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APPLIED LINEAR ALGEBRA TEST 2

VARIANT 8

№1

Problem

For a matrix

$$A = \begin{pmatrix} 42 & -86 & -46 & 8 \\ -39 & 68 & -110 & 40 \\ -18 & 52 & 83 & -64 \end{pmatrix}$$

find the best approximation A_1 of rank 2 in the norm $\|\cdot\|_2$. Find $\|A - A_1\|_2$.

Solution

First of all, we perform a singular value decomposition for A: $A = U\Sigma V^*$, where

$$U = \begin{pmatrix} -0.333333 & -0.666667 & 0.666667 \\ -0.666667 & 0.666667 & 0.333333 \\ 0.666667 & 0.333333 & 0.666667 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} 162 & 0 & 0 & 0 \\ 0 & 135 & 0 & 0 \\ 0 & 0 & 27 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} 0 & -0.444444 & 0.111111 & 0.888889 \\ 0.111111 & 0.888889 & 0 & 0.444444 \\ 0.888889 & -0.111111 & -0.444444 & 0 \\ -0.444444 & 0 & -0.888889 & 0.111111 \end{pmatrix}.$$

Secondly, to construct an approximation A_1 we consider Σ_2 instead of Σ, where

$$\Sigma_2 = \begin{pmatrix} 162 & 0 & 0 & 0 \\ 0 & 135 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To get an explicit form of A_1, we build it up from its SVD $A_1 = U\Sigma_2 V^*$:

$$A_1 = \begin{pmatrix} 40 & -86 & -38 & 24 \\ -40 & 68 & -106 & 48 \\ -20 & 52 & 91 & -48 \end{pmatrix}.$$

In the end, we calculate the error of approximation.

$$\|A - A_1\|_2 = \left\| \begin{pmatrix} 2 & 0 & -8 & -16 \\ 1 & 0 & -4 & -8 \\ 2 & 0 & -8 & -16 \end{pmatrix} \right\|_2 = \left\| \widehat{U} \begin{pmatrix} 27 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \widehat{V}^* \right\|_2 = 27.$$

Answer

$$A_1 = \begin{pmatrix} 40 & -86 & -38 & 24 \\ -40 & 68 & -106 & 48 \\ -20 & 52 & 91 & -48 \end{pmatrix};$$

$\|A - A_1\|_2 = 27.$

№2

Problem

Estimate the relative error of the approximate solution $\hat{x} = (1,1)^T$ of the system $Ax = b$ in the norms $|\cdot|_1$ and $|\cdot|_2$ using the condition number of matrix A, where

$$A = \begin{pmatrix} 4.0 & 0.18 \\ 3.94 & -7.92 \end{pmatrix},$$

$$b = \begin{pmatrix} 4.17 \\ -4.09 \end{pmatrix}.$$

Solution

Approximate solution $\hat{x}$ can be got by solving an approximation of the original system with

$$\hat{A} = \begin{pmatrix} 4 & 0 \\ 4 & -8 \end{pmatrix},$$

$$\hat{b} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}.$$

Let us find relative errors for A and b.

$$\delta_1 A = \frac{\|A - \hat{A}\|_1}{\|A\|_1} = \frac{\max\{0.06, 0.26\}}{\max\{7.94, 8.1\}} = \frac{0.26}{8.1} = 0.032099$$

$$\delta_1 b = \frac{|b - \hat{b}|_1}{|b|_1} = \frac{0.26}{8.26} = 0.031477$$

$$\delta_2 A = \frac{\|A - \hat{A}\|_2}{\|A\|_2} = \frac{\left\| U' \begin{pmatrix} 0.198602 & 0 \\ 0 & 0.05438 \end{pmatrix} V'^* \right\|_2}{\left\| U'' \begin{pmatrix} 9.021845 & 0 \\ 0 & 3.590086 \end{pmatrix} V''^* \right\|_2} = \frac{0.198602}{9.021845} =$$

$$= 0.022013$$

$$\delta_2 b = \frac{|b - \hat{b}|_2}{|b|_2} = \frac{0.192354}{5.840976} = 0.032932$$

To find $\varkappa(A)$ we calculate A^{-1}.

