Midterm-Q3
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1 The Perron-Frobenius Theorem

The Perron–Frobenius theorem was derived by Oskar Perron and later generalized by Georg Frobenius. This theorem asserts that a real square matrix with positive entries has a unique largest real eigenvalue and that the corresponding eigenvector can be chosen to have strictly positive components, and also asserts a similar statement for certain classes of non-negative matrices.

The theorem was proved for matrices with strictly positive entries by Perron in 1907 and extended by Frobenius to matrices which have non-negative entries and are irreducible in 1912.

There are widely used application of this theorem such as at probability theory for Markov chains, dynamical systems, economics, demography of population, social networking and internet search engines [1].

1.1 Positive Matrices

Let $\mathbf{A}$ be a matrix having entries a_{ij}.

$\mathbf{A}$ is said to be positive matrix if all entries are positive.

i.e., $a_{ij} > 0$ where a_{ij} represents the (i, j)th entry of A

1.2 Primitive Matrices

Let $\mathbf{A}$ be a nonnegative matrix whose entries a_{ij} are nonnegative numbers.

$\mathbf{A}$ is said to be primitive if, for some integer m_0, A^{m_0} is a positive matrix.

i.e., $a_{ij}^{(m_0)} > 0$ where $a_{ij}^{(m_0)}$ represents the (i, j)th entry of A^{m_0}

For example, the square matrix $\mathbf{P}$ below is primitive for all $p_i > 0$, since P_2 is a positive matrix. [2].

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 & \cdots & p_m \\ 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

1.3 Irreducible Matrices

A non-negative matrix square $\mathbf{A}$ is called **irreducible** if for any i, j there is a $k = k(i, j)$ such that $A_{ij}^k > 0$. In graph perspective, this means any node i, j is connected to each other in k steps.

1.4 Spectral Radius

The spectral radius of a matrix $\mathbf{A}$ represents the maximum of the absolute values of the eigenvalues of A. An eigenvalue and eigenvector pair of the matrix A satisfies the equation $A\underline{x} = \lambda\underline{x}$, where λ and $\underline{x}$ represent the eigenvalue and the corresponding eigenvector, respectively.Thus, spectral radius of A, $\rho(A)$, is as follows: [2]

$$\rho(A) = \max_i |\, \lambda_i(A)\, |$$

1.5 Statement of Theorem

Let $\mathbf{A}$ be a an irreducible matrix with non-negative entries, i.e. $a_{ij} \in \mathbb{R}_{\geq 0}$. Then, [1], [3]

1. There exists a unique eigenvalue pf, called Perron root or the Perron–Frobenius eigenvalue, of A , where $pf \in \mathbb{R}_{\geq 0}$, whose absolute value is bigger than those of other eigenvalues : **The leading eigenvalue**

2. Up to scalars, there is a unique eigenvector PF with entries from $\mathbb{R}_{\geq 0}$, and it has eigenvalue pf : **The leading eigenvector**

3. The only eigenvectors with the same absolute value of pf are on the same circle of pf : **Symmetry of eigenvalue**

4. pf is an eigenvalue of A and any other eigenvalue λ (possibly complex) in absolute value is strictly smaller than pf , $|\, \lambda\, | < pf$. Thus, the spectral radius $\rho(A)$ is equal to pf.

5. pf is a simple root of the characteristic polynomial of A. Consequently, the eigenspace associated to pf is one-dimensional. (The same is true for the left eigenspace, i.e., the eigenspace for $\mathbf{A}^\top$, the transpose of $\mathbf{A}$.)

1.6 Applications of Theorem

This theorem is used various studies. Some of the well-know studies are:

1. **The Leslie model of population growth** : In 1945 Leslie introduced a model for the growth of a stratified population. The population to consider consists of the females of a species, and the stratification is by age group.

2. **Markov Chains** : A non-negative matrix $\mathbf{M}$, a stochastic matrix, having each of the row sums equal to 1

3. **The Google ranking** : Pagerank is the algorithm to define the ranking of webpages based on "importance", which is defined by highest probability to be clicked from previous pages (higher number of links from other pages)

References. [1] Wikipedia, *Perron–Frobenius theorem*, https://en.wikipedia.org/wiki/Perron-Frobenius_theorem [2] S. Unnikrishna Pillai, Torsten Suel, and Seunghun Cha, *The Perron-Frobenius Theorem [Some of its applications]*, IEEE SIGNAL PROCESSING MAGAZINE, March 2005 [3] VisualMath, *What is...the Perron-Frobenius theorem?*, YouTube, https://www.youtube.com/watch?v=jMmagF4IWrY&t