Tarea 2 Probabilidad 2

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1. (X, Y) es un vector aleatorio con distribución hipergeométrica bivariada. Si su función de probabilidad está
   dada por: $f(x,y) = \dfrac{\binom{N_1}{x}\binom{N_2}{y}\binom{N-N_1-N_2}{n-x-y}}{\binom{N}{n}}$; donde $\begin{array}{lll} x = 0,1,2,...,n; & y = 0,1,2,...,n; & x+y \le n \\ N1 > 0; & N2 > 0; & N1+N2 \le N \\ 0 < n < N \end{array}$

   (a) Calcula $f_X(x)$ y $f_Y(y)$

   $f_X(x) = \displaystyle\sum_{y=0}^{n-x} \dfrac{\binom{N_1}{x}\binom{N_2}{y}\binom{N-N_1-N_2}{n-x-y}}{\binom{N}{n}} = \dfrac{\binom{N_1}{x}}{\binom{N}{n}} \displaystyle\sum_{y=0}^{n-x} \binom{N_2}{y}\binom{N-N_1-N_2}{n-x-y} = \dfrac{\binom{N_1}{x}\binom{N-N_1}{n-x}}{\binom{N}{n}}$ (Ver anexo)

   De forma análoga

   $f_Y(y) = \dfrac{\binom{N_2}{y}\binom{N-N_2}{n-y}}{\binom{N}{n}}$

   (b) Calcula $Cov(X,Y)$ y $\rho(X,Y)$

2.

3. X y Y tienen función de densidad conjunta:

$$f(x,y) = \frac{1}{x^2 y^2} \mathbf{I}_{(1,\infty)}^{(x)} \mathbf{I}_{(1,\infty)}^{(y)}$$

   (a) Calcula la función de densidad conjunta de $U = XY$ y $V = \dfrac{X}{Y}$

   $UV = X^2 \Rightarrow X = \sqrt{UV}$

   $\dfrac{U}{V} = Y^2 \Rightarrow Y = \sqrt{\dfrac{U}{V}}$

   $\left| \dfrac{\partial(X,Y)}{\partial(U,V)} \right| = \left| \det \begin{pmatrix} \dfrac{\sqrt{V}}{2\sqrt{U}} & \dfrac{1}{2\sqrt{UV}} \\ \dfrac{\sqrt{U}}{2\sqrt{V}} & -\dfrac{\sqrt{U}}{2\sqrt{V^3}} \end{pmatrix} \right| = \dfrac{1}{4\sqrt{V}} \left| \det \begin{pmatrix} \sqrt{V} & \dfrac{1}{\sqrt{V}} \\ 1 & -\dfrac{1}{V} \end{pmatrix} \right| = \dfrac{1}{2V}$

   Así

   $f(u,v) = \dfrac{1}{2v} \dfrac{1}{(uv)(\frac{u}{v})} \mathbf{I}_{(1,\infty)}^{(\sqrt{uv})} \mathbf{I}_{(1,\infty)}^{(\frac{\sqrt{u}}{\sqrt{v}})} = \dfrac{1}{2u^2 v} \mathbf{I}_{(1,\infty)}^{(\sqrt{uv})} \mathbf{I}_{(1,\infty)}^{(\frac{\sqrt{u}}{\sqrt{v}})}$

   $1 < uv < \infty$ y $1 < \dfrac{u}{v} < \infty$

   $\dfrac{1}{v} < u < \infty$ y $v < u < \infty$

   Por lo tanto

   $f(u,v) = \dfrac{1}{2u^2 v} (\mathbf{I}_{(v,\infty)}^{(u)} \mathbf{I}_{(1,\infty)}^{(v)} + \mathbf{I}_{(\frac{1}{v},\infty)}^{(u)} \mathbf{I}_{(0,1)}^{(v)}) = \dfrac{1}{2u^2 v} \mathbf{I}_{(\frac{1}{u},u)}^{(v)} \mathbf{I}_{(1,\infty)}^{(u)}$

   (b) Calcula $f_U(u)$ y $f_V(v)$

$$f_U(u) = \int_{\frac{1}{u}}^{u} \frac{1}{2u^2 v} dv \mathbf{I}_{(1,\infty)}^{(u)} = \frac{\ln u^2}{2u^2} \mathbf{I}_{(1,\infty)}^{(u)}$$

$$f_V(v) = \int_{v}^{\infty} \frac{1}{2u^2 v} du \mathbf{I}_{(1,\infty)}^{(v)} + \int_{\frac{1}{v}}^{\infty} \frac{1}{2u^2 v} du \mathbf{I}_{(0,1)}^{(v)} = \frac{1}{2v^2} \mathbf{I}_{(1,\infty)}^{(v)} + \frac{1}{2} \mathbf{I}_{(0,1)}^{(v)}$$

4. Sean X y Y independientes ambas con distribución uniforme en $(0,1)$. Sea $U = X + Y$ y $V = X - Y$. Tenemos que $f_{X,Y}(x,y) = 1\mathbf{I}_{(0,1)}^{(x)}\mathbf{I}_{(0,1)}^{(y)}$

