Enhanced Rotation Scenarios for Quantum-Classical Benchmarking

A Reformulation for Computational Hardness

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1 Introduction

This report describes a fundamental reformulation of the crop allocation optimization problem, transforming it from a simple assignment problem into a **multi-period crop rotation optimization** with spatial interactions and frustrated synergies. The new formulation addresses two critical objectives:

1. **Classical hardness**: Create instances that challenge state-of-the-art MIP solvers
2. **Quantum tractability**: Maintain bounded degree for quantum annealer embedding

2 Original Formulation: Static Crop Allocation

2.1 Problem Structure

The original formulation (e.g., `full_family`) solves a **single-period assignment problem**:

$$\max_{Y_{f,c}} \sum_{f \in \mathcal{F}} \sum_{c \in \mathcal{C}} B_c \cdot L_f \cdot Y_{f,c} \tag{1}$$

subject to:

$$\sum_{c \in \mathcal{C}} Y_{f,c} \leq 1 \quad \forall f \in \mathcal{F} \qquad \text{(at most one crop per farm)}$$

$$Y_{f,c} \in \{0,1\} \quad \forall f, c \tag{2}$$

where:

- $\mathcal{F}$: Set of farms (plots)
- $\mathcal{C}$: Set of crops (27 individual crops)
- $Y_{f,c}$: Binary decision variable (farm f grows crop c)
- B_c: Weighted benefit of crop c (nutrition, sustainability, etc.)
- L_f: Land availability at farm f

2.2 Computational Properties

Classical complexity:

- Linear objective with binary variables
- LP relaxation finds integer solutions directly (0% integrality gap)
- Gurobi solves optimally at root node in <1 second
- **Too easy** for classical solvers

Quantum complexity:

- Dense coupling: $|\mathcal{C}| = 27$ crops per farm
- Max degree: $26 + k \cdot 27$ (unbounded with spatial interactions)
- **Not embeddable** on current quantum annealers (Pegasus graph)

3 New Formulation: Multi-Period Rotation with Synergies

3.1 Temporal Extension: 3-Period Rotation

The reformulation introduces **temporal dynamics** by optimizing crop rotations over $T = 3$ periods:

$$\max_{Y_{f,c,t}} \sum_{t=1}^{T} \underbrace{\sum_{f,c} B_c \cdot L_f \cdot Y_{f,c,t}}_{\text{Linear benefits}} + \underbrace{\sum_{t=2}^{T} \sum_{f} \sum_{c,c'} \gamma \cdot R_{c,c'} \cdot L_f \cdot Y_{f,c,t-1} \cdot Y_{f,c',t}}_{\text{Rotation synergies (QUADRATIC)}} \tag{3}$$

where:

- $Y_{f,c,t} \in \{0, 1\}$: Farm f grows crop family c in period t
- $R_{c,c'}$: Rotation synergy matrix (how well crop c follows c')
- γ: Rotation synergy weight (0.15–0.35)
- $T = 3$: Number of periods (years)

3.2 Crop Family Aggregation

To achieve quantum tractability, we aggregate 27 individual crops into $|\mathcal{C}| = 6$ crop families:

Crop Family	**Examples**
Fruits	Mango, Papaya, Orange, Banana, Apple
Grains	Corn, Potato (staple crops)
Legumes	Tofu, Tempeh, Peanuts, Chickpeas (nitrogen-fixing)
Leafy Vegetables	Spinach, Cabbage
Root Vegetables	Pumpkin, Eggplant, Tomatoes
Proteins	Egg, Beef, Lamb, Pork, Chicken

Table 1: Crop family aggregation scheme

This reduces the decision space from $|\mathcal{F}| \times 27 \times T$ to $|\mathcal{F}| \times 6 \times T$ variables.

3.3 Frustrated Rotation Synergies

The rotation matrix $R \in \mathbb{R}^{6\times 6}$ encodes agronomic interactions:

$$R_{c,c'} = \begin{cases} -\beta \cdot 1.5 & \text{if } c = c' \text{ (monoculture penalty)} \\ \text{Unif}(\beta \cdot 1.2, \beta \cdot 0.3) & \text{with prob. } p_{\text{frust}} \text{ (disease, competition)} \\ \text{Unif}(0.02, 0.20) & \text{otherwise (beneficial rotation)} \end{cases} \tag{4}$$

where:

- $\beta \in [-0.8, -1.5]$: Negative synergy strength
- $p_{\text{frust}} \in [0.70, 0.88]$: Frustration ratio (70%–88% negative edges)

Agronomic justification:

- **Monoculture penalty** ($c = c'$): Same crop depletes specific nutrients
- **Disease carryover**: Pathogens persist in soil (e.g., tomato $\rightarrow$ potato)
- **Allelopathy**: Some plants inhibit others chemically
- **Beneficial rotations**: Nitrogen-fixing legumes improve soil for grains

3.4 Spatial Neighbor Interactions

Farms are arranged on a grid and interact with their $k = 4$ nearest neighbors:

$$\text{Spatial term} = \sum_{t=1}^{T} \sum_{(f,f')\in\mathcal{E}} \sum_{c,c'} \gamma_s \cdot S_{c,c'} \cdot Y_{f,c,t} \cdot Y_{f',c',t} \tag{5}$$

where:

- $\mathcal{E}$: Edge set of k-nearest neighbor graph
- $S_{c,c'} = 0.3 \cdot R_{c,c'}$: Spatial compatibility (dampened rotation matrix)
- $\gamma_s = 0.5\gamma$: Spatial coupling strength

This models:

- Positive: Pollination, beneficial insects, wind breaks
- Negative: Pest/disease spread, resource competition

3.5 Soft One-Hot Constraint (Key Innovation)

Original (too strong):

$$\sum_{c\in\mathcal{C}} Y_{f,c,t} = 1 \quad \forall f, t \qquad \text{(hard constraint)}$$

Enhanced (creates integrality gap):

$$\text{Objective penalty} = -P \sum_{f,t} \left(\sum_{c} Y_{f,c,t} - 1 \right)^2 \tag{6}$$

with upper bound constraint:

$$\sum_{c\in\mathcal{C}} Y_{f,c,t} \leq 2 \quad \forall f, t \tag{7}$$

where $P \in [1.5, 3.0]$ (penalty strength).
Why this works:

- LP relaxation can fractionally satisfy: $Y_{f,c_1,t} = Y_{f,c_2,t} = 0.5$
- This achieves higher objective (diversity bonus + synergies)
- But MIP **must choose** $Y \in \{0, 1\}$
- Choosing creates conflicts with frustrated synergies
- Result: **massive integrality gap**

3.6 Diversity Bonus (Competing Objective)

$$\text{Diversity bonus} = \delta \sum_{f,c} \sum_{t=1}^{T} Y_{f,c,t} \tag{8}$$

where $\delta \in [0.15, 0.25]$.

This encourages using many crop families, competing with rotation quality. LP can fractionally use all crops; MIP forced to make discrete choices.

4 Computational Complexity Analysis

4.1 Problem Size

Scenario	Farms	Families	Periods	Variables	Max Degree
rotation_micro_25	5	6	3	90	∼29
rotation_small_50	10	6	3	180	∼29
rotation_medium_100	20	6	3	360	∼29
rotation_large_200	50	6	3	900	∼29

Table 2: Problem sizes for rotation scenarios

Max degree calculation:

$$d_{\max} = \underbrace{(|\mathcal{C}| - 1)}_{\text{same farm}} + \underbrace{k \cdot |\mathcal{C}|}_{\text{spatial neighbors}} = 5 + 4 \times 6 = 29 \tag{9}$$

4.2 Classical Hardness: Empirical Results

Scenario	Gurobi Status	Time (s)	BB Nodes	MIP Gap
rotation_micro_25	TIME_LIMIT	300	2,476,215	>700%
rotation_small_50	TIME_LIMIT	300	1,233,138	>700%
rotation_medium_100	TIME_LIMIT	300	315,378	>700%
rotation_large_200	TIME_LIMIT	300	77,483	>700%

Table 3: Gurobi 12.0.3 benchmark results (5-minute timeout)

Key finding: All instances timeout – Gurobi cannot solve optimally within 5 minutes, exploring millions of branch-and-bound nodes.

Metric	Original (full_family)	Enhanced (rotation)
Objective type	Linear	Quadratic
Time periods	1 (static)	3 (dynamic)
Crop choices	27 individual crops	6 crop families
Synergies	None	70–88% frustrated
One-hot	Hard constraint	Soft penalty
Variables (100 farms)	2700	360
Max degree	Unbounded (>100)	Bounded (29)
Classical:		
Integrality gap	0%	>700%
Gurobi time	<1s	>300s (timeout)
BB nodes	1	77K–2.5M
Quantum:		
Embeddable?	No (too dense)	Yes (degree 29)
QPU chains	N/A	∼2–3

Table 4: Comprehensive comparison of formulations

4.3 Comparison: Original vs. Enhanced

5 The Quantum Advantage Sweet Spot

The enhanced rotation formulation achieves:

$$\boxed{\begin{array}{rl} \text{Classical:} & \text{VERY HARD (timeout, unmeasurable gap)} \\ \text{Quantum:} & \text{FEASIBLE (bounded degree, embeddable)} \end{array}} \tag{10}$$

This is precisely the regime where quantum annealers may demonstrate advantage over classical solvers.

5.1 Why Quantum Annealers May Succeed

1. **Native QUBO structure**: Quadratic synergies map directly to qubit couplings
2. **Frustrated landscape**: Quantum tunneling may escape local minima
3. **Sparse coupling**: $k = 4$ neighbors creates embeddable topology
4. **No branch-and-bound**: Quantum evolution explores solution space differently

6 Summary

The rotation scenarios represent a **paradigm shift** from the original formulation:

- **From static to dynamic**: Single period → 3-period rotation
- **From linear to quadratic**: Simple benefits → Synergistic interactions
- **From unfrustrated to frustrated**: Positive-only → 70–88% negative couplings
- **From hard to soft constraints**: Exact one-hot → Penalized deviation
- **From dense to sparse**: 27 crops → 6 families, bounded degree

These modifications transform an easy assignment problem into a challenging combinatorial optimization suitable for quantum-classical benchmarking, while preserving **agronomic realism** (crop rotations are standard practice in sustainable agriculture).

Next step: Benchmark these instances on D-Wave quantum annealers to measure time-to-solution and solution quality compared to classical methods.