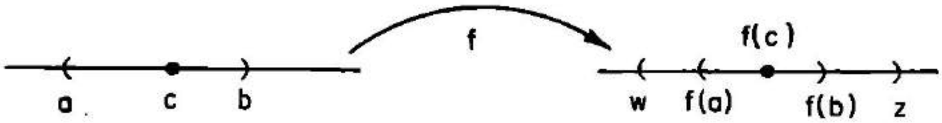
Chapter 1

THE BASIC CONSTRUCTS

A. TOPOLOGICAL SPACES AND CONTINUITY

To a significant extent, topology may be viewed as the study of the metamorphosis that sets or spaces undergo when they fall under the influence of continuous functions. The idea of continuity should be familiar to the student from his experience with calculus, but the notion of space, or more accurately topological space, perhaps remains a rather vague concept. The reader is certainly aware of some special sets such as the real line $\mathbf{R}^1$, the plane $\mathbf{R}^2$, closed intervals, spheres, balls, doughnuts (tori), etc. Our first task is to give each of these sets a precise structure that will enable us to define the continuity of functions between any two of them.

The reader undoubtedly remembers the $\varepsilon - \delta$ concept of continuity employed in calculus, wherein a function $f : \mathbf{R}^1 \to \mathbf{R}^1$ is said to be continuous at a point $c \in \mathbf{R}^1$ if and only if for each positive number ε, there is a positive number δ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. This definition can be stated in more geometric terms as follows: a function f is continuous at a point c if and only if for each open interval (w,z) containing $f(c)$, there is an open interval (a,b) containing c such that $f((a,b)) \subset (w,z)$.



The reader should verify that the two definitions are indeed equivalent. The advantage of the latter definition is that if we could establish the notion of "open interval" or, more generally, open subset of a given set, then our definition could be generalized to functions between arbitrary sets.

We begin by defining an open set in $\mathbf{R}^1$ to be any set that can be written as the union of a family of open intervals (in particular, this will make the empty set open, since it is the union over an empty family of open intervals). It is easy to verify that

(i) the union of any collection of open sets is open, and
(ii) the intersection of any finite number of open sets is open.

We now say that a function $f : \mathbf{R}^1 \to \mathbf{R}^1$ is continuous at a point c if and only if for each open set U containing $f(c)$, there is an open set V containing c such that $f(V) \subset U$. The reader should check that this definition of continuity is compatible with the previous ones. The appropriate designation of open subsets of a set X determines what is called a topological structure for X. Much of the beauty of topology stems from the fact that properties (i) and (ii) given above are all that is needed to determine whether or not a subfamily $\mathscr{U}$ of $\mathscr{P}(X)$ (the collection of all subsets of X) is a topological structure, and that the idea of continuity is so easily defined using this structure. Formally, then, we have the following definition.

(1.A.1) Definition. Suppose that X is a set and that $\mathscr{U}$ is a collection of subsets of X. The pair $(X,\mathscr{U})$ is a *topological space* if and only if the following conditions are satisfied:

(i) if $\{U_\alpha \mid \alpha \in \Lambda\}$ is a collection of members of $\mathscr{U}$, then $(\bigcup_{\alpha\in\Lambda} U_\alpha) \in \mathscr{U}$;
(ii) if $\{U_\alpha \mid \alpha \in K\}$ is a finite subcollection of members of $\mathscr{U}$, then $(\bigcap_{\alpha\in K} U_\alpha) \in \mathscr{U}$;
(iii) $\varnothing \in \mathscr{U}$ and $X \in \mathscr{U}$.

The family $\mathscr{U}$ is called a *topology* for X, and the elements of $\mathscr{U}$ are called the *open sets* of $(X,\mathscr{U})$.

Note that if each point in a set U is contained in an open set $V \subset U$, then U is open.

The reader should observe that the family of unions of open intervals in $\mathbf{R}^1$ yields a topology for $\mathbf{R}^1$.

(1.A.2) Definition. If $\mathscr{U}$ and $\mathscr{V}$ are topologies for a set X, then $\mathscr{U}$ is *smaller* than $\mathscr{V}$ (or $\mathscr{V}$ is *larger* than $\mathscr{U}$) if and only if $\mathscr{U} \subset \mathscr{V}$.

(1.A.3) Definition. Suppose that $(X,\mathscr{U})$ and $(Y,\mathscr{V})$ are topological spaces and that $f : X \to Y$. Then f is *continuous at a point* $c \in X$ if and only if whenever $f(c) \in V \in \mathscr{V}$, there is a $U \in \mathscr{U}$ containing c such that $f(U) \subset V$. If f is continuous at each point $c \in X$, we say that f is *continuous.*

It is often convenient to express continuity in the following terms.

(1.A.4) Theorem. Suppose that $(X,\mathscr{U})$ and $(Y,\mathscr{V})$ are topological spaces and that $f : X \to Y$. Then f is continuous if and only if for each $V \in \mathscr{V}$, $f^{-1}(V) \in \mathscr{U}$.

Proof. Suppose that f is continuous and that $V \in \mathscr{V}$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$, and since f is continuous, there is a set $U_x \in \mathscr{U}$ such that $x \in U_x$ and $f(U_x) \subset V$. Hence we have that $U_x \subset f^{-1}(V)$. It follows from (i) of (1.A.1) that $f^{-1}(V)$ is open, since $f^{-1}(V) = \bigcup \{U_x \mid x \in f^{-1}(V)\}$.

Conversely, suppose that $f^{-1}(V) \in \mathscr{U}$ for each $V \in \mathscr{V}$. Let $x \in X$ and suppose that $f(x) \in V \in \mathscr{V}$. Then we have that $f^{-1}(V) \in \mathscr{U}$, $x \in f^{-1}(V)$, and $f(f^{-1}(V)) \subset V$. Consequently, f is continuous at x, and since x was arbitrary, we conclude that f is continuous.

Frequently, the topology $\mathscr{U}$ for a set X proves to be rather unwieldy; this situation is rendered more tractable by the introduction of the notion of a basis.

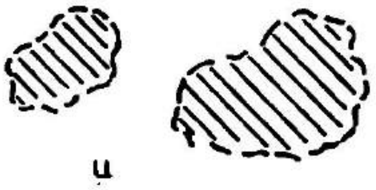
(1.A.5) Definition. If $(X,\mathscr{U})$ is a topological space, then a *basis* for $\mathscr{U}$ is a subcollection $\mathbf{B}$ of $\mathscr{U}$ with the property that if $x \in X$ and U is an open set containing x, then there is a $V \in \mathbf{B}$ such that $x \in V \subset U$. In other words, each $U \in \mathscr{U}$ can be written as a union of sets in $\mathbf{B}$.

For example, the open intervals of $\mathbf{R}^1$ constitute a basis for the topology determined by the open sets in $\mathbf{R}^1$ discussed earlier. This topology for $\mathbf{R}^1$ will henceforth be referred to as the *usual* topology for $\mathbf{R}^1$ (other topological structures for the set of real numbers are to be introduced shortly).

Let us next define a reasonable topology for the plane $\mathbf{R}^2$. Here it is particularly advantageous to define a topology in terms of a basis. For each point $z = (x,y) \in \mathbf{R}^2$ and each $\varepsilon > 0$, let

$$S_\varepsilon(z) = \{(u,v) \in \mathbf{R}^2 \mid \sqrt{(x-u)^2 + (y-v)^2} < \varepsilon\}$$

Then $\mathbf{B} = \{S_\varepsilon(z) \mid z \in \mathbf{R}^2 \text{ and } \varepsilon > 0\}$ is a basis for a topology $\mathscr{U}$ for $\mathbf{R}^2$, where $\mathscr{U}$ consists of all possible unions of members of $\mathbf{B}$. Typical elements from $\mathbf{B}$ and $\mathscr{U}$ are indicated below. The sets $S_\varepsilon(x)$ are called *open disks in* $\mathbf{R}^2$.



That $\mathbf{B}$ is a basis for a unique topology for $\mathbf{R}^2$ is a consequence of the following theorem and exercise.

(1.A.6) Theorem. Suppose that $\mathbf{B}$ is a collection of subsets of a set X. Then $\mathbf{B}$ is a basis for some topology for X if and only if $X = \bigcup \{B \mid B \in \mathbf{B}\}$, and whenever $B_1, B_2 \in \mathbf{B}$ and $x \in B_1 \cap B_2$, there is a $B \in \mathbf{B}$ such that $x \in B \subset B_1 \cap B_2$.

Proof. Suppose that $\mathbf{B}$ is a basis for a topology $\mathscr{U}$ on X. If $x \in X$, there is a $U \in \mathscr{U}$ such that $x \in U$. Since $\mathbf{B}$ is a basis, there is a $B \in \mathbf{B}$ such that $x \in B \subset U$, and hence $X = \bigcup \{B \mid B \in \mathbf{B}\}$. If $B_1, B_2 \in \mathbf{B}$ and $x \in B_1 \cap B_2$, then since B_1 and B_2 also belong to $\mathscr{U}$, we have that $B_1 \cap B_2 \in \mathscr{U}$. Consequently, there is a $B \in \mathbf{B}$ such that $x \in B \subset B_1 \cap B_2$.

Conversely, let $\mathscr{U} = \{U \subset X \mid U \text{ is a union of members of } \mathbf{B}\}$. Clearly, $X, \varnothing \in \mathscr{U}$ and arbitrary unions of members of $\mathscr{U}$ are in $\mathscr{U}$. If $U, V \in \mathscr{U}$ and $x \in U \cap V$, then there are sets $B, B_1, B_2 \in \mathbf{B}$ such that $x \in B \subset B_1 \cap B_2 \subset U \cap V$, and hence $U \cap V$ is a union of members of $\mathbf{B}$.

(1.A.7) Exercise. Suppose that $\mathbf{B}$ is a basis for two topologies $\mathscr{U}$ and $\mathscr{U}'$ on a set X. Show that $\mathscr{U} = \mathscr{U}'$, and hence that a basis determines a unique topology.

A given topology, however, may have distinct bases. For example, for each $z = (x,y) \in \mathbf{R}^2$ and each $\varepsilon > 0$, let $T_\varepsilon(z) = \{(u,v) \in \mathbf{R}^2 \mid |x - u| < \varepsilon \text{ and } |y - v| < \varepsilon\}$. Then $\mathbf{B}' = \{T_\varepsilon(z) \mid z \in \mathbf{R}^2 \text{ and } \varepsilon > 0\}$ is a basis for precisely the same topology generated by $\mathbf{B} = \{S_\varepsilon(z) \mid z \in \mathbf{R}^2 \text{ and } \varepsilon > 0\}$. This follows immediately from the next exercise.

(1.A.8) Exercise. Show that two bases $\mathbf{B}$ and $\mathbf{B}'$ generate the same topology on a set X if and only if whenever $x \in B \in \mathbf{B}$, there is a $B' \in \mathbf{B}'$ such that $x \in B' \subset B$, and whenever $x \in B' \in \mathbf{B}'$, there is a $B \in \mathbf{B}$ such that $x \in B \subset B'$.

(1.A.9) Definition. Bases that generate the same topology on a set X are said to be *equivalent*.

It follows from the previous exercise that the families of open intervals $\mathbf{B} = \{(x,y) \mid x,y \text{ are rational}\}$ and $\mathbf{B}' = \{(x,y) \mid x,y \text{ are irrational}\}$ are equivalent bases for the usual topology for $\mathbf{R}^1$.

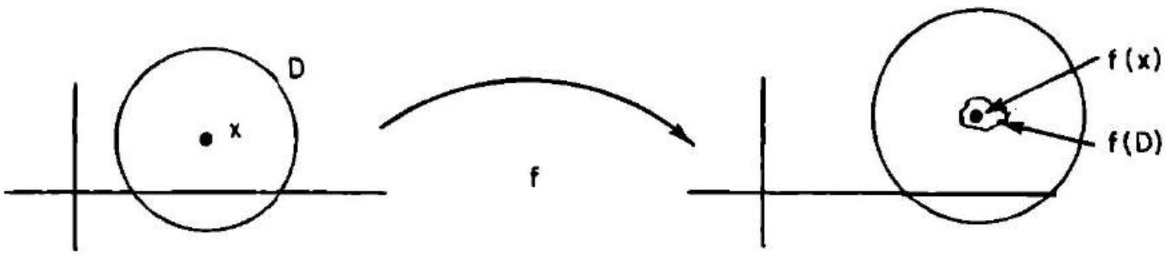
Continuity may also be expressed in terms of basis elements.

(1.A.10) Theorem. Suppose that $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are topological spaces and $\mathbf{B}$ is a basis for $\mathcal{U}$ and $\mathbf{B}'$ is a basis for $\mathcal{V}$. Then a function $f: X \to Y$ is continuous at a point c if and only if whenever $f(c) \in B' \in \mathbf{B}'$, there is a $B \in \mathbf{B}$ such that $c \in B$ and $f(B) \subset B'$.

