there are continuous functions mapping the unit interval onto the unit square. By definition, such functions are paths, even though they are probably not quite what the reader had in mind when we first described paths as a means of traveling from one point to another in a topological space. A notion undoubtedly much closer to the reader's preconception of what a path should be is that of an arc. An *arc* is an embedding of $\mathbf{I}$ into a space X. We shall eventually prove that in T_2 spaces, an arc is merely a path which is 1–1 (2.G.11.). As was the case with paths, if $h : \mathbf{I} \to X$, is an arc, then $h(0)$ is referred to as the *initial point* of h and $h(1)$ is called the *terminal point*. Spaces in which any two distinct points may be connected by an arc are called *arc connected*, and *local arc connectedness* is defined in a manner analogous to local path connectedness.

(2.C.10) Exercise. Prove or disprove: If a space X is path connected, then X is always arc connected.

(2.C.11) Exercise. Prove (2.C.8) with paths replaced by arcs.

(2.C.12) Exercise. Prove that connected n-manifolds are arc connected.

D. COMPONENTS

Spaces that are not connected may be viewed as consisting of a (possibly infinite) number of connected pieces. This leads us to the following notion.

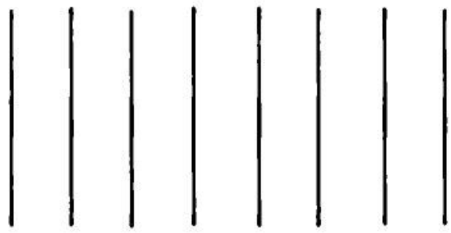
(2.D.1) Definition. Suppose that X is a topological space and that

A is a subspace of X. Then A is a *component* of X if and only if (i) A is connected and (ii) if B is a connected subspace of X containing A, then $B = A$.

Consequently, components are maximal connected subspaces.

Of course, if X is connected, then the only component of X is X itself. At the other extreme, we have that if X is a discrete space, then each point of X is a component.

(2.D.2) Example. Let X be the subspace of the plane $\mathscr{E}^2$ consisting of vertical line segments, as shown. Then each vertical line segment is a component of X.



(2.D.3) Exercise. Find the components of the rational numbers with the relative topology. Do the same for the irrational numbers.

The basic properties of components are easily established.

(2.D.4) Theorem.

(i) Components of a space X are closed.
(ii) Distinct components of a space X are disjoint.
(iii) Each point of a space X belongs to exactly one component of X.
(iv) If spaces X and Y are homeomorphic, then there is a 1–1, onto correspondence between the set of components of X and those of Y. Thus, the cardinality of the set of components is a topological invariant.

Proof. (i) This is immediate from (2.A.13). (ii) This is immediate from (2.A.10). (iii) Note that the component containing x is simply the union of all connected sets that contain x. (iv) This follows from the fact that images of connected sets under continuous maps are connected (2.A.14).

(2.D.5) Definition. A space X is *totally disconnected* if and only if each of its components consists of a point.

By (2.D.3) the set of rationals (or irrationals) with the relative topology is a totally disconnected space.

(2.D.6) Exercise. A connected space X is said to have an *explosion point* p (sometimes called a *dispersion point*) if and only if $X \setminus \{p\}$ is totally disconnected. Find such a space and show that a space can have at most one explosion point. [Hint: Use (2.B.10).]

(2.D.7) Definition. A subspace C of a topological space X is a *path component* of X if and only if C is path connected and is not contained in a larger path connected subspace.

(2.D.8) Exercise. Show that for open subsets of $\mathscr{E}^n$, path components and components coincide. Is the modifier "open" necessary?

E. LOCALLY CONNECTED SPACES

Consider the following example. In $\mathscr{E}^2$, for each $n \in \mathbf{Z}^+$, let $A_n = \{(x,1/n) \mid 0 \leq x \leq 1\}$ and let $A_0 = \{(x,0) \mid 0 \leq x \leq 1\}$. The components of $X = A_0 \cup (\bigcup_{n=1}^{\infty} A_n)$ are precisely the sets $A_0, A_1, \ldots$. Although, by (2.D.4), A_0 must be closed, A_0 is clearly not open in X, and thus we see that components need not be open. (Another example of this was given in (2.D.3).) For certain types of spaces, however, components will be both open and closed. These are known as locally connected spaces, and they include the n-manifolds. The formal definition of such spaces is couched in somewhat different terms (emphasizing the local nature of the concept), but the fact that the two notions are essentially equivalent is readily established in (2.E.2).

(2.E.1) Definition. A space X is *locally connected* if and only if for each point $x \in X$ and each open set U containing x, there is a connected open set V such that $x \in V \subset U$.

Note that the foregoing definition is equivalent to saying that there is a neighborhood base at each point consisting of connected open sets.

(2.E.2) Theorem. A space X is locally connected if and only if the components of every open subset of X are open. (In particular, the components of locally connected spaces are open.)

Proof. Suppose that X is locally connected and that C is a component of an open subset U. Let c be a point in C. By (2.E.1), there is a connected

X-open subset V of U that contains c. Then V must lie in C (C is a maximal connected subset), and therefore C is open.

Conversely, let $x \in X$ and let U be any open set containing x. Then the component V of U that contains x is open and $x \in V \subset U$. Hence, X is locally connected.

(2.E.3) ***Theorem.*** The (finite) product of locally connected spaces is locally connected.

Proof. Suppose that $X_1, X_2, \ldots, X_n$ are locally connected and let $X = \prod_{i=1}^{n} X_i$. Suppose that $x = (x_1, x_2, \ldots, x_n) \in U$. Let $U_1 \times U_2 \times \quad \times U_n$ be a basic open set in X that contains x and is contained in U. Since each X_i is locally connected, there are connected open sets V_i such that $x_i \in V_i \subset U_i$. Consequently, by (2.B.2), $V = V_1 \times V_2 \times \quad \times V_n$ is a connected open set containing x and contained in U. Therefore, X is locally connected.

(2.E.4) ***Corollary 1.*** The space $\mathscr{E}^n$ is locally connected.

(2.E.5) ***Corollary 2.*** All n-manifolds are locally connected.

(2.E.6) ***Exercise.*** Show that local connectedness is a topological invariant, but that local connectedness is not necessarily preserved by continuous functions.

(2.E.7) ***Theorem.*** If X is locally connected and $f : X \to Y$ is continuous, onto, and closed, then Y is locally connected.

Proof. Suppose that U is open in Y and C is a component of U. We show that C is open. For each $x \in f^{-1}(C)$, let C_x be the component of x in $f^{-1}(U)$. By (2.E.2), C_x is open, and since $f(x) \in C$, the connected set $f(C_x)$ also lies in C. Therefore, $f^{-1}(C) = \bigcup \{C_x \mid x \in f^{-1}(C)\}$ and consequently $f^{-1}(C)$ is open. Thus, $f(X \setminus f^{-1}(C)) = Y \setminus C$ is closed (f is closed and onto), and this, of course, implies that C is open.