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} -7.92 & -0.18 \\ -3.94 & 4.0 \end{pmatrix} = \frac{1}{-32.3892}\begin{pmatrix} -7.92 & -0.18 \\ -3.94 & 4.0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0.244526 & 0.005557 \\ 0.121645 & -0.123498 \end{pmatrix}$$

Thus,

$$\varkappa_1(A) = \|A\|_1 \|A^{-1}\|_1 = 8.1 \cdot \max\{0.366171, 0.129055\} = 8.1 \cdot 0.366171 =$$

$$= 2.965985$$

$$\varkappa_2(A) = \|A\|_2 \|A^{-1}\|_2 = 9.021845 \cdot \left\| U''' \begin{pmatrix} 0.278545 & 0 \\ 0 & 0.110842 \end{pmatrix} V'''^* \right\|_2 =$$

$$= 9.021845 \cdot 0.278545 = 2.51299$$

In the very end, for $|\cdot|_1$:

$$\frac{\delta_1 A + \delta_1 b}{\varkappa_1(A)} \le \delta_1 x \le (\delta_1 A + \delta_1 b) \cdot \varkappa_1(A)$$

$$\frac{0.032099 + 0.031477}{2.965985} \le \delta_1 x \le (0.032099 + 0.031477) \cdot 2.965985$$

$$0.021435 \le \delta_1 x \le 0.188565$$

For $|\cdot|_2$:

$$\frac{\delta_2 A + \delta_2 b}{\varkappa_2(A)} \le \delta_2 x \le (\delta_2 A + \delta_2 b) \cdot \varkappa_2(A)$$

$$\frac{0.022013 + 0.032932}{2.51299} \le \delta_2 x \le (0.022013 + 0.032932) \cdot 2.51299$$

$$0.021864 \le \delta_2 x \le 0.138076$$

Answer

$\delta_1 x \in [0.021435, 0.188565]$;

$\delta_2 x \in [0.021864, 0.138076]$.

№3

Problem

Solve the system

$$\begin{cases} 4(-8+\varepsilon_1)\ x \ + \ 2(-5+\varepsilon_2)\ y \ = \ \ 4 \ + \ \varepsilon_3 \\ \quad 4 \qquad x \ + \quad (3+\varepsilon_1) \quad y \ = -1 \ + \ \varepsilon_4 \end{cases}$$

approximately and estimate the error of the solution in the norms $|\cdot|_1$, $|\cdot|_2$ and $|\cdot|_\infty$. All unknown numbers satisfy the condition $|\varepsilon_i| < 0.05$.

Solution

This system can be represented in vector-matrix form $Av = b$, where

$$A = \underbrace{\begin{pmatrix} -32 & -10 \\ 4 & 3 \end{pmatrix}}_{\hat{A}} + \underbrace{\begin{pmatrix} 4\varepsilon_1 & 2\varepsilon_2 \\ 0 & \varepsilon_1 \end{pmatrix}}_{\Delta A}$$

$$b = \underbrace{\begin{pmatrix} 4 \\ -1 \end{pmatrix}}_{\hat{b}} + \underbrace{\begin{pmatrix} \varepsilon_3 \\ \varepsilon_4 \end{pmatrix}}_{\Delta b}$$

Let us solve the approximate system $\hat{A}\hat{v} = \hat{b}$.

$$\begin{pmatrix} -32 & -10 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -32 & -10 & 4 \\ 4 & 3 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 4 & 3 & -1 \\ 0 & 14 & -4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3/4 & -1/4 \\ 0 & 1 & -2/7 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -1/28 \\ 0 & 1 & -2/7 \end{array}\right) \Rightarrow$$

$$\Rightarrow \hat{v} = \begin{pmatrix} -1/28 \\ -2/7 \end{pmatrix} \approx \begin{pmatrix} -0.035714 \\ -0.285714 \end{pmatrix}$$

Let us now estimate relative errors of $\hat{A}$ and $\hat{b}$.