Proof. Suppose that $f(c) \in B' \in \mathbf{B}'$. Since B' is also in $\mathcal{V}$ and f is continuous, there are sets $U \in \mathcal{U}$ and $B \in \mathbf{B}$ such that $c \in B \subset U$ and $f(c) \in f(B) \subset f(U) \subset B'$.

Conversely, suppose that $f(c) \in V \in \mathcal{V}$. Then there is a $B' \in \mathbf{B}'$ such that $f(c) \in B' \subset V$, and consequently, there exists $B \in \mathbf{B} \subset \mathcal{U}$ such that $c \in B$ and $f(B) \subset B' \subset V$.

Thus, it follows from (1.A.10) that if one wants to determine whether or not a function mapping $\mathbf{R}^2$ into itself is continuous, it suffices to check that f carries appropriate open disks into given open disks.



B. ADDITIONAL EXAMPLES OF TOPOLOGICAL SPACES

1. *Discrete topology.* Let X be arbitrary and $\mathcal{U} = \mathcal{P}(X)$.
2. *Indiscrete topology.* Let X be arbitrary and $\mathcal{U} = \{\varnothing, X\}$.
3. *Half-open interval topology.* Let X be $\mathbf{R}^1$. The sets of the form $[a,b)$ form a basis for a topology $\mathcal{U}$, which is distinct from the usual topology for $\mathbf{R}^1$.
4. *Open ray topology.* Let X be $\mathbf{R}^1$ and $\mathcal{U}$ be the topology determined by a basis consisting of sets of the form (a,∞) for $a \in \mathbf{R}^1$.
5. *Usual topology for* $\mathbf{R}^n$. For $z = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ and $\varepsilon > 0$, let $S_\varepsilon(z) = \{(u_1, u_2, \ldots, u_n) \in \mathbf{R}^n \mid \sqrt{\sum_{i=1}^{n} (x_i - u_i)^2} < \varepsilon\}$. Then $\mathbf{B} = \{S_\varepsilon(z) \mid z \in \mathbf{R}^n$ and $\varepsilon > 0\}$ is a basis for the usual topology for $\mathbf{R}^n$.

6. *Order topology*. Let $(X, \leq)$ be a linearly ordered set. For each $x,y \in X$, with $x < y$, let $(x,y) = \{c \in X \mid x < c < y\}$. The sets of the form (x,y) together with sets of the form $\{c \in X \mid c < x\}$ and sets of the form $\{c \in X \mid c > x\}$ constitute a basis for a topology for X. Note that the usual topology for $\mathbf{R}^1$ is an order topology.

7. *Finite complement topology*. Let X be an infinite set. Define a topology $\mathscr{U}$ to be the collection that consists of the empty set together with all subsets A of X with the property that $X \setminus A$ is finite.

8. Let X be any uncountable set and p be a particular point of X. Then $\mathscr{U} = \{U \subset X \mid X \setminus U \text{ is countable or } p \in X \setminus U\}$ is a topology for X.

9. Let X be an arbitrary set and p be a particular point in X. Then $\mathscr{U} = \{U \subset X \mid U = X \text{ or } p \notin U\}$ is a topology for X.

(1.B.1) Notation. Henceforth, we shall let $\mathscr{E}^n$ denote $\mathbf{R}^n$ with the usual topology.

We conclude this section with a brief introduction to a few of the more common properties that a topological space may satisfy.

(1.B.2) Definition. A topological space is a *Hausdorff* (or T_2) space if and only if for each pair of distinct points a and b in X, there are disjoint open sets U and V that contain a and b respectively.

(1.B.3) Exercise. Determine which of the above spaces are Hausdorff.

(1.B.4) Definition. A topological space $(X,\mathscr{U})$ is *second countable* if and only if $\mathscr{U}$ has a basis consisting of a countable number of sets.

For instance, $\mathscr{E}^1$ is second countable, since the family of open intervals with rational endpoints constitutes a countable basis for the usual topology for $\mathbf{R}^1$. An uncountable discrete space is obviously not second countable.

(1.B.5) Exercise. Determine which of the foregoing spaces are second countable.

(1.B.6) Definition. If $(X,\mathscr{U})$ is a topological space and $x \in X$, then a *basis for $\mathscr{U}$ at x* (often called a *neighborhood basis for $\mathscr{U}$ at x*) is a subcollection $\mathbf{B}_x \subset \mathscr{U}$ with the property that whenever $x \in U \in \mathscr{U}$, there exists $V \in \mathbf{B}_x$ such that $x \in V \subset U$.

(1.B.7) Definition. A topological space $(X,\mathscr{U})$ is *first countable* if and only if there is a countable neighborhood basis at each point $x \in X$.

Thus $\mathscr{E}^n$ is first countable, since for each $z \in \mathscr{E}^n$, $\{S_{1/i}(z) \mid i \in \mathbf{Z}^+\}$ forms a countable neighborhood basis at z.

(1.B.8) Exercise. Find a space that is first countable but is not second countable.

C. METRIC SPACES: A PREVIEW

A very important class of topological spaces is that in which the concept of distance between two points is defined. These spaces are known as metric spaces, and they will play a fundamental role throughout the book.

(1.C.1) Definition. A *metric space* is a pair (X,d), where X is a set and $d : X \times X \to [0,\infty)$ is a function with the property that for all $x,y,z \in X$:

(i) $d(x,y) = 0$ if and only if $x = y$ (*reflexive property*);
(ii) $d(x,y) = d(y,x)$ (*symmetric property*);
(iii) $d(x,z) \leq d(x,y) + d(y,z)$ (*triangle inequality*).

The function d is called the *distance function* or the *metric* for the metric space (X,d).

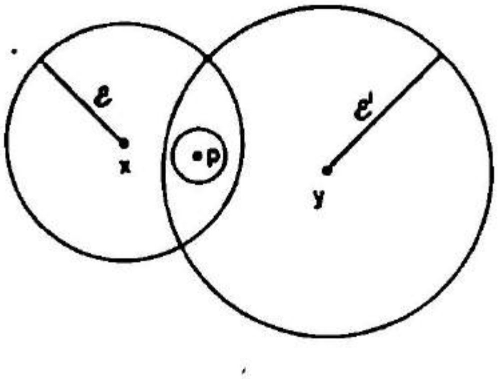
The first condition implies that the distance from a point to itself is 0 and that the distance between distinct points is positive. Condition (ii) states that the distance from a point x to a point y is the same as the distance from y to x. The third condition can be conceived of as a generalization of the idea that the shortest distance between two points is a straight line.

(1.C.2) Notation. Suppose that (X,d) is a metric space, $x \in X$, and $\varepsilon > 0$. We denote the set $\{y \in X \mid d(x,y) < \varepsilon\}$ by $S_\varepsilon^d(x)$. Whenever the metric is clear from the context, "d" is omitted and we simply write $S_\varepsilon(x)$.

The distance function d is used to define a topology for X. In the next theorem, it is shown that if (X,d) is a metric space, then $\mathbf{B} = \{S_\varepsilon^d(x) \mid x \in X, \varepsilon > 0\}$ is a basis for a (unique) topology for X. This topology is called the *metric topology* or the *topology induced by the metric d.*

(1.C.3) Theorem. If (X,d) is a metric space, then $\mathbf{B} = \{S_\varepsilon^d(x) \mid x \in X, \varepsilon > 0\}$ is a basis for a topology for X.

Proof. By (1.A.6) it suffices to show that if $p \in S_\varepsilon^d(x) \cap S_{\varepsilon'}^d(y)$, then there is a positive number ε'' such that $S_{\varepsilon''}^d(p) \subset S_\varepsilon^d(x) \cap S_{\varepsilon'}^d(y)$. Let $\lambda_1 = d(x,p)$ and $\lambda_2 = d(y,p)$. Then if ε'' is any positive number less than both $\varepsilon - \lambda_1$ and $\varepsilon' - \lambda_2$, it is easy to see that $S_{\varepsilon''}^d(p)$ is the required basis element containing p.



We list some examples of metric spaces. The reader should verify that each example does satisfy properties (i), (ii), and (iii) of (1.C.1).

(1.C.4) Examples.

1. Let X be $\mathbf{R}^1$ and define $d(x,y) = |x - y|$ (*usual metric* for $\mathbf{R}^1$).

2. Let X be $\mathbf{R}^2$ and define $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ where $x = (x_1,x_2)$ and $y = (y_1,y_2)$ (*usual metric* for $\mathbf{R}^2$). Use the Minkowski inequality to show that d satisfies the triangle inequality.

3. Let X be $\mathbf{R}^2$ and define $d(x,y) = |x_1 - x_2| + |y_1 - y_2|$ (*the taxicab metric*).

4. Let X be $\mathbf{R}^n$ and define $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$, where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ (*usual metric* for $\mathbf{R}^n$). Use the Minkowski inequality to show that d satisfies the triangle inequality.

5. Let X be arbitrary and define $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$ (*discrete metric*).

6. Let X be $\{f \mid f : y \to \mathbf{R}^1$ and f is bounded$\}$ and define $d(f,g) = \sup\{|f(y) - g(y)| \mid y \in Y\}$. Here Y may be any set. (A function $f : Y \to \mathbf{R}^1$ is *bounded* if and only if there is a positive number M such that $|f(y)| \leq M$ for each $y \in Y$.)

Note that the metrics in examples 2 and 3 induce the same topology; that is, if $\mathscr{U}$ is the topology generated by the usual metric for $\mathbf{R}^2$ and $\mathscr{U}'$ is the topology arising from the taxicab metric, then $\mathscr{U} = \mathscr{U}'$. To see this, suppose that d is the usual metric and d' is the metric described in example 3. Let $x \in X$, and let $B' = \{y \in X \mid d'(x,y) < \varepsilon\}$ be a typical basis element for $\mathscr{U}'$. Suppose that $d(x,y) < \varepsilon/2$. Then $d'(x,y) = |x_1 - x_2| + |y_1 - y_2| \leq 2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 2d(x,y) < \varepsilon$. Hence, $x \in S_{\varepsilon/2}(x) \subset B'$, and it follows that $\mathscr{U}' \subset \mathscr{U}$. On the other hand, if $B = S_\varepsilon^d(x)$ is a member of the basis for $\mathscr{U}$, then since $d(x,y) \leq d'(x,y)$ for all $x,y \in X$, it is clear that $x \in S_\varepsilon^{d'}(x) \subset S_\varepsilon^d(x)$ and consequently $\mathscr{U} \subset \mathscr{U}'$.

(1.C.5) ***Exercise.*** Prove or find a counterexample: (i) all metric spaces are second countable; (ii) all metric spaces are first countable.

(1.C.6) ***Exercise.*** Show that all metric spaces are Hausdorff.

We have just observed that a topology can be derived in a natural way from a metric space. An interesting and actually quite important question (which will be treated in some detail in Chapter 10) concerns the reverse implication—given a topological space $(X,\mathscr{U})$, when does there exist a metric for X that induces $\mathscr{U}$? A topological space $(X,\mathscr{U})$ is *metrizable* if and only if it is possible to define a metric d on X such that the metric topology induced by d coincides with $\mathscr{U}$. Note that example 5 above guarantees the existence of at least one metric for any set, but clearly the topology associated with this metric is discrete. There is a great advantage in having a metric function available, although fairly stringent conditions are necessary to guarantee the existence of one.

In the following exercises, determine whether or not there is a metric function whose induced topology coincides with the given topology, and if there is, find one.

(1.C.7) ***Exercise.*** Let X be an arbitrary set with the indiscrete topology. (Assume X has more than one point.)

(1.C.8) ***Exercise.*** Let X be an arbitrary set with the discrete topology.

(1.C.9) ***Exercise.*** Let X be $\mathbf{R}^1$ with the open ray topology.

It is possible that two distinct metrics d and d' on a set X will induce or generate the same topology. In this case, d and d' are said to be *equivalent metrics.*

(1.C.10) ***Example.*** Let (X,d) be any metric space. Define $d': X \times X \to [0,\infty)$ by setting $d'(x,y) = \min\{d(x,y), 1\}$. It is easy to show that d' is a metric; that d and d' are equivalent follows immediately from the next theorem.

(1.C.11) ***Theorem.*** Suppose that d and d' are metrics for a set X. Then d and d' are equivalent if and only if for each $x \in X$ and each $r > 0$, there exist numbers $s > 0$ and $s' > 0$ such that $S_s^d(x) \subset S_r^{d'}(x)$ and $S_{s'}^{d'}(x) \subset S_r^d(x)$.

Proof. Suppose that d and d' are equivalent, $x \in X$, and $r > 0$. Since $S_r^{d'}(x)$ is an open set containing x, there is a basis element $S_s^d(x)$ in the topology generated by d such that $S^d_s(x) \subset S^{d'}_r(x)$. Similarly, an $s' > 0$ can be found so that $S^{d'}_{s'}(x) \subset S^d_r(x)$.