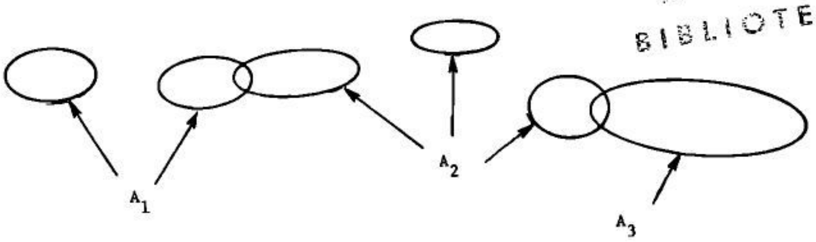
F. CHAINS

Paths were introduced as a means of mathematically traveling from one point to another in a topological space. A geometrically pleasing concept somewhat akin to the notion of path is that of a chain.

(2.F.1) ***Definition.*** A family $\{A_1, A_2, \ldots, A_n\}$ of subsets of a space

X is a *simple chain* in X in case $A_i \cap A_j \neq \varnothing$ if and only if $|i - j| \leq 1$. A simple chain $C = \{A_1, A_2, \ldots, A_n\}$ is said to *connect points a and b in X* if and only if $a \in A_1$ and $b \in A_n$.

It should be observed that the links of a simple chain need not be connected; thus the accompanying figure illustrates a perfectly acceptable simple chain.



Most frequently, simple chains are forged from open links; a common procedure for their construction consists in starting with an arbitrary entanglement of open sets and then extracting a suitable simple chain from the confusion. That this may be done is a result of the following elegant and useful theorem.

(2.F.2) Theorem. Suppose that X is a connected space and that a and b are any two points in X. Let $\mathscr{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ be a family of open sets whose union is X. Then there is a simple chain with links from $\mathscr{U}$ that connects a and b.

Proof. Let D be the set of all points x in X such that there is a simple chain (with links in $\mathscr{U}$) that runs from a to x. The set D is certainly nonempty, since a itself is found there. We show that D is both open and closed; since X is connected, this will imply, by (2.A.4), that D is all of X.

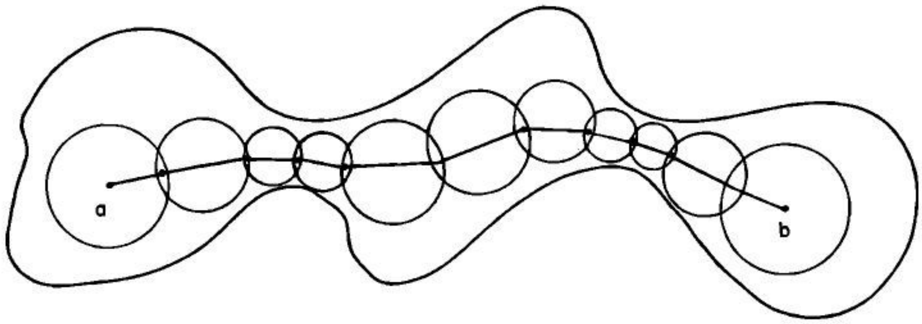
Suppose that $x \in D$ and that $\{U_1, U_2, \ldots, U_n\}$ is a simple chain with $a \in U_1$ and $x \in U_n$. Clearly, then $U_n \subset D$, and it follows that D is open.

To see that D is also closed, we show that D contains all of its accumulation points (recall (1.E.14)). Suppose that x is an accumulation point of D and that $U \in \mathscr{U}$ contains x. Since x is an accumulation point of D, U must intersect D, and hence if z is any point in the intersection, there is a simple chain $\{U_1, \ldots, U_n\}$ of elements in $\mathscr{U}$ connecting z to x. Let r be the first integer such that $U_r \cap U \neq \varnothing$. Then $\{U_1, U_2, \ldots, U_r, U\}$ is a simple chain between a and x, and consequently $x \in D$.

(2.F.3) Definition. A *polygonal arc* in $\mathscr{E}^n$ is an arc whose image consists of a finite number of straight line segments.

(2.F.4) Corollary. Any two points contained in a connected open subset U of $\mathscr{E}^n$ may be joined by a polygonal arc in U.

Proof. For each $x \in U$, choose ε_x small enough so that the open ball $S_{\varepsilon_x}(x)$ is contained in U. Then $\mathscr{V} = \{S_{\varepsilon_x}(x) \mid x \in U\}$ is an open cover of U, and consequently, members of $\mathscr{V}$ may be found to form a simple chain $\{V_1, V_2, \ldots, V_n\}$ running from a to b. For each i, let $z_i \in V_i \cap V_{i+1}$. Let L_1 be a line segment in V_1 that runs from a to z_1. For $i = 2, 3, \ldots, n-1$, let L_i be a line segment in V_i from z_{i-1} to z_i. Finally, let L_n be a line segment in V_n connecting z_{n-1} to b. These segments yield a polygonal arc extending from a to b.



G. COMPACTNESS

In calculus, a great deal of emphasis is placed on the fact that continuous functions defined on a closed interval, say [0,1], are bounded and attain their maximum and minimum on the interval, whereas functions such as $f(x) = 1/x$ defined on (0,1] fail to be bounded. It is natural to inquire what intrinsic quality [0,1] possesses that (0,1] lacks. In topological terms, the answer turns out to be compactness—one of the most important of all topological concepts.

(2.G.1) Definition. Suppose that X is a set and that B is a subset of X. A *cover* of B is a collection of subsets of X, $\mathscr{C} = \{C_\alpha \mid \alpha \in \Lambda\}$, whose union contains B. If X is a topological space and the sets C_α are open, then the cover is said to be *open*. If Λ is a finite set, then the collection $\mathscr{C}$ is a *finite cover* of B. A *subcover* of B is a subcollection of $\mathscr{C}$ that is also a cover for B.

The utilization of covers provides the topologist with one of his most successful tools. Covers play a prominent role in the study of such diverse areas as metrizability, dimension theory, and Čech homology. We begin by defining compactness in terms of open covers.

(2.G.2) Definition. A topological space X is *compact* if and only if every open cover of X has a finite subcover.

Any finite space is compact, no matter what its topology. More generally, any space with only a finite number of open sets is compact.

(2.G.3) Exercise. Show that every closed subspace of a compact space is compact.

(2.G.4) Example. Let $X = \mathbf{R}^1$ with the open ray topology, and let $A = [0,1]$. Then A is a compact subset of X, but A is not closed.

