1.

2. Demostrar que $||f + g||_\infty \leq ||f||_\infty + ||g||_\infty$
   Sabemos que para $x \in R^n$
   $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq ||f||_\infty + ||g||_\infty$
   Así $||f + g||_\infty \leq ||f||_\infty + ||g||_\infty$

3. Si $f \in L^1$ y $g \in L^\infty$, entonces
$$\int |fg| \leq ||f||_1 \cdot ||g||_\infty$$

   Por monotonía, tenemos:
$$|fg| = |f||g| \leq |f| \cdot ||g||_\infty$$
$$\int |fg| \leq \int |f| \cdot ||g||_\infty = ||f||_1 \cdot ||g||_\infty$$

4. (a) Demostrar la desigualdad de Minkowski para $0 < p < 1$.
   Lema: Sea $0 < p < 1$ y $q = 1 - p$, entonces
$$\int |fg| \geq ||f||_p \cdot ||g||_q$$

   Sean $\frac{1}{q} = 1 - \frac{1}{p} \Rightarrow q = \frac{p}{p-1} < 0$
   $p' = \frac{1}{p}$ y $q' = 1 - q = -\frac{1}{p-1}$
   Además $\frac{1}{p'} + \frac{1}{q'} = p + \frac{1}{1-q} = p + \frac{1}{1-\frac{p}{p-1}} = p - p + 1 = 1$
   Así tenemos:
$$\int |f|^p = \int |fg|^p \cdot |g|^{-p} \leq \text{(Hölder) } ||(|fg|^p)||_{p'} \cdot ||(|g|^{-p})||_{q'}$$
$$= \left(\int (|fg|^p)^{p'} \right)^{1/p'} \cdot \left(\int |g|^{-pq'} \right)^{1/q'}$$
$$= \left(\int |fg| \right)^p \cdot \left(\int |g|^{\frac{p}{p-1}} \right)^{1-p}$$

   Así tenemos:
$$\left(\int |f|^p \right) \left(\int |g|^q \right)^{p-1} \leq \left(\int |fg| \right)^p$$

   Sacando raíz p
$$\int |fg| \geq ||f||_p ||g||_q$$

   Supongamos que si $f, g \in L^p$, entonces $(f + g) \in L^p$
   Sea $q = \frac{p}{p-1}$, entonces $|f + g|^{p-1} \in L^p$ y
$$\left| \left| |f + g|^{p-1} \right| \right|_q = \left(\int (|f + g|^{p-1})^q \right)^{1/q} = \left(\int |f + g|^p \right)^{(p-1)/p} = ||f + g||_p^{p-1}$$

Así tenemos que:

$$||f + g||_p^p = \int |f + g|^p = \int |f + g| \cdot |f + g|^{p-1}$$

$$= \int (f + g) \cdot |f + g|^{p-1}$$

$$= \int f|f + g|^{p-1} + \int g|f + g|^{p-1}$$

$$\text{Lema } \geq ||f||_p ||f + g||_p^{p-1} + ||g||_p ||f + g||_p^{p-1}$$

Así tenemos:

$$||f + g||_p \geq ||f||_p + ||g||_q$$

Cumpliendose la igualdad si $||f + g||_p = 0$

(b) Demostrar que si $f \in L^p$, $g \in L^p$ entonces $f + g \in L^p$ para $0 < p < 1$. Tenemos que:

$$|f(x) + g(x)|^p \leq (|f(x)| + |g(x)|)^p \leq (2 \max\{|f(x)|, |g(x)|\})^p = 2^p \max\{|f(x)|^p, |g(x)|^p\}$$

$$\leq 2^p (|f(x)|^p + |g(x)|^p) \quad \square$$

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