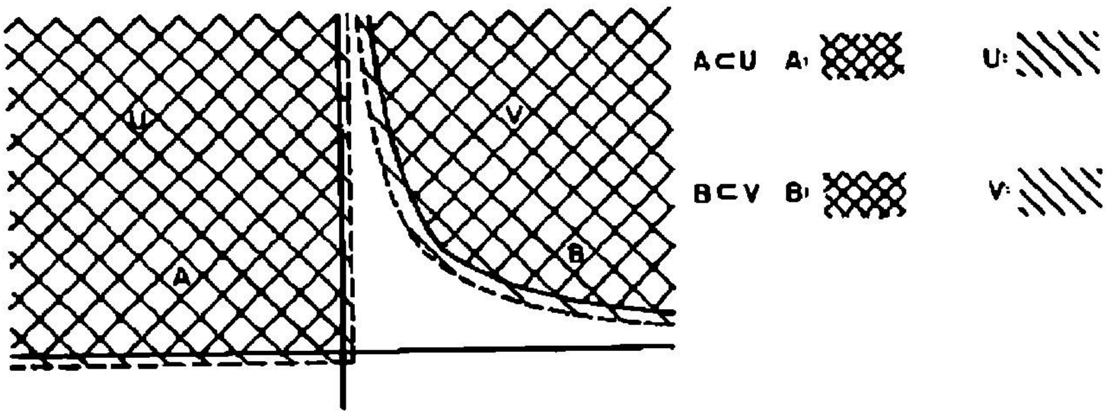
Chapter 4

NORMALITY AND OTHER SEPARATION PROPERTIES

In this chapter, a variety of spaces are studied—spaces that possess some, but in general not all, of the attributes of metric spaces. Normal spaces may lack a distance function, but they nevertheless have sufficient topological structure to yield some of topology's major theorems. A number of other spaces that we shall investigate are of interest precisely because they fail to have many of the properties which are inherent to metric spaces.

A. NORMAL SPACES

We have previously seen that in $\mathscr{E}^2$, the distance between closed subsets A and B may be 0, even if A and B are disjoint.



Note that in this case, however, open sets U and V can be found that serve to keep A and B apart. In fact, in metric spaces, all that is needed for such "protection" to exist is that the closure of A (respectively, B) does not intersect B (respectively, A). This is the content of the next theorem.

(4.A.1) ***Theorem.*** If A and B are subsets of a metric space X such that $\bar{A} \cap B = \bar{B} \cap A = \varnothing$, then there are disjoint open sets U and V of X such that $A \subset U$ and $B \subset U$.

Proof. For each $a \in A$, there is a positive number r_a such that $S_{r_a}(a) \subset X \setminus \bar{B}$. Cover A with sets of the form $S_{r_a/2}(a)$ and B with similarly defined open sets of the form $S_{r_b/2}(b)$. Let U be the union of sets of the former type and V the union of sets of the latter form. It is easy to verify that U and V are the desired open sets.

This property of metric spaces turns out to be very useful, and, as is frequently the case with useful concepts, it is made the basis for a definition.

(4.A.2) ***Definition.*** A topological space X is *completely normal* if and only if for every pair of subsets, A and B, with $\bar{A} \cap B = A \cap \bar{B} = \varnothing$, there are disjoint open sets U and V, containing A and B respectively.

A slightly weaker, but no less important notion is the following.

(4.A.3) ***Definition.*** A topological space X is *normal* if and only if for every pair of disjoint closed subsets A and B of X there are disjoint open subsets in X that contain A and B respectively.

Clearly, any completely normal space is normal; however, the converse does not hold (an example, the Tihonov Plank, will be given later in section C of this chapter).

(4.A.4) ***Exercise.*** Show that a space X is normal if and only if for each closed subset A and for each open set U with $A \subset U$, there is an open set V such that $A \subset V \subset \bar{V} \subset U$.

The reader might note that if X is a T_1 space that is not T_2, then X fails to be normal.

(4.A.5) ***Exercise.*** Show that a space X is completely normal if and only if every subspace of X is normal. [Hint: Consider $X \setminus (\bar{A} \cap \bar{B})$.]

Metric spaces are of course normal. However, in view of the example cited above, there exist normal spaces that are not metrizable. A number of such examples may be derived from the following result whose proof is an almost classic illustration of the use of compactness.

(4.A.6) ***Theorem.*** If X is a compact Hausdorff space, then X is normal.

Proof. Suppose that A and B are disjoint closed subsets of X. Then A and B are compact. For each pair of points a and b with $a \in A$ and $b \in B$ there are disjoint open sets $U_{(a,b)}$ and $V_{(a,b)}$ that contain a and b respectively. Fix a point $a \in A$. The family of open sets $\{V_{(a,b)} \mid b \in B\}$ forms an open cover of B from which a finite subcover $\{V_{(a,b_1)}, V_{(a,b_2)}, \ldots, V_{(a,b_n)}\}$ may be extracted. Let U_a be the intersection of the corresponding sets $U_{(a,b_1)}$, $U_{(a,b_2)}, \ldots, U_{(a,b_n)}$ and set $V_a = \bigcup_{i=1}^{n} V_{(a,b_i)}$. Repeat this procedure for each fixed $a \in A$ to obtain an open cover $\{U_a \mid a \in A\}$ of A. Let $\{U_{a_1}, \ldots, U_{a_m}\}$ be a finite subcover. For each U_{a_i} there corresponds an open set V_{a_i} that contains B. Then $\bigcup_{i=1}^{m} U_{a_i}$ and $V = \bigcap_{i=1}^{m} V_{a_i}$ are disjoint open sets that contain A and B respectively.

The following result is used in the proof of the next theorem.

(4.A.7) ***Exercise.*** Show that if $\{x_i\}$ is a sequence in a compact space X, then $\{x_i\}$ has a cluster point in X (compare with the remarks following (2.I.7)).

(4.A.8) ***Theorem.*** Suppose that $A_1 \supset A_2 \supset \cdots$ is a countable family of closed, connected, nonempty subsets of a compact Hausdorff space X. Then $A = \bigcap_{i=1}^{\infty} A_i$ is compact, connected, and nonempty.

Proof. Since A is closed it is compact. Now suppose that $A \subset U \cup V$, where U and V are disjoint open sets. By (2.H.3), there is an integer N such that if $i > N$, then $A_i \subset U \cup V$. However each A_i is connected; hence $A_i \subset U$ or $A_i \subset V$ for every $i > N$, which implies that $A \subset U$ or $A \subset V$. Thus A is connected.

We now show that $A \neq \varnothing$. For each i, let $x_i \in A_i$. Since X is compact, the sequence $\{x_i\}$ has a cluster point a in X (4.A.7). If $a \notin A$, then by (4.A.6) there are disjoint open sets U and V containing $\{a\}$ and A respectively. However it follows from (2.H.3) that there is an integer N such that if $i > N$, then $A_i \subset V$. Hence for each $i > N$, we have that $a_i \notin U$, which contradicts the fact that a is a cluster point of the sequence $\{x_i\}$.

(4.A.9) ***Exercise.*** Generalize the previous theorem as follows. Let Λ be an index set that is partially ordered by $\preccurlyeq$. Suppose furthermore that for

each $\alpha,\beta \in \Lambda$, there is $\lambda \in \Lambda$ such that $\alpha \preccurlyeq \lambda$ and $\beta \preccurlyeq \lambda$ (such a set is said to be *directed*). Suppose that $\{A_\alpha \mid \alpha \in \Lambda\}$ is a family of closed, connected, non-empty subsets of a compact T_2 space X and that $A_\alpha \subset A_\beta$ if and only if $\beta \preccurlyeq \alpha$. Show that $\bigcap_{\alpha \in \Lambda} A_\alpha$ is compact and connected.

Suppose that A and B are disjoint closed subsets of a space X. Then of course if X is normal, there are disjoint open subsets U and V such that $A \subset U$ and $B \subset V$. Can the open sets U and V be chosen so that $X = U \cup V$? To do so obviously requires that X be disconnected. A sufficient condition for the existence of such a dramatic cleavage is given in our next result.

(4.A.10) Definition. Disjoint subsets A and B of a topological space X are *separated* in X if and only if there is a separation (U,V) of X such that $A \subset U$ and $B \subset V$.

(4.A.11) Theorem. Suppose that X is a compact T_2 topological space and that A and B are disjoint closed subsets of X. If no connected subset of X intersects both A and B, then A and B can be separated in X.