(2.G.5) Exercise. Suppose that $A \subset X$, A is compact, and X is T_2. Show that A is closed.

We give a nontrivial example of a compact space, by proving the previous assertion that closed and bounded intervals in $\mathscr{E}^1$ are compact. This result of course has numerous applications in elementary analysis.

(2.G.6) Theorem. Intervals that are closed and bounded in $\mathscr{E}^1$ are compact.

Proof. Let $\mathscr{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ be an open cover of $[a,b]$. Set $\mathscr{V} = \{V_\beta \mid \text{for some } \alpha \in \Lambda,\ V_\beta \text{ is a component of } U_\alpha\}$. Clearly $\mathscr{V}$ also covers $[a,b]$, and since $[a,b]$ is locally connected, each member of $\mathscr{V}$ is open. By (2.F.2) there is a simple chain $\{V_1, \ldots, V_n\}$ of sets from $\mathscr{V}$ that connects a and b. Since $V = \bigcup_{i=1}^{n} V_i$ is a connected subset of $\mathscr{E}^1$, it follows from (2.A.8) that V is an interval. Furthermore, since $a,b \in V$, we have that $[a,b] = V$. For each i, $1 \leq i \leq n$, select $\alpha_i \in \Lambda$ such that $V_i \subset U_{\alpha_i}$. Then $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ is a finite subcover of $\mathscr{U}$.

(2.G.7) Exercise. Show that compact subsets of $\mathscr{E}^1$ are bounded and closed. Note that $\mathscr{E}^1$ itself is not compact, since a cover consisting of sets of the form $(-n,n)$ for $n \in \mathbf{Z}^+$ has no finite subcover.

The next exercise is one of the most important in the entire book. The result will be generalized at a later time with the aid of the axiom of choice, but as it stands, it represents a very substantial if not formidable task, which should more than adequately test the reader's understanding of both compactness and the product topology.

(2.G.8) Exercise. Show that if X_1 and X_2 are compact spaces, then $X_1 \times X_2$ is compact.

Compactness, like connectedness, is preserved by continuous functions.

(2.G.9) Theorem. If X is a compact topological space and $f : X \to Y$ is continuous and onto, then Y is compact.

Proof. Suppose that $\mathscr{U}$ is an open cover of Y. Then $\{f^{-1}(U) \mid U \in \mathscr{U}\}$ is an open cover of X, and the compactness of X allows us to select a finite subcover $\{f^{-1}(U_i) \mid i = 1, 2, \ldots, n\}$ from $\{f^{-1}(U) \mid U \in \mathscr{U}\}$. Since f is onto, the collection $\{U_i \mid i = 1, 2, \ldots, n\}$ is a finite subcover of Y.

(2.G.10) Corollary. Compactness is a topological (and a continuous) invariant.

(2.G.11) Exercise. Let f be a continuous function from a compact space into a T_2 space. Show that f is closed. If, in addition, f is 1–1, show that f is an embedding.

We now establish a theorem that gives a complete (and very applicable) characterization of compact subsets of $\mathscr{E}^n$.

(2.G.12) Theorem. Suppose that $A \subset \mathscr{E}^n$. Then A is compact if and only if A is closed and bounded.

Proof. Suppose that A is closed and bounded. Let $\mathscr{E}_i^1$ $(= \mathscr{E}^1)$ denote the i-th coordinate space of $\mathscr{E}^n$ and let $p_i : \mathscr{E}^n \to \mathscr{E}_i^1$ be the natural projection. It is easily seen that $p_i(A)$ is bounded in $\mathscr{E}_i^1$ for each i, and hence, for each $i = 1, 2, \ldots, n$ there is a closed interval $[w_i,z_i] \subset \mathscr{E}_i^1$ such that $p_i(A) \subset [w_i,z_i]$. However, this implies that $A \subset [w_1,z_1] \times [w_2,z_2] \times \quad \times [w_n,z_n]$. Since the product of closed and bounded intervals is compact by (2.G.6) and (2.G.8), it follows from (2.G.3) that the closed set A is compact.

Suppose now that A is compact. By (2.G.9), $p_i(A)$ is compact for each i, and thus by (2.G.7), $p_i(A)$ must be bounded. Since this is true in each coordinate, then A is bounded also. That A is closed follows from (2.G.5).

As an immediate corollary we have another familiar theorem from calculus.

(2.G.13) Corollary. Suppose that A is compact and $f : A \to \mathbb{R}$ is continuous. Then f attains its maximum and minimum.

(2.G.14) Theorem. Suppose that $A \times B$ is a compact subset of a product space $X \times Y$ and W is an open subset of $X \times Y$ which contains $A \times B$. Then there are open sets U and V in X and Y, respectively, such that $A \times B \subset U \times V \subset W$.

Proof. Let $a \in A$ and for each $b \in B$, let $U_b \times V_b$ be an open set in $X \times Y$ such that $(a,b) \in U_b \times V_b \subset W$. Then $\{V_b \mid b \in B\}$ is an open cover of B, and hence there is a finite subcover $\{V_{b_1}, V_{b_2}, \ldots, V_{b_n}\}$. Let $U^a = \bigcap_{i=1}^{n} U_{b_i}$ and $V^a = \bigcup_{i=1}^{n} V_{b_i}$. Repeat this procedure for each $a \in A$ to obtain an open cover $\{U^a \mid a \in A\}$ of A and a corresponding family of open sets $\{V^a \mid a \in A\}$ each of which contains B. Once more, we may use compactness (this time of A) to obtain a finite subcover $\{U^{a_1}, U^{a_2}, \ldots, U^{a_m}\}$ of A. Set $U = \bigcup_{i=1}^{m} U^{a_i}$ and $V = \bigcap_{i=1}^{m} U^{a_i}$ to complete the proof.

H. TWO ALTERNATE CHARACTERIZATIONS OF COMPACTNESS

Compactness may be characterized in a number of ways, one of the most important of which is given in the next theorem.

(2.H.1) Definition. A collection of sets $\{C_\alpha \mid \alpha \in \Lambda\}$ has the *finite intersection property* if and only if for each nonempty finite subset $K \subset \Lambda$, $\bigcap \{C_\alpha \mid \alpha \in K\} \neq \varnothing$.

(2.H.2) Theorem. A space X is compact if and only if for each collection $\mathscr{C} = \{C_\alpha \mid \alpha \in \Lambda\}$ of closed subsets of X with the finite intersection property, $\bigcap \{C_\alpha \mid \alpha \in \Lambda\} \neq \varnothing$.