(a) Calcula $f_{U,V}(u,v)$

$X = \dfrac{U+V}{2}$ y $Y = \dfrac{U-V}{2}$

$\left|\dfrac{\partial(X,Y)}{\partial(U,V)}\right| = \left|\det(\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{smallmatrix})\right| = \dfrac{1}{2}$

Así

$$f_{U,V}(u,v) = \frac{1}{2}\mathbf{I}_{(0,1)}^{(\frac{u+v}{2})}\mathbf{I}_{(0,1)}^{(\frac{u-v}{2})}$$

$0 < u + v < 2$ y $0 < u - v < 2$

Entonces

$(0 < u < 2)$ y $(-u < v < 2 - u$ y $u - 2 < v < u)$ o bien

$(-1 < v < 1)$ y $(-v < u < 2 - v$ y $v < u < 2 + v)$

Por lo tanto

$$f_{U,V}(u,v) = \frac{1}{2}(\mathbf{I}_{(-v,2+v)}^{(u)}\mathbf{I}_{(-1,0)}^{(v)} + \mathbf{I}_{(v,2-v)}^{(u)}\mathbf{I}_{(0,1)}^{(v)}) = \frac{1}{2}\mathbf{I}_{(-u,u)}^{(v)}\mathbf{I}_{(0,1)}^{(u)} + \mathbf{I}_{(u-2,2-u)}^{(v)}\mathbf{I}_{(1,2)}^{(u)})$$

(b) Demuestra que $Cov(U,V) = 0$, pero no son independientes.

$$E(U,V) = \frac{1}{2}(\int_{-1}^{0}\int_{-v}^{2+v}uv\,du\,dv + \int_{0}^{1}\int_{v}^{2-v}uv\,du\,dv) = \frac{1}{2}(\int_{-1}^{0}4(v+v^2)dv + \int_{0}^{1}4(v-v^2) = 0$$

$$E(U) = \frac{1}{2}(\int_{-1}^{0}\int_{-v}^{2+v}u\,du\,dv + \int_{0}^{1}\int_{v}^{2-v}u\,du\,dv) = \frac{1}{2}(\int_{-1}^{0}4(1+v)dv + \int_{0}^{1}4(1-v)) = 2$$

$$E(V) = \frac{1}{2}(\int_{-1}^{0}\int_{-v}^{2+v}v\,du\,dv + \int_{0}^{1}\int_{v}^{2-v}v\,du\,dv) = \int_{-1}^{0}v + v^2\,dv + \int_{0}^{1}v - v^2\,dv = 0$$

Por lo tanto

$$Cov(U,V) = E(U,V) - E(U)E(V) = 0$$

Y claramente U y V no son independientes.

5.

6.

7.

8.

9. Sea $U \sim U(0,2\pi)$ y $Z \sim exp(1)$ tal que U y Z son independientes. Sea $X = \sqrt{2Z}\cos U$ y $Y = \sqrt{2Z}\sin U$. Demuestra que X y Y son variables aleatorias independientes con distribución normal estándar.

$$f_{U,Z}(u,z) = \frac{1}{2\pi}e^{-z}\mathbf{I}_{(0,2\pi)}^{(u)}\mathbf{I}_{(0,\infty)}^{(z)}$$

$X = \sqrt{2Z}\cos U$ y $Y = \sqrt{2Z}\sin U$

$\dfrac{X^2 + Y^2}{2} = Z$ y $\arctan(\dfrac{Y}{X}) = U$

$\left|\dfrac{\partial(U,Z)}{\partial(X,Y)}\right| = \left|\det(\begin{smallmatrix} \frac{-X^2Y}{X^2(X^2+Y^2)} & X \\ \frac{X^2}{X(X^2+Y^2)} & Y \end{smallmatrix})\right| = 1$

$$f_{X,Y}(x,y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}\mathbf{I}_{(0,2\pi)}^{(\arctan\frac{y}{x})}\mathbf{I}_{(0,\infty)}^{(\frac{x^2+y^2}{2})} = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}\mathbf{I}_{(-\infty,\infty)}^{(x)}\mathbf{I}_{(-\infty,\infty)}^{(y)}$$

Así

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathbf{I}_{(-\infty,\infty)}^{(x)} \Rightarrow X \sim N(0,1)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \mathbf{I}_{(-\infty,\infty)}^{(y)} \Rightarrow Y \sim N(0,1)$$

Ambas son normal standar.

1 Anexo

Sabemos que:

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

Además

$$\left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{r=0}^{n+m} \left(\sum_{i=0}^{r} a_i b_{r-i}\right) x^r$$

Así

$$(1+x)^{n+m} = (1+x)^n (1+x)^m = \left(\sum_{i=0}^{n} \binom{n}{i} x^i\right)\left(\sum_{j=0}^{m} \binom{m}{j} x^j\right) = \sum_{r=0}^{n+m} \left(\sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}\right) x^r = \sum_{r=0}^{n+m} \binom{n+m}{r} x^r$$

Por lo tanto

$$\sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i} = \binom{n+m}{r}$$