Proof. We first prove the theorem for the case where A and B are singletons, i.e., $A = \{x\}$ and $B = \{y\}$. Suppose that $\{x\}$ and $\{y\}$ cannot be separated. Let $\mathscr{K} = \{K_\alpha \mid \alpha \in \Lambda\}$ be the collection of all closed subsets of X in which $\{x\}$ and $\{y\}$ are not separated. Note that $X \in \mathscr{K}$. Partially order $\mathscr{K}$ by defining $K_\alpha \preceq K_\beta$ if and only if $K_\alpha \subset K_\beta$. By the Kuratowski lemma (0.D.4), there is a maximal nest $\mathscr{K}^* = \{K_\alpha \mid \alpha \in L \subset A\}$ in $\mathscr{K}$. Let $K = \bigcap_{\alpha \in L} K_\alpha$. We first show that $K \in \mathscr{K}$ (and hence by the maximality of $\mathscr{K}^*$, we will have that $K \in \mathscr{K}^*$). If $K \notin \mathscr{K}$, there are disjoint closed subsets K_1 and K_2 of K, such that $x \in K_1$, $y \in K_2$, and $K_1 \cup K_2 = K$. Since K itself is closed in the normal space X, so must be K_1 and K_2. Thus, there are disjoint open sets U and V in X with $K_1 \subset U$ and $K_2 \subset V$. By (2.H.4), some member K_β of $\mathscr{K}^*$, is contained in $U \cup V$. However, this is impossible, since it implies that $\{x\}$ and $\{y\}$ can be separated in K_β by $U \cap K_\beta$ and $V \cap K_\beta$. Consequently, we have that $K \in \mathscr{K}$.

Since x and y lie in K, it follows from the hypothesis that K cannot be connected. Therefore, K can be written as the union of disjoint closed subsets H_1 and H_2, and since K is "minimal" in $\mathscr{K}^*$, both x and y must be contained in either H_1 or in H_2, say H_1. If $\{x\}$ and $\{y\}$ could be separated in H_1 by sets H_1' and H_2'', then H_1' and $H_1'' \cup H_2$ would separate $\{x\}$ and $\{y\}$ in K, which we have just seen to be impossible. Thus, an impasse has been reached, for if $\{x\}$ and $\{y\}$ cannot be separated in H_1, then H_1 belongs to the maximal chain $\mathscr{K}^*$. It does not, and hence the special case of the theorem is settled.

Now suppose that $A = \{x\}$, B is a closed set disjoint from A, and A and B satisfy the hypothesis of the theorem. For each $b \in B$, the special case just proved is applied to obtain a separation (U_b, V_b) of X such that $x \in U_b$ and $b \in V_b$. The family $\{V_b \mid b \in B\}$ is an open cover of the compact space B and consequently yields a finite subcover $\{V_{b_1}, V_{b_2}, \ldots, V_{b_n}\}$. Then $U = \bigcap_{i=1}^{n} U_{b_i}$ and $V = \bigcup_{i=1}^{n} V_{b_i}$ form a separation of X, where $x \in U$ and $B \subset V$.

By employing the method used in the preceding paragraph, the reader should have no difficulty in establishing the general case.

A particularly elegant application of the foregoing theorem is found in the proof of the following innocent appearing, but nevertheless deep result.

(4.A.12) Theorem. Suppose that U is an open subset of a compact connected T_2 space X and that C is a component of U. Then $\bar{C}^X \cap \operatorname{Fr} U \neq \varnothing$. (Recall that $\bar{C}^X$ is the closure of C in X.)

Proof. Suppose that $\bar{C}^X \cap \operatorname{Fr} U = \varnothing$. Then $\bar{C}^X = C$, and hence C is closed in X. The space X is normal; therefore, there is an open set V such that $C \subset V \subset \bar{V} \subset U$. The space $\bar{V}$ is of course compact. Since no connected subset of $\bar{V}$ intersects both C and $\operatorname{Fr} V$ (recall that C is a component of U), by the previous theorem there is a separation (F,G) of $\bar{V}$ such that $C \subset F$ and $\operatorname{Fr} V \subset G$. Then $(F, G \cup (X \setminus \bar{V}))$ is a separation of the connected space X, which is impossible.

B. URYSON'S LEMMA AND TIETZE'S THEOREM

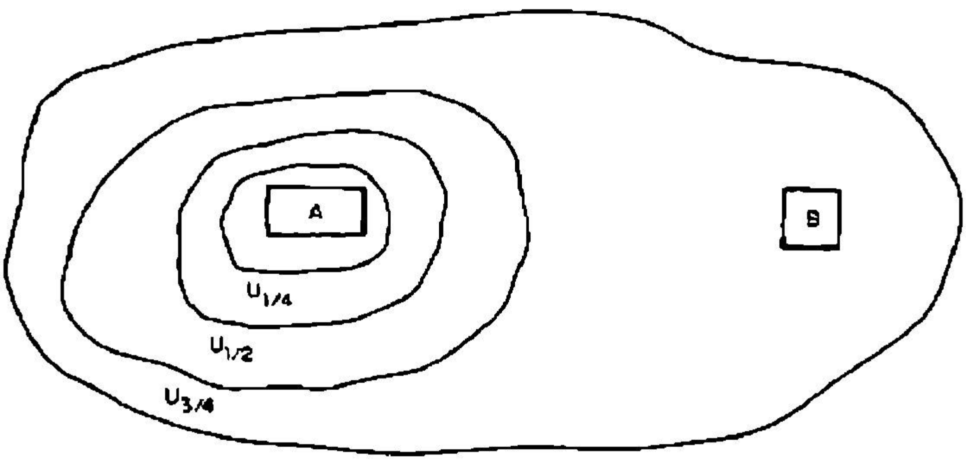
• We now prove two of the great classical theorems in general topology, Uryson's lemma and Tietze's extension theorem. Suppose that X is a space with the property that for each pair of disjoint closed subsets A and B, there is a continuous function $f_{AB} : X \to [0,1]$ such that $f_{AB}(A) = 0$ and $f_{AB}(B) = 1$. Then X is normal ($f^{-1}([0,1/3))$ and $f^{-1}((2/3,1])$ are disjoint open subsets of X that contain A and B respectively). Uryson's remarkable lemma gives us the following converse.

(4.B.1) Theorem (Uryson's Lemma). A space X is normal if and only if for each pair A and B of disjoint closed subsets of X, there is a continuous function $f : X \to [0,1]$ such that $f(a) = 0$ for each $a \in A$ and $f(b) = 1$ for each $b \in B$.

Proof. We first prove the following lemma,

(4.B.2) Lemma. Let D be the set of all dyadic fractions in $(0,1)$, i.e., $D = \{p/2^n \mid n,p \in \mathbf{Z}^+ \text{ and } p < 2^n\}$. Suppose that A and B are disjoint closed subsets of a normal space X. Then there is a collection of open sets $\{U_d \mid d \in D\}$ such that if $d_1 < d_2$, then $A \subset U_{d_1} \subset \overline{U_{d_1}} \subset U_{d_2} \subset \overline{U_{d_2}} \subset X \setminus B$.

Proof. By (4.A.4) there is an open set $U_{1/2}$ such that $A \subset U_{1/2} \subset \overline{U}_{1/2} \subset X \setminus B$. Another application of this exercise yields open sets $U_{1/4}$ and $U_{3/4}$ with the properties that $A \subset U_{1/4} \subset \overline{U_{1/4}} \subset U_{1/2} \subset \overline{U_{1/2}} \subset U_{3/4} \subset \overline{U_{3/4}} \subset X \setminus B$.



The inductive step should now be apparent. Suppose that the sets U_d (satisfying the lemma) have been defined for all members of D with denominators less than or equal to 2^k. Once again, by (4.A.4), there are open sets $U_{2j-1/2^{k+1}}$, $j = 1, 2, \ldots, 2^k$, that have the property that $A \subset U_{1/(2^{k+1})} \subset \overline{U_{1/(2^{k+1})}} \subset U_{1/2^k} \subset$ etc. The U_d's are now defined for all d's of the form $p/2^{k+1}$, and this completes the induction.

We now return to the proof of Uryson's lemma. Let $\{U_d \mid d \in D\}$ be the collection of open sets obtained in the preceding lemma. Define a function $F : X \to [0,1]$ by

$$F(x) = \begin{cases} \operatorname{lub}\{d \mid x \notin U_d\} & \text{if } x \notin U_d \text{ for some } d \in D \\ 0 & \text{otherwise} \end{cases}$$

Note that if $x \in A$, then $x \in U_d$ for each $d \in D$ and hence $F(x) = 0$; furthermore, if $x \in B$, then $x \notin U_d$ for all $d \in D$ and therefore $F(x) = 1$. Thus, to complete the proof it suffices to establish the continuity of F.

Suppose that $x \in X$ and that $\varepsilon > 0$ is given. Assume first that $0 < F(x) < 1$. Since the dyadic rationals are dense in $(0,1)$, there are numbers $d_1,d_2 \in D$

such that $F(x) - \varepsilon < d_1 < F(x) < d_2 < F(x) + \varepsilon$. Let $V = U_{d_2} \setminus \overline{U_{d_1}}$. Then V is an open set and V contains x, since $F(x) > d_1$ implies that $x \notin \overline{U_{d_1}}$ and $F(x) < d_2$ implies that $x \in U_{d_2}$. We assert that $F(V)$ is contained in the ε-neighborhood about $F(x)$. To see this, let $y \in V$. Then since $y \in U_{d_2}$, $F(y)$ must be at most equal to d_2 and hence strictly less than $F(x) + \varepsilon$. Since $y \notin U_{d_1}$, $F(y)$ must be greater than or equal to d_1 and thus strictly greater than $F(x) - \varepsilon$. Consequently, F is continuous at x whenever $F(x)$ lies in the interval $(0,1)$. A similar proof may be supplied by the reader for the case where $F(x) = 0$ or $F(x) = 1$.