To prove the converse, let $\mathscr{U}$ be the topology induced by d and $\mathscr{U}'$ be the topology induced by d'. Suppose that $U \in \mathscr{U}$ and that x is an arbitrary point of U. Then there is $r > 0$ such that $S^d_r(x) \subset U$. By the hypothesis there exists an $s' > 0$ such that $x \in S^{d'}_{s'}(x) \subset S^d_r(x) \subset U$; thus, U is a union of open sets in $\mathscr{U}'$, and consequently $\mathscr{U} \subset \mathscr{U}'$. An analogous argument shows that $\mathscr{U}' \subset \mathscr{U}$.

(1.C.12) Definition. A metric space (X,d) is *bounded* if and only if $\sup\{d(x,y) \mid x,y \in X\}$ is finite.

(1.C.13) Corollary. Every metric space is equivalent to a bounded metric space.

Proof. The proof is immediate from (1.C.10) and (1.C.11).

D. BUILDING NEW TOPOLOGICAL SPACES FROM OLD ONES

The Relative Topology

Frequently we will wish to consider a subset A of a space $(X,\mathscr{U})$ as a topological space in its own right. This is possible once we have determined which subsets of A are to be designated as open (in A). The next definition handles this in a natural manner.

(1.D.1) Definition. Suppose that $(X,\mathscr{U})$ is a topological space and that A is a subset of X. Declare a set $V \subset A$ to be *open in A* (or *A-open*) if and only if $V = U \cap A$ for some $U \in \mathscr{U}$. The collection of all A-open sets forms a topology for A, called the *relative topology*. Henceforth, whenever a subset A of a space $(X,\mathscr{U})$ is considered as a topological space, it is assumed that A has the relative topology. In this context, A is often called a *subspace* of $(X,\mathscr{U})$.

(1.D.2) Exercise. Show that the relative topology is in fact a topology.

It should be noted that A-open sets need not be open in the original space X. For instance, if $X = \mathscr{E}^1$ and $A = [0,1)$, then $[0,\frac{1}{2})$ is open in A but not in X. In fact, if $A = [0,1]$, then A is an open subset of itself, but not open in $\mathscr{E}^1$. However, whenever A is itself open in X, then A-open sets are also open in X.

(1.D.3) Theorem. Suppose that $U \subset A \subset X$ where A is open in X and U is open in A. Then U is open in X.

Proof. Since U is open in A, there is an open set V in X such that $V \cap A = U$. However, since A is open in X, we have that $A \cap V$ is open in X, and the theorem follows.

A second trivial but useful result is the following.

(1.D.4) Theorem. Suppose that W and Z are subspaces of a topological space X, and that $U \subset W \cap Z$ is open in both W and Z. Then U is open in $W \cup Z$.

Proof. Since U is open in both W and Z, there are open sets (in X), W' and Z', such that $W' \cap W = U$ and $Z' \cap Z = U$. Then $(W' \cap Z') \cap (W \cup Z) = U$, which implies that U is an open set in the relative topology for $W \cup Z$.

(1.D.5) Exercise. Show that if X and Y are topological spaces, $f: X \to Y$ is continuous, and $A \subset X$, then $f_{|A}$ is continuous.

(1.D.6) Exercise. Show that if (X,d) is a metric space and $A \subset X$, then the relative topology for A is metrizable. In fact, $d_{|A \times A}$ is a metric that induces the relative topology.

The (Finite) Product Topology

The point set $\mathbf{R}^n$ is the Cartesian product of $\mathbf{R}^1$ with itself n times. Thus, it is not unreasonable to expect that the usual topology defined for $\mathbf{R}^n$ might be related in some way to the usual topology for $\mathbf{R}^1$. This relationship is perhaps best seen if we generalize somewhat and consider the notion of a finite Cartesian product of arbitrary spaces. In a later chapter, infinite Cartesian products will be treated.

(1.D.7) Definition. Suppose that $(X_1, \mathscr{U}_1)$, $(X_2, \mathscr{U}_2), \ldots, (X_n, \mathscr{U}_n)$ are topological spaces and $\prod_{i=1}^{n} X_i$ is the Cartesian product of the sets $X_1, X_2, \ldots, X_n$. The *product topology* for $\prod_{i=1}^{n} X_i$ is the topology which has a basis consisting of sets of the form $U = U_1 \times U_2 \times \cdots \times U_n$, where $U_i \in \mathscr{U}_i$ for $i = 1, 2, \ldots, n$.

(1.D.8) ***Remark.*** If $(X_1,\mathscr{U}_1)$, $(X_2,\mathscr{U}_2), \dots, (X_n,\mathscr{U}_n)$ are topological spaces, then, unless otherwise stated, $\prod_{i=1}^{n} X_i$ will be assumed to have the product topology.

(1.D.9) ***Exercise.*** Show that the product topology is in fact a topology.

(1.D.10) ***Theorem.*** If $\mathscr{U}$ is the usual topology for $\mathbf{R}^n$ and $\mathscr{V}$ is the product topology for $\mathscr{E}^1 \times \mathscr{E}^1 \times \cdots \times \mathscr{E}^1$, then $\mathscr{U} = \mathscr{V}$.

Proof. To see that $\mathscr{U} \subset \mathscr{V}$, let $U = S_\varepsilon^d(x)$ be a member of the usual basis for $\mathscr{U}$, where $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$ and $\varepsilon > 0$. For each x_i, let $U_i = \{y \in \mathscr{E}^1 \mid |x_i - y| < \varepsilon/\sqrt{n}\}$. Then $x \in U_1 \times U_2 \times \cdots \times U_n$, and if $u = (u_1, u_2, \dots, u_n) \in U_1 \times U_2 \times \cdots \times U_n$, we have that

$$d(u,x) = \sqrt{\sum_{i=1}^{n} (x_i - u_i)^2} < \sqrt{\frac{n\varepsilon^2}{n}} = \varepsilon$$

Hence, $U \in \mathscr{V}$.

Now suppose that $U_1 \times U_2 \times \cdots \times U_n$ is a typical member of the basis for the product topology, and let

$$x = (x_1, x_2, \dots, x_n) \in U_1 \times U_2 \times \cdots \times U_n.$$

Then for each x_i, there is an $\varepsilon_i > 0$ such that $\{y \in \mathscr{E}^1 \mid |x_i - y| < \varepsilon_i\} \subset U_i$. Let $\varepsilon = \min\{\varepsilon_i \mid i = 1, 2, \dots, n\}$. We show that $S_\varepsilon^d(x) \subset U_1 \times U_2 \times \cdots \times U_n$. If $z = (z_1, z_2, \dots, z_n) \in S_\varepsilon^d(x)$, then

$$|x_i - z_i| \leq \sqrt{\sum_{i=1}^{n} (x_i - z_i)^2} < \varepsilon \leq \varepsilon_i$$

and therefore $z_i \in U_i$ for each i, which completes the proof.

(1.D.11) ***Definition.*** For $i = 1, 2, \dots, n$, the function $p_i : (X_1 \times \cdots \times X_i \times \cdots \times X_n) \to X_i$ defined by $p_i(x_1, \dots, x_i, \dots, x_n) = x_i$ is called the ith *projection map.*

(1.D.12) ***Theorem.*** Suppose that $(X_1,\mathscr{U}_1)$, $(X_2,\mathscr{U}_2), \dots, (X_n,\mathscr{U}_n)$ are topological spaces and that $\prod_{i=1}^{n} X_i$ is the associated product space. Then the projection maps p_i are continuous. Furthermore, the product topology is the smallest topology for which all the p_i are continuous, i.e., if $\mathscr{U}$ denotes the product topology and $\mathscr{V} \subset \mathscr{U}$ is any other topology for the set $\prod_{i=1}^{n} X_i$ for which each p_i is continuous, then $\mathscr{V} = \mathscr{U}$.

Proof. Suppose that U_i is an open set in X_i. Then $p_i^{-1}(U_i) = X_1 \times X_2 \times \cdots \times U_i \times X_{i+1} \times \cdots \times X_n$ is clearly a member of $\mathscr{U}$, and therefore p_i is continuous (1.A.4).

Suppose now that $\mathscr{V} \subset \mathscr{U}$ and that each p_i is continuous with respect to $\mathscr{V}$. To demonstrate that $\mathscr{V} = \mathscr{U}$, it suffices to show that each basis set $U = U_1 \times U_2 \times \cdots \times U_n$ in $\mathscr{U}$ belongs to $\mathscr{V}$ (why?). Since for each i, p_i is continuous, it follows that $p_i^{-1}(U_i)$ is open with respect to $\mathscr{V}$. But $U = U_1 \times U_2 \times \cdots \times U_n = \bigcap_{i=1}^{n} p_i^{-1}(U_i)$, and hence $U \in \mathscr{V}$, which concludes the proof.

(1.D.13) Exercise. Suppose that for $i = 1, 2, \ldots, n$, $f_i : X_i \to Y_i$ is continuous. Show that the map $\prod f_i : \prod_{i=1}^{n} X_i \to \prod_{i=1}^{n} Y_i$ defined by $\prod f_i(x_1, x_2, \ldots, x_n) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$ is continuous.

The Disjoint Union Topology

We now give one further example of how new topologies may be generated from old ones. Suppose that $\{(X_\alpha, \mathscr{U}_\alpha) \mid \alpha \in \Lambda\}$ is a collection of pairwise disjoint topological spaces, i.e., if $\alpha, \beta \in \Lambda$, then $X_\alpha \cap X_\beta = \varnothing$. Let $X = \bigcup_{\alpha \in \Lambda} X_\alpha$. Define a topology $\mathscr{U}$ for X by declaring a set $U \subset X$ to be open if and only if $U \cap X_\alpha$ is open in X_α for each $\alpha \in \Lambda$.

(1.D.14) Exercise. Show that the family $\mathscr{U}$ defined above forms a topology.

The topological space obtained in this matter is customarily called the *free union* of the spaces X_α, and the corresponding topology is frequently referred to as the *disjoint union topology*. Requiring that the spaces in question be pairwise disjoint can be avoided, since any collection of sets may be replaced by a disjoint collection of sets as follows. Suppose that $\{X_\alpha \mid \alpha \in \Lambda\}$ is a family of sets. For each $\alpha \in \Lambda$, define $\hat{X}_\alpha = X_\alpha \times \{\alpha\}$. Then clearly for $\alpha \neq \beta$, $\hat{X}_\alpha \cap \hat{X}_\beta = \varnothing$. Now suppose that $\{(X_\alpha, \mathscr{U}_\alpha) \mid \alpha \in \Lambda\}$ is a collection of topological spaces. If $\alpha \in \Lambda$, let $\hat{X}_\alpha = X_\alpha \times \{\alpha\}$ and define a topology $\hat{\mathscr{U}}_\alpha$ for $\hat{X}_\alpha$ by declaring a subset $U_\alpha \times \{\alpha\} \subset X_\alpha \times \{\alpha\}$ open if and only if U_α is open in X_α. Then the *free union* of the X_α is defined to be the free union of the disjoint spaces $(\hat{X}_\alpha, \hat{\mathscr{U}}_\alpha)$ described previously.

(1.D.15) Exercise. Suppose that $\{A_\alpha \mid \alpha \in \Lambda\}$ is a collection of disjoint subsets of a topological space X. Let $\mathscr{U}$ denote the relative topology for

$\bigcup \{A_\alpha \mid \alpha \in \Lambda\}$ and let $\mathscr{V}$ denote the disjoint union topology for the same set, where each A_α is endowed with the relative topology. Discuss the relationship between $\mathscr{U}$ and $\mathscr{V}$.

E. A POTPOURRI OF FUNDAMENTAL CONCEPTS

Although the concepts to be introduced in this section are relatively simple, the reader should give them careful consideration. A firm grasp of these ideas is needed before one can successfully proceed to the more intriguing aspects of topology.

Closed Sets

(1.E.1) Definition. If $(X,\mathscr{U})$ is a topological space, then a subset A of X is *closed* in X if and only if $X \setminus A$ is open.

Note that if U is open in X, then $X \setminus A$ is closed. It follows easily from De Morgan's rules and (1.A.1.) that both finite unions and arbitrary intersections of closed sets are closed. Of course, subsets of a space may be neither open nor closed, e.g., $[0,1) \subset \mathscr{E}^1$. Furthermore, infinite unions of closed subsets may fail to be closed, e.g., the set of rational numbers is a countable union of singletons (each of which is closed in $\mathscr{E}^1$), and yet the rationals are not closed in $\mathscr{E}^1$.

(1.E.2) Exercise. Suppose that A is a subspace of a topological space X and that $F \subset A$. Show that F is closed in A if and only if there is a closed set F' in X such that $F = F' \cap A$.

(1.E.3) Exercise. Show that if F is closed in G and G is closed in X, then F is closed in X.

All subsets of a topological space may be "closed off" in the following way.

(1.E.4) Definition. Let $(X,\mathscr{U})$ be a topological space and suppose that $A \subset X$. Then the *closure of* A (in X) is defined to be $\bigcap \{F \mid F \text{ is closed in } X \text{ and } A \subset F\}$, and is denoted by $\bar{A}^X$ (or when the context is clear, simply by $\bar{A}$).