$$\delta_1 A \approx \frac{\|\Delta A\|_1}{\|\hat{A}\|_1} = \frac{\max\{4|\varepsilon_1|, |\varepsilon_1| + 2|\varepsilon_2|\}}{\max\{36,13\}} < \frac{\max\{0.2, 0.15\}}{36} = \frac{0.2}{36} = 0.005556$$

$$\delta_2 A \approx \frac{\|\Delta A\|_2}{\|\hat{A}\|_2} = \frac{\sqrt{\lambda_{max}\begin{pmatrix} 16\varepsilon_1^2 + 4\varepsilon_2^2 & 2\varepsilon_1\varepsilon_2 \\ 2\varepsilon_1\varepsilon_2 & \varepsilon_1^2 \end{pmatrix}}}{\left\| U \begin{pmatrix} 33.856523 & 0 \\ 0 & 1.654039 \end{pmatrix} V^* \right\|_2} =$$

$$= \frac{\sqrt{\operatorname{rootmax}\left((16\varepsilon_1^2 + 4\varepsilon_2^2 - \lambda)(\varepsilon_1^2 - \lambda) - 4\varepsilon_1^2\varepsilon_2^2\right)}}{33.856523} =$$

$$= \frac{\sqrt{\operatorname{rootmax}(16\varepsilon_1^4 + 4\varepsilon_1^2\varepsilon_2^2 - \varepsilon_1^2\lambda - 16\varepsilon_1^2\lambda - 4\varepsilon_2^2\lambda + \lambda^2 - 4\varepsilon_1^2\varepsilon_2^2)}}{33.856523} =$$

$$= \frac{\sqrt{\operatorname{rootmax}(\lambda^2 + (-17\varepsilon_1^2 - 4\varepsilon_2^2)\lambda + 16\varepsilon_1^4)}}{33.856523} =$$

$$= \frac{\sqrt{\frac{17\varepsilon_1^2 + 4\varepsilon_2^2 + \sqrt{D}}{2}}}{33.856523}$$

Here $D = (17\varepsilon_1^2 + 4\varepsilon_2^2)^2 - 64\varepsilon_1^4 = 225\varepsilon_1^4 + 136\varepsilon_1^2\varepsilon_2^2 + 16\varepsilon_2^4$. Since D is monotonously increasing, its values can be estimated: $D \in [0, 0.002356)$. Hence, $\sqrt{D} \in [0, 0.048541)$. By using this, we derive that

$$\delta_2 A < \frac{\sqrt{0.050521}}{33.856523} = \frac{0.237904}{33.856523} = 0.007027$$

$$\delta_\infty A \approx \frac{\|\Delta A\|_\infty}{\|\hat{A}\|_\infty} = \frac{\max\{4|\varepsilon_1| + 2|\varepsilon_2|, |\varepsilon_1|\}}{\max\{42,7\}} < \frac{\max\{0.3,0.05\}}{42} = \frac{0.3}{42} = 0.007143$$

Let us find the relative errors of the right-had side vector.

$$\delta_1 b \approx \frac{|\Delta b|_1}{|\hat{b}|_1} = \frac{|\varepsilon_3| + |\varepsilon_4|}{5} < \frac{0.1}{5} = 0.02$$

$$\delta_2 b \approx \frac{|\Delta b|_2}{|\hat{b}|_2} = \frac{\sqrt{\varepsilon_1^2 + \varepsilon_2^2}}{4.123106} < \frac{0.070711}{4.123106} = 0.01715$$

$$\delta_\infty b \approx \frac{|\Delta b|_\infty}{|\hat{b}|_\infty} = \frac{\max\{|\varepsilon_3|, |\varepsilon_4|\}}{4} < \frac{0.05}{4} = 0.0125$$

For further analysis we assume that $\varkappa(A) \approx \varkappa(\hat{A})$. To find $\varkappa(\hat{A})$ we first need to find $\hat{A}^{-1}$.