The following corollary is useful in the proof of Tietze's extension theorem.

(4.B.3) Corollary. Suppose that A is a closed subset of a normal space X and that $f : A \to \mathscr{E}^1$ is a continuous function such that $|f(a)| \leq c$ for each $a \in A$. Then there is a continuous function $h : X \to \mathscr{E}^1$ such that

(i) $|h(x)| \leq (1/3)c$ for each $x \in X$, and
(ii) $|f(a) - h(a)| \leq (2/3)c$ for each $a \in A$.

Proof. Consider the sets $A_+ = \{a \in A \mid f(a) \geq (1/3)c\}$ and $A_- = \{a \in A \mid f(a) \leq -(1/3)c\}$. Since A_+ and A_- are closed and disjoint, there is a map h $X \to [-(1/3)c,(1/3)c] \subset \mathscr{E}^1$ with $h(A_+) = (1/3)c$ and $h(A_-) = (-1/3)c$ (why?). A routine verification shows that h has all the requisite properties.

Uryson's lemma does not guarantee that $f^{-1}(0) = A$ and $f^{-1}(1) = B$. For this to be the case, an additional condition is imposed on the sets A and B.

(4.B.4) Definition. A subset A of a space X is a G_δ subset of X if and only if A may be expressed as the countable intersection of open subsets of X. (G stems from *Gebiet*, the German word for "region"; δ stems from *Durchschnitt*, the German word for "intersection.")

Note that closed subsets as well as open subsets of a metric space X are G_δ sets. (If $A \subset (X,d)$ is closed, then $A = \bigcap_{n=1}^{\infty} S_{1/n}(A)$ where $S_{1/n}(A) = \{x \in X \mid d(x,A) < 1/n\}$.)

(4.B.5) Definition. A normal T_1 space X is *perfectly normal* if and only if each closed subset of X is a G_δ.

(4.B.6) Exercise. Show that if X is perfectly normal and A and B are disjoint closed subsets of X, then there is a map $f: X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$. [Hint: Let $A = \bigcap_{n=1}^{\infty} U_n$, where U_n is open in X and $U_{n+1} \subset U_n$. For each n, find $f_n : X \to [0,1]$ such that $f_n(A) = 0$ and $f_n(X \setminus U_n) = 1$. Define $f_A(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n}$. Define a function f_B in a similar fashion and set $f = f_A/(f_A + f_B)$.]

(4.B.7) Definition. A subset A of a space X is an F_σ subset of X if and only if A may be expressed as the countable union of closed subsets of X. (F is from *Fermé*, the French word for "closed"; σ is from *Summe*, the French word for "union.")

Note that a subset A of a space X is a G_δ set if and only if $X \setminus A$ is an F_σ set.

A great number of problems in mathematics can be reduced to the question: If A is a subset of a space X and $f: A \to Y$ is a continuous map into a space Y, can f be extended to a continuous map with domain X? Uryson's lemma is one (rather trivial) solution to the extension problem. Another partial solution is given in the next theorem. For a discussion of some of the problems that are equivalent to the extension problem, see Hu [1959, pp. 1–34].

(4.B.8) Theorem (Tietze's Extension Theorem). Suppose that X is a normal space, A is a closed subset of X, and $f: A \to \mathscr{E}^1$ is a continuous function. Then there is a continuous map $F: X \to \mathscr{E}^1$ such that $F(a) = f(a)$ for each $a \in A$. Furthermore,

(i) if $|f(a)| < c$ for each $a \in A$, then F may be chosen so that $|F(x)| < c$ for each $x \in X$, and
(ii) if $|f(a)| \leq c$ for each $a \in A$, then F may be chosen so that $|F(x)| \leq c$ for each $x \in X$.

Proof. The proof is based on repeated applications of (4.B.3). Three cases are considered.

Case 1. Suppose that $|f(a)| \leq c$ for each $a \in A$. By (4.B.3), there is a map $g_0 : X \to \mathscr{E}^1$ such that $|f(a) - g_0(a)| \leq (2/3)c$ for each $a \in A$ and $|g_0(x)| \leq (1/3)c$, for each $x \in X$. Application of (4.B.3) to the function $f - g_0$ yields a function $g_1 : X \to \mathscr{E}^1$, where $|g_1(x)| \leq (1/3)(2/3)c$ for $x \in X$ and $|f(a) - g_0(a) - g_1(a)| \leq (2/3)(2/3)c$ for $a \in A$. Continuing inductively, we obtain a sequence of functions $g_0, g_1, g_2,$ such that for each $n \in \mathbb{N}$,

$|g_n(x)| \leq (1/3)(2/3)^n c$ on X and $|f(a) - g_0(a) - \quad - g_n(a)| \leq (2/3)(2/3)^n c$ on A. Define $F : X \to \mathscr{E}^1$ by $F(x) = \sum_{n=0}^{\infty} g_n(x)$. The function F is seen to be continuous as follows. For each n, let $s_n(x) = \sum_{i=0}^{n} g_i(x)$. Note that $\sum_{n=0}^{\infty} g_n(x) \leq \sum_{n=0}^{\infty} (1/3)(2/3)^n c = c$, and hence the sequence of partial sums $s_n(x)$ converges uniformly to $F(x)$, i.e., for each $\varepsilon > 0$, there is an integer N such that for $n > N$, $|s_n(x) - F(x)| < \varepsilon$ for each $x \in X$. We give the usual advanced calculus argument to show that F is continuous. Let $x_0 \in X$ be given. Choose N such that for $n \geq N$, $|s_n(x) - F(x)| < \varepsilon/3$. Since $s_N(x)$ is continuous, there is a $\delta > 0$ such that if $|x - x_0| < \delta$, then $|s_N(x) - s_N(x_0)| < \varepsilon/3$. Consequently, we have that $|F(x) - F(x_0)| \leq |F(x) - s_N(x)| + |s_N(x) - s_N(x_0)| + |s_N(x_0) - F(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$, which establishes the continuity of F. That F is an extension of f and $|F(x)| \leq c$ for each $x \in X$ may be readily verified by the reader.

Case 2. Suppose that $|f(a)| < c$ for $a \in A$. By Case 1 there is a continuous extension F of f that at least satisfies $|F(x)| \leq c$. Let $B = \{x \mid F(x) = c\}$. Note that B is closed and disjoint from A. Apply (4.B.1) to obtain a function $h : X \to \mathscr{E}^1$ such that $h(A) = 1$ and $h(B) = 0$. The desired extension G of f is defined by pointwise multiplying the functions F and h, i.e., $G(x) = h(x) \quad F(x)$.

Case 3. Suppose that f is not bounded. Let $h : \mathscr{E}^1 \to (-1,1)$ be an arbitrary homeomorphism. By Case 2, the map $hf : A \to (-1,1)$ may be extended to $G : X \to (-1,1)$; then $F = h^{-1}G$ extends f to all of X.

Euclidean 1-space $\mathscr{E}^1$ is not the only range space for which the foregoing theorem is valid. In fact, quite a number of spaces may be used in place of $\mathscr{E}^1$, and such spaces are called Absolute Retracts.

(4.B.9) Definition. A normal space Y is an *Absolute Retract* (AR) if and only if for each normal space X, for each closed subspace A of X, and for each continuous map $f : A \to Y$, there is a continuous function $F : X \to Y$ such that $F(a) = f(a)$ for each $a \in A$.

This unusual terminology will make more sense shortly, but first let us prove a basic result concerning AR's.

(4.B.10) Theorem. If the spaces $X_1, \ldots, X_n$ are AR's and if the product space $\prod_{i=1}^{n} X_i$ is normal, then the product space is an AR

Proof. Suppose that X is a normal space and that A is a closed subset of X. Let f be a continuous map from A into $\prod_{i=1}^{n} X_i$. For each i, the map $p_i f : A \to X_i$ may be extended to a map $F_i : X \to X_i$. Define $F : X \to \prod_{i=1}^{n} X_i$ by $F(x) = (F_1(x), \ldots, F_n(x))$.

(4.B.11) ***Corollary.*** The space $\mathscr{E}^n$ is an AR.

(4.B.12) ***Exercise.*** Show that $\mathbf{I} \times \quad \times \mathbf{I}$ is an AR.

(4.B.13) ***Exercise.*** Show that the property of being an AR is a topological invariant.

(4.B.14) ***Definition.*** A subset A of a space X is a *retract* of X if and only if there is a continuous function $r : X \to A$ such that $r(a) = a$ for each $a \in A$ (i.e., the identity map on A has a continuous extension r). In this case, r is called a *retraction*.