(1.E.5) Theorem. If A and B are subsets of a topological space X, then

(i) $A \subset \bar{A}$,
(ii) $\bar{A}$ is closed,

(iii) if A is closed, then $A = \overline{A}$,
(iv) $\overline{A}$ is the smallest closed subset of X containing A, in the sense that if $A \subset B \subset \overline{A}$ and B is closed, then $B = \overline{A}$,
(v) if $A \subset B$, then $\overline{A} \subset \overline{B}$,
(vi) $\overline{A \cup B} = \overline{A} \cup \overline{B}$, and
(vii) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

Proof. (i) Trivial. (ii) Recall that the intersection of closed sets is closed. (iii) Since A is closed, $\overline{A}$ is contained in A; on the other hand, A is always contained in $\overline{A}$. (iv) Since B is closed, we have that $\overline{A} \subset B$. (v) Trivial. (vi) Since the finite union of closed sets is closed, we have by (iv) that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. On the other hand, $A \subset (A \cup B)$, and hence by (v), $\overline{A} \subset \overline{A \cup B}$. Similarly, $\overline{B}$ is contained in $\overline{A \cup B}$ and therefore $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. (vii) Since $A \cap B \subset A$ and $A \cap B \subset B$, it follows from (v) that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

(1.E.6) Exercise. Find an example to show that $\overline{A \cap B}$ is not necessarily equal to $\overline{A} \cap \overline{B}$.

(1.E.7) Theorem. Let $X_1, X_2, \ldots, X_n$ be topological spaces and suppose that for $i = 1, 2, \ldots, n$, $A_i \subset X_i$. Then $\overline{A_1 \times \cdots \times A_n} = \overline{A_1} \times \cdots \times \overline{A_n}$.

Proof. We prove the theorem for $n = 2$; an easy induction argument yields the more general result. First, observe that if F and G are closed subsets of spaces X and Y respectively, then $F \times G$ is closed in $X \times Y$. To see this, suppose that $(x,y) \notin F \times G$, and without loss of generality assume that $x \notin F$. Then $(X \setminus F) \times Y$ is an open set in $X \times Y$ that contains (x,y) and lies in the complement of $F \times G$. Consequently, $(X \times Y) \setminus (F \times G)$ is open, and therefore $F \times G$ is closed. It now follows immediately from part (iv) of the previous theorem that $\overline{A_1 \times A_2} \subset \overline{A_1} \times \overline{A_2}$.

To establish the converse we show that $X \setminus (\overline{A \times B}) \subset X \setminus (\overline{A} \times \overline{B})$. Let $(x,y) \in X \setminus (\overline{A \times B})$. Note that $X \setminus (\overline{A \times B})$ is an open set and clearly $(X \setminus (\overline{A \times B})) \cap (\overline{A \times B}) = \varnothing$. Hence there is a basic open set $U \times V$ such that $(x,y) \in U \times V \subset X \setminus (\overline{A \times B})$ and thus $(U \times V) \cap (A \times B) = \varnothing$. To finish the proof it suffices to show that $U \cap A = \varnothing$ or $V \cap B = \varnothing$ (why?). But this is clear since $(U \cap A) \times (V \cap B) \subset (U \times V) \cap (A \times B) = \varnothing$.

(1.E.8) Definition. A topological space $(X,\mathscr{U})$ is T_1 if and only if each point in X is closed.

(1.E.9) Exercise. Show that a space X is T_1 if and only if for each $x,y \in X$ such that $x \neq y$, there is an open set containing x that does not contain y.

Accumulation Points

Probably no concept in either topology or analysis is of more importance than the notion of an accumulation point. It will become increasingly apparent as we proceed that the entire fabric of topology is permeated with this idea.

(1.E.10) Definition. Suppose that X is a topological space and that A is a subset of X. Then a point $x \in X$ is an *accumulation point* of A if and only if each open set containing x has nonempty intersection with $A \setminus \{x\}$.

It should be noted that in the literature, accumulation points are often referred to as limit points.

(1.E.11) Examples.

1. If $X = \mathscr{E}^1$ and $A = [0,1)$, then any x such that $0 \leq x \leq 1$ is an accumulation point of A.
2. If $X = \mathscr{E}^1$ and A is the set of rationals, then every point of X is an accumulation point of A.
3. If $X = \mathscr{E}^1$ and $A = \mathbf{Z}$, then no point of X is an accumulation point of A.

(1.E.12) Exercise. In $\mathbf{R}^1$ let $A = (0,1)$. Find the accumulation points of A with respect to each of the different topologies we have imposed on $\mathbf{R}^1$.

(1.E.13) Definition. If A is a subset of a space X, then the *derived set* of A is $\{x \mid x \text{ is an accumulation point of } A\}$. The derived set of A is denoted by A'.

(1.E.14) Theorem. Suppose that A is a subset of a space X. Then (i) $\bar{A} = A \cup A'$, and (ii) A is closed if and only if $A' \subset A$.

Proof. (i) If a point x does not lie in $A \cup A'$, then there is an open set containing x that misses A. Consequently, the complement of $A \cup A'$ is open and $A \cup A'$ is closed. Thus, we have that $\bar{A} \subset A \cup A'$. If $x \notin \bar{A}$, then since $\bar{A}$ is closed, x belongs to an open set that does not intersect A. Thus, $x \notin A \cup A'$, and hence $\bar{A} = A \cup A'$. (ii) If A is closed, then $A = \bar{A} = A \cup A'$ and therefore $A' \subset A$. Conversely, if $A' \subset A$, then $A = A \cup A' = \bar{A}$ and hence A is closed.

Interior, Exterior, and Frontier

If A is any subset of a topological space X, then X may be split in a natural way into three distinct sets as follows.

(1.E.15) Definition. Suppose that A is a subset of a space X. The *interior* of A denoted by A° or int A, is the union of all open sets of X that are contained in A. The *exterior* of A, denoted by ext A, is the union of all open sets in X that are contained in $X \setminus A$. The *frontier* of A, denoted by Fr A, consists of all points $x \in X$ with the property that each open set containing x intersects both A and $X \setminus A$.

For example, if $A = (0,4] \subset \mathscr{E}^1$, then int $A = (0,4)$, ext $A = (-\infty,0) \cup (4,\infty)$, and Fr $A = \{0,4\}$.

(1.E.16) Theorem. If A is a subset of a space X, then $X = (\text{int } A) \cup (\text{ext } A) \cup (\text{Fr } A)$ and these sets are pairwise disjoint.

Proof. Suppose that $x \in X$ fails to be in either int A or ext A. Then it is clear from the foregoing definition that each open set that contains x must intersect both A and $X \setminus A$. Therefore, $x \in \text{Fr } A$. That the three sets are pairwise disjoint is also immediate from the definition.

Although any of the three sets int A, ext A, and Fr A may be empty, it should be obvious that int A is the largest open set contained in A and ext A is the largest open set contained in $X \setminus A$.

(1.E.17) Exercise. Show that if $A \subset X$, then $\text{Fr } A = \bar{A} \cap (\overline{X \setminus A})$.

Frontiers and products are related as follows.

(1.E.18) Theorem. Suppose that $A \subset X$ and $B \subset Y$. Then $\text{Fr}(A \times B) = ((\text{Fr } A) \times \bar{B}) \cup (\bar{A} \times (\text{Fr } B))$.

Proof. Suppose that $(x,y) \in \text{Fr}(A \times B)$. Then $x \in \bar{A}$ and $y \in \bar{B}$ (why?). If $x \notin \text{Fr } A$ and $y \notin \text{Fr } B$, then by the previous exercise $x \notin \overline{X \setminus A}$ and $y \notin \overline{Y \setminus B}$. Let $U = X \setminus (\overline{X \setminus A})$ and $V = Y \setminus (\overline{Y \setminus B})$. Then $(x,y) \in U \times V$ and $U \times V$ fails to intersect $(X \times Y) \setminus (A \times B)$. Consequently, either $x \in \text{Fr } A$ or $y \in \text{Fr } B$, and we have that $\text{Fr}(A \times B) \subset ((\text{Fr } A) \times \bar{B}) \cup (\bar{A} \times (\text{Fr } B))$.

Suppose now that $(x,y) \in (\text{Fr } A) \times \bar{B}$ and let $U \times V$ be a basic open set such that $(x,y) \in U \times V$. We want to show that $(U \times V) \cap (A \times B) \neq \varnothing$ and $(U \times V) \cap ((X \times Y) \setminus (A \times B)) \neq \varnothing$. Since $x \in \text{Fr } A$ and $y \in \bar{B}$, we have that $U \cap A \neq \varnothing$ and $V \cap B \neq \varnothing$; hence it follows that $(U \times V) \cap (A \times B) \neq \varnothing$.

Since $x \in \text{Fr } A$, we have that $U \cap (X \setminus A) \neq \varnothing$. Then for any $z \in U \cap (X \setminus A)$, it is clear that $\{z\} \times V \subset (U \times V) \cap ((X \times Y) \setminus (A \times B))$. Consequently, $((\text{Fr } A) \times \bar{B}) \subset \text{Fr } (A \times B)$.

In a similar manner, one may show that $\bar{A} \times (\text{Fr } B) \subset \text{Fr } (A \times B)$.

(1.E.19) Exercise. Generalize (1.E.18) to finite products.

We conclude this section with an important definition.

(1.E.20) Definition. If X is a topological space and $x \in X$, then a subset N of X is a *neighborhood* of x if and only if there is an open set U in X such that $x \in U \subset N$. (The reader should check to see in which of the previous definitions the words "open set containing x" may be replaced by the words "neighborhood of x" without changing the meaning of the definition).

F. CONTINUITY

Recall that we used the notion of continuity to motivate the definition of a topology. We now reexamine the concept of continuity in light of our newly developed topological ideas. We first establish some useful generalizations of presumably familiar results.

(1.F.1) Theorem. Suppose that X, Y, and Z are topological spaces and that $f: X \to Y$ and $g: Y \to Z$ are continuous functions. Then $gf: X \to Z$ is continuous.

Proof. We show continuity at each point $x \in X$. Let $x \in X$ and suppose that V is an open set containing $gf(x)$. Since g is continuous, there is an open set W in Y that contains $f(x)$ and such that $g(W) \subset V$. Similarly, from the continuity of f at x, we can find an open set U in X such that $x \in U$ and $f(U) \subset W$. Then $fg(U) \subset V$.

The solutions to the next exercise embody nothing more than manipulating ε's and δ's.

(1.F.2) Exercise. Show that the following functions are continuous:

(i) $f: \mathscr{E}^1 \to \mathscr{E}^1$, where f is defined by $f(x) = x^2$;
(ii) $f: \mathscr{E}^1 \setminus \{0\} \to \mathscr{E}^1$, where f is defined by $f(x) = 1/x$;
(iii) $f: \mathscr{E}^1 \to \mathscr{E}^1$, where f is defined by $f(x) = |x|$;
(iv) $f: \mathscr{E}^1 \to \mathscr{E}^1$, where f is defined by $f(x) = ax$ where $a \in \mathbf{R}^1$.

(1.F.3) Theorem. Suppose that f and g are continuous functions mapping a topological space X into $\mathscr{E}^1$. Let $h = f + g$, $k = f \cdot g$, and if $g(x) \neq 0$ for all x, let $j = f/g$. Then h, k, and j are continuous. (Here, of course, $(f + g)(x) = f(x) + g(x)$, $(f \cdot g)(x) = f(x)\, g(x)$, and $f/g(x) = f(x)/g(x)$.)

Proof. Suppose that $x \in X$. We first establish the continuity of h at x. Let U be an open interval of radius ε with $h(x)$ as its center. Since f and g are continuous at x, there are open sets V and W containing x such that $f(V)$ is contained in the interval $(f(x) - \varepsilon/2, f(x) + \varepsilon/2)$ and $g(W)$ is contained in $(g(x) - \varepsilon/2, g(x) + \varepsilon/2)$. If $Y = V \cap W$, it is easily checked that $h(Y) \subset U$.

In order to show that k is continuous at x, one begins by demonstrating the continuity of $f^2 = f \cdot f$. This, however, follows immediately from the exercises above and the previous theorem, since f^2 can be written as a composition of f and the function $p(x) = x^2$. Since $f \cdot g = (1/4)((f + g)^2 - (f - g)^2)$, (1.F.2) may again be applied along with the first part of this theorem to see that k is continuous.

The function $j = f/g$ may be considered as the product of f and $1/g$. The continuity of $1/g$ is immediate from the observation that $1/g$ is merely a composition of g and $q(x) = 1/x$. Then, since j is a product of continuous functions, the proof of the theorem is complete.

The next theorem shows that continuity may be expressed equally as well in terms of closed sets as in terms of open sets.

(1.F.4) Theorem. Suppose that $(X,\mathscr{U})$ and $(Y,\mathscr{V})$ are topological spaces and that $f : X \to Y$. Then the following conditions are equivalent:

(i) f is continuous;
(ii) for each $V \in \mathscr{V}, f^{-1}(V) \in \mathscr{U}$;
(iii) for each closed set $D \subset Y, f^{-1}(D)$ is closed in X.

Proof.