$$\hat{A}^{-1} = \frac{1}{\det \hat{A}}\begin{pmatrix} 3 & 10 \\ -4 & -32 \end{pmatrix} = \frac{1}{-56}\begin{pmatrix} 3 & 10 \\ -4 & -32 \end{pmatrix} = \begin{pmatrix} -0.053571 & -0.178571 \\ 0.071429 & 0.571429 \end{pmatrix}$$

Now let us calculate $\varkappa(\hat{A})$.

$$\varkappa_1(\hat{A}) = \|\hat{A}\|_1 \|\hat{A}^{-1}\|_1 = 36 \cdot \max\{0.125,0.75\} = 36 \cdot 0.75 = 27$$

$$\varkappa_2(\hat{A}) = \|\hat{A}\|_2 \|\hat{A}^{-1}\|_2 = 33.856523 \cdot \left\| U' \begin{pmatrix} 0.604581 & 0 \\ 0 & 0.990217 \end{pmatrix} V'^* \right\|_2 =$$

$$= 33.856523 \cdot 0.604581 = 20.469011$$

$$\varkappa_\infty(\hat{A}) = \|\hat{A}\|_\infty \|\hat{A}^{-1}\|_\infty = 42 \cdot \max\{0.232142, 0.642858\} = 42 \cdot 0.642858 =$$

$$= 27.000036$$

Thus, we are able to estimate the overall relative error of the solution $\hat{v}$. For $|\cdot|_1$:

$$\frac{\delta_1 A + \delta_1 b}{\varkappa_1(A)} \le \delta_1 v \le (\delta_1 A + \delta_1 b) \cdot \varkappa_1(A)$$

$$0 \le \frac{\delta_1 A + \delta_1 b}{\varkappa_1(A)} \le \delta_1 v \le (\delta_1 A + \delta_1 b) \cdot \varkappa_1(A) < (0.005556 + 0.02) \cdot 27$$

$$0 \le \delta_1 v < 0.690012$$

For $|\cdot|_2$:

$$\frac{\delta_2 A + \delta_2 b}{\varkappa_2(A)} \le \delta_2 v \le (\delta_2 A + \delta_2 b) \cdot \varkappa_2(A)$$

$$0 \le \delta_2 v < (0.007027 + 0.01715) \cdot 20.469011$$

$$0 \le \delta_2 v < 0.494879$$

For $|\cdot|_\infty$:

$$\frac{\delta_\infty A + \delta_\infty b}{\varkappa_\infty(A)} \le \delta_\infty v \le (\delta_\infty A + \delta_\infty b) \cdot \varkappa_\infty(A)$$

$$0 \le \delta_\infty v < (0.007143 + 0.0125) \cdot 27.000036$$

$$0 \le \delta_\infty v < 0.530362$$

Answer

$$\hat{v} = \begin{pmatrix} -1/28 \\ -2/7 \end{pmatrix} \approx \begin{pmatrix} -0.035714 \\ -0.285714 \end{pmatrix};$$

$\delta_1 v \in [0, 0.690012)$;

$\delta_2 v \in [0, 0.494879)$;

$\delta_\infty v \in [0, 0.530362)$.

№4

Problem

Find the approximate inverse for the matrix A, where

$$A \approx \begin{pmatrix} 6 & -1 \\ -4 & 9 \end{pmatrix}.$$

Evaluate the approximation error with respect to $\|\cdot\|_1$, knowing that each element of A is given with an absolute error of 0.01.

Solution

The real matrix A looks like

$$A = \underbrace{\begin{pmatrix} 6 & -1 \\ -4 & 9 \end{pmatrix}}_{\hat{A}} + \underbrace{\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}}_{\Delta A},$$

where each $|\varepsilon_{ij}| < 0.01$.