(4.B.15) ***Exercise.*** (a) Show that a subset A of a space X is a retract of X if and only if the following diagram commutes, where i is the inclusion map.

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ A \arrow[ur, "i"] \arrow[rr, "id"'] & & A \end{tikzcd}$$

(b) Show that a subset A of X is a retract of X if and only if each continuous map $f : A \to Z$ can be extended to a continuous map $f : X \to Z$.

(4.B.16) ***Examples.***

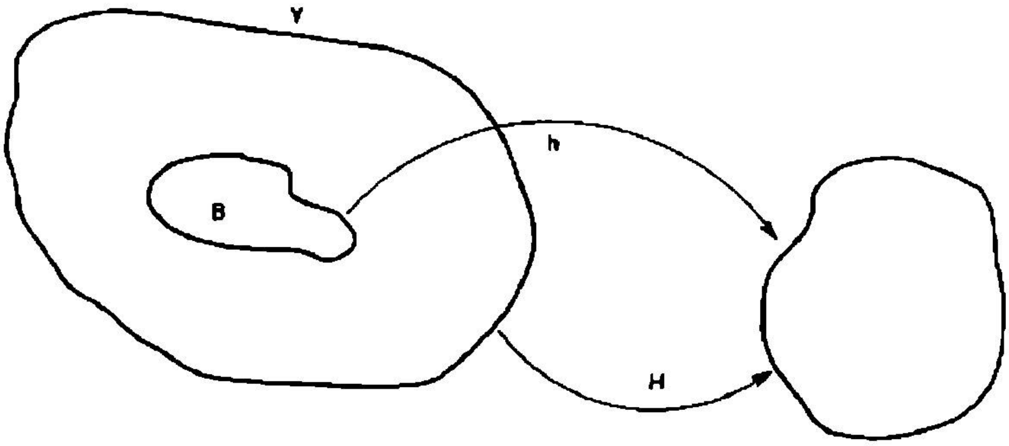
1. Any point in a space X is a retract of X.
2. Any closed interval of $\mathscr{E}^1$ is a retract of $\mathscr{E}^1$.
3. If $x \in \mathscr{E}^n$, and $\varepsilon > 0$, then the closed ball $\overline{S_\varepsilon(x)}$ in $\mathscr{E}^n$ is a retract of $\mathscr{E}^n$.

(4.B.17) ***Exercise.*** Show that if X has the fixed point property and A is a retract of X, then A has the fixed point property.

(4.B.18) Definition. A normal space X is an *absolute retract* (*ar*) if and only if whenever X is homeomorphic to a closed subset B of a normal space Y, then B is a retract of Y.

(4.B.19) Theorem. A space X is an AR if and only if X is an ar.

Proof. Suppose that X is an AR. Let B be a closed subset of a normal space Y, and suppose that B and X are homeomorphic. We seek a retraction r, from Y onto B. Let $h : B \to X$ be a homeomorphism.



Since X is an AR, h may be extended to a continuous map $H\ \ Y \to X$, where $H_{|B} = h$. Then $r = h^{-1}H$ is the desired retraction. The converse is considerably more difficult, and its proof will be deferred until Chapter 8.

Closely related to AR's are Absolute Neighborhood Retracts, ANR's.

(4.B.20) Definition. A normal space X is an *Absolute Neighborhood Retract* (ANR) if and only if for each normal space Y, for each closed $A \subset Y$, and for each continuous map $f : A \to X$, there is a neighborhood U of A and a continuous extension $F : U \to X$ such that $F_{|A} = f$.

Although AR's are obviously ANR's, there are numerous examples of ANR's that are not AR's. Perhaps the most prominent of these are the Euclidean n-spheres $\mathscr{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathscr{E}^{n+1} \mid x_1^2 + \quad + x_{n+1}^2 = 1\}$.

(4.B.21) Example. The n-sphere $\mathscr{S}^n$ is an ANR. Suppose that Y is normal and that A is a closed subset of Y. Let $f : A \to \mathscr{S}^n$ be a continuous function. Since $\mathscr{S}^n \subset \mathscr{E}^{n+1}$, f may be regarded as a map from A into the AR, $\mathscr{E}^{n+1}$. Consequently, there is a continuous extension of f, $F : Y \to \mathscr{E}^{n+1}$. Set $U = F^{-1}(\mathscr{E}^{n+1} \setminus \{0\})$ and on U define a function F' by $F'(y) = F(y)/\|F(y)\|$, where $\|F(y)\|$ denotes the usual distance between $F(y)$ and the origin. Then F' is the desired extension of f to a neighborhood U of A. That $\mathscr{S}^n$ is not an AR follows from the $(n + 1)$-dimensional Brouwer fixed point theorem (14.C.22); see problem 4.B.9.

The following result will be used in later sections.

(4.B.22) Lemma (Brouwer Reduction Theorem). Suppose that X is a second countable space and that $\mathscr{C} = \{C_\alpha \mid \alpha \in \Lambda\}$ is a family of closed subsets of X with the property that whenever $C_1 \supset C_2 \supset C_3$ is a countable decreasing family of members of $\mathscr{C}$, then $\bigcap_{i=1}^{\infty} C_i \in \mathscr{C}$. Then there is an irreducible set C^* in $\mathscr{C}$, i.e., no proper subset of C^* belongs to $\mathscr{C}$.

Proof. Let $U_1, U_2, \ldots$ be a countable basis for X. Let C_1 be an arbitrary member of $\mathscr{C}$ and inductively select a sequence of sets from $\mathscr{C}$ by choosing $C_n \in \mathscr{C}$ such that $C_n \subset C_{n-1} \cap (X \setminus U_n)$. (If no such set in $\mathscr{C}$ exists, then define $C_n = C_{n-1}$.)

We assert that $C^* = \bigcap_{i=1}^{\infty} C_i$ is irreducible. If this is not the case, then there is a proper subset $\hat{C}$ of C^* that belongs to $\mathscr{C}$. Clearly, there is a basis member U_n with the property that $U_n \cap \hat{C} = \varnothing$ and $U_n \cap C^* \neq \varnothing$. Since $\hat{C} \in \mathscr{C}$, and $\hat{C} \subset C^* \cap (X \setminus U_n)$, it follows that we have a situation where $C_n \subset C_{n-1} \cap (X \setminus U_n)$; however, this is impossible, since $U_n \cap C^* \neq \varnothing$ implies that $U_n \cap C_n$ is nonempty.

That $\mathscr{S}^n$ is an ANR plays a key role in the proof of the following extension result, which will be used in Chapter 15.

(4.B.23) Theorem. Suppose that K is a compact metric space, C is a closed subset of K, and f is a continuous map from C into $\mathscr{S}^n$ that cannot be extended to all of K. Then there is a closed subset C^* of K such that f cannot be extended continuously to $C^* \cup C$ but can be extended continuously to $C \cup D$, where D is any proper closed subset of C^*.

Proof. Let $\mathscr{D} = \{D \mid D$ is closed and f can not be extended continuously to $C \cup D\}$. We show that $\mathscr{D}$ has an irreducible member. Suppose that $D_1 \supset D_2 \supset$ is a countable decreasing family of elements of $\mathscr{D}$, and let $D = \bigcap_{i=1}^{\infty} D_i$. If $D \notin \mathscr{D}$, then there is a continuous extension $F : (C \cap D) \to \mathscr{S}^n$ of f. Since $\mathscr{S}^n$ is an ANR, F can be extended continuously to an open neighborhood U containing $C \cup D$. However, by (2.H.3) for some integer i, we have that $D_i \subset U$, contradicting the fact that f cannot be extended continuously to $C \cup D_i$. Thus, $D \in \mathscr{D}$, and hence by the Brouwer reduction theorem (4.B.22), there is an irreducible element C^* in Y. Clearly, C^* is the required subset of X.

The notion of ANR is also frequently defined in a metric space context.

(4.B.24) Definition. A metric space X is a *metric absolute neighborhood retract*, denoted by ANR_M, if and only if for each closed subspace B of a metric space Y and for each continuous function $f : B \to X$, there is a neighborhood U of B and a continuous extension $g : U \to X$ of f.

(4.B.25) Theorem. Suppose that X is an ANR_M, Y is a metric space, and $h : X \to Y$ is an embedding such that $h(X)$ is closed in Y. Then there is a neighborhood U of $h(X)$ and a retraction $r : U \to h(X)$.

Proof. Consider $h^{-1} : h(X) \to X$. Since X is an ANR_M, there is a neighborhood U of $h(X)$ and an extension $g \quad U \to X$ of h^{-1}. Then hg is the desired retraction.

(4.B.26) Remark. The converse of (4.B.25) holds if X is compact (10.C.3), but its proof requires material from Chapters 8 and 10.

The following rather technical lemma is needed in the proof of (4.B.28).