Proof. Suppose that X is compact and $\mathscr{C} = \{C_\alpha \mid \alpha \in \Lambda\}$ is a collection of closed subsets of X with the finite intersection property. For each $\alpha \in \Lambda$, let $U_\alpha = X \setminus C_\alpha$. If $\bigcap \{C_\alpha \mid \alpha \in \Lambda\} = \varnothing$, then it is immediate from one of the De Morgan rules that $\bigcup_{\alpha \in \Lambda} U_\alpha = X$, and hence $\{U_\alpha \mid \alpha \in \Lambda\}$ is an open cover of X. Since X is compact, there is a finite subcover, $U_1, \ldots, U_n$. But an application of the other De Morgan rule yields $\bigcap_{i=1}^{n} C_i = \varnothing$, a contradiction. Thus, the intersection of the members of $\mathscr{C}$ must be nonempty.

Now suppose that X is not compact. Then there is an open cover $\{U_\alpha \mid \alpha \in \Lambda\}$ of X that has no finite subcover. For each $\alpha \in \Lambda$ let $C_\alpha = X \setminus U_\alpha$. It is easy to see that the collection $\mathscr{C} = \{C_\alpha \mid \alpha \in \Lambda\}$ has the finite intersection property, but that the intersection of all the members of $\mathscr{C}$ is empty.

The proof of the next theorem illustrates how the previous result may be used.

(2.H.3) Theorem. Suppose that $\{A_i \mid i \in \mathbf{Z}^+\}$ is a countable family of compact subsets of a T_2 space X, such that for each i, $A_{i+1} \subset A_i$. If there is an open set U such that $\bigcap_{i=1}^{\infty} A_i = A \subset U$, then there is an integer N such that for $i > N$, $A_i \subset U$.

Proof. Suppose to the contrary that for each i, $C_i = A_i \setminus U$ is nonempty. Then $\{C_i \mid i \in \mathbf{Z}^+\}$ is a collection of closed sets (2.G.5) with the finite intersection property. This follows easily from the observation that if $n < m$, then $A_m \subset A_n$ and hence $A_m \setminus U \subset A_n \setminus U$. Since for each i, $C_i \subset A_1$ and A_1 is compact, we have from (2.H.2) that $\bigcap_{i=1}^{\infty} C_i \neq \varnothing$. But $\bigcap_{i=1}^{\infty} C_i = \bigcap_{i=1}^{\infty} (A_i \setminus U) = (\bigcap_{i=1}^{\infty} A_i) \setminus U = \varnothing$, a contradiction.

(2.H.4) Exercise. We give two generalizations of the previous theorem:

(i) Let Λ be a partially ordered set with the property that if $\alpha, \beta \in \Lambda$, then there is a $\lambda \in \Lambda$ such that $\lambda \geq \alpha$ and $\lambda \geq \beta$. Suppose that $\{A_\alpha \mid \alpha \in \Lambda\}$ is a collection of compact subsets of a T_2 space such that $A_\alpha \subset A_\beta$ if and only if $\alpha \geq \beta$. Show that if $\bigcap \{A_\alpha \mid \alpha \in \Lambda\} \subset U$, where U is open, then there is a $\partial \in \Lambda$ such that $A_\alpha \subset U$ for each $\alpha \geq \partial$.

(ii) Suppose that $\{A_\alpha \mid \alpha \in \Lambda\}$ is a family of closed compact sets such that $\bigcap \{A_\alpha \mid \alpha \in \Lambda\}$ is a subset of an open set U. Show that there is a finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \Lambda$ such that $\bigcap \{A_{\alpha_i} \mid i = 1, \ldots, n\} \subset U$.

Our second characterization of compactness, based on nests of closed subsets, is somewhat subtler than its predecessor, and some equivalent of the axiom of choice is apparently needed for its proof.

(2.H.5) Theorem. A space X is compact if and only if each nest of nonempty closed subsets in X has nonempty intersection.

Proof (*Catlin* [1968]). The proof of the "only if" part of the theorem is rendered trivial by (2.H.2).

Suppose then that in X each nest of closed nonempty subsets has nonempty intersection. We show that X satisfies the finite intersection property, criterion for compactness.

To this end, let $\mathscr{C}$ be a family of closed subsets of X with the finite intersection property and let $\mathscr{K} = \{K \subset X \mid K$ is closed and $\mathscr{C} \cap \{K\}$ has the finite intersection property$\}$. The Kuratowski lemma (0.D.4) may be employed to extract a maximal nest $\mathscr{N}$ (with respect to inclusion) in $\mathscr{K}$. Let $D = \bigcap \{N \mid N \in \mathscr{N}\}$. By hypothesis, D is not empty; thus, to complete the proof it suffices to show that $D \subset \bigcap \{C \mid C \in \mathscr{C}\}$.

If D is not a subset of $\bigcap \{C \mid C \in \mathscr{C}\}$, then there is a $C \in \mathscr{C}$ with $C \cap D \subsetneq D$. We obtain a contradiction by showing that $C \cap D \in \mathscr{K}$, which of course negates the maximality of $\mathscr{N}$. Note that $C \cap D \neq \varnothing$, since $C \cap D = \bigcap \{C \cap N \mid N \in \mathscr{N}\}$ and $\{C \cap N \mid N \in \mathscr{N}\}$ is a nest of closed sets ($C \cap N \neq \varnothing$, since $\mathscr{C} \bigcup \{N\}$ has the finite intersection property). To see that $C \cap D \in \mathscr{K}$, observe that whenever $C_1, C_2, \ldots, C_n \in \mathscr{C}$, then $C_1 \cap C_2 \cap \cdots \cap C_n \cap C \cap D = \bigcap \{(C_1 \cap C_2 \cap \cdots \cap C_n \cap C \cap N) \mid N \in \mathscr{N}\}$, and the latter set is nonempty, since it is an intersection of a nest of closed nonempty subsets of X. Thus, $\mathscr{C} \cup \{C \cap D\}$ has the finite intersection property, and consequently $C \cap D \in \mathscr{K}$.

I. LOCAL COMPACTNESS, COUNTABLE COMPACTNESS, AND SEQUENTIAL COMPACTNESS

Although the varieties of compactness are legion, we shall concentrate for the present on just three such mutations.

(2.I.1) Definition. A topological space X is *locally compact* if and only if for each $x \in X$ and for each open set U containing x, there is an open set V such that $x \in V \subset \overline{V} \subset U$ and $\overline{V}$ is compact.

(2.I.2) Examples.

1. For each $n \in \mathbf{Z}^+$, $\mathscr{E}^n$ is locally compact.
2. The set $\mathbf{R}^1$ with the open ray topology is not locally compact.
3. For each $i \in \mathbf{Z}^+$, let $A_i = \{(x,1/i) \mid 0 \leq x \leq 1\}$ and let $A_0 = \{(x,0) \mid 0 \leq x \leq 1\}$. Then $\bigcup_{i=0}^{\infty} A_i$ is locally compact.