Let us first estimate the relative error for A.

$$\delta_1 A \approx \frac{\|\Delta A\|_1}{\|\hat{A}\|_1} = \frac{\max\{|\varepsilon_{11}| + |\varepsilon_{21}|, |\varepsilon_{12}| + |\varepsilon_{22}|\}}{\max\{10,10\}} < \frac{0.02}{10} = 0.002$$

Secondly, we will find $\hat{A}^{-1}$.

$$\hat{A}^{-1} = \frac{1}{\det \hat{A}} \begin{pmatrix} 9 & 1 \\ 4 & 6 \end{pmatrix} = \frac{1}{50} \begin{pmatrix} 9 & 1 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 0.18 & 0.02 \\ 0.08 & 0.12 \end{pmatrix}$$

Using this, we may find $\varkappa(A)$ assuming that $\varkappa(A) \approx \varkappa(\hat{A})$.

$$\varkappa_1(\hat{A}) = \|\hat{A}\|_1 \|\hat{A}^{-1}\|_1 = 10 \cdot \max\{0.26, 0.14\} = 10 \cdot 0.26 = 2.6$$

Thus,

$$\delta_1 A^{-1} \leq \frac{\delta_1 A \cdot \varkappa_1(A)}{1 - \delta_1 A \cdot \varkappa_1(A)} = \frac{0.002 \cdot 2.6}{1 - 0.002 \cdot 2.6} = \frac{0.0052}{0.9948} = 0.005227$$

Answer

$$A^{-1} \approx \begin{pmatrix} 0.18 & 0.02 \\ 0.08 & 0.12 \end{pmatrix};$$

$$\delta_1 A^{-1} \leq 0.005227.$$

№5

Problem

Use simple iteration method to find the solution of the system

$$\begin{cases} 26x + 4y + 5z = 5 \\ 7x + 24y + 6z = 6 \\ 5x + 3y + 27z = 2 \end{cases}$$

Determine the sufficient number of iterations after which the approximation error for each variable does not exceed 0.01 and find the corresponding solution. Start with $v_0 = (0,0,0)^T$.

Solution

This system can be rewritten in a vector-matrix form $Av = b$, where

$$A = \begin{pmatrix} 26 & 4 & 5 \\ 7 & 24 & 6 \\ 5 & 3 & 27 \end{pmatrix},$$

$$b = \begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix}.$$

Since A has an obvious diagonal dominance, it is appropriate to utilise the Jacobi method in order to find the approximate solution with the help of an iterative process.

$$A = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 7 & 0 & 0 \\ 5 & 3 & 0 \end{pmatrix}}_{L} + \underbrace{\begin{pmatrix} 26 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 27 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 4 & 5 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}}_{U}$$

Here we can easily deduce that

$$D^{-1} = \begin{pmatrix} 0.038462 & 0 & 0 \\ 0 & 0.041667 & 0 \\ 0 & 0 & 0.037037 \end{pmatrix}$$

The iterative process will be defined as

$$v^{n+1} = -D^{-1}(L + U)v^n + D^{-1}b$$

Since the approximation error for *each* variable should not exceed 0.01, we will use the $|\cdot|_\infty$ and $\|\cdot\|_\infty$ for further calculations. Let us find the sufficient number of steps to achieve the aforementioned goal.

$$k \geq \log_{\|D^{-1}(L+U)\|_\infty} \frac{0.01(1-\|D^{-1}(L+U)\|_\infty)}{|v^1 - v^0|_\infty}$$

Here

$\|D^{-1}(L+U)\|_\infty =$

$$= \left\| \begin{pmatrix} 0.038462 & 0 & 0 \\ 0 & 0.041667 & 0 \\ 0 & 0 & 0.037037 \end{pmatrix} \begin{pmatrix} 0 & 4 & 5 \\ 7 & 0 & 6 \\ 5 & 3 & 0 \end{pmatrix} \right\|_\infty =$$

$$= \left\| \begin{pmatrix} 0 & 0.153848 & 0.19231 \\ 0.291669 & 0 & 0.250002 \\ 0.185185 & 0.111111 & 0 \end{pmatrix} \right\|_\infty =$$

$$= \max\{0.346158, 0.541671, 0.296296\} = 0.541671$$

Hence,

$$k \geq \log_{0.541671} \frac{0.01(1-0.541671)}{|D^{-1}b|_\infty} =$$

$$= \log_{0.541671} \frac{0.004583}{|(0.19231, 0.250002, 0.074074)^T|_\infty} = \log_{0.541671} \frac{0.004583}{0.250002} =$$

$$= \log_{0.541671} 0.018332 = 6.522802 \Rightarrow k_{min} = 7$$

Let us now apply this iterative method 7 times.