(4.B.27) Lemma. Suppose that C_1 and C_2 are closed subspaces of a topological space $X = C_1 \cup C_2$. Let A be a subset of X and for $i = 1,2$ let U_i be an open subset of C_i that satisfies $A \cap C_i \subset U_i$. Then $U_1 \cup U_2$ is a neighborhood (in X) of A.

Proof. There are open sets V_1 and V_2 in X such that $U_1 = V_1 \cap C_1$ and $U_2 = V_2 \cap C_2$. Note that $V_1 \cap (C_1 \setminus C_2) \subset U_1$, $V_2 \cap (C_2 \setminus C_1) \subset U_2$, and $V_1 \cap V_2 = V_1 \cap V_2 \cap (C_1 \cup C_2) \subset U_1 \cup U_2$. Hence, we have that $A = (A \cap (C_1 \setminus C_2)) \cup (A \cap (C_2 \setminus C_1)) \cup (A \cap C_1 \cap C_2) \subset (V_1 \cap (C_1 \setminus C_2)) \cup (V_2 \cap (C_2 \setminus C_1)) \cup (V_1 \cap V_2) \subset U_1 \cup U_2$. Since $V_1 \cap (C_1 \setminus C_2)$, $V_2 \cap (C_2 \setminus C_1)$, and $V_1 \cap V_2$ are open subsets of X, so is their union.

(4.B.28) Theorem [*Borsuk, 1932*]. Suppose that C_1 and C_2 are closed subspaces of a metric space X such that $X = C_1 \cup C_2$. Suppose further that C_1, C_2, and $C_1 \cap C_2$ are ANR_M's. Then for each embedding h of X as a closed subset of a metric space Y, there is a neighborhood U of $h(X)$ and a retract $r \quad U \to h(X)$.

Proof. (*Borsuk* [*1932*]). Let Y be a metric space and $h \quad X \to Y$ be an embedding such that $h(X)$ is closed. By (4.B.25), there is a neighborhood U of

$h(C_1 \cap C_2)$ and a retraction $r\colon U \to h(C_1 \cap C_2)$. Since metric spaces are completely normal, there are open sets $\hat{U}_1$ and $\hat{U}_2$ in X such that $h(C_1) \setminus h(C_2) \subset \hat{U}_1$, $h(C_2) \setminus h(C_1) \subset \hat{U}_2$, and $\hat{U}_1 \cap \hat{U}_2 = \varnothing$. Let $U_1 = \hat{U}_1 \setminus h(C_2)$ and $U_2 = \hat{U}_2 \setminus h(C_1)$. Then we have that $h(C_1) \setminus h(C_2) \subset U_1$, $h(C_2) \setminus h(C_1) \subset U_2$, $U_1 \cap U_2 = \varnothing$, and $(U_1 \cup U_2) \cap h(C_1) \cap h(C_2) = \varnothing$. For $i = 1,2$ define sets $P_i = (U \setminus U_1 \setminus U_2) \cup h(C_i)$ and functions $r_i\colon P_i \to h(C_i)$ by

$$r_i(x) = \begin{cases} r(x) & \text{if} \quad x \in U \setminus U_1 \setminus U_2 \\ x & \text{if} \quad x \in h(C_i) \end{cases}$$

Clearly r_i is continuous.

Now let $E_1 = (U \cup U_1 \cup U_2) \setminus U_2$ and $E_2 = (U \cup U_1 \cup U_2) \setminus U_1$. Then for $i = 1,2$ we have that $P_i = (U \setminus U_1 \setminus U_2) \cup h(C_i) = (E_i \setminus (U_1 \cup U_2)) \cup h(C_i)$. Hence, P_i is a closed subset of E_i. By (4.B.24), there are open subsets $V_1 \subset E_1$ and $V_2 \subset E_2$ such that $P_1 \subset V_1 \subset E_1$ and $P_2 \subset V_2 \subset E_2$, and there are extensions $\hat{r}_1$ and $\hat{r}_2$ of r_1 and r_2 respectively, such that for $i = 1,2$, $\hat{r}_i\colon V_i \to h(C_i)$. We have that $h(C_1 \cup C_2) \subset C \cup U_1 \cup U_2 = E_1 \cup E_2$, $E_1 = (E_1 \cup E_2) \setminus U_2$, and $E_2 = (E_1 \cup E_2) \setminus U_1$, where the E_i's are closed in $E_1 \cup E_2$. Furthermore, $h(C_1 \cup C_2) \cap E_i = h(C_i \cap C_2) \subset P_i \subset V_i$. It follows from (4.B.27) that $V = V_1 \cup V_2$ is a neighborhood of $h(C_1 \cup C_2)$ in $E_1 \cup E_2 = U \cup U_1 \cup U_2$, and since this latter set is open, V is a neighborhood of $h(C_1 \cup C_2)$ in Y. Note now that $U \setminus U_1 \setminus U_2 \subset P_1 \cap P_2 \subset V_1 \cap V_2 \subset E_1 \cap E_2 = ((U \setminus U_2) \setminus U_1) \cap ((U \setminus U_1) \cup U_2) = U \setminus U_1 \setminus U_2$, and hence $V_1 \cap V_2 = E_1 \cap E_2 = U \setminus U_1 \setminus U_2$. Since the E_i's are closed in $E_1 \cup E_2$, we have that $\overline{V}_1 \cap (V_1 \cup V_2) = V_1 \cup (\overline{V}_1 \cap V_2) \subset V_1 \cup (E_1 \cap E_2) = V_1$. Finally, since the V_i's are closed in V and by the definition of the $\hat{r}_i$'s, it follows that the function $R\colon V \to h(C_1 \cup C_2)$ defined by

$$R(x) = \begin{cases} \hat{r}_1(x) & \text{if} \quad x \in V_1 \\ \hat{r}_2(x) & \text{if} \quad x \in V_2 \end{cases}$$

is a retraction of V onto $h(C_1 \cup C_2)$.

(4.B.29) Remark. Once we have (10.C.3), it will follow that (4.B.28) may be restated so that if X is a compact metric space, then the conclusion will be that $X = C_1 \cup C_2$ is an ANR_M.

C. FURTHER RESULTS CONCERNING NORMAL SPACES

(4.C.1) Definition. If $\mathscr{A} = \{A_\alpha \mid \alpha \in \Lambda\}$ is a cover of a space X, then an *open shrinkage* of $\mathscr{A}$ is an open cover $\{U_\alpha \mid \alpha \in \Lambda\}$ of X such that $\overline{U}_\alpha \subset A_\alpha$ for each $\alpha \in \Lambda$.

Certain open covers of normal spaces are amenable to being shrunk. The next theorem is not only a remarkable result, but is actually quite useful in a number of distinct contexts. Its proof is dependent on the axiom of choice, which appears in its well-ordering guise.

(4.C.2) Theorem. Suppose that $\mathscr{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ is an open cover of a normal space X such that each point in X lies in at most a finite number of the U_α (such a cover is called *point finite*). Then there is an open cover $\mathscr{V} = \{V_\alpha \mid \alpha \in \Lambda\}$ that is an open shrinkage of $\mathscr{U}$.

Proof. We assume that Λ has been well ordered with first element α_0, second element α_1, etc. We shall use transfinite induction to construct a family $\{V_\alpha \mid \alpha \in \Lambda\}$ of open sets such that

(i) for each $\alpha \in \Lambda$, $\overline{V}_\alpha \subset U_\alpha$, and

(ii) for each $\alpha \in \Lambda$, the family $\{V_\lambda \mid \lambda < \alpha\}$ together with the family $\{U_\delta \mid \delta \geq \alpha\}$ cover X. We then show that the family $\{V_\alpha \mid \alpha \in \Lambda\}$ is an open cover of X.

For purposes of motivation observe that $X \setminus \bigcup_{\alpha_0 < \alpha} U_\alpha$ is closed and is contained in U_{α_0}. Hence by (4.A.4), there is an open set V_{α_0} such that $(X \setminus \bigcup_{\alpha_0 < \alpha} U_\alpha) \subset V_{\alpha_0} \subset \overline{V}_{\alpha_0} \subset U_{\alpha_0}$. In order to replace U_{α_1} by a similar construction, we make use of the fact that $\{V_{\alpha_0}, U_{\alpha_1}, \ldots\}$ is an open cover of X.