(i) $\leftrightarrow$ (ii) This is just (1.A.4).

(ii) $\rightarrow$ (iii) Suppose that D is a closed subset of Y and let $V = Y \setminus D$. Then V is open, and hence, $f^{-1}(V)$ is open in X. But $f^{-1}(D) = X \setminus f^{-1}(V)$, which proves that $f^{-1}(D)$ is closed in X.

(iii) $\rightarrow$ (ii) Suppose that $V \in \mathscr{V}$. Then $Y \setminus V$ is closed and hence $f^{-1}(Y \setminus V)$ is closed in X. But $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ and therefore $f^{-1}(V)$ is open in X (why?).

We next give a criterion for continuity that is expressed in terms of the closure operator.

(1.F.5) Theorem. Suppose that $(X,\mathscr{U})$ and $(Y,\mathscr{V})$ are topological spaces and that $f : X \to Y$. Then f is continuous if and only if $f(\bar{A}) \subset \overline{f(A)}$ for each subset A of X.

Proof. Suppose that f is continuous. By part (iii) of the previous theorem, $f^{-1}(\overline{f(A)})$ is a closed set. Since $A \subset f^{-1}f(A)$, it follows that $\bar{A} \subset f^{-1}(\overline{f(A)})$, and consequently $f(\bar{A}) \subset ff^{-1}(\overline{f(A)}) \subset \overline{f(A)}$.

Conversely, suppose that $f(\bar{A}) \subset \overline{f(A)}$ for every subset A of X. First we observe that under the hypothesis, if $B \subset Y$, then $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$. This is immediate, since if $A = f^{-1}(B)$, then $f(\bar{A}) \subset \overline{f(A)} \subset \bar{B}$, which implies that $\overline{f^{-1}(B)} = \bar{A} \subset f^{-1}(\bar{B})$. To establish the continuity of f, one now shows that inverse images of closed sets are closed. If $B \subset Y$ is closed, then $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B}) = f^{-1}(B)$, and hence $f^{-1}(B)$ is closed.

The next result will see continual service during succeeding chapters.

(1.F.6) Theorem (Map Gluing Theorem). Suppose that A and B are closed subsets of a topological space X. Let Y be an arbitrary space and suppose that $f: A \to Y$ and $g: B \to Y$ are continuous functions such that $f_{|A \cap B} = g_{|A \cap B}$. Then the function $h: (A \cup B) \to Y$, defined by

$$h(x) = \begin{cases} f(x) & \text{if} \quad x \in A \\ g(x) & \text{if} \quad x \in B \end{cases}$$

is continuous.

Proof. We apply (1.F.4). Suppose that C is a closed subset of Y. Then $f^{-1}(C)$ is closed in A and hence also in X, since A is closed in X. Similarly, $g^{-1}(C)$ is closed in both B and X. However, $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$, and thus $h^{-1}(C)$ is closed in X, which completes the proof.

(1.F.7) Theorem. Suppose that X_1, X_2, and Y are topological spaces and that $f: Y \to X_1 \times X_2$, where $X_1 \times X_2$ is assumed to have the product topology. Let $f_1 = p_1 f$ and $f_2 = p_2 f$, where p_1 and p_2 are the projection maps. Then f is continuous if and only if f_1 and f_2 are continuous.

Proof. By (1.F.1), if f is continuous, then so are f_1 and f_2. To prove the converse, suppose that $U \times V$ is a typical basis element in the product topology. We are to show that $f^{-1}(U \times V)$ is open in Y. Note that $U \times V = (U \times X_2) \cap (X_1 \times V) = p_1^{-1}(U) \cap p_2^{-1}(V)$, and consequently we have that $f^{-1}(U \times V) = f^{-1}(p_1^{-1}(U) \cap p_2^{-1}(V)) = f^{-1}p_1^{-1}(U) \cap f^{-1}p_2^{-1}(V) = f_1^{-1}(U) \cap f_2^{-1}(V)$, which is an open set under the assumption that f_1 and f_2 are continuous.

In metric spaces, continuity may be expressed neatly in terms of sequences. We remind the reader that a *sequence* in a set X is a function $f: D \to X$, where D is a subset of $\mathbf{N}$. If for each $i \in D, f(i) = x_i$, then the sequence f is usually denoted by $\{x_i\}$. Unless otherwise stated, it will be assumed that $D = \mathbf{Z}^+$.

(1.F.8) Definition. A sequence $\{x_i\}$ in a topological space X *converges to a point* $x \in X$ if and only if for every neighborhood U of x, there is a positive integer N_U such that $x_i \in U$ whenever $i > N_U$. In this case, x is called a *limit point of the sequence* $\{x_i\}$.

Thus, if U is any neighborhood of x, members of a sequence $\{x_i\}$ converging to x are eventually trapped in U.

(1.F.9) Theorem. Suppose that (X,d) and (Y,d') are metric spaces. A function $f : X \to Y$ is continuous at a point $x \in X$ if and only if whenever a sequence $\{x_i\}$ in X converges to x, then the sequence $\{f(x_i)\}$ converges to $f(x)$.

Proof. Suppose that f is continuous at x, and let V be any open set containing $f(x)$. Since f is continuous, there is a neighborhood U of x such that $f(U) \subset V$. The convergence of $\{x_i\}$ implies that there is some integer N such that $x_i \in U$ whenever $i > N$. Thus, for each $i > N$, we have that $f(x_i) \in V$, and therefore $\{f(x_i)\}$ converges to $f(x)$.

To prove the converse, suppose that f is not continuous at $x \in X$. Then there must be an $\varepsilon > 0$ such that no neighborhood of x is mapped by f into $S_\varepsilon(f(x))$. For each positive integer n, select a point $x_n \in S_{1/n}(x)$ such that $f(x_n) \notin S_\varepsilon(f(x))$. Clearly, the sequence $\{x_n\}$ converges to x, but the corresponding sequence $\{f(x_n)\}$ fails to converge to $f(x)$.

The preceding theorem gives rigor to the concept of continuity as often taught in elementary calculus: a function is continuous at a point $x \in \mathscr{E}^1$ if and only if whenever points in $\mathscr{E}^1$ approach x, then their images under f approach $f(x)$.

(1.F.10) Definition. Suppose that (X,d) is a metric space and that A and B are nonempty subsets of X. Then the *distance between A and B, $d(A,B)$*, is defined to be $\text{glb}\{d(a,b) \mid a \in A \text{ and } b \in B\}$.

In the two exercises that conclude this section, we see that both d and a distance function closely related to d are continuous.

(1.F.11) Exercise. Suppose that (X,d) is a metric space. Show that the metric function $d : X \times X \to [0,\infty)$ is continuous.

(1.F.12) Exercise. Suppose that (X,d) is a metric space, and let $A \subset X$. Show that the function $\Psi : X \to [0,\infty)$ is continuous, where $\Psi(x) = d(\{x\},A)$.

G. HOMEOMORPHISMS

Suppose that $f : X \to Y$ is a continuous function from one topological space onto another. In spite of the continuity of f, the space X may differ considerably from Y: a flagrant example of this occurs when X is arbitrary and Y

consists of a single point. Not only may Y be a good deal simpler than X, but, surprisingly, as we shall see later, the reverse situation can happen as well. In Chapter 9, a continuous function is constructed that maps the unit interval $\mathbf{I} = [0,1]$ onto the unit square $\mathbf{I} \times \mathbf{I}$. In fact, maps can be found that carry $\mathbf{I}$ onto cubes as well as onto a host of far more exotic spaces.

In these cases, the continuous function in question will not be 1–1; nevertheless, even with this further imposition, the spaces X and Y may still have little similarity. To see this, let a set X be given the discrete topology, and let Y be any space with the same cardinality as X. Then all 1–1, onto mappings from X to Y are continuous; and hence, although X might have uncountably many open sets, Y might have only two.

Suppose now, however, that $f: X \to Y$ is a continuous bijection such that f^{-1} (which makes sense, since f is a bijection) is also continuous. The reader should verify that not only is there a 1–1 correspondence between the points of X and Y, but also between the open sets of the respective topologies. Thus, the topological structure of X has an almost exact counterpart in Y. Bijective, continuous mappings with continuous inverses are called *homeomorphisms*, and the corresponding spaces are said to be *homeomorphic* or *topologically equivalent*. An *embedding* is a 1–1 map $f: X \to Y$ such that $f^{-1}: f(X) \to X$ is continuous. At least a germinal feel for topology should arise from our discussion of homeomorphisms.

(1.G.1) Definition. A function $f: X \to Y$ is *open* (respectively, *closed*) if and only if f maps open (respectively, closed) sets onto open (respectively, closed) sets.

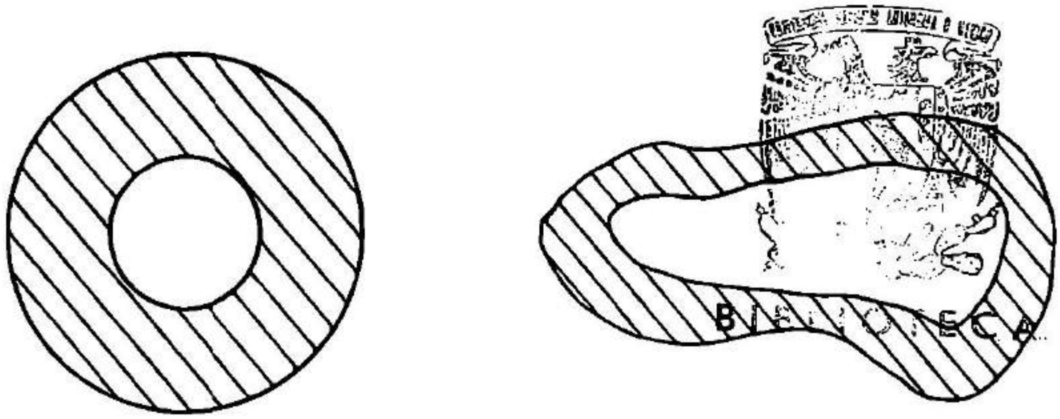
(1.G.2) Exercise. Show that homeomorphisms are both open and closed.

(1.G.3) Exercise. Show that a continuous open (or closed) bijection is a homeomorphism.

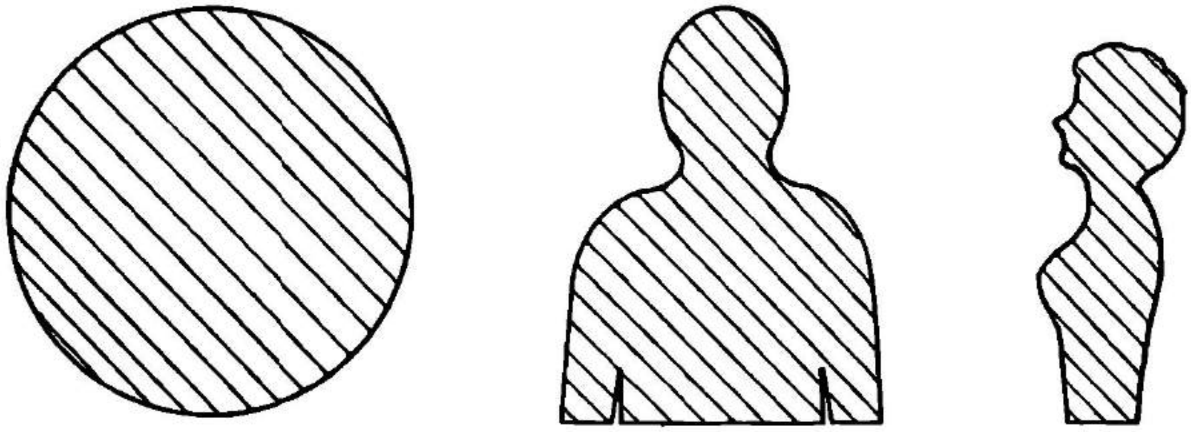
(1.G.4) Exercise. Show that if $X_1 \times \cdots \times X_n$ has the product topology, then the projection maps, p_i, are open.

(1.G.5) Exercise. Show that two metrics d and d' for a set X are equivalent if and only if the identity map $id: (X,d) \to (X,d')$ is a homeomorphism.

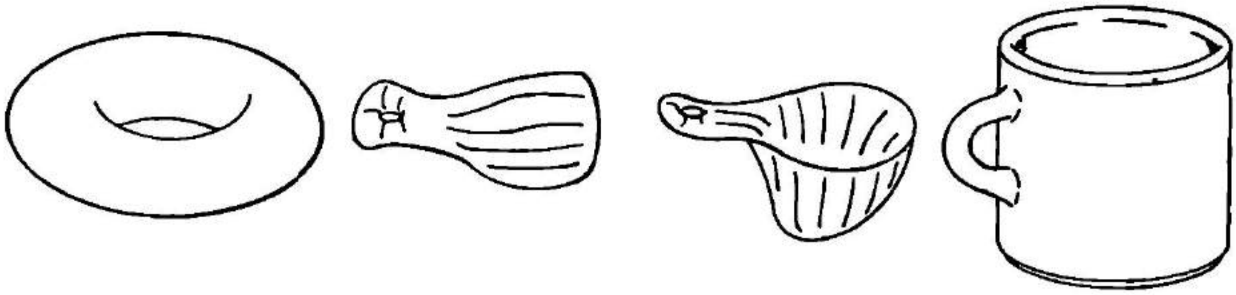
For a topologist, homeomorphic spaces are essentially identical. Thus, little distinction (if any) would be made between the following pair of spaces (considered as subspaces of $\mathscr{E}^2$), since one can be mapped homeomorphically onto the other.