Note that in example 3, if $A_0 = \{(0,0), (1,0)\}$, then $\bigcup_{i=0}^{\infty} A_i$ is not locally compact.

Suppose that $X = \mathbf{R}^1$ with the finite complement topology. Then X is compact but not locally compact (the closure of any open set is X). In order to obviate this problem, local compactness is sometimes defined by merely requiring that every point have a compact neighborhood. In a T_2 space, the two concepts coincide, as the reader is asked to establish in the next exercise.

(2.I.3) ***Exercise.*** Show that a T_2 space X is locally compact if and only if each point in X has a compact neighborhood.

(2.I.4) ***Exercise.*** Suppose that $X_1, X_2, \ldots, X_n$ are locally compact spaces. Show that $\prod_{i=1}^{n} X_i$ is locally compact.

(2.I.5) ***Exercise.*** Show that local compactness is a topological invariant. Is it a continuous invariant?

(2.I.6) ***Definition.*** Suppose that $f : \mathbf{Z}^+ \to X$ is a sequence. A sequence $g : \mathbf{Z}^+ \to X$ is a *subsequence* of f if and only if there is a strictly increasing function $h : \mathbf{Z}^+ \to \mathbf{Z}^+$ such that $fh = g$.

(2.I.7) ***Definition.*** A topological space X is *sequentially compact* if and only if every (infinite) sequence in X has a convergent subsequence.

In the next chapter, it is shown that the notions of compactness and sequential compactness coincide in metric spaces. However, in general there is no precise relationship between these two concepts. In Chapter 4, an example is given of a compact space that is not sequentially compact. The set $[0,\Omega)$ with the order topology is sequentially compact (by (0.E.4), all sequences in $[0,\Omega)$ have a least upper bound in $[0,\Omega)$, but $[0,\Omega)$ is not compact (why?)).

Somewhat akin to the idea of sequential compactness is the notion of countable compactness.

(2.I.8) ***Definition.*** A topological space X is *countably compact* if and only if every countable open cover of X has a finite subcover.

Compactness clearly implies countable compactness; however, it follows from the next theorem and the foregoing remarks that $[0,\Omega)$ is countably compact, but not compact.

(2.I.9) ***Theorem.*** If X is sequentially compact, then X is countably compact.

Proof. Suppose that $\mathscr{U} = \{U_1, U_2, \ldots\}$ is a countable open cover of X. If no finite subcover of $\mathscr{U}$ exists, then for each n, there is a point $x_n \in X \setminus$

$\bigcup_{i=1}^{n} U_i$. The sequence $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ that converges to some point, $x^* \in X$. Since $\mathscr{U}$ is a cover of X, there is a $U_k \in \mathscr{U}$ such that $x^* \in U_k$. However, for i sufficiently large (and greater than k), we have that $x_{n_i} \in U_k$, which is of course impossible.

We now introduce a property of topological spaces that might be viewed as a very weak form of compactness.

(2.I.10) Definition. A topological space X is *Lindelöf* if and only if every open cover of X has a countable subcover.

A principal theorem involving Lindelöf spaces is the following.

(2.I.11) Theorem (Lindelöf). If X is a second countable space, then X is Lindelöf.

Proof. Let $\mathscr{U}$ be an open cover of X and let $\mathscr{B}$ be a countable base for X. Then for each $x \in X$, there is a set $U_x \in \mathscr{U}$ and $B_x \in \mathscr{B}$ such that $x \in B_x \subset U_x$. Since $\{B_x \mid x \in X\}$ is actually a countable family and covers X (many B_x's may be duplicated), there is a corresponding countable collection of the U_x's that covers X.

Note that it follows from (1.G.10) that separable metric spaces are Lindelöf.

(2.I.12) Exercise. Show that a metric space X is Lindelöf if and only if X is second countable. [Hint: For each positive integer n, cover X with sets of the form $S_{1/n}(x)$.]

(2.I.13) Exercise. Show that the concepts of Lindelöf, second countability, and separability coincide in a metric space.

J. ONE POINT COMPACTIFICATIONS

As we proceed, the virtues of compactness will become increasingly apparent. Compactness is such a powerful property that one frequently finds it useful to embed noncompact spaces as dense subsets of compact spaces.

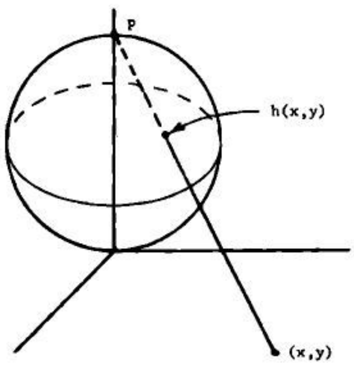
(2.J.1) Definition. A *compactification* of a space X is a pair (Y,h) where Y is a compact topological space and $h : X \to Y$ is an embedding whose image is a dense subset of Y.

Compactifications were originally inspired by problems in analysis where it was sometimes found desirable to add appropriate "boundary points" to a region. Actually, there are many ways of doing this; for instance, we will see in Chapter 17 (17.E.1) that every compact connected n-manifold may be viewed as a compactification of $\{x \in \mathscr{E}^n \mid |x| < 1\}$.

In this section, we are concerned with what is probably the simplest and most common means of compactifying a space, the one point (or Aleksandrov) compactification. The following situation can be considered as the motivation for the definition that eventually follows.

In $\mathscr{E}^3$ consider the sphere $S = \{(x,y,z) \mid x^2 + y^2 + (z - 1/2)^2 = (1/2)^2\}$ which rests on top of the xy plane. Denote the north pole of S by p, i.e., $p = (0,0,1)$. The following homeomorphism "wraps" the plane around the sphere by embedding $\mathscr{E}^2$ as $S \setminus \{p\}$.

$$h(x,y) = \left(\frac{x}{1 + x^2 + y^2}, \frac{y}{1 + x^2 + y^2}, \frac{x^2 + y^2}{1 + x^2 + y^2}\right)$$



Clearly, (S,h) is a compactification of the plane.

It is instructive to consider the relation between open sets in S and the corresponding open sets in $\mathscr{E}^2$ resulting from the homeomorphism h. If U is open in $\mathscr{E}^2$, then $h(U)$ is open in S, and conversely, open sets of $S \setminus \{p\}$ have homeomorphic images in $\mathscr{E}^2$. Consider, however, an open set V of S that contains p; its complement is compact, and hence $h^{-1}(S \setminus V)$ is compact in $\mathscr{E}^2$. Thus, V may be described as being the complement of a compact set in $\mathscr{E}^2$ together with the point p.

(2.J.2) Definition. Suppose that X is a topological space, and let ∞ be a point not in X. Define a topology for $X^* = X \cup \{\infty\}$ as follows: a set $U \subset X^*$ is open if and only if either

(i) $\infty \notin U$ and U is open in X, or
(ii) $\infty \in U$ and $X^* \setminus U$ is a closed compact subset of X.