Now assume that for each $\beta < \alpha$, we have constructed an open set V_β such that $\overline{V}_\beta \subset U_\beta$ and that $\{V_\beta \mid \beta < \alpha\}$ and $\{U_\gamma \mid \delta \geq \alpha\}$ form a cover of X. We show that this permits the construction of an open set V_α such that

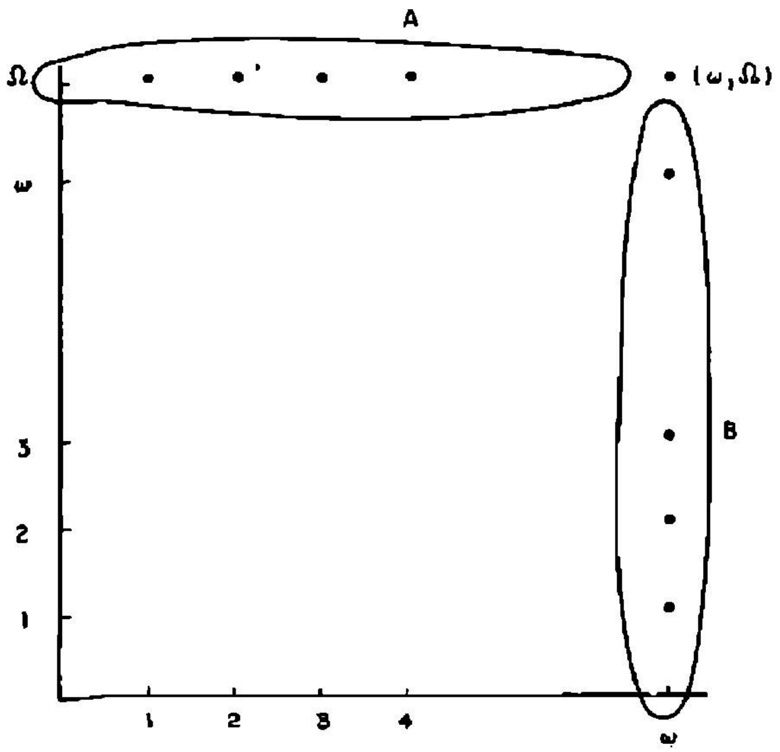
(i) $\overline{V}_\alpha \subset U_\alpha$, and

(ii) the families $\{V_\beta \mid \beta \leq \alpha\}$ and $\{U_\gamma \mid \delta > \alpha\}$ form a cover of X.

Note that $X \setminus (\cup\{V_\beta \mid \beta < \alpha\} \bigcup \cup \{U_\gamma \mid \gamma \geq \alpha\}) \subset U_\alpha$; hence, by the normality of X, there is an open set V_α such that $X \setminus (\cup\{V_\beta \mid \beta < \alpha\} \bigcup \cup \{U_\gamma \mid \gamma > \alpha\}) \subset V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$. Then the combined collections $\{V_\beta \mid \beta \leq \alpha\}$ and $\{U_\gamma \mid \gamma < \alpha\}$ cover X, and this completes the inductive step. To see that the family $\{V_\beta \mid \beta \in \Lambda\}$ covers X, suppose that $x \in X$ and let $U_{\gamma_1}, U_{\gamma_2}, \ldots, U_{\gamma_n}$ be the sets in $\mathscr{U}$ that contain x. Assume that $\gamma_n > \gamma_i$ for $i = 1, 2, \ldots, n - 1$. Then $\{V_{\alpha_0}, \ldots, V_{\gamma_1}, \ldots, V_{\gamma_n}, U_{\gamma_{n+1}}, \ldots\}$ covers X, which implies that $x \in V_{\alpha_0} \cup \cdots \cup V_{\gamma_n}$.

A property of a given space is *hereditary* if and only if every subspace of the space also has the property. For example, any subspace of a metrizable space is still metrizable, subspaces of Hausdorff spaces are Hausdorff, and subspaces of first countable spaces are first countable. As we see next, normality is not hereditary.

Ordinal numbers are useful in the construction of several important examples in topology. Certain basic features of ordinals are given in the Chapter 0, and a more thorough introduction may be found in Monk [1969]. We denote the first infinite ordinal by ω and the first uncountable ordinal by Ω. Let $X = [0,\omega] = \{\alpha \mid \alpha$ is an ordinal and $0 \leq \alpha \leq \omega\}$, and let $Y = [0,\Omega] = \{\alpha \mid \alpha$ is an ordinal and $0 \leq \alpha \leq \Omega\}$. Each of these spaces is given the order topology. The reader may verify (see exercise below) that both X and Y are compact T_2 spaces. Consequently, $X \times Y$ is a compact Hausdorff space and hence by (4.A.6) is normal. We shall exhibit a subspace of $X \times Y$ that is not normal. (This, incidentally, implies that $X \times Y$ is not completely normal (4.A.5) and hence not metrizable (4.A.1).) For a simple example, see Problem C-5. The space $X \times Y$ with the product topology is called the *Tihonov plank*. The desired subspace is obtained by simply removing the point (ω,Ω) from the product space. Let $W = (X \times Y) \setminus \{(\omega,\Omega)\}$. Then $A = \{(\alpha,\Omega) \mid 0 \leq \alpha < \omega\}$ and $B = \{(\omega,\beta) \mid 0 \leq \beta < \Omega\}$ are disjoint closed subsets of W. We show that it is impossible to enclose A and B in disjoint open subsets. Suppose that U is any open set containing A. For each $\alpha \in X \setminus \{\omega\}$, let β_α be the least element of Y such that $(\alpha,\beta) \in U$ whenever $\beta > \beta_\alpha$. Since the collection $S = \{\beta_\alpha \mid \alpha \in X \setminus \{\omega\}\}$ is countable, the least upper bound of S, s_0, will be strictly less than Ω (0.E.4). However, $\{(\alpha,\beta) \mid \beta > s_0, \alpha < \omega\}$ is a subset of U. Now suppose that V is an open set containing B. Choose an ordinal β such that $s_0 < \beta < \Omega$.



Then of course, $(\omega,\beta) \in B$, and it should now be clear that points "close to" (ω,β) must lie in both U and V; hence, W is not normal.

(4.C.3) Exercise. Show that $[0,\omega]$ and $[0,\Omega]$ are compact. (Hint: Consider $A = \{\alpha \in [0,\omega] \mid [0,\alpha] \text{ is compact}\}$ and show that if $\lambda = \sup A$, then $\lambda \in A$, and $\lambda = \omega$.)

D. THE SEPARATION AXIOMS

A number of properties other than normality have been given sufficient attention to warrant special mention. We examine a few of these.

(4.D.1) Definition. A topological space X is *regular* if and only if whenever $x \in X$ and F is a closed subset of X not containing x, there are disjoint open sets U and V such that $x \in U$ and $F \subset V$.

(4.D.2) Exercise. Show that a space X is regular if and only if for each point $x \in X$ and for each open set U containing x, there is an open set V such that $x \in V \subset \overline{V} \subset U$.

Note the similarity between this definition (4.D.1) and that for normal spaces. In fact, it should be clear that normal T_1 spaces are regular.

Requiring that points be closed is obviously not a particularly strong imposition on a topological space, but a still weaker condition is all that is needed to define an even lower form of topological life. A space X is T_0 if and only if whenever $x,y \in X$, an open set may be found that contains either x or y but not the other; there is no guarantee, however, as to whether the open set will contain specifically x or y.

A heirarchy of spaces (in increasingly restrictive structure) may be set up as follows:

T_0 space—defined above
T_1 space—defined in Chapter 1
T_2 space—defined in Chapter 1
T_3 space—a regular T_1 space
T_4 space—a normal T_1 space
T_5 space—a completely normal T_1 space
Metric spaces

The T_i's are frequently referred to as separation axioms; basically, they reflect the degree to which points or sets may be kept apart by open sets. Strangely enough, an inordinate amount of mathematical energy has been expended in defining separation conditions that lie somewhere between the ones listed above ($1\frac{1}{2}$, $2\frac{3}{8}$, etc.); however, apart from a few notable exceptions, this sort of exercise would appear to be about as significant as it is interesting.

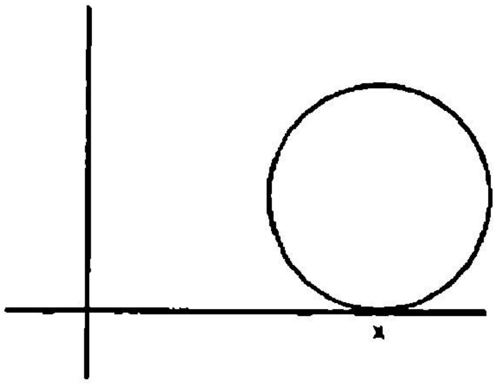
(4.D.3) Exercise. Show that metric $\Rightarrow T_5 \Rightarrow T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.

In the remainder of this chapter we shall see that none of the above implications is reversible.

(4.D.4) Examples.

1. Any set (consisting of more than one point) with the indiscrete topology fails to be T_0.
2. The set $\mathbf{R}^1$ with the open ray topology is T_0 but not T_1.
3. The set $\mathbf{R}^1$ with the finite complement topology is T_1 but not T_2.
4. Let $X = \mathbf{R}^1$. For each point $x \in \mathbf{R}^1$ and each open interval U containing x, set $\hat{U} = \{y \in U \mid y \text{ is rational}\} \cup \{x\}$. The family of all such sets $\hat{U}$ forms a basis for a topology that is T_2 but not regular.
5. (*Tangent disk or bubble topology*). Let $X = \{(x,y) \in \mathbf{R}^2 \mid y \geq 0\}$ and let $L = \{(x,y) \in \mathbf{R}^2 \mid y = 0\}$. To form a topology for X, let $\mathscr{B}$ be the set of all $U \subset X$ such that either

(i) $U \subset X \setminus L$ and U is a member of the usual topology for $\mathscr{E}^2$, or
(ii) U is of the form $\{x\} \cup D$ where $x \in L$ and D is an open disk tangent to L at the point x.