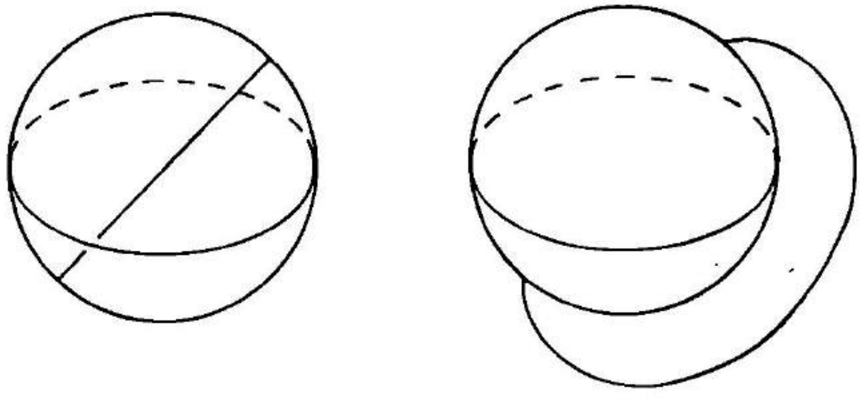
Similarly, a topologist is resigned to the fact that no distinction is to be made between the following trio of spaces.



In a like vein, since a doughnut can be transformed into a coffee cup via a homeomorphism (see figure), a topologist has been classically defined to be an otherwise enlightened individual who fails to note any difference between these two objects.



A word of caution, however. At this stage it might appear that two spaces A and B are homeomorphic if, intuitively, A could be molded into B without either A or B being torn apart. While it is true that a homeomorphism will result whenever one space is stretched, bent, or otherwise deformed into another, this does not constitute the only manner in which homeomorphisms may arise. For instance, it should be clear that the two spaces shown in the following figure are homeomorphic even though no amount of tugging (in $\mathscr{E}^3$) will transform one of them into the other.



Thus, in order for two spaces to be homeomorphic, it suffices that we be able to jump immediately from one to another without any intermediate steps.

(1.G.6) Exercise. Show that $(-1,1)$ and $\mathscr{E}^1$ are homeomorphic via the map $h : \mathscr{E}^1 \to (-1,1)$ defined by $h(x) = x/(1 + |x|)$.

The problem of establishing the existence or nonexistence of a homeomorphism between two topological spaces is by no means a trivial one. The reader might enjoy discovering this for himself by attempting to show that $\mathscr{E}^1$ and $\mathscr{E}^2$ are not homeomorphic. In order to attack this problem in general, we shall investigate in the ensuing chapters certain *topological invariants*, i.e., properties of topological spaces that are preserved by homeomorphisms. If a space X possesses a certain property that is topologically invariant, and a space Y does not, then clearly X and Y cannot be homeomorphic. One obvious invariant (which does not turn out to be very useful) is the cardinality of the sets in question; another is the cardinality of their respective topologies. Note that neither of these invariants is of any help in deciding whether or not $\mathscr{E}^1$ and $\mathscr{E}^2$ are homeomorphic.

Metrizability is a prime example of topological invariance.

(1.G.7) Theorem. Suppose that $(X,\mathscr{U})$ is homeomorphic to $(Y,\mathscr{V})$ and that $(X,\mathscr{U})$ is metrizable. Then $(Y,\mathscr{V})$ is metrizable.

Proof. Let d be a metric for X which induces the topology $\mathscr{U}$ and let

$h : X \to Y$ be a homeomorphism. Define $d' : Y \times Y \to [0,\infty)$ by $d'(y_1,y_2) = d(h^{-1}(y_1),h^{-1}(y_2))$. Since d' is clearly a metric, it remains to be shown that the topology $\mathscr{W}$ generated by d' coincides with $\mathscr{V}$. Suppose that $y \in V \in \mathscr{V}$. Then $h^{-1}(y) \in h^{-1}(V) \in \mathscr{U}$, and hence there is an $\varepsilon > 0$ such that $S_\varepsilon^d(h^{-1}(y)) \subset h^{-1}(V)$. Note that if $d'(z,y) < \varepsilon$, then we have $d(h^{-1}(z),h^{-1}(y)) < \varepsilon$ and consequently $h^{-1}(z) \in h^{-1}(V)$, so $z \in V$. Therefore, $S_\varepsilon^{d'}(y) \subset V$ and $\mathscr{V} \subset \mathscr{W}$.

Now consider $S_\varepsilon^{d'}(y)$. If $z \in S_\varepsilon^{d'}(y)$, then we have $h^{-1}(z) \in S_\varepsilon^d(h^{-1}(y))$, and therefore $h(S_\varepsilon^d(h^{-1}(y)) \subset S_\varepsilon^{d'}(y)$. Since $S_\varepsilon^d(h^{-1}(y)) \in \mathscr{U}$, it follows that $h(S_\varepsilon^d(h^{-1}(y)) \in \mathscr{V}$, and hence $\mathscr{W} \subset \mathscr{V}$.

(1.G.8) Exercise. Show that the properties T_2, first countability, and second countability are topological invariants. Use the topological invariance of T_2 to show that $\mathscr{E}^1$ and $\mathbf{R}^1$ with the finite complement topology are not homeomorphic.

(1.G.9) Definition. A subset D of a space X is *dense* in X if and only if every nonempty open set in X intersects D. A space X is *separable* if and only if X contains a countable dense subset.

The set of points in $\mathscr{E}^n$ with rational coordinates is easily seen to be a countable dense subset, and hence $\mathscr{E}^n$ is a separable metric space. Separability is obviously a topological invariant.

(1.G.10) Theorem. If (X,d) is a separable metric space, then X is second countable.

Proof. Let $x_1, x_2, \ldots$ be a countable dense subset of X. Then the collection $\{S_{1/n}(x_i) \mid n \in \mathbf{Z}^+, i \in \mathbf{Z}^+\}$ is a countable basis for the topology generated by d.

The following theorem has numerous applications.

(1.G.11) Theorem. Suppose that $f,g : X \to Y$ are continuous and that Y is a T_2 space. If $f(x) = g(x)$ for each x in a dense subset D of X, then $f = g$.

Proof. Suppose that for some $z \in X$, we have $f(z) \neq g(z)$. Then there are disjoint open subsets U and V in Y that contain $f(z)$ and $g(z)$, respectively. However, this is impossible, since $W = f^{-1}(U) \cap g^{-1}(V)$ is an open set that clearly lies in the complement of the dense set D. Therefore, f equals g.

We conclude this chapter with the definition of a class of spaces whose properties will serve as a focus for much of our attention during the remainder of the text.

(1.G.12) Definition. A separable metric space X is an *n-manifold* if and only if each point of X is contained in a neighborhood that is homeomorphic to $\mathscr{E}^n$.

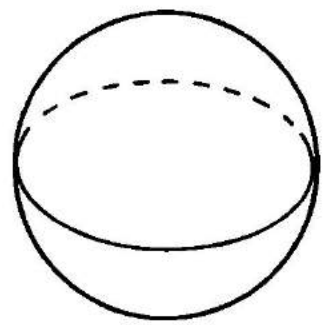
In general, n-manifolds are strongly geometric and have considerable visual appeal; nevertheless, they may become exceedingly complex. Observe that a point with little wanderlust living in an exotic n-manifold, might well consider its domicile to be nothing more exciting than $\mathscr{E}^n$.

(1.G.13) Examples of 1-Manifolds.

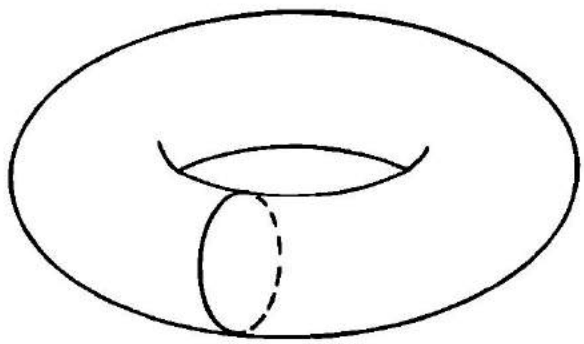
1. $\mathscr{E}^1$.
2. For each $n \in \mathbf{Z}$, let $A_n = \{(x,y) \in \mathscr{E}^2 \mid y = n\}$. Then $M = \bigcup \{A_n \mid n \in \mathbf{Z}\}$ is a 1-manifold.
3. The unit circle, $\mathscr{S}^1$, considered as a subspace of $\mathscr{E}^2$.

(1.G.14) Examples of 2-Manifolds.

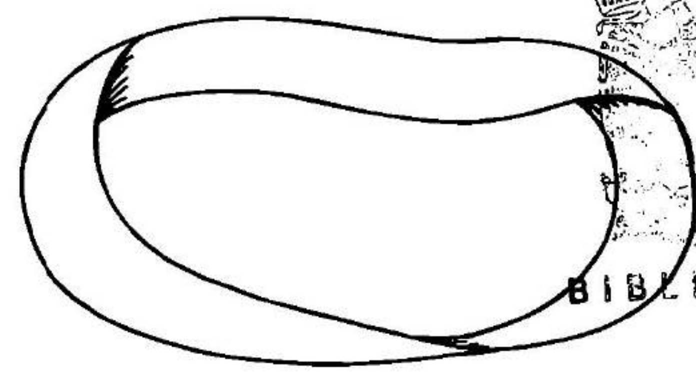
1. $\mathscr{E}^2$.
2. $\mathscr{S}^2 = \{(x,y,z) \in \mathscr{E}^3 \mid x^2 + y^2 + z^2 = 1\}$.



3. A torus (the outside of a tire or, alternatively, $\mathscr{S}^1 \times \mathscr{S}^1$).



4. A Moebius strip (without the edge).



5. Any open subset of $\mathscr{E}^2$.

(1.G.15) Examples of 3-Manifolds.

1. $\mathscr{E}^3$.
2. $\mathscr{S}^3 = \{(x,y,z,w) \in \mathscr{E}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$.
3. The inside of a torus.
4. The inside of a tin can.
5. Any open subset of $\mathscr{E}^3$.

(1.G.16) Exercise. Determine which of the following subspaces of $\mathscr{E}^2$ are manifolds ($\mathbf{O} = (0,0)$ and $S_r(\mathbf{O}) = \{y \in \mathscr{E}^2 \mid d(y,\mathbf{O}) < r\}$):

(i) $S_1(\mathbf{O})$;
(ii) $\overline{S_1(\mathbf{O})}$;
(iii) $S_2(\mathbf{O}) \setminus S_1(\mathbf{O})$;
(iv) $S_2(\mathbf{O}) \setminus \overline{S_1(\mathbf{O})}$;
(v) $\{(x,y) \mid -1 < x < 1, y = 0\} \cup \{(x,y) \mid x = 0, 0 \le y < 1\}$;

(1.G.17) Exercise.

Let X be the union of the rays in $\mathbf{R}^2$:
$A = \{(x,1) \mid x \ge 0\}$,
$B = \{(x,-1) \mid x \ge 0\}$, and
$C = \{(x,0) \mid x < 0\}$.

Points x in X other than $(0,-1)$ and $(0,1)$ are given neighborhood bases consisting of open intervals centered at x. Base neighborhoods of $(0,1)$ are of the form $\{(x,y) \mid b < x < 0$ and $y = 0$, or $0 \le x < a$ and $y = 1\}$, and base neighborhoods of $(0,-1)$ are of the form $\{(x,y) \mid b < x < 0$ and $y = 0$, or $0 \le x < a$ and $y = -1\}$. Show that X with the resulting topology is separable, T_1, first countable, and locally Euclidean, but that X is not T_2 and, hence, is not a 1-manifold.