$v^0 = (0,0,0)^T$

$v^1 = (0.19231, 0.250002, 0.074074)^T$

$v^2 = (0.139603, 0.175392, 0.010683)^T$

$v^3 = (0.163272, 0.206613, 0.028734)^T$

$v^4 = (0.154997, 0.195197, 0.020881)^T$

$v^5 = (0.158264, 0.199574, 0.023682)^T$

$v^6 = (0.157052, 0.197921, 0.022591)^T$

$v^7 = (0.157516, 0.198547, 0.022999)^T$

Answer

7 iterations are sufficient to find an approximate solution;

$$\hat{v} = \begin{pmatrix} 0.157516 \\ 0.198547 \\ 0.022999 \end{pmatrix}.$$

№6

Problem

Find the most influential vertex of the graph using PageRank algorithm with $\beta = 0.15$, where graph is given with its adjacency matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Solution

The ranking vector r is defined as

$$r = \left(\alpha P + \frac{\beta}{N}\mathbb{1}\right) r\,,$$

where $\alpha = 1 - \beta$, P is a transition matrix, $\mathbb{1}$ is an $N \times N$ matrix with all ones, N is a number of vertices in a graph. Here $\alpha = 0.85$, $N = 5$ and

$$P = \begin{pmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \end{pmatrix}$$

Thus,

$$\alpha P + \frac{\beta}{N}\mathbb{1} = 0.85 \begin{pmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \end{pmatrix} + \frac{0.15}{5} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0.425 & 0 & 0 & 0 & 0 \\ 0.425 & 0 & 0 & 0 & 0.425 \\ 0 & 0.85 & 0 & 0 & 0.425 \\ 0 & 0 & 0.425 & 0.425 & 0 \\ 0 & 0 & 0.425 & 0.425 & 0 \end{pmatrix} + \begin{pmatrix} 0.03 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.03 & 0.03 \end{pmatrix} =$$

$$= \begin{pmatrix} 0.455 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.455 & 0.03 & 0.03 & 0.03 & 0.455 \\ 0.03 & 0.88 & 0.03 & 0.03 & 0.455 \\ 0.03 & 0.03 & 0.455 & 0.455 & 0.03 \\ 0.03 & 0.03 & 0.455 & 0.455 & 0.03 \end{pmatrix}$$

Thus, we have an iterative procedure to find r:

$$r^{n+1} = \left(\alpha P + \frac{\beta}{N}\mathbb{1}\right) r^n$$

Starting with $r^0 = (0.2, 0.2, 0.2, 0.2, 0.2)^T$, we get the following results:

$r^0 = (0.2, 0.2, 0.2, 0.2, 0.2)^T$

$r^1 = (0.115, 0.2, 0.285, 0.2, 0.2)^T$

$r^2 = (0.078875, 0.163875, 0.285, 0.236125, 0.236125)^T$

$r^3 = (0.063522, 0.163875, 0.269647, 0.251478, 0.251478)^T$

$r^4 = (0.056997, 0.163875, 0.276172, 0.251478, 0.251478)^T$

Hence, the most influential vertex is vertex number 3.

Answer

Vertex 3.

№7

Problem

Find the value $f(A)$ of the function $f(x) = e^{x+1}$, where

$$A = \begin{pmatrix} 0 & 0 & -1 \\ -14 & -10 & -13 \\ 20 & 16 & 20 \end{pmatrix}$$

Solution

Let us first represent A in its Jordan form. First, we find all eigenvalues of A.

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 0 & -1 \\ -14 & -10-\lambda & -13 \\ 20 & 16 & 20-\lambda \end{pmatrix} =$$

$$= (\lambda - 2)^2(\lambda - 6) = 0$$

Thus, $\lambda_1 = 2$ (algebraic multiplicity 2), $\lambda_2 = 6$ (algebraic multiplicity 1). Let us then find a set of linearly independent eigenvectors.