Then $\mathscr{B}$ is a basis for a topology for X that is T_3 but not T_4. The Baire Category Theorem is used to show that X is not normal. Let $A = \{(x,0) \in L \mid x \text{ is rational}\}$ and $B = \{(x,0) \in L \mid x \text{ is irrational}\}$. Certainly, A and B are disjoint closed subsets of X (any subset of L is closed in X). Suppose that U and V are open subsets of X containing A and B, respectively. We show that U and V must intersect. For each $(x,0) \in B$, let D_x be a disk of radius r_x that lies entirely in V and is tangent to L at $(x,0)$. For $n = 1, 2, \ldots$, set $W_n =$

$\{(x,0) \in B \mid r_x > 1/n\}$. Then $L = A \cup (\bigcup_{n=1}^{\infty} W_n)$. Since L (with the usual topology for $\mathscr{E}^1$) is a complete metric space, it follows from [illegible] that some W_n contains an open interval. However, since every open interval in L contains a rational point, U and V must intersect.

(4.D.5) Exercise. Show that the space X in Example 5 is regular, T_1, and separable. Show that L with the relative topology is not separable.

Under certain conditions regular spaces are normal.

(4.D.6) Theorem. A regular, Lindelöf space X is normal.

Proof. Suppose that A and B are disjoint closed subsets of X. By regularity, for each $a \in A$, there is an open set U_a containing a such that $\overline{U_a} \cap B = \varnothing$ and for each $b \in B$, there is an open set V_b containing b, whose closure misses A. Clearly, we have that $A \subset \bigcup\{U_a \mid a \in A\}$ and $B \subset \bigcup\{V_b \mid b \in B\}$. Since both A and B are Lindelöf (closed subspaces of a Lindelöf space are Lindelöf), countable subcovers $\{U_1, U_2, \ldots\}$ and $\{V_1, V_2, \ldots\}$ can be found for A and B respectively. Now we separate the U's from the V's.

Let $W_1 = U_1$ and $Z_1 = V_1 \setminus \overline{W_1}$. Let $W_2 = U_2 \setminus \overline{Z_1}$ and $Z_2 = V_2 \setminus (\overline{W_1 \cup W_2})$. In general, set $W_n = U_n \setminus (\overline{Z_1 \cup \cdots \cup Z_{n-1}})$ and $Z_n = V_n \setminus (\overline{W_1 \cup \cdots \cup W_n})$. It should be apparent that $W = \bigcup_{i=1}^{\infty} W_i$ and $Z = \bigcup_{i=1}^{\infty} Z_i$ are disjoint and open.

(4.D.7) Exercise. Let X be any uncountable set and let $x^* \in X$. Define a subset U of X to be open if and only if (i) $X \setminus U$ is countable, or (ii) $x^* \in X \setminus U$. Show that X with this topology is T_5, but that X is not first countable and, hence, is not metrizable.

We mention one other type of space that is of special interest in analysis. Its definition is reminiscent of Uryson's lemma.

(4.D.8) Definition. A T_1 space X is a *Tihonov space* (or $T_{3\frac{1}{2}}$) if and only if for each closed subset K of X and for each point $x \in X \setminus K$, there is a continuous function $f : X \to \mathbf{I}$ such that $f(x) = 0$ and $f(K) = 1$.

If the requirement that X be T_1 is removed, then X is said to be *completely regular*.

(4.D.9) Exercise. Show that $T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3$.

We shall presently describe a $T_{3\frac{1}{2}}$ space that is not T_4. For a T_3 space that fails to be $T_{3\frac{1}{2}}$, the reader may consult Steen and Seebach [1970] or attempt to devise an easier example.

(4.D.10) ***Exercise.*** Suppose that X and Y are topological spaces. Show that $X \times Y$ has property Q if and only if both X and Y possess Q, where Q ranges over the following properties: (1) T_0; (2) T_1; (3) T_2; (4) T_3; (5) $T_{3\frac{1}{2}}$.

(4.D.11) ***Exercise.*** Show that the following properties are hereditary: T_0; T_1; T_2; T_3; $T_{3\frac{1}{2}}$.

Observe that in both of the preceding exercises, we came to a halt before T_4 was reached. That "normal" space is perhaps a misnomer might well be argued from the standpoint that normality is neither hereditary nor does it behave decently under product formation. To illustrate the latter point, we consider two of the most pleasing (in their simplicity) and rewarding examples in topology.

(4.D.12) ***Example (Half-Open Interval Topology).*** Let $X = \mathbf{R}^1$ and let the topology for X be determined by a basis consisting of all half-open intervals of the form $[a,b)$, where $a < b$.

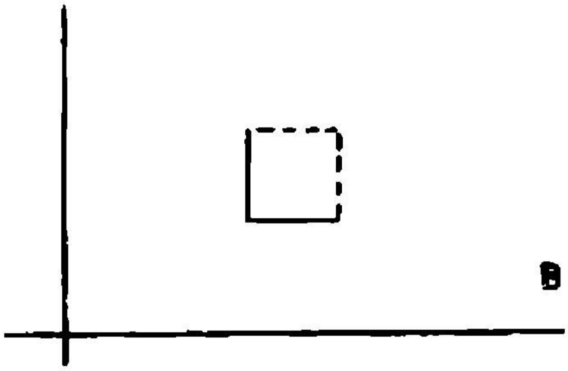
We show that X is T_5. That X is T_1 is clear. Suppose, then, that A and B are subsets of X such that $\bar{A} \cap B = \varnothing$ and $\bar{B} \cap A = \varnothing$. For each $x \in A$, there is a half-open interval $[x,e_x)$ contained entirely in $X \setminus \bar{B}$, and similarly for each $y \in B$ there is a half-open interval $[y,e_y) \subset X \setminus \bar{A}$. It is readily seen that the sets $U = \bigcup \{[x,e_x) \mid x \in A\}$ and $V = \bigcup \{[y,e_y) \mid y \in B\}$ are disjoint open sets containing A and B, respectively.

(4.D.13) ***Exercise.*** Show that $\mathbf{R}^1$ with the half-open interval topology is first countable, separable, and Lindelöf.

(4.D.14) ***Exercise.*** Show that $\mathbf{R}^1$ with the half-open interval topology is not second countable, and hence, by (4.D.13) and (1.G.10), is not metrizable.

Even more interesting is the next example.

(4.D.15) ***Example (Sorgenfrey's Half-Open Square Topology).*** With X defined to be the topological space described in the previous example, let $Y = X \times X$ (with the product topology). A typical basis element for Y is illustrated on page 121. Since X was seen to be T_5, it follows from (4.D.3), (4.D.9), and (4.D.10) that Y is $T_{3\frac{1}{2}}$.



(4.D.16) ***Exercise.*** Let Y be the space defined in the previous example and let $L \subset Y$ be the diagonal line $\{(x,y) \mid y = -x\}$. Show that any subset of L is closed, and use an argument similar to that which was employed in connection with the bubble topology to demonstrate that Y is not normal. Consequently, Y is a $T_{3\frac{1}{2}}$ space that is not T_4.

PROBLEMS

Section A

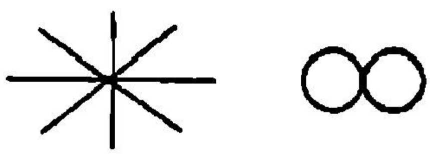
1. (a) Show that if X is a normal space, Y is a topological space and if $f: X \to Y$ is continuous, closed, and onto, then Y is normal.
   (b) Is normality a continuous invariant?
   (c) Suppose that $X_1, X_2, \ldots, X_n$ are topological spaces. Show that if $\prod_{i=1}^{n} X_i$ is normal, then X_i is normal for each i.
2. Suppose that $C_1, C_2, \ldots$ are closed subsets of a normal space X such that for each $k = 1, 2, \ldots, C_k \cap (\overline{\bigcup\{C_n \mid n \neq k\}}) = \varnothing$. Show that there are disjoint open sets $U_1, U_2, \ldots$ such that $C_n \subset U_n$ for each n.
3.* Does there exist a countable connected normal space that is T_1?
4. Show that a topological space X is normal if and only if for each pair of disjoint closed subsets A and B there are open sets U and V containing A and B respectively such that $\overline{U} \cap \overline{V} = \varnothing$.
5. Suppose that X is a 0-dimensional second countable set and that A and B are closed disjoint subsets of X. Show that there is a separation of X, (U,V), such that $A \subset U$ and $B \subset V$.
6. Suppose that (X,d) is a metric space. The metric d is said to be *normal* if and only if whenever A and B are closed disjoint subsets, then $d(A,B) > 0$. Show that if d is normal, then (X,d) is complete.
7.* Suppose that (X,d) is a metric space. Show that d is normal (see problem

6) if and only if every continuous function $f : (X,d) \to (Y,d')$ is uniformly continuous.

8. Show that if a space X is the union of a finite number of closed, normal subspaces, then X is normal.

Section B

1. Show that subspaces of perfectly normal spaces are perfectly normal.
2. Show that $[0,1)$ is an AR.
3. Prove Uryson's lemma using Tietze's extension theorem.
4. Suppose that X is a normal space and that A and B are disjoint closed subsets of X. Show that there is a continuous map $f : X \to [a,b]$ such that $f(A) = a$ and $f(B) = b$.
5. Let $A = \{0,1\}$. Show that a space X is path connected if and only if each continuous map $f : A \to X$ can be extended to $[0,1]$.
6. Suppose that (X,d) is a metric space. Show that (X,d) is perfectly normal.
7.* Show that a retract of a T_2 space is closed.
8. Show that if a (finite) product of ANR's is normal, then it is an ANR.
9. Assume the $(n+1)$-dimensional Brouwer fixed point theorem. Use (4.B.17) to show that $\mathcal{S}^n$ is not an AR.
10. Find a closed subset of a normal space that is not a G_δ.
11.* Show that a pseudometric space is perfectly normal.
12. Suppose that A is a retract of X and that B is a retract of Y. Show that $A \times B$ is a retract of $X \times Y$.
13. Show that a retract of an AR is an AR.
14. Show that an open normal subset of an ANR is an ANR.
15. Do the following subspaces of the plane $\mathscr{E}^2$ have the fixed point property?



Section C

1. Prove the converse of (4.C.2).
2. Show that closed subsets of a normal space are normal.
3. Show that a space X is normal if and only if for each finite open cover $\{U_1, U_2, \ldots, U_n\}$ of X there is an open cover of X, $\{V_1, V_2, \ldots, V_n\}$, with the property that $\bar{V}_i \subset U_i$ for $i = 1, 2, \ldots, n$. Avoid using the axiom of choice in your proof.

4. Suppose that X is a normal space and that $A_1, A_2, \ldots, A_n$ are closed subsets of X such that $\bigcap_{i=1}^{n} A_i = \varnothing$. Show that there are open sets $V_1, V_2, \ldots, V_n$ such that $A_i \subset V_i$ and $\bigcap_{i=1}^{n} \bar{V}_i = \varnothing$.
5. Let X_1 be an uncountable set and X_2 be any infinite set. Suppose that X_1 and X_2 have discrete topologies and let $X_1^* = X_1 \cup \{\infty_1\}$ and $X_2^* = X_2 \cup \{\infty_2\}$ be the corresponding one-point compactifications. Show that $X_1^* \times X_2^*$ is normal, but $(X_1^* \times X_2^*) \setminus \{\infty_1, \infty_2\}$ is not.

Section D

1. Show that a regular second countable space is completely normal.
2. Show that a normal space is completely regular if and only if it is regular.
3. Suppose that X is a T_2 space such that each $x \in X$ has a neighborhood V with the property that $\bar{V}$ is regular. Show that X is regular.
4.* Show that there is no countable connected $T_{3\frac{1}{2}}$ space containing more than one point.
5.* Let $X = \mathbf{Z}^+$. For each $a,b \in \mathbf{Z}^+$ which are relatively prime, let $B_{(a,b)} = \{an + b \mid n \in \mathbf{Z}^+ \cup \{0\}\}$.
   (i) Show that the collection of all such $B_{(a,b)}$ forms a basis for a T_2 topology $\mathscr{U}$ for $\mathbf{Z}^+$.
   (ii) Show that if $U = B_{(a,b)}$ and $V = B_{(c,d)}$, then $a \cdot c$ is an accumulation point of both U and V.
   (iii) Show that $(\mathbf{Z}^+, \mathscr{U})$ is connected.
6. Suppose that X is a first countable space. Show that X is T_2 if and only if every convergent sequence has a unique limit.
7. Show that every F_σ subset of a compact space is Lindelöf.
8. (i) Show that locally compact T_1 spaces are regular. [Hint: Use the one-point compactification.]
   (ii) Show that locally compact T_0 spaces are regular. [Hint: Locally compact T_0 spaces are T_2.]
9.* Suppose that X is a completely regular space, A is a compact subset of X, and B is a closed subset of X disjoint from A. Show that there is a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.
10. Show that F_σ subsets of a T_4 space are T_4.
11.* Suppose that X is an infinite T_2 space. Show that X has an infinite discrete subspace.
12. Suppose that X is a T_3 space and f is a continuous, open and closed function from X onto Y. Show that Y is T_2.

13. A space X is an *Uryson space* if and only if whenever $x,y \in X$ and $x \neq y$, there are open subsets U and V of X such that $x \in U$, $y \in V$, and $\overline{U} \cap \overline{V} = \varnothing$. Show that T_3 spaces are Uryson spaces and find an Uryson space that is not regular.
14.* Show that perfectly normal spaces are completely normal.
15.* Is Example (4.D.12) perfectly normal?
16. Show that every T_1 space $(X,\mathscr{U})$ can be embedded in a separable T_1 space. [Hint: Let $Y = X \cup \mathbf{Z}^+$ (disjoint union) and define a topology $\mathscr{V}$ for Y by $\mathscr{V} = \{\varnothing\} \cup \{U \cup A \mid U \in \mathscr{U},\ A \subset \mathbf{Z}^+, \text{ and } \mathbf{Z}^+ \setminus A \text{ is finite}\}$.]
17. A topological space X is $T_{1\frac{1}{2}}$ if and only if for each pair of disjoint compact subspaces A and B of X, there are open sets U and V such that $A \subset U$, $B \subset V$, and $A \cap V = \varnothing$ and $B \cap U = \varnothing$. Show that $T_2 \Rightarrow T_{1\frac{1}{2}} \Rightarrow T_1$.
18. Suppose that X is a set. A *uniform structure* for X is a nonempty family $\mathscr{U}$ of subsets of $X \times X$ (relations on X) such that:

    (i) for each $U \in \mathscr{U}$, $id_X \subset U$,
    (ii) if $U \in \mathscr{U}$, then $U^{-1} \in \mathscr{U}$,
    (iii) if $U \in \mathscr{U}$ then there is a $V \in \mathscr{U}$ such that $V \circ V \subset U$,
    (iv) if $U,V \in \mathscr{U}$, then $U \cap V \in \mathscr{U}$, and
    (v) if $U \in \mathscr{U}$ and $U \subset V \subset X \times X$, then $V \in \mathscr{U}$.

    (Inverses and compositions of relations are defined in an analogous manner to that of functions).

    The *uniform topology* $T_{\mathscr{U}}$ on X determined by the uniform structure $\mathscr{U}$ is defined by: $T_{\mathscr{U}} = \{W \subset X \mid \text{for each } x \in W, \text{ there is a } U \in \mathscr{U} \text{ such that } U(x) \subset W\}$, where $U(x) = \{z \in X \mid (x,z) \in U\}$.

    (a) Show that if X is a set with a uniform structure $\mathscr{U}$, then $T_{\mathscr{U}}$ is a topology.
    (b) Find a uniform structure that yields the usual topology for $\mathbf{R}^1$, the discrete topology, and the indiscrete topology.
19. Show that if $\mathscr{U}$ is a uniform structure (see preceding problem) for a set X, then the topological space $(X,T_{\mathscr{U}})$ is completely regular. [Hint: Suppose that $x \in X$ and that C is a closed subset of X not containing x. Then there is a sequence $\{U_n \mid n \in \mathbf{Z}^+\}$ of elements of $\mathscr{U}$ such that $U_n = U_n^{-1}$, and $U_n \circ U_n \subset U_{n-1}$. Let $V_0 = id_X$. For each dyadic rational $r \in (0,1]$ expressed in the form $\sum_{k=1}^{N} 2^{-n_k}$ where $0 \le n_1 < n_2 < \quad < n_N$, let $V_r = U_{n_N} \circ U_{n_{N-1}} \circ \quad U_{n_2} \circ U_{n_1}$. If $r \le s$, then $V_r \subset V_s$ (first show that $U_n \circ V_{m2^{-n}} \subset V_{(m+1)2^{-n}}$). Define $\phi : X \to [0,1]$ by

    $$\phi(y) = \begin{cases} \sup\{r \mid x \notin V_r(x)\}, & \text{if } y \neq x \\ 0, & \text{if } y = x \end{cases}$$

    and show that ϕ is the desired function.]

20. Suppose that X and Y are topological spaces and that $f : X \to Y$. The function f satisfies property * if and only if for each open cover $\mathscr{U}$ of Y, $\{\operatorname{int} f^{-1}(U) \mid U \in \mathscr{U}\}$ is an open cover of X.
    (a) Show that every continuous function $g : X \to Y$ satisfies property *.
    (b) Show that if Y is T_1, then a function $g : X \to Y$ is continuous whenever g satisfies property *.
    (c) Show that property * is not equivalent to continuity.