PROBLEMS

Section A

1. (a) Suppose that $\{\mathscr{U}_\alpha \mid \alpha \in \Lambda\}$ is a collection of topologies for a set X. Show that $\bigcap \{\mathscr{U}_\alpha \mid \alpha \in \Lambda\}$ is a topology for X.
   (b) Is the union of topologies for a set X necessarily a topology?
   (c) Suppose that $\{\mathscr{U}_\alpha \mid \alpha \in \Lambda\}$ is a collection of topologies for a set X. Establish the existence of a unique largest topology that is smaller than each $\mathscr{U}_\alpha$ and a unique smallest topology that is larger than each $\mathscr{U}_\alpha$.
2. Find all possible topologies for the set $X = \{a,b,c\}$.
3. True or false? If $(X,\mathscr{U})$ is a topological space and $\mathscr{C}$ is a family of open sets in X, then $\bigcap \{C \mid C \in \mathscr{C}\}$ is open.
4. For each subset A of a set X, let $\mathscr{T}(A)$ be the topology on X whose open sets are $\varnothing$, X, and all subsets of X containing A. Assume that X has at least two elements.
   (i) Show that $A \subset B$ if and only if $\mathscr{T}(B) \subset \mathscr{T}(A)$.
   (ii) Suppose that $A_1, A_2, \ldots, A_n$ are subsets of X, and that $\mathscr{T}$ is a topology for X such that $\mathscr{T}(A_i) \subset \mathscr{T}$ for each i. Show that $\mathscr{T}(\bigcap_{i=1}^{n} A_i) \subset \mathscr{T}$.
5. Find four equivalent bases for $\mathscr{E}^2$.
6. Suppose that $\mathscr{U}_1$ and $\mathscr{U}_2$ are topologies for a set X and that $id : (X,\mathscr{U}_1) \to (X,\mathscr{U}_2)$ is the identity map. Show that id is continuous if and only if $\mathscr{U}_2 \subset \mathscr{U}_1$.
7. Show that a function $f : \mathscr{E}^1 \to \mathscr{E}^1$ is continuous if and only if $f^{-1}(-\infty,a)$ and $f^{-1}(a,\infty)$ are open for each $a \in \mathscr{E}^1$.
8. Show that a subset A of $\mathscr{E}^1$ (with the usual topology) is open if and only if A can be written as the countable union of pairwise disjoint open intervals (intervals such as $(-\infty,a)$, (b,∞), $(-\infty,\infty)$ are allowed).
9. Suppose that $(X,\mathscr{U})$ and $(Y,\mathscr{V})$ are topological spaces. A map $f : X \to Y$ is *open* if and only if $f(U) \in \mathscr{V}$ whenever $U \in \mathscr{U}$. Find an open map that is not continuous.

Section B

1. Given a topological space $(X,\mathscr{U})$, show that $\mathbf{B} \subset \mathscr{U}$ is a basis for $\mathscr{U}$ if and only if for each $x \in X$, $\mathbf{B}_x = \{B \in \mathbf{B} \mid x \in B\}$ is a neighborhood basis at x.

2. Suppose that X is a first countable space, Y is an arbitrary space, and $f: X \to Y$ is continuous and onto. Is Y necessarily first countable?
3. Determine which subsets A of $\mathscr{E}^1$ (with the usual topology) have the property that both A and $\mathscr{E}^1 \setminus A$ are open.
4. Suppose that $\mathscr{U}$ and $\mathscr{U}'$ are topologies for a set X such that $(X,\mathscr{U})$ is T_2 and $\mathscr{U} \subset \mathscr{U}'$. Show that $(X,\mathscr{U}')$ is T_2.
5. Suppose that $(X,\mathscr{U})$ is second countable. Show that any basis for $(X,\mathscr{U})$ contains a countable subcollection that is a basis for $\mathscr{U}$.
6. Show that $(X,\mathscr{U})$ is a topological space, and determine if it is T_2, first countable, or second countable where:
   (a) X is any uncountable set and $\mathscr{U}$ is $\varnothing$ together with $\{A \subset X \mid X \setminus A$ is countable$\}$;
   (b) X is any uncountable set and $\mathscr{U}$ consists of $\varnothing$ together with those subsets A of X such that either $X \setminus A$ is countable or $p \in (X \setminus A)$, where p is a fixed point in X;
   (c) $X = [-1,1]$, and $\mathscr{U}$ is the topology generated by a basis consisting of sets of the form $[-1,b)$, $(a,1]$, and (a,b), where $a < 0$ and $0 < b$.
7.* Let $X = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ and define an order on X by declaring $(a,b) \le (u,v)$ if and only if $a < u$, or $a = u$ and $b \le v$. This ordering is called the *lexicographic ordering*. Show that this ordering is a linear ordering and determine if X (with the order topology) is T_2, first countable, or second countable.
8. For each $x \in \mathbf{Z}$ and each $n \in \mathbf{N}$, let $B_x^n = \{y \mid y = rn + x, r \in \mathbf{Z}\}$. Show that $\mathbf{B} = \{B_x^n \mid x \in \mathbf{Z}, n \in \mathbf{N}\}$ is a basis for a topology on $\mathbf{Z}$.
9. Suppose that X is first countable (respectively, second countable) and that $f: X \to Y$ is open and onto. Show that Y is first countable (respectively, second countable).
10. True or false? Suppose that X is second countable. Then every nest of distinct open sets in X is countable.

Section C

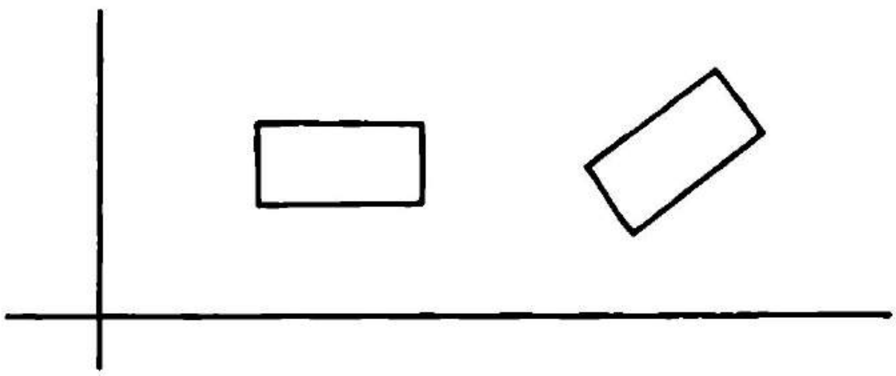
1. Suppose that (X,d) is a metric space and that p is a point of X. Show that for each $r > 0$, $\{x \in X \mid d(x,p) > r\}$ is open.
2. Give an $\varepsilon - \delta$ definition of continuity for a function between two metric spaces that is equivalent to (1.A.3).
3. Suppose that (X,d) is a metric space and $d' : X \times X \to [0,\infty)$ is defined by $d'(x,y) = d(x,y)/(1 + d(x,y))$. Show that d' is a metric for X and that d and d' are equivalent. [Hint: If $a \ge b \ge 0$ and $m \ge n \ge 0$, then $a/(1 + a) \ge b/(1 + b)$ and $(b + m)/(a + n) \ge b/a$.]
4. Suppose that X is a set consisting of a finite number of points. Describe

all possible topological structures that X can have in order to be metrizable.

5. *The post office metric.* Let $X = \mathbf{R}^2$ and d be the usual metric. Denote $(0,0)$ by $\mathbf{O}$. Define $\hat{d} : X \times X \to [0,\infty)$ by $\hat{d}(p,q) = d(\mathbf{O},p) + d(\mathbf{O},q)$ for $p,q \in X$ and $p \neq q$, and $d(p,p) = 0$ for all $p \in X$.
   (a) Show that $\hat{d}$ is a metric.
   (b) Show that all points other than $\mathbf{O}$ are open.
   (c) What are the neighborhoods of $\mathbf{O}$?
6. If X is a set, a function $d : X \times X \to [0,\infty)$ is called a *pseudometric* if and only if

   (i) $d(x,y) = d(y,x)$ for all $(x,y) \in X \times X$,
   (ii) $d(x,x) = 0$ for all $x \in X$, and
   (iii) $d(x,z) \leq d(x,y) + d(y,z)$ for $x,y,z \in X$.

   Let $X = \{f \mid f : \mathscr{E}^1 \to \mathscr{E}^1 \text{ and } f \text{ is integrable on } [0, 1]\}$. Define $d : X \times X \to [0,\infty)$ by $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$. Show that d is a pseudometric but not a metric. Is there a theorem analogous to (1.C.3) for pseudometric spaces?
7. Let $X = \mathbf{R}^2$, and for points $x = (x_1,x_2)$ and $y = (y_1,y_2)$ define
$$d(x,y) = \begin{cases} 1/2 & \text{if } x_1 = y_1,\ x_2 \neq y_2, \text{ or } x_1 \neq y_1 \text{ and } x_2 = y_2 \\ 1 & \text{if } x_1 \neq y_1 \text{ and } x_2 \neq y_2 \\ 0 & \text{otherwise} \end{cases}$$
   Show that d is a metric and that the sets shown in the figure possess different "area" using d to measure the length of sides.



   Describe the topology induced by this metric.
8. Suppose that is V a vector space over $\mathbf{R}^1$. Then a function $\phi : V \to [0,\infty)$ is a *norm* for V if and only if

   (i) $\phi(x) = 0$ if and only if $x = \mathbf{O}$,
   (ii) $\phi(x + y) \leq \phi(x) + \phi(y)$,
   (iii) $\phi(\alpha x) = |\alpha|\phi(x)$, where α is a real number.

The pair (V,ϕ) is called a *normed vector space* and $\phi(x)$ is frequently denoted by $\|x\|$.
Suppose that (V,ϕ) is a normed vector space and $d : V \times V \to [0,\infty)$ is defined by $d(x,y) = \|x - y\|$. Show that d is a metric for V.

9. *French railway metric.* Let $X = \mathbf{R}^2$. Suppose that x and y are points in X. If x and y are on a line passing through the origin, define their distance to be the usual Euclidean distance. If x and y do not lie on such a line, then define their distance as in the post office metric (problem 5). Show that the French railway metric is a metric and describe the induced topology.

Section D

1. Determine the relative topology that is induced by $\mathscr{E}^1$ for the integers.
2. Suppose that $\mathbf{B}$ is a basis for a topology on a set X and that $A \subset X$. Show that $\{B \cap A \mid B \in \mathbf{B}\}$ is a basis for the relative topology on A.
3. Suppose that $\mathbf{B}$ is a basis for a topology on a space X and $\mathbf{B}'$ is a basis for a topology on a space Y. Show that $\{B \times B' \mid B \in \mathbf{B},\ B' \in \mathbf{B}'\}$ is a basis for the product topology on $X \times Y$.
4. Show that if X and Y are T_2 spaces, then $X \times Y$ is a T_2 space.
5*. Suppose that $(X_1,d_1), (X_2,d_2), \ldots, (X_n,d_n)$ are metric spaces. Let $Y = X_1 \times X_2 \times \cdots \times X_n$ and define $\rho : Y \times Y \to [0,\infty)$ by setting

$$\rho((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sqrt{\sum_{i=1}^{n} (d_i(x_i,y_i))^2}$$

Show that ρ is a metric and the topology induced by ρ coincides with the product topology on Y.

6. Consider the following diagram where the p_i's and q_i's are projection maps, and the f_i's are continuous. Show that a unique continuous function $\phi : X_1 \times X_2 \to Y_1 \times Y_2$ may be defined which yields the following commutative diagram.

$$\begin{tikzcd}
X_1 \arrow[d, "f_1"'] & X_1 \times X_2 \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\phi"'] & X_2 \arrow[d, "f_2"] \\
Y_1 & Y_1 \times Y_2 \arrow[l, "q_1"] \arrow[r, "q_2"'] & Y_2
\end{tikzcd}$$

7. Suppose that $X_1, X_2, \ldots, X_n$ are topological spaces and that $Y = \prod_{i=1}^{n} X_i$. Let $g : Z \to Y$ be a function from a topological space Z into Y.

Show that g is continuous if and only if $p_i g$ is continuous for each projection map p_i (cf. 1.F.7.).

8. Is the product of a finite number of first countable spaces first countable? Is the product of a finite number of second countable spaces second countable?
9. Is the free union of a family of first countable spaces first countable? Is the free union of a family of second countable spaces second countable?
10.* Show that the free union of metrizable spaces is metrizable.
11.* Suppose that X and Y are topological spaces and $V \subset X \times Y$ is open. Show that $V[x] = \{y \mid (x,y) \in V\}$ and $V[y] = \{x \mid (x,y) \in V\}$ are open in Y and X respectively. Is the converse true: If $A \subset X \times Y$ and $A[x]$ and $A[y]$ are open for each $x \in X$ and $y \in Y$, then is A open in $X \times Y$?

Section E

1. Show that a subset U of a space X is open if and only if $A \cap U = \varnothing$ implies that $\bar{A} \cap U = \varnothing$ for each $A \subset X$.
2. Suppose that X is a space and that $A \subset B \subset X$. Show that $\bar{A}^B = \bar{A}^X \cap B$.
3. Find a topological space which has neither the discrete nor indiscrete topology in which subsets are open if and only if they are closed.
4. Suppose that f and g are continuous maps from a space X into a T_2 space Y. Show that $\{x \mid f(x) = g(x)\}$ is closed.
5. Show that $(A \cap B)^\circ = A^\circ \cap B^\circ$, and $(A^\circ)^\circ = A^\circ$.
6. Suppose that X is a finite T_1 space. Show that X must have the discrete topology.
7. Show that a set $U \subset X$ is open if and only if Fr $U \subset (X \setminus U)$.
8. Show that a set A is open and closed if and only if Fr $A = \varnothing$.
9. Find an example to show that if $A \subset B \subset X$, then int A in B is not necessarily the same as int A in X.
10. Show that $(A \cup B)' = A' \cup B'$.
11. Suppose that $(X,\mathcal{U})$ is a topological space, $A \subset X$, and x is an accumulation point of A. Discuss whether or not x must be an accumulation point of A with respect to topologies $\mathcal{U}'$ and $\mathcal{U}''$, where $\mathcal{U} \subset \mathcal{U}'$ and $\mathcal{U}'' \subset \mathcal{U}$.
12. Let $X = \{a,b,c\}$ and $\mathcal{U} = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}\}$ be a topology for X. Find the derived sets of all subsets of X.
13. Let X be an infinite set with the finite complement topology. Find the derived set, interior, exterior, frontier, and closure for each of the subsets of X.