For $\lambda_1 = 2$:

$$\begin{pmatrix} -2 & 0 & -1 \\ -14 & -12 & -13 \\ 20 & 16 & 18 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 16 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = -0.5C \\ x_2 = -0.5C \\ x_3 = C \end{cases}$$

Hence,

$$v_1 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

Since $(A - 2I)^{1-1} \cdot v_1 = Iv_1 = v_1 \neq 0$, v_1 is a generalised eigenvector that also represent the first and the last element of its Jordan chain. Next, we consider $(A - 2I)^2$.

$$\begin{pmatrix} -16 & -16 & -16 \\ -64 & -64 & -64 \\ 96 & 96 & 96 \end{pmatrix} \sim \begin{pmatrix} -1 & -1 & -1 \\ -4 & -4 & -4 \\ 6 & 6 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = -C_1 - C_2 \\ x_2 = C_1 \\ x_3 = C_2 \end{cases}$$

Hence,

$$v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Since $(A - 2I)^{2-1} \cdot v_2 = (A - 2I) \cdot v_2 \neq 0$, v_2 is a generalised eigenvector. Let us construct a Jordan chain for it.

$$v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_2' = (A - 2I) \cdot v_2 = \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}$$

As we see, this Jordan chain provides us with linearly independent vectors for λ_1 and there is no need to construct any more Jordan chains for λ_1.

For $\lambda_2 = 6$:

$$\begin{pmatrix} -6 & 0 & -1 \\ -14 & -16 & -13 \\ 20 & 16 & 14 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1/6 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = -1/6C \\ x_2 = -2/3C \\ x_3 = C \end{cases}$$

Hence,

$$v_4 = \begin{pmatrix} -1 \\ -4 \\ 6 \end{pmatrix}$$

Since $(A - 6I)^{1-1} \cdot v_4 = Iv_4 = v_4 \neq 0$, v_4 is a generalised eigenvector that also represent the first and the last element of its Jordan chain. As we see, this Jordan chain provides us with linearly independent vectors for λ_2 and there is no need to construct any more Jordan chains for λ_2.

Thus, the Jordan basis matrix is

$$T = \begin{pmatrix} 2 & -1 & -1 \\ 2 & 1 & -4 \\ -4 & 0 & 6 \end{pmatrix}$$

and

$$J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

For T we may find T^{-1}.

$$T^{-1} = \begin{pmatrix} 1.5 & 1.5 & 1.25 \\ 1 & 2 & 1.5 \\ 1 & 1 & 1 \end{pmatrix}$$

Now let us find $f(A) = e \cdot e^{A}$ as $Tf(J)T^{-1}$.

$$e^{A} = \begin{pmatrix} e^{2} & e^{2} & 0 \\ 0 & e^{2} & 0 \\ 0 & 0 & e^{6} \end{pmatrix} \Rightarrow f(J) = e \cdot e^{J} = \begin{pmatrix} e^{3} & e^{3} & 0 \\ 0 & e^{3} & 0 \\ 0 & 0 & e^{7} \end{pmatrix}$$

Thus,

$$f(A) = \begin{pmatrix} 2 & -1 & -1 \\ 2 & 1 & -4 \\ -4 & 0 & 6 \end{pmatrix} \begin{pmatrix} e^{3} & e^{3} & 0 \\ 0 & e^{3} & 0 \\ 0 & 0 & e^{7} \end{pmatrix} \begin{pmatrix} 1.5 & 1.5 & 1.25 \\ 1 & 2 & 1.5 \\ 1 & 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -1016.29 & -996.205 & -1016.29 \\ -4266.02 & -4205.76 & -4245.93 \\ 6378.94 & 6298.6 & 6358.86 \end{pmatrix}$$

Answer

$$\begin{pmatrix} -1016.29 & -996.205 & -1016.29 \\ -4266.02 & -4205.76 & -4245.93 \\ 6378.94 & 6298.6 & 6358.86 \end{pmatrix}.$$