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APPLIED LINEAR ALGEBRA TEST 1

VARIANT 8

№1

Problem

Find an interpolation polynomial in the Lagrange form that passes through the four points with the following coordinates:

x_i	-3	-2	0	3
y_i	0	-20	19	20

Solution

Apply the Lagrange formula

$$p(x) = \sum_{i=1}^{n} y_i \cdot \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j},$$

where n is a number of given points. This results in:

$$p(x) = -20 \cdot \frac{(x+3)(x-0)(x-3)}{(-2+3)(-2-0)(-2-3)} + 19 \cdot \frac{(x+3)(x+2)(x-3)}{(0+3)(0+2)(0-3)} +$$

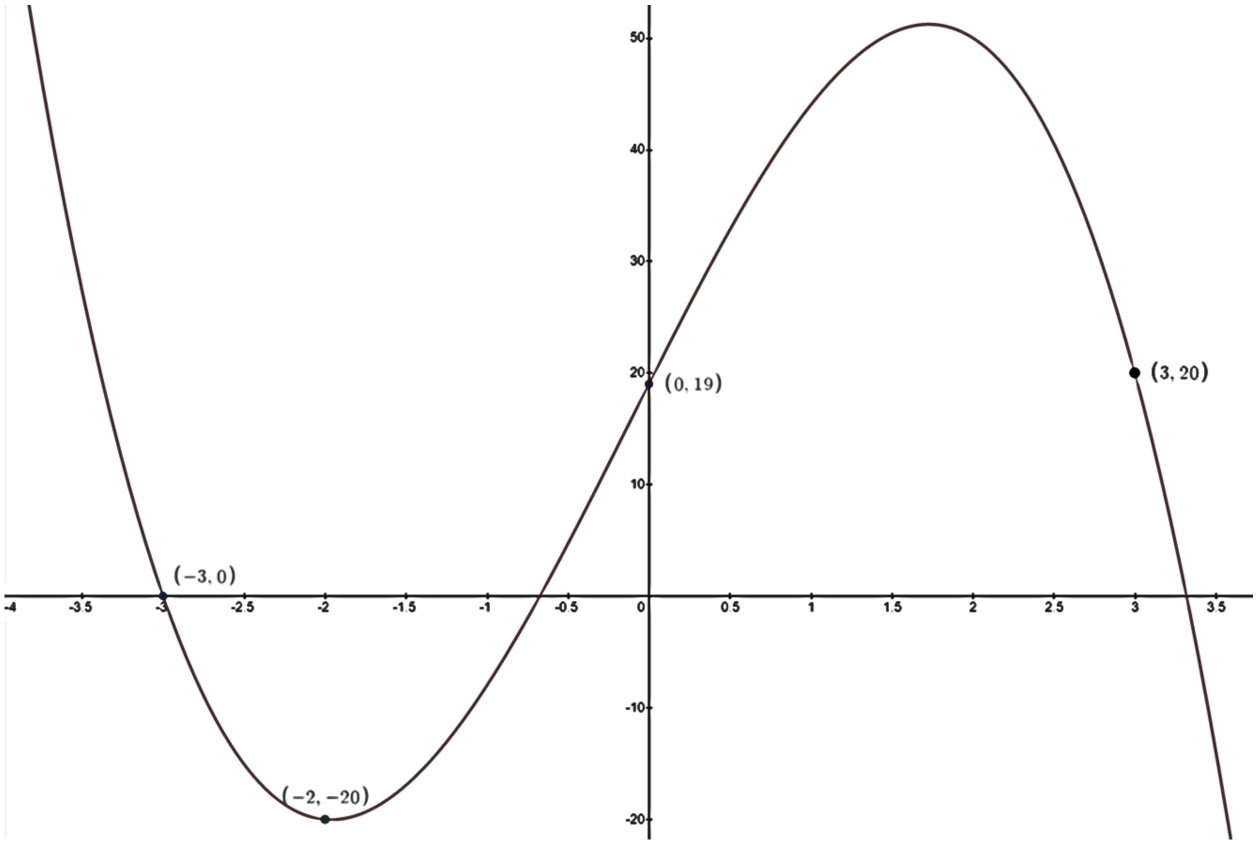
$$+20 \cdot \frac{(x+3)(x+2)(x-0)}{(3+3)(3+2)(3-0)} =$$

$$= -20 \cdot \frac{x^3 - 9x}{10} + 19 \cdot \frac{x^3 + 2x^2 - 9x - 18}{-18} + 20 \cdot \frac{x^3 + 5x^2 + 6x}{90} =$$

$$= -2x^3 + 18x - \frac{19}{18}x^3 - \frac{19}{9}x^2 + \frac{19}{2}x + 19 + \frac{2}{9}x^3 + \frac{10}{9}x^2 + \frac{4}{3}x =$$

$$= -\frac{17}{6}x^3 - x^2 + \frac{173}{6}x + 19$$

Plot of $p(x)$ is provided below.



Answer

$$p(x) = -\frac{17}{6}x^3 - x^2 + \frac{173}{6}x + 19.$$

№2

Problem

Find a parametric equation defining the Bezier curve defined by the four points with the following coordinates:

x_i	1	3	5	7
y_i	3	4	0	8

Solution

One can express the parametric formula for Bezier curve via Bernstein polynomials:

$$B(t) = \sum_{k=0}^{n-1} P_n^k(t) \cdot A_k \ , \quad P_n^k(t) = \binom{n-1}{k} \cdot (1-t)^{n-k-1} \cdot t^k \ ,$$

where n is a number of given points, A_k is the k-th given point. This results in the following.

$$P_4^0(t) = \binom{3}{0} \cdot (1-t)^3 \cdot t^0 = -t^3 + 3x^2 - 3x + 1$$

$$P_4^1(t) = \binom{3}{1} \cdot (1-t)^2 \cdot t^1 = 3t \cdot (1 - 2t + t^2) = 3t^3 - 6t^2 + 3t$$

$$P_4^2(t) = \binom{3}{2} \cdot (1-t)^1 \cdot t^2 = 3t^2 \cdot (1-t) = -3t^3 + 3t^2$$

$$P_4^3(t) = \binom{3}{3} \cdot (1-t)^0 \cdot t^3 = t^3$$

Combining everything together:

$$B(t) = (-t^3 + 3t^2 - 3t + 1)\binom{1}{3} + (3t^3 - 6t^2 + 3t)\binom{3}{4} +$$

$$+(-3t^3 + 3t^2)\binom{5}{0} + t^3\binom{7}{8} =$$

$$= \begin{pmatrix} -t^3 + 3t^2 - 3t + 1 \\ -3t^3 + 9t^2 - 9t + 3 \end{pmatrix} + \begin{pmatrix} 9t^3 - 18t^2 + 9t \\ 12t^3 - 24t^2 + 12t \end{pmatrix} + \begin{pmatrix} -15t^3 + 15t^2 \\ 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} 7t^3 \\ 8t^3 \end{pmatrix} =$$

$$= \begin{pmatrix} 6t + 1 \\ 17t^3 - 15t^2 + 3t + 3 \end{pmatrix}, t \in [0,1]$$

One may perform a substitution $x = 6t + 1$ to obtain a functional representation of the same curve.

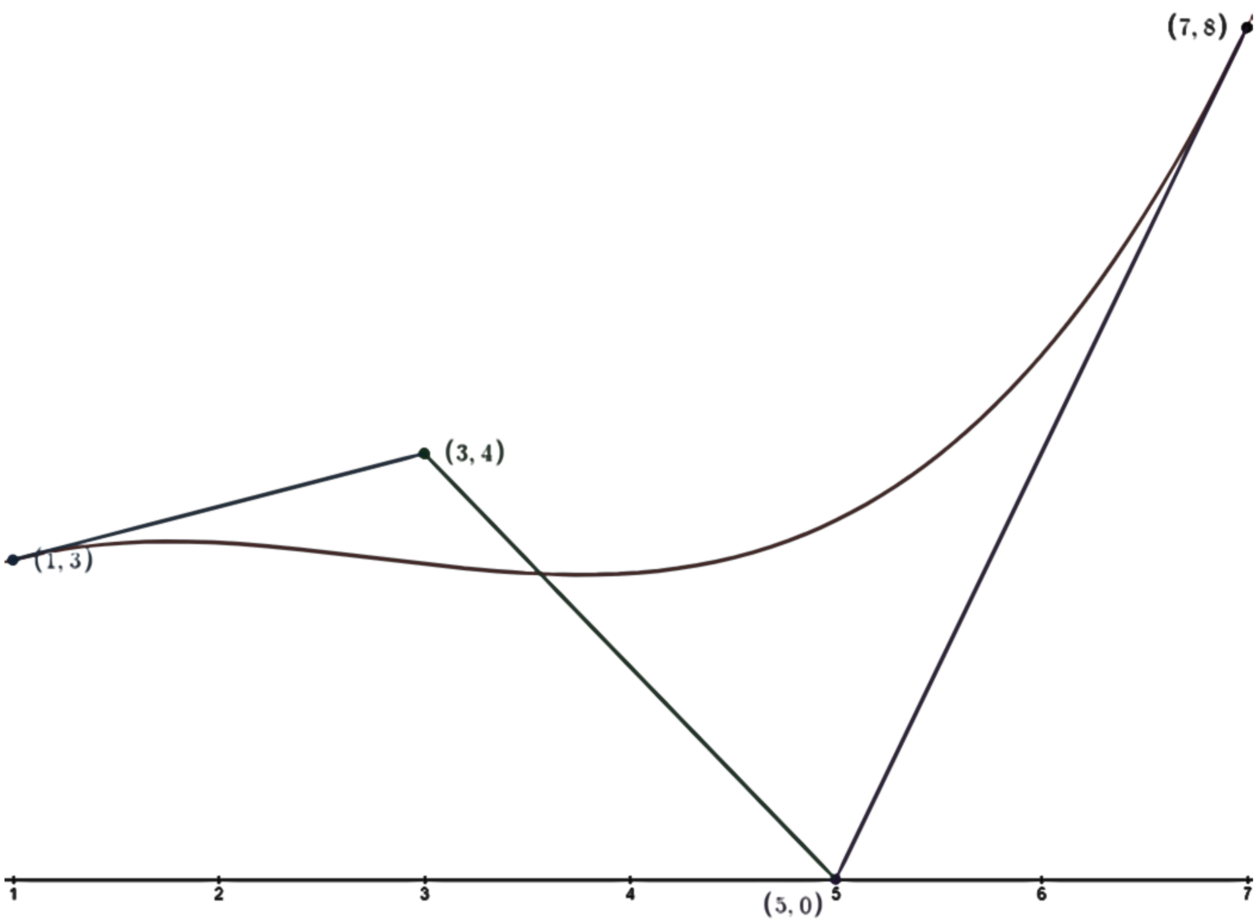
$$B(t) = \begin{pmatrix} 6t + 1 \\ 17t^3 - 15t^2 + 3t + 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 6t + 1 \\ \frac{17}{216}(6t + 1)^3 - \frac{47}{72}(6t + 1)^2 + \frac{113}{72}(6t + 1) + \frac{433}{216} \end{pmatrix} =$$

$$= \begin{pmatrix} x \\ \frac{17}{216}x^3 - \frac{47}{72}x^2 + \frac{113}{72}x + \frac{433}{216} \end{pmatrix}$$

$$B(x) = \frac{17}{216}x^3 - \frac{47}{72}x^2 + \frac{113}{72}x + \frac{433}{216}$$

Plot of Bezier curve is provided below.



Answer

$$B(t) = \begin{pmatrix} 6t + 1 \\ 17t^3 - 15t^2 + 3t + 3 \end{pmatrix}, \quad B(x) = \frac{17}{216}x^3 - \frac{47}{72}x^2 + \frac{113}{72}x + \frac{433}{216}.$$

№3

Problem

Find a full rank decomposition and the pseudoinverse for the matrix A.

$$A = \begin{pmatrix} 13 & 15 & -2 \\ 3 & 3 & 4 \\ 8 & 9 & 1 \\ -2 & -3 & 7 \end{pmatrix}$$

Solution

Apply Gaussian elimination to find a full rank decomposition.

$$A = \begin{pmatrix} 13 & 15 & -2 \\ 3 & 3 & 4 \\ 8 & 9 & 1 \\ -2 & -3 & 7 \end{pmatrix} \sim \begin{pmatrix} -1 & -3/2 & 7/2 \\ 3 & 3 & 4 \\ 8 & 9 & 1 \\ 13 & 15 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3/2 & -7/2 \\ 0 & -3/2 & 29/2 \\ 0 & -3 & 29 \\ 0 & -9/2 & 87/2 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 3/2 & -7/2 \\ 0 & 1 & -29/3 \\ 0 & -3 & 29 \\ 0 & -9/2 & 87/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3/2 & -7/2 \\ 0 & 1 & -29/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} \overbrace{\boxed{\begin{matrix} 1 & 0 & 11 \\ 0 & 1 & -29/3 \end{matrix}}}^{G} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{pmatrix}$$

$$F = (A^1 \quad A^2) = \begin{pmatrix} 13 & 15 \\ 3 & 3 \\ 8 & 9 \\ -2 & -3 \end{pmatrix}$$

Thus, $A = FG = \begin{pmatrix} 13 & 15 \\ 3 & 3 \\ 8 & 9 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 11 \\ 0 & 1 & -\frac{29}{3} \end{pmatrix}$ is a full rank decomposition of A.

Now one may find pseudoinverses for both F and G.

$F^{+} = (F^{*}F)^{-1}F^{*} =$

$$= \begin{pmatrix} 246 & 282 \\ 282 & 324 \end{pmatrix}^{-1} \begin{pmatrix} 13 & 3 & 8 & -2 \\ 15 & 3 & 9 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{5} & -\frac{47}{30} \\ -\frac{47}{30} & \frac{41}{30} \end{pmatrix} \begin{pmatrix} 13 & 3 & 8 & -2 \\ 15 & 3 & 9 & -3 \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{1}{10} & \frac{7}{10} & \frac{3}{10} & \frac{11}{10} \\ \frac{2}{15} & -\frac{3}{5} & -\frac{7}{30} & -\frac{29}{30} \end{pmatrix}$$

$G^+ = G^*(GG^*)^{-1} =$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 11 & -\frac{29}{3} \end{pmatrix} \begin{pmatrix} 122 & -\frac{319}{3} \\ -\frac{319}{3} & \frac{850}{9} \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 11 & -\frac{29}{3} \end{pmatrix} \begin{pmatrix} \frac{850}{1939} & \frac{957}{1939} \\ \frac{957}{1939} & \frac{1098}{1939} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{850}{1939} & \frac{957}{1939} \\ \frac{957}{1939} & \frac{1098}{1939} \\ \frac{99}{1939} & -\frac{87}{1939} \end{pmatrix}$$

Knowing this, one may deduce using $A^+ = G^+F^+$ that

$$A^+ = \begin{pmatrix} \frac{850}{1939} & \frac{957}{1939} \\ \frac{957}{1939} & \frac{1098}{1939} \\ \frac{99}{1939} & -\frac{87}{1939} \end{pmatrix} \begin{pmatrix} -\frac{1}{10} & \frac{7}{10} & \frac{3}{10} & \frac{11}{10} \\ \frac{2}{15} & -\frac{3}{5} & -\frac{7}{30} & -\frac{29}{30} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{213}{9695} & \frac{104}{9695} & \frac{317}{19390} & \frac{99}{19390} \\ \frac{507}{19390} & \frac{111}{19390} & \frac{309}{19390} & -\frac{87}{19390} \\ -\frac{43}{3878} & \frac{243}{3878} & \frac{50}{1939} & \frac{193}{1939} \end{pmatrix}$$

Answer

Full rank decomposition:

$$A = \begin{pmatrix} 13 & 15 \\ 3 & 3 \\ 8 & 9 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 11 \\ 0 & 1 & -\frac{29}{3} \end{pmatrix};$$

Pseudoinverse:

$$A^{+} = \begin{pmatrix} \frac{213}{9695} & \frac{104}{9695} & \frac{317}{19390} & \frac{99}{19390} \\ \frac{507}{19390} & \frac{111}{19390} & \frac{309}{19390} & -\frac{87}{19390} \\ -\frac{43}{3878} & \frac{243}{3878} & \frac{50}{1939} & \frac{193}{1939} \end{pmatrix}.$$

№4

Problem

Find the minimal length least squares solution of the system of linear equations:

$$\begin{cases} 8x+9y+1z+3t=9 \\ 13x+14y+2z-2t=8 \\ 7x+8y+4z+5t=5 \\ 3x+4y+0z+8t=6 \end{cases}$$

Solution

The given system of linear equations may be rewritten in a vector-matrix form:

$$\underbrace{\begin{pmatrix} 8 & 9 & 1 & 3 \\ 13 & 14 & 2 & -2 \\ 7 & 8 & 4 & 5 \\ 3 & 4 & 0 & 8 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 9 \\ 8 \\ 5 \\ 6 \end{pmatrix}}_{B}$$

Then the minimal length least squares solution can be constructed as $X = A^{+}B$. One may find a full rank decomposition of A in order to find A^{+}. Apply Gaussian elimination for this.

$$A = \begin{pmatrix} 8 & 9 & 1 & 3 \\ 13 & 14 & 2 & -2 \\ 7 & 8 & 4 & 5 \\ 3 & 4 & 0 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 4/3 & 0 & 8/3 \\ 0 & -10/3 & 2 & -110/3 \\ 0 & -4/3 & 4 & -41/3 \\ 0 & -5/3 & 1 & -55/3 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 4/5 & -12 \\ 0 & 1 & -3/5 & 11 \\ 0 & 0 & 16/5 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} \overbrace{\boxed{\begin{matrix} 1 & 0 & 0 & -49/4 \\ 0 & 1 & 0 & 179/16 \\ 0 & 0 & 1 & 5/16 \end{matrix}}}^{G} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} \end{pmatrix}$$

$$F = (A^1 \quad A^2 \quad A^3) = \begin{pmatrix} 8 & 9 & 1 \\ 13 & 14 & 2 \\ 7 & 8 & 4 \\ 3 & 4 & 0 \end{pmatrix}$$

Thus, $A = FG = \begin{pmatrix} 8 & 9 & 1 \\ 13 & 14 & 2 \\ 7 & 8 & 4 \\ 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -\frac{49}{4} \\ 0 & 1 & 0 & \frac{179}{16} \\ 0 & 0 & 1 & \frac{5}{16} \end{pmatrix}$ is a full rank decomposition of A.

Now one may find pseudoinverses for both F and G.

$$F^+ = (F^*F)^{-1}F^* = \begin{pmatrix} 291 & 322 & 62 \\ 322 & 357 & 69 \\ 62 & 69 & 21 \end{pmatrix}^{-1} \begin{pmatrix} 8 & 13 & 7 & 3 \\ 9 & 14 & 8 & 4 \\ 1 & 2 & 4 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{57}{32} & -\frac{207}{128} & \frac{7}{128} \\ -\frac{207}{128} & \frac{2267}{1536} & -\frac{115}{1536} \\ \frac{7}{128} & -\frac{115}{1536} & \frac{203}{1536} \end{pmatrix} \begin{pmatrix} 8 & 13 & 7 & 3 \\ 9 & 14 & 8 & 4 \\ 1 & 2 & 4 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{1}{4} & \frac{5}{8} & -\frac{1}{4} & -\frac{9}{8} \\ \frac{13}{48} & -\frac{49}{96} & \frac{3}{16} & \frac{101}{96} \\ -\frac{5}{48} & -\frac{7}{96} & \frac{5}{16} & -\frac{13}{96} \end{pmatrix}$$

$$G^+ = G^*(GG^*)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{49}{4} & \frac{179}{16} & \frac{5}{16} \end{pmatrix} \begin{pmatrix} \frac{2417}{16} & -\frac{8771}{64} & -\frac{245}{64} \\ -\frac{8771}{64} & \frac{32297}{256} & \frac{895}{256} \\ -\frac{245}{64} & \frac{895}{256} & \frac{281}{256} \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{49}{4} & \frac{179}{16} & \frac{5}{16} \end{pmatrix} \begin{pmatrix} \frac{16161}{35369} & \frac{17542}{35369} & \frac{490}{35369} \\ \frac{17542}{35369} & \frac{38697}{70738} & -\frac{895}{70738} \\ \frac{490}{35369} & -\frac{895}{70738} & \frac{70713}{70738} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{16161}{35369} & \frac{17542}{35369} & \frac{490}{35369} \\ \frac{17542}{35369} & \frac{38697}{70738} & -\frac{895}{70738} \\ \frac{490}{35369} & -\frac{895}{70738} & \frac{70713}{70738} \\ -\frac{1568}{35369} & \frac{1432}{35369} & \frac{40}{35369} \end{pmatrix}$$

Knowing this, one may deduce using $A^+ = G^+F^+$ that

$$A^+ = \begin{pmatrix} \frac{16161}{35369} & \frac{17542}{35369} & \frac{490}{35369} \\ \frac{17542}{35369} & \frac{38697}{70738} & -\frac{895}{70738} \\ \frac{490}{35369} & -\frac{895}{70738} & \frac{70713}{70738} \\ -\frac{1568}{35369} & \frac{1432}{35369} & \frac{40}{35369} \end{pmatrix} \begin{pmatrix} -\frac{1}{4} & \frac{5}{8} & -\frac{1}{4} & -\frac{9}{8} \\ \frac{13}{48} & -\frac{49}{96} & \frac{3}{16} & \frac{101}{96} \\ -\frac{5}{48} & -\frac{7}{96} & \frac{5}{16} & -\frac{13}{96} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1979}{106107} & \frac{59}{1878} & -\frac{598}{35369} & \frac{1249}{212214} \\ \frac{2704}{106107} & \frac{119}{3756} & -\frac{1795}{70738} & \frac{8185}{424428} \\ -\frac{11780}{106107} & -\frac{217}{3756} & \frac{21685}{70738} & -\frac{69719}{424428} \\ \frac{2327}{106107} & -\frac{91}{1878} & \frac{673}{35369} & \frac{19591}{212214} \end{pmatrix}$$

Now one may construct the pseudosolution.

$$X = A^{+}B = \begin{pmatrix} \frac{1979}{106107} & \frac{59}{1878} & -\frac{598}{35369} & \frac{1249}{212214} \\ \frac{2704}{106107} & \frac{119}{3756} & -\frac{1795}{70738} & \frac{8185}{424428} \\ -\frac{11780}{106107} & -\frac{217}{3756} & \frac{21685}{70738} & -\frac{69719}{424428} \\ \frac{2327}{106107} & -\frac{91}{1878} & \frac{673}{35369} & \frac{19591}{212214} \end{pmatrix} \begin{pmatrix} 9 \\ 8 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{39256}{106107} \\ \frac{50045}{106107} \\ -\frac{97003}{106107} \\ \frac{48679}{106107} \end{pmatrix}$$

Answer

$$\begin{cases} x = \frac{39256}{106107} \\ y = \frac{50045}{106107} \\ z = -\frac{97003}{106107} \\ t = \frac{48679}{106107} \end{cases}.$$

№5

Problem

For the polynomial $p(x) = -2x^3 - 3x^2 - 4x + 5$ find the best approximation with respect to $\mathcal{L}_\infty$-norm by a polynomial of degree 2 on a line segment $[-2,2]$.

Solution

Consider the Chebyshev polynomial of the first kind of degree 3.

$$T_3(x) = 4x^3 - 3x$$

Using Chebyshev theorem, one may construct the least deviating from zero monic polynomial on the segment $[-1,1]$ with respect to $\mathcal{L}_\infty$ by dividing T_3 by 4.

$$\hat{T}_3(x) = \frac{T_3(x)}{4} = x^3 - \frac{3}{4}x$$

Now one may use transformations to construct from $\hat{T}_3$ the least deviating from zero polynomial on the segment $[-2,2]$ with leading coefficient -2 with respect to $\mathcal{L}_\infty$.

$$\bar{T}_3(x) = -2 \cdot 8 \cdot \hat{T}_3\left(\frac{x}{2}\right) = -16\left(\left(\frac{x}{2}\right)^3 - \frac{3}{4}\cdot\frac{x}{2}\right) = -16\left(\frac{1}{8}x^3 - \frac{3}{8}x\right) =$$
$$= -2x^3 + 6x$$

It is known that $\bar{T}_3$ is a solution of optimisation problem $\|r(x)\|_\infty \to \min$ for $x \in [-2,2]$ where r is a polynomial of degree 3 with leading coefficient -2. Furthermore, the best approximation of $p(x)$ by a polynomial $q(x)$ of degree 2 should be a solution of optimisation problem $\|p(x) - q(x)\|_\infty \to \min$. One may notice that $\big(p(x) - q(x)\big)$ is going to be a polynomial of degree 3 with leading coefficient -2, hence, $\|p(x) - q(x)\|_\infty \to \min \Leftrightarrow \|r(x)\|_\infty \to \min$ and, hence,

$$p(x) - q(x) = \bar{T}_3(x)$$
$$-2x^3 - 3x^2 - 4x + 5 - q(x) = -2x^3 + 6x$$
$$q(x) = -3x^2 - 10x + 5$$

Answer

$q(x) = -3x^2 - 10x + 5$.

№6

Problem

Find all values of q such that the equation

$$2x^2 + y^2(-4q+1) + yz(2q+2) + z^2(-4q+1) = 1$$

defines a unit circle with respect to some norm. Find the value of this norm for the vector $(1,1,1)^T$ as a function of q.

Solution

By Minkowski theorem, a subset of $\mathbb{R}^3$ can be a unit ball iff it is closed, bounded, convex, centrally symmetric and contains some Euclidian ball inside. Given equation defines a second-degree three-dimensional surface that might be a bound of such a unit ball. To ensure that this surface is bounded, one may check the positive definiteness of the respective symmetric matrix using Sylvester criterion.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1-4q & 1+q \\ 0 & 1+q & 1-4q \end{pmatrix}$$

$$\Delta_1 = |2| = 2 > 0$$

$$\Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 1-4q \end{vmatrix} = 2 - 8q > 0$$

$$\Delta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1-4q & 1+q \\ 0 & 1+q & 1-4q \end{vmatrix} = 2(1-4q)^2 - 2(1+q)^2 = 30q^2 - 20q > 0$$

This results in a system of inequalities.

$$\begin{cases} 2 - 8q > 0 \\ 30q^2 - 20q > 0 \end{cases} \Leftrightarrow \begin{cases} q < \dfrac{1}{4} \\ q(30q - 20) > 0 \end{cases} \Leftrightarrow \begin{cases} q < \dfrac{1}{4} \\ \left[\begin{array}{l} q < 0 \\ q > \dfrac{2}{3} \end{array}\right. \end{cases} \Leftrightarrow q \in (-\infty, 0)$$

Thus, at $q \in (-\infty, 0)$ the given equation defines an ellipsoid which bounds a body that is closed, bounded, convex, and contains some Euclidian ball inside. Since all monomials in the left-hand side are of degree 2, the ellipsoid is centrally symmetric. Hence, the given equation defines a unit circle with respect to some norm $\mathcal{N}_q$ at $q \in (-\infty, 0)$.

To calculate the $\mathcal{N}_q((1,1,1)^T)$ one may first find a codirectional vector that lies on the described above unit circle. Let $v = (1,1,1)^T$. Hence, it is desirable to construct $v^\circ = (t,t,t)^T$ such that $\mathcal{N}_q(v^\circ) = 1$. One may substitute t's into the given equation.

$$2t^2 + t^2(-4q+1) + t^2(2q+2) + t^2(-4q+1) = 1$$

$$(-6q+6)t^2 = 1$$

$$t^2 = \frac{1}{6-6q}$$

$$t = \frac{1}{\sqrt{6-6q}}$$

Positive number is selected as t because the coordinates of v are positive. Hence,

$$v^\circ = \begin{pmatrix} \frac{1}{\sqrt{6-6q}} \\ \frac{1}{\sqrt{6-6q}} \\ \frac{1}{\sqrt{6-6q}} \end{pmatrix}$$

and $v = \sqrt{6-6q} \cdot v^\circ$. Thus, by definition of norm, $\mathcal{N}_q(v) = \left|\sqrt{6-6q}\right| \cdot \mathcal{N}_q(v^\circ)$ or $\mathcal{N}_q(v) = \sqrt{6-6q}$.

Answer

$q \in (-\infty, 0)$, $\mathcal{N}_q((1,1,1)^T) = \sqrt{6-6q}$.