Total: 95/100 (A).

Problem 1

(a) For $X>0$ essentially and $u>0$, for any $\lambda>0$,

$$E[e^{\lambda\frac{X}{u}}] = E[\sum_{i=0}^{\infty} (\frac{\lambda X}{u})^i \frac{1}{i!}]$$

$$\underset{\text{Convergence Thm}}{\overset{\text{Monotone}}{=}} \sum_{i=0}^{\infty} E[(\frac{X}{u})^i] \frac{\lambda^i}{i!}$$

Yes $\rightarrow$

$$\geq \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \inf_{\substack{k\geq 0 \\ k\in\mathbb{Z}}} E[(\frac{X}{u})^k]$$

$$= e^{\lambda} \inf_{\substack{k\geq 0, \\ k\in\mathbb{Z}}} E[(\frac{X}{u})^k], \quad \text{For all } \lambda>0.$$

Therefore

$$\inf_{\lambda>0} E[e^{\lambda\frac{X}{u}}] \geq \inf_{\lambda>0} e^{\lambda} \inf_{\substack{k\geq 0 \\ k\in\mathbb{Z}}} E[(\frac{X}{u})^k]$$

$$\geq \inf_{\substack{k\geq 0, \\ k\in\mathbb{Z}}} E[(\frac{X}{u})^k] \quad \square.$$

+

(b) $M_X(\lambda)e^{-\lambda u} = \frac{1}{2}M_X(\lambda)e^{-\lambda u} + \frac{1}{2}M_X(\lambda)e^{-\lambda u}$ by symmetric.

$$= E[\frac{1}{2}\sum_{i=0}^{\infty}(-\frac{\lambda X}{u})^i\frac{1}{i!} + \frac{1}{2}\sum_{i=0}^{\infty}(\frac{\lambda X}{u})^i\frac{1}{i!}]$$

$$= E[\sum_{i=0}^{\infty}(\frac{\lambda X}{u})^{2i}\frac{1}{(2i)!}]$$

Now that the summarands are all ≥ 0

$$\underset{\text{Theorem}}{\overset{\text{Monotone Convergence}}{=}} \sum_{i=0}^{\infty} \frac{\lambda^{2i}}{(2i)!} E[(\frac{X}{u})^{2i}].$$

I wonder if they still teach cosh & sinh in high school :)

$$\geq \sum_{i=0}^{\infty} \frac{\lambda^{2i}}{(2i)!} \inf_{\substack{k\in\mathbb{Z} \\ k\geq 0}} E[(\frac{X}{u})^{2k}].$$

$$\geq \frac{1}{2} \inf_{\substack{k\in\mathbb{Z} \\ k\geq 0}} E[(\frac{X}{u})^{2k}] \quad \text{for all } \lambda>0. \quad \square$$

+

Problem 2

Assume X is a discrete random variable, point mass $\{p_i\}_{i=1}^{k}$

$$\frac{d\, K_X(t)}{dt} = \frac{d \log E[e^{tX}]}{dt} = \frac{1}{E[e^{tX}]} E[Xe^{tX}].$$

$$\frac{d^2 K_X(t)}{dt} = \frac{E[X^2e^{tX}]E[e^{tX}] - E[Xe^{tX}]^2}{E[e^{tX}]^2}.$$

Apply Young's inequality with conjugage pair $(\frac{1}{2},\frac{1}{2}) = (\frac{1}{p},\frac{1}{q})$

$$\sum_{i=1}^{k} |p_i x_i e^{tx_i}| = \sum_{i=1}^{k} |p_i^{\frac{1}{2}} x_i e^{\frac{1}{2}tx_i} p_i^{\frac{1}{2}} e^{\frac{1}{2}tx_i}|$$

[Also works, yes] $$\leq \left(\sum_{i=1}^{k} p_i x_i^2 e^{tx_i}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{k} p_i e^{tx_i}\right)^{\frac{1}{2}}$$

therefore $E[Xe^{tX}]^2 \leq E[|Xe^{tX}|]^2$

$$\leq \left(\sum_{i=1}^{k} p_i e^{tx_i} x_i^2\right)\left(\sum_{i=1}^{k} p_i e^{tx_i}\right)$$

$$= E[X^2 e^{tX}]\, E[e^{tX}].$$

we have $\frac{d^2 K_X(t)}{dt} \geq 0$ for all t. $\Rightarrow K_X(t)$ is convex.

+

Problem 3

(a) $$\int_u^{\infty} \phi(t)dt = \int_u^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{t \geq u} \frac{1}{2t} e^{-\frac{t^2}{2}} dt^2$$

(since $u \geq 0$) $$\leq \frac{1}{\sqrt{2\pi}} \int_{t \geq u} \frac{1}{2u} e^{-\frac{t^2}{2}} dt^2 = \frac{1}{\sqrt{2\pi}} \frac{1}{u} e^{-\frac{u^2}{2}}$$

$$= \frac{1}{u} \phi(u)$$

Now show the other bound.

$$\int_u^{\infty} \phi(t) dt = