The ortogonality relations

$$\langle T_m, T_n \rangle = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ \pi & m = n = 0 \end{cases}$$

The best approximation of a function f by polynomial of degree $\leq n$

$$f(x) = \sum_{i=0}^{n} \frac{\langle T_i, f \rangle}{\langle T_i, T_i \rangle} T_i(x)$$

Семинар 6

$$\left\| f - \sum_{i=0}^{n} c_i \varphi_i \right\| \longrightarrow 0$$

norm is chosen

$c_i \in R$

$\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$

$\|x\| = \sqrt{(x, x)}$

$g(x, x)$

1) $g(x, y) = g(y, x)$

2) $g(\alpha x, y) = \alpha g(x, y)$

$g(x + y, z) = g(x, z) + g(y, z)$

3) $g(x, x) \geq 0 \quad g(x, x) = 0 \iff x = 0$

orthonormal system

$\varphi_0, \varphi_1, \ldots \varphi_n,$

$(\varphi_i, \varphi_j) = 0$

$\forall i \neq j \quad (\varphi_i, \varphi_i) = 1$

$$\left(f-\sum_{i=0}^{n} c_i\varphi_i,\ f-\sum_{i=0}^{n} c_i\varphi_i\right) \overset{2)}{=}$$

$$=\underset{\substack{\shortparallel \\ \|f\|^2}}{(f,f)}-2\sum_{k=0}^{n} c_k\underbrace{(f,\varphi_k)}_{\alpha_k}+\sum_{k=0}^{n} c_k^2\underbrace{(\varphi_k,\varphi_k)}_{1}=$$

$$=\|f\|^2+\sum_{k=0}^{n} c_k^2-2\sum_{k=0}^{n} c_k\alpha_k=\|f\|^2+\sum_{k=0}^{n}(\alpha_k-c_k)^2-\sum_{k=0}^{n}\alpha_k^2 \to \min$$

$c_k=\alpha_k=(f,\varphi_k)$ - Fourier coef.

$c_k=(f,\varphi_k)$ - solution

P1.

$$\underbrace{\|x^3-P_2(x)\|_\infty}_{x\in[-1,1]} \to \min \qquad \|f\|_\infty=\max_{x\in[-1,1]}|f(x)|$$

$T_3(x)=4x^3-3x$

$\tilde{T}_3(x)=x^3-\frac{3}{4}x$ - нормирован

$P_2=\frac{3}{4}x$

P2.

$$\|x^3-P_2\|_\infty \to \min, \quad x\in[2,3]$$

$$\boxed{\overline{T}_n(x)=\frac{(b-a)^n}{2^{2n-1}}\cdot T_n\left(\frac{2x-(b+a)}{b-a}\right)}$$

$$\overline{T}_3(x)=\frac{1}{32}T_3\left(\frac{2x-5}{1}\right)=\frac{1}{32}\left(4(2x-5)^3-3(2x-5)\right)$$

$x \in [-1, 1]$

$\downarrow L$

$\tilde{x} \in [a, b]$

$$x = \frac{\tilde{x} - \frac{a+b}{2}}{b-a}$$

$$U_n(x) = \frac{1}{n+1} \left(T_{n+1}(x)\right)'$$

$$\|f\|_1 = \int_{-1}^{1} |f(x)|\,dx$$

P3.

$P_2(x) - ?$

$$\|x^3 - P_2\|_1 \longrightarrow \min_{x \in [0,2]}$$

$$U_3(x) = 8x^3 - 4x$$

$$\bar{U}_3(x) = \frac{1}{8} U_3(x-1) = \frac{1}{8}\left[8(x-1)^3 - 4x - 4\right] =$$

$$= \frac{1}{8}\left[8(x^3 - 3x^2 + 3x - 1) - 4x + 4\right] = x^3 - 3x^2 + \frac{5}{2}x - \frac{1}{2}$$

P4. $\quad b \quad c_i = \frac{(f, \varphi_i)}{(\varphi_i, \varphi_i)}$

$$\|e^x - P_2(x)\|_2 \longrightarrow \min$$

$$\|f\| = \sqrt{\int_a^b f^2(x)\,dx} \leftarrow (f, g) = \int_a^b f(x) g(x)\,dx$$

$x \in [0, 1]$

$a_1, \ldots, a_n$ given

$b_1, b_2, \ldots b_n$

$$b_n = a_n - \sum_{i=1}^{n-1} \mathrm{proj}_{b_i} a_n = \boxed{a_n - \sum_{i=1}^{n-1} \frac{\langle b_i, a_n \rangle}{\langle b_i, b_i \rangle} b_i}$$

$$\varphi_0 = 1$$

$$\varphi_1 = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} 1 = x - \frac{\int_0^1 x\,dx}{\int_0^1 1\,dx} = x - \frac{\frac{x^2}{2}\Big|_0^1}{1 \cdot (1-0)} = x - \frac{1}{2}$$

$$\varphi_2 = x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle x^2, x - \frac{1}{2} \rangle}{\langle x - \frac{1}{2}, x - \frac{1}{2} \rangle}\left(x - \frac{1}{2}\right) =$$

$$= x^2 - x + \frac{1}{6}$$