Then X^* with this topology is the *one point compactification (Aleksandrov compactification)* of X.

(2.J.3) Exercise. Show that X^* has the following properties:

(i) X^* is a topological space;
(ii) X^* is compact;
(iii) X is locally compact and T_2 if and only if X^* is T_2;
(iv) X is dense in X^* if and only if X is not compact.

A considerably more sophisticated compactification, the Stone-Čech compactification, is considered in problem A-B.10 of Chapter 7.

PROBLEMS

Sections A and B

1. (Riesz-Lennes-Hausdorff Separation Criteria). Suppose that A is a subspace of a topological space X. Show that A is disconnected if and only if there are subsets of C and D of X satisfying

   (i) $A \subset C \cup D$,
   (ii) $A \cap C \neq \varnothing$, $A \cap D \neq \varnothing$, and
   (iii) $(\bar{C} \cap D) \cap (C \cap \bar{D}) = \varnothing$.

2. Suppose that $\{U_n \mid n \in \mathbf{Z}^+\}$ is a countable collection of connected subspaces of a space X such that $U_n \cap U_{n+1} \neq \varnothing$ for each n. Show that $\bigcup \{U_n \mid n \in \mathbf{Z}^+\}$ is connected.
3. Show that an infinite set with the finite complement topology is connected.
4. Prove or disprove: if $A_1, A_2, \ldots$ is a collection of closed connected subsets of the plane such that $A_{n+1} \subset A_n$ for $n = 1, 2, \ldots$, then $\bigcap_{n=1}^{\infty} A_n$ is connected.
5. Suppose that E is a proper subset of a space X, and suppose that A is a connected subset of X such that $A \cap E \neq \varnothing$ and $A \cap (X \setminus E) \neq \varnothing$. Show that $A \cap \operatorname{Fr} E \neq \varnothing$ and $A \cap \operatorname{Fr}(X \setminus E) \neq \varnothing$.
6. Suppose that $S \subset \mathscr{E}^1$ has the fixed point property. Show that S is a point or a closed and bounded interval.
7.* The intermediate value theorem has something of a converse. Suppose

that $f : [a,b] \to \mathscr{E}^1$. Show that f is continuous if and only if

(i) whenever $x_1, x_2 \in [a,b]$ and $f(x_1) \leq c \leq f(x_2)$, there is an $x \in [x_1,x_2]$ such that $f(x) = c$ and
(ii) $f^{-1}(y)$ is closed for each $y \in \mathscr{E}^1$.

8. Find an example where the intermediate value theorem holds, but the function in question is not continuous.
9.* Show that every connected open subset of $\mathscr{E}^2$ can be written as a disjoint union of open line segments.
10. Suppose that X is a T_1 space. Show that every connected subset of X containing more than one point is infinite.
11. Suppose that $A \subset \mathscr{E}^1$ is an interval and $f : A \to \mathscr{E}^1$ is continuous and 1–1. Show that f is either strictly increasing or strictly decreasing.
12. Suppose that X is a space such that each pair of points in X lie in a connected subset of X. Show that X is connected.
13. Suppose that $(X,\mathscr{U}_1)$ and $(X,\mathscr{U}_2)$ are connected. Is $(X,\mathscr{U}_1 \cap \mathscr{U}_2)$ necessarily connected?
14. Show that $\mathscr{E}^1$ with the open ray topology is connected.
15.* Suppose that $h : \mathscr{E}^1 \to \mathscr{E}^1$ is a homeomorphism and has no fixed point. Show that h^n (the composition of h with itself n times) is also fixed point free. Find a homeomorphism $h : \mathscr{E}^2 \to \mathscr{E}^2$ that is fixed point free but for which h^2 has a fixed point.
16. A metric space is said to be *well chained* if and only if for each $\varepsilon > 0$ and each pair of points x,y in X there is a finite subset $\{x_1, x_2, \ldots, x_n\}$ in X such that $x_1 = x$, $x_n = y$, and $d(x_i,x_{i+1}) < \varepsilon$ for $i = 1, 2, \ldots, n - 1$. Show that if X is connected, then X is well chained but that the converse is false.
17. Suppose that $f : \mathscr{E}^1 \to \mathscr{E}^1$ is a homeomorphism and $f^n = id$. Show that if f is order preserving, then $f = id$, and if f is order reversing, then $f^2 = id$.
18. Find a continuous function $f : \mathbf{I} \to \mathbf{I}$ with precisely two fixed points, 0 and 1.
19. Show that if a continuous function f maps a half-open interval onto itself, then f has a fixed point.
20. Show that each open subset of $\mathscr{E}^n$ can be written as a countable union of disjoint connected open sets.

Section C

1. Show that path connectivity is a topological invariant (in fact, a continuous invariant).

2. Suppose that $X_1, X_2, \ldots, X_n$ are topological spaces. Show that $\prod_{i=1}^{n} X_i$ is path connected if and only if each X_i is path connected.
3. Prove or disprove: if A is a path connected subset of X, then $\bar{A}$ is path connected.
4. Prove or disprove: if A and B are path connected subsets of a space X and $A \cap \bar{B} \neq \varnothing$, then $A \cup B$ is path connected.
5. A topological space is *contractible* if and only if there is a continuous map $H : X \times I \to X$ and a point $x^* \in X$ such that $H(x,0) = x$ and $H(x,1) = x^*$ for each $x \in X$. Prove that every contractible space is path connected.

Section D

1. Define an equivalence relation on a space X by setting $x \sim y$ if and only if for any separation (U,V) of X, x and y are both in U or both in V.
   (a) Show that $\sim$ is an equivalence relation. The equivalence classes are called *quasicomponents*.
   (b) Show that quasicomponents are closed.
   (c) Show that each component of X is contained in some quasicomponent.
   (d) Describe the quasicomponents of the following subset of $\mathscr{E}^2$: $X = (\bigcup_{n=1}^{\infty} A_n) \cup \{(0,0), (1,0)\}$, where $A_n = \{(x,y) \mid 0 \leq x \leq 1,\ y = 1/n\}$.
2. Show that a space X has an infinite number of components if and only if there is a countably infinite family of nonempty disjoint sets in X that are both open and closed.
3. Find an example in the plane where components and path components do not coincide.
4. Show that every component of a space X is a union of path components.
5. Suppose that a topological space X has the property that for every open set U and for each point $p \in U$, there is an open set V, $p \in V \subset U$ such that if $y \in V$, then y may be connected to x by a path lying in U. Show that for every open set W containing p, there is a path connected neighborhood Z with $p \in Z \subset W$.
6.* It is possible that spaces X and Y may fail to be homeomorphic even though there are continuous bijective maps $f : X \to Y$ and $g : Y \to X$. For example, let $X = (\bigcup_{n=0}^{\infty} [3n,\ 3n+1)) \cup (\bigcup_{n=0}^{\infty} \{3n+2\})$ and $Y =$

$(\bigcup_{n=0}^{\infty}(3n, 3n+1)) \cup (\bigcup_{n=0}^{\infty}\{3n+2\})$. Find continuous bijective maps $f: X \to Y$ and g $Y \to X$, and show that X and Y are not homeomorphic.

7.* Let $X = \{x \in \mathbf{R}^1 \mid x \text{ is a dyadic rational}\}$, let $Y = \{y \in \mathbf{R}^1 \mid y \text{ is rational and } y \notin X\}$, and let $Z = \{z \in \mathbf{R}^1 \mid z \text{ is irrational}\}$. Then $\mathbf{R}^1 = X \cup Y \cup Z$. Define a topology for $\mathbf{R}^1$ whereby

(i) X and Y are open,
(ii) if $U \subset X$ or $U \subset Y$, then U is open if it is open in the usual relative topology for X or Y, and
(iii) a neighborhood basis for a point $z \in Z$ is of the form $\{z\} \cup \{w \in X \cup Y \mid |w - z| < r\}$ $(r > 0)$.

Show that X and Y are totally disconnected and that Z is discrete. Let $A = \mathbf{R}^1 \setminus X$ and $B = \mathbf{R}^1 \setminus Y$. Show that A and B are totally disconnected but that their union is not.

8. A function $f: X \to Y$ is *connected* if and only if the image under f of every connected set is connected. Prove or disprove: if $f: \mathscr{E}^1 \to \mathscr{E}^1$ is connected, then f is continuous.
9. Suppose that $f: X \to Y$ is connected (see problem 8) and $f^{-1}(y)$ is connected for each $y \in Y$. Show that if X and Y are T_2, then for each $y \in Y$, $f^{-1}(y)$ is closed.
10. Show that if $f: \mathscr{E}^1 \to \mathscr{E}^1$ is connected (see problem 8), and $f^{-1}(y)$ is connected for each $y \in \mathscr{E}^1$, then f is continuous.
11. Show that a path component of an n-manifold is an n-manifold.
12.* Suppose that a topological space X has an explosion point p, and that $f: X \to X$ is continuous. Show that if $f^{-1}(p)$ is finite (or empty), then f has a fixed point. [Hint: if $\{x_1, x_2, \ldots, x_n\} \subset X \setminus \{p\}$, then $X \setminus \{x_1, x_2, \ldots, x_n\}$ is connected.]
13. A function $f: X \to Y$ is *super continuous* if and only if $f^{-1}(B)$ is open for each $B \subset Y$. Show that if f is super continuous, then f is constant on each quasi-component of X.

Section E

1. Show that open subsets of locally connected spaces are locally connected. Is the same true for arbitrary subsets of locally connected spaces?
2. Suppose that $X = A \cup B$, where A and B are locally connected. Is X locally connected? If A and B are also closed, is X locally connected? If, in addition, $A \cap B$ is locally connected, is X locally connected?
3. Show that if x and y are points belonging to different components of a

locally connected topological space X, then there is a separation (U,V) of X such that $a \in U$ and $b \in V$.

4. Does the conclusion of problem 3 hold if the hypothesis "locally connected" is dropped?
5. Show that in a locally connected T_2 space, components and quasi-components coincide.
6. Prove or disprove: If a space X is locally path connected, then X is locally connected.
7.* Suppose that X is connected and locally connected and $f: X \to \mathbf{I}$ is is continuous and bijective. Show that X and $\mathbf{I}$ are homeomorphic.

Section F

1. Show that a space X is connected if and only if for each $x,y \in X$ and for each open cover $\mathscr{U}$ of X, $\mathscr{U}$ contains a simple chain from x to y.
2. Show that every two points in a connected n-manifold can be joined by a simple chain whose links are open n-cells.
3. Suppose that X is a locally connected T_1 space and p, x, and y are points in X. Show that x and y are in distinct components of $X \setminus \{p\}$ if and only if every simple chain of connected open sets from x to y has a link containing p.
4. A metric space K is *chainable* if and only if for each $\varepsilon > 0$, there is a simple chain of open sets covering K such that each link has diameter less than ε. Are the following spaces chainable?
   (a) $\mathbf{I}$
   (b) $\mathbf{I} \times \mathbf{I}$
   (c) The unit circle
   (d) The topologist's sine curve

Sections G and H

1. Show that if X is an infinite set with the finite complement topology $\mathscr{U}$, then $(X,\mathscr{U})$ is compact.
2. Suppose that $\mathscr{B}$ is a basis for a topological space $(X,\mathscr{U})$. Show that X is compact if and only if every cover of X by basic open sets has a finite subcover.
3.* Find examples to show that the intersection of two compact subsets of a topological space may fail to be compact and that the closure of a compact subset may fail to be compact.

4.* Find a space in which points are closed, but where the intersection of compact subsets is not necessarily compact.

5.* Show that in a compact space X, each compact subset is closed if and only if each continuous bijection from a compact space onto X is a homeomorphism.

6. Show that $[0,1] \subset \mathscr{E}^1$ may be covered by a family of closed intervals that has no finite subcover.

7. Suppose that C is a compact subset of a T_2 space X and that $x \in X \setminus C$. Show that there are disjoint open subsets of U and V with $x \in U$ and $C \subset V$.

8. Suppose that C and D are disjoint compact subsets of a T_2 space X. Show that there are disjoint open subsets U and V of X such that $C \subset U$ and $D \subset V$.

9.* Show that if X is T_2 and $\mathscr{C}$ is a family of compact subsets of X such that every finite intersection of members of $\mathscr{C}$ is connected, then $\bigcap \{C \mid C \in \mathscr{C}\}$ is connected.

10. Show that any first countable space is T_2 if and only if every compact subset is closed.

11.* Show that X is compact if and only if every open cover of X has an irreducible subcover (a subcover that fails to be a cover if any element is removed).

12. Suppose that $A \subset X$ and let Y be a compact space. Show that if U is a neighborhood of $A \times Y$ in $X \times Y$ then there is a neighborhood $V \subset A$ such that $V \times Y \subset U$.

13.* Show that if $f : \mathbf{I} \to X$ is continuous, onto, and open, while X is T_2 and has at least 2 points, then $\mathbf{I}$ and X are homeomorphic. [Hint: Find the smallest closed interval $[0,a]$ such that $f([0,a]) = X$.]

14. Suppose that X is T_2 and that Y is compact and T_2. Show that $f : X \to Y$ is continuous if and only if $\{(x,f(x)) \mid x \in X\}$ is closed in $X \times Y$.

15. Let $(X,\mathscr{U})$ be a compact T_2 space. Show that
(a) If $\mathscr{U} \subsetneq \mathscr{U}'$, then $(X,\mathscr{U}')$ is not compact.
(b) If $\mathscr{U}'' \subsetneq \mathscr{U}$, then $(X,\mathscr{U}'')$ is not T_2.

16. Suppose that X is a compact space and that $\mathscr{C}$ is a family of continuous functions from X into $\mathbf{I}$

(i) if $f,g \in \mathscr{C}$, then $f - g \in \mathscr{C}$, and
(ii) for each $x \in X$ there is a neighborhood U_x of x and an $f \in \mathscr{C}$ such that $f(U_x) = 0$.

Show that $f(x) = 0$ for each $f \in \mathscr{C}$ and each $x \in X$.

17. Let $X = \mathbf{Z}^+$ and let $\mathscr{U}$ be a topology for $\mathbf{Z}^+$ that consists of $\varnothing$, X, and sets of the form $\{1, 2, \ldots, n\}$. Show that $(X,\mathscr{U})$ contains no nonempty compact closed subset.

18. Let $X = X_1 \times X_2 \times \quad \times X_n$ and suppose that a subspace A of X is

compact if and only if A is closed. Show that each X_i has the same property.

19. Suppose that X is a compact metric space. Show that X is connected if and only if X is well chained (see problem A-B.16).

20. Show that the following space X is an example of a noncompact subspace of $\mathscr{E}^2$ that has the fixed point property. For each $n \in \mathbf{Z}^+$, let $A_n = \{(x,y) \in \mathscr{E}^2 \mid x = 1/n,\ 0 \leq y \leq 1\}$. Let $B = \{(x,y) \in \mathscr{E}^2 \mid 0 \leq x \leq 1, y = 0\}$. Then $X = (\bigcup_{n=1}^{\infty} A_n) \cup B$.

21. Suppose that Y is a compact topological space and that $f: Y \to Y$ is a continuous map. Show that there is a nonempty closed set $A \subset Y$ such that $f(A) = A$.

22. Show that a compact locally connected space has only a finite number of components.

Section I

1. Suppose that X is a connected, locally connected, locally compact T_2 space. Show that if $x,y \in X$, then there is a compact connected set containing both x and y.

2.* Suppose that X is a locally compact space, Y is a T_2 space, and $f: X \to Y$ is continuous, open, and onto. Show that if C is a compact subset of Y, then there is a compact subset D of X such that $f(D) = C$.

3. Let $X = [-1,1]$ and define a topology for X by letting sets of the form $[-1,b)$, (a,b), and $(a,1]$ be open, where $a < 0 < b$ (include of course X and $\varnothing$). Show that X is compact, but that no open set (a,b) is locally compact.

4.* Suppose that X is a T_2 space and Y is a dense locally compact subspace. Show that Y is open.

5. Suppose that $A \subset X$ and define a topology for X by declaring $\varnothing$ and all sets that contain A to be open. Is this space locally compact?

6. Let C be a compact subspace of a locally compact metric space (X,d). Show that there is an $\varepsilon > 0$ such that $\{y \in X \mid d(y,C) \leq \varepsilon\}$ is compact.

7. Show that every countably compact metric space is separable.

8.* Show that if every closed ball in a metric space is compact, then the space is locally compact and separable.

9.* Let $\mathscr{C} = \{\{x_n\} \mid \{x_n\}$ is a convergent sequence in $\mathscr{E}^1\}$. Define $d(\{x_n\},\{y_n\}) = \sup\{|x_n - y_n|\}$. Show that d is a metric and that $(\mathscr{C},d)$ is separable but not locally compact.

10. A space X is *pseudocompact* if and only if each continuous function $f: X \to \mathscr{E}^1$ is bounded.

(i) Show that pseudocompactness is a continuous invariant.

(ii) Show that compact implies pseudocompact, but that the converse does not hold.

11. Let $X = \mathbf{R}^1$ and let $\mathscr{U}$ be a topology for X that consists of $\varnothing$, X, and sets of the form $(-n,n)$ where $n \in \mathbf{Z}^+$. Show that each $x \in X$ has a neighborhood base consisting of compact sets, but that not all points have a closed compact neighborhood.
12.* Show that the product of a compact space and a countably compact space is countably compact.
13.* Show that a T_1 space is countably compact if and only if every infinite subset has an accumulation point.
14. Show that the continuous image of a countably compact space is countably compact.
15. Show that a second countable T_1 space is compact if and only if it is sequentially compact.
16. Suppose that X is a countably compact T_1 space and that $U_1, U_2, \ldots$ are open subsets of X such that $\bigcap_{i=1}^{\infty} U_i = \{x\}$. Show that the sets U_i need not form a neighborhood basis at x.
17. Let $(\mathbf{Z}^+,\mathscr{U})$ have for a basis $\{\{2n-1, 2n\} \mid n = 1, 2, \ldots\}$ and let $(\mathbf{Z}^+,\mathscr{V})$ have for a basis $\{\{1,n\} \mid n = 1, 2, \ldots\}$. Show that $(\mathbf{Z}^+,\mathscr{U})$ is locally compact and $(\mathbf{Z}^+,\mathscr{V})$ is not. Define $f : (\mathbf{Z}^+,\mathscr{U}) \to (\mathbf{Z}^+,\mathscr{V})$ by $f(2n-1) = 1$ and $f(2n) = n$. Show that f is open but not continuous, and consequently local compactness is not preserved under open maps.
18. Show that a T_1 space X is countably compact if and only if every infinite open cover of X has a proper subcover.

Section J

1. Let X be the rationals with the relative topology and let X^* denote the 1-point compactification of X. Show directly that X^* is not T_2.
2. What is the 1-point compactification of $(0,1]$?
3. Show that the 1-point compactification of $\mathbf{Z}^+ \cup \{0\}$ is homeomorphic to $\{0\} \cup \{1/n \mid n \in \mathbf{Z}^+\}$.
4. Suppose that X is a locally compact T_2 space and that $f : X \to \mathscr{E}^1$ is continuous. Show that there is a continuous function $\hat{f} : X^* \to \mathscr{E}^1$ such that $\hat{f}(x) = f(x)$ for each $x \in X$ if and only if for each $\varepsilon > 0$, there is a compact subset K_ε of X such that $|f(x) - f(y)| < \varepsilon$ whenever $x,y \notin K_\varepsilon$.
5.* Suppose that X is a T_2 space and X^* is first countable. Is X necessarily locally compact?
6. Let $X = (0,1) \cup (1,2) \cup (2,3) \cup (9,10)$ be given the relative topology (with respect to $\mathscr{E}^1$). Describe the 1-point compactification of X.