1.

2. Demostrar que $||f+g||_\infty \leq ||f||_\infty + ||g||_\infty$
Sabemos que para $x \in R^n$
$|f(x)+g(x)| \leq |f(x)| + |g(x)| \leq ||f||_\infty + ||g||_\infty$
Así $||f+g||_\infty \leq ||f||_\infty + ||g||_\infty$

3. Si $f \in L^1$ y $g \in L^\infty$, entonces
$$\int |fg| \leq ||f||_1 \cdot ||g||_\infty$$
Por monotonía, tenemos:
$$|fg| = |f||g| \leq |f| \cdot ||g||_\infty$$
$$\int |fg| \leq \int |f| \cdot ||g||_\infty = ||f||_1 \cdot ||g||_\infty$$

4. (a) Demostrar la desigualdad de Minkowski para $0 < p < 1$.
Lema: Sea $0 < p < 1$ y $q = 1 - p$, entonces
$$\int |fg| \geq ||f||_p \cdot ||g||_q$$
Sean $\frac{1}{q} = 1 - \frac{1}{p} \Rightarrow q = \frac{p}{p-1} < 0$
$p' = \frac{1}{p}$ y $q' = 1 - q = -\frac{1}{p-1}$
Además $\frac{1}{p'} + \frac{1}{q'} = p + \frac{1}{1-q} = p + \frac{1}{1-\frac{p}{p-1}} = p - p + 1 = 1$
Así tenemos:
$$\begin{aligned}\int |f|^p = \int |fg|^p \cdot |g|^{-p} &\leq \text{(Hölder) } ||(|fg|^p)||_{p'} \cdot ||(|g|^{-p})||_{q'} \\ &= \left(\int (|fg|^p)^{p'}\right)^{1/p'} \cdot \left(\int |g|^{-pq'}\right)^{1/q'} \\ &= \left(\int |fg|\right)^p \cdot \left(\int |g|^{\frac{p}{p-1}}\right)^{1-p}\end{aligned}$$
Así tenemos:
$$\left(\int |f|^p\right)\left(\int |g|^q\right)^{p-1} \leq \left(\int |fg|\right)^p$$
Sacando raíz p
$$\int |fg| \geq ||f||_p ||g||_q$$
Supongamos que si $f, g \in L^p$, entonces $(f+g) \in L^p$
Sea $q = \frac{p}{p-1}$, entonces $|f+g|^{p-1} \in L^p$ y
$$\left|\left||f+g|^{p-1}\right|\right|_q = \left(\int (|f+g|^{p-1})^q\right)^{1/q} = \left(\int |f+g|^p\right)^{(p-1)/p} = ||f+g||_p^{p-1}$$

Así tenemos que:

$$\begin{aligned} ||f+g||_p^p = \int |f+g|^p &= \int |f+g| \cdot |f+g|^{p-1} \\ &= \int (f+g) \cdot |f+g|^{p-1} \\ &= \int f|f+g|^{p-1} + \int g|f+g|^{p-1} \\ \text{Lema } &\geq ||f||_p ||f+g||_p^{p-1} + ||g||_p ||f+g||_p^{p-1} \end{aligned}$$

Así tenemos:

$$||f+g||_p \geq ||f||_p + ||g||_q$$

Cumpliendose la igualdad si $||f+g||_p = 0$

(b) Demostrar que si $f \in L^p$, $g \in L^p$ entonces $f+g \in L^p$ para $0 < p < 1$. Tenemos que:

$$\begin{aligned} |f(x)+g(x)|^p \leq& (|f(x)|+|g(x)|)^p \leq (2\max\{|f(x)|,|g(x)|\})^p = 2^p \max\{|f(x)|^p, |g(x)|^p\} \\ \leq& 2^p(|f(x)|^p + |g(x)|^p) \quad \square \end{aligned}$$

5. Sea E medible con medida finita y $1 \leq p_1 \leq p_2 \leq \infty$. Entonces $L^{p_2} \subset L^{p_1}$. Más aún

$$||f||_{p_1} \leq c||f||_{p_2}$$

para toda $f \in L^{p_2}$ con $c = (m(E))^{\frac{p_2-p_1}{p_1p_2}}$ si $p_2 < \infty$ y $c = (m(E))^{\frac{1}{p_1}}$ si $p_2 = \infty$.
Caso 1: $p_2 < \infty$
Si $f \in L^{p_2}$, entonces $|f|^{p_1} \in L^{p_2/p_1}$ y

$$\Big|\Big| |f|^{p_1} \Big|\Big|_{p_2/p_1} = \left(\int |f|^{p_2} \right)^{p_1/p_2} = ||f||_{p_2}^{p_1}$$

Por Hölder tenemos

$$||f||_{p_1}^{p_1} = \int_E |1 \cdot f|^{p_1} \leq ||1_E||_{p_2/(p_2-p_1)} \Big|\Big| |f|^{p_1} \Big|\Big|_{p_2/p_1} = (m(E))^{\frac{p_2-p_1}{p_2}} ||f||_{p_2}^{p_1}$$

Así tenemos

$$||f||_{p_1} \leq (m(E))^{\frac{p_2-p_1}{p_1p_2}} ||f||_{p_2}$$

Caso 2: $p_2 = \infty$

$$\begin{aligned} ||f||_{p_1} =& \left(\int_E |f|^{p_1} \right)^{1/p_1} \\ \text{monotonia } \leq& \left(\int_E 1 \cdot ||f||_\infty^{p_1} \right)^{1/p_1} = (m(E))^{\frac{1}{p_1}} ||f||_\infty \end{aligned}$$

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