14.* A space X is called a *door space* if and only if each subset of X is either open or closed.
   (a) Find a door space that does not have either the discrete or the indiscrete topology.
   (b) Suppose that X is a T_2 door space. Show that X has at most one accumulation point and that nonaccumulation points are open.

15.* A space X is called a *semi-door space* if and only if for each $A \subset X$, there is an open set U such that either $U \subset A \subset \overline{U}$ or $U \subset (X \setminus A) \subset \overline{U}$.
   (a) Find a semi-door space which is not a door space.
   (b) Show that a T_2 semi-door space is a door space. [Hint: Show that there is at most one point that is not open.]

16. Find a metric space to illustrate that $\overline{S_\varepsilon(x)} \neq \{y \in X \mid d(x,y) \leq \varepsilon\}$.

17. (a) Find a topological space that is not T_1.
   (b) Find a T_1 space that is not T_2.
   (c) Determine which of the spaces listed in the examples given in Section B are T_1.

18. Show that in a T_1 space, derived sets are closed.

19. Let $(X,\mathscr{U})$ have the finite complement topology, where X is infinite. Show that if $\mathscr{U}' \subset \mathscr{U}$ and $(X,\mathscr{U}')$ is T_1, then $\mathscr{U} = \mathscr{U}'$. Furthermore, show that if $(X,\mathscr{V})$ is T_1 then $\mathscr{U} \subset \mathscr{V}$. (Hence, $\mathscr{U}$ is the smallest T_1 topology for X.) If $\mathscr{T}$ is the family of all T_1 topologies for X, show that $\mathscr{U} = \bigcap \{T \mid T \in \mathscr{T}\}$.

20. Show that an uncountable subset of a second countable space has an accumulation point.

21.* Show that there exist closed subsets of [0,1] (usual topology) which are uncountable and consist only of irrational points.

22. Suppose that X is a set and $c : \mathscr{P}(X) \to \mathscr{P}(X)$ is a function with the following properties:

   (i) $c(\varnothing) = \varnothing$;
   (ii) $A \subset c(A)$ for each $A \in \mathscr{P}(X)$;
   (iii) $c(c(A)) = c(A)$ for each $A \in \mathscr{P}(X)$;
   (iv) $c(A \cup B) = c(A) \cup c(B)$ for each A and B in $\mathscr{P}(X)$.

   Let $\mathscr{U} = \{(X \setminus c(A)) \mid A \in \mathscr{P}(X)\}$. Show that $\mathscr{U}$ is a topology for X with the property that for each $A \in \mathscr{P}(X)$, $\overline{A} = c(A)$. A function with the properties of c is called a *closure operator*.

23. Let X be a set and $I : \mathscr{P}(X) \to \mathscr{P}(X)$ have the properties that:
   (i) $I(X) = X$;
   (ii) $I(I(A)) = I(A)$;
   (iii) $I(A) \subset A$;
   (iv) $I(A \cap B) = I(A) \cap I(B)$.

Let $\mathscr{U} = \{I(A) \mid A \in \mathscr{P}(X)\}$. Show that $\mathscr{U}$ is a topology for X, and that for each $A \in \mathscr{P}(X)$, $I(A) = A^\circ$. A function with the properties of I is called an *interior operator*.

24.* Formulate and prove a similar theorem for the derived operator, and for the frontier operator.

25. Prove that in a T_1 space, a point x is an accumulation point of a set A if and only if every neighborhood of x contains infinitely many points of A.

26. Suppose that A and B are disjoint closed subsets of a metric space. Show that there are disjoint open sets U and V containing A and B respectively.

27. Suppose that A is an open subset of a topological space X. Prove or disprove $\operatorname{int}(\bar{A}) = A$.

Section F

1. Use (1.A.4) to give an alternate proof of (1.F.1).
2. Suppose that X is a set and that $A \subset X$. The *characteristic function f_A associated with A* is a map from X into $\{0,1\}$ which assumes the value 0 for points not in A and 1 for points in A. If X is a topological space and $A \subset X$, show that f_A is continuous if and only if A is open and closed (assume that $\{0,1\}$ has the discrete topology).
3. Show that a function $f : X \to Y$ is continuous if and only if $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ for each $B \subset Y$.
4. Suppose that $f,g : X \to \mathscr{E}^1$ are continuous. Show that $h : X \to \mathscr{E}^1$ defined by $h(x) = \max\{f(x),g(x)\}$ is continuous.
5.* A map $f : X \to \mathscr{E}^1$ is *upper semicontinuous* if and only if for each $b \in \mathscr{E}^1$, $\{x \mid f(x) < b\}$ is open.
   (a) Suppose that $\{f_\alpha\}_{\alpha\in\Lambda}$ is a family of continuous functions each mapping a space X into $(0,1) \subset \mathscr{E}^1$. Show that $h : X \to \mathscr{E}^1$ defined by $h(x) = \inf\{f_\alpha(x) \mid \alpha \in \Lambda\}$ is upper semicontinuous.
   (b) Suppose that $f : X \to \mathscr{E}^1$ has the property that for each rational r, $\{x \mid f(x) < r\}$ is open. Show that f is upper semicontinuous.
6. Prove that the following are equivalent:
   (a) $f : X \to Y$ is continuous;
   (b) $f(A') \subset (f(A))'$ for each $A \subset X$;
   (c) $\operatorname{Fr} f^{-1}(B) \subset f^{-1}(\operatorname{Fr} B)$ for each $B \subset Y$.
7. Let $f : X \to Y$ and $A \subset X$. Show that it is possible for $f_{|A}$ to be continuous even though f is not continuous at any point of A.
8. Construct a function $f : \mathscr{E}^1 \times \mathscr{E}^1 \to \mathscr{E}^1$ such that f is continuous in each variable separately, but is not continuous on $\mathscr{E}^1 \times \mathscr{E}^1$.
9. Suppose that $(X,\mathscr{U})$ and $(Y,\mathscr{V})$ are topological spaces. A function

$f : X \to Y$ is *weakly continuous at a point* $x \in X$ if and only if whenever $f(x) \in V \in \mathscr{V}$, there is a $U \in \mathscr{U}$ such that $x \in U$ and $f(U) \subset \overline{V}$. The function f is *weakly continuous* if and only if it is weakly continuous at each $x \in X$. Show that f is weakly continuous if and only if for each $V \in \mathscr{V}$, $f^{-1}(V) \subset \operatorname{int}(f^{-1}(\overline{V}))$.

10.* Suppose that $(X,\mathscr{U})$ and $(Y,\mathscr{V})$ are topological spaces. A function $f : X \to Y$ is *feebly continuous* if and only if whenever $V \in \mathscr{V}, f^{-1}(\operatorname{Fr} V)$ is closed in X. Show that weak continuity and feeble continuity are unrelated, but that a function f is continuous if and only if it is both feebly and weakly continuous.

11. Suppose that $(X,\mathscr{U})$ and $(Y,\mathscr{V})$ are topological spaces. A function $f : X \to Y$ is *strongly continuous* if and only if $f(\overline{A}) \subset f(A)$ for each $A \subset X$. Show that f is strongly continuous if and only if $f^{-1}(B)$ is closed for each $B \subset Y$.

12. (a) Find an example of a topological space X and a sequence $\{x_i\}$ in X such that $\{x_i\}$ has more than one limit point in X.
    (b) Show that if X is Hausdorff then a sequence can have at most one limit point.

13. Find a counterexample to the following proposition: Suppose that X and Y are topological spaces and $f : X \to Y$ is a function with the property that whenever $\{x_n\}$ is a sequence in X and x is a point in X to which $\{x_n\}$ converges, the sequence $\{f(x_n)\}$ converges to $f(x)$. Then f is continuous.

Section G

1. Show that $\mathscr{E}^n$ is homeomorphic with $\{x \in \mathscr{E}^n \mid d(x,\mathbf{O}) < 1\}$.
2. A subset A of $\mathbf{R}^n$ is *convex* if and only if the line segment $\{tx + (1 - t)y \mid 0 \leq t \leq 1\}$ is contained in A whenever $x,y \in A$. Show that bounded convex open subsets of $\mathscr{E}^n$ are homeomorphic. Is the modifier "bounded" necessary?
3. Suppose that X and Y are topological spaces and that $f : X \to Y$ is bijective. Show that f is a homeomorphism if and only if for each $A \subset X$, $f(\overline{A}) = \overline{f(A)}$ holds.
4. Fill in the blank: Let $id : (X,\mathscr{U}) \to (X,\mathscr{U}')$. Then id is open if and only if $\mathscr{U}$ ______ $\mathscr{U}'$.
5. Suppose that X and X' are homeomorphic and that Y and Y' are homeomorphic. Show that $X \times Y$ and $X' \times Y'$ are homeomorphic.
6. Find mappings that are open but not closed, closed but not open, continuous but not open or closed.
7. Let A be the set of irrationals in $\mathscr{E}^1$ with the relative topology. Is A separable?

8. Suppose that A is a dense subset of X and G is a dense open subset of X. Show that $A \cap G$ is dense in X.
9. Suppose that A is dense in X and B is dense in Y. Show that $A \times B$ is dense in $X \times Y$.
10. A map $f : X \to Y$ is a *local homeomorphism* if and only if for each point $x \in X$, there is an open set U containing x that is mapped homeomorphically by f onto an open subset of Y. Show that local homeomorphisms are continuous and open.
11.* Prove the following converse of (1.G.11). Suppose that Y has the property that for any space X, any dense subset D of X, and any two continuous functions $f,g : X \to Y$ which agree on D, then $f = g$. Then Y is T_2.
12. A metric space X is *totally bounded* if and only if for each $\varepsilon > 0$, there are points $x_1, x_2, \ldots, x_n$ such that $X = \bigcup \{S_\varepsilon(x_i) \mid i = 1, 2, \ldots, n\}$. Show that a totally bounded metric space is separable.
13.* Give an example of a nonseparable metric space that does not have the discrete topology.
14. Suppose that (X,d) is a separable metric space and that $A \subset X$. Show that $(A,d_{|A})$ is separable. In general, is it true that a subspace of a separable space is separable?
15.* Suppose that $\mathbf{R}^1$ has the half-open interval topology $\mathscr{U}$, and $A = [0,1)$ is given the relative topology (with respect to $\mathscr{U}$). Show that $\mathbf{R}^1$ and A are homeomorphic.
16. True or false? If U is an open subset of X and $h : U \to X$ is an embedding, then $h(U)$ is open in X.
17.* Assume that the statement in problem 16 is true whenever $X = \mathscr{E}^n$, and prove that it is true for n-manifolds.
18.* A subset A of a space X is *semiopen* if and only if there is an open set U in X such that $U \subset A \subset \overline{U}$.
    (a) Show that A is semiopen if and only if $\overline{A} = \overline{A \cap D}$ for each dense subset D of X.
    (b) Suppose that $D \subset X$. Show that D is dense if and only if $\overline{A} = \overline{A \cap D}$ for each semiopen set A in X.
19. (a) A topological space is *0-dimensional* if and only if whenever $x \in V$, and V is open, there is an open set U with empty frontier such that $x \in U \subset V$. Show that the rationals and the irrationals (with the relative topology) are 0-dimensional sets.
    (b) The higher dimensions are defined inductively. A space X is said to have *dimension* $\leq n$, if and only if for each point $x \in X$ and for each open set V containing x there is an open set U such that $x \in U \subset V$, and dimension Fr $U \leq n - 1$; X has *dimension* n if and only if dimension $X \leq n$, but it is false that dimension $X \leq n - 1$. Show that $\mathscr{E}^1$ has dimension 1.

(c) Show that dimension is a topological invariant.

20. Are restrictions of open (closed) mappings open (closed)? Is the restriction of an open (closed) mapping to an open (closed) subset open (closed)?

21. Find a set X and distinct topologies $\mathscr{U}_1$ and $\mathscr{U}_2$ for X such that there is a homeomorphism $h : (X,\mathscr{U}_1) \to (X,\mathscr{U}_2)$.

22. Let $(X,\mathscr{U})$ and $(Y,\mathscr{V})$ be topological spaces. Show that a map $f : (X,\mathscr{U}) \to (Y,\mathscr{V})$ is open if and only if for each $A \subset X$, $f(\text{int } A) \subset \text{int}(f(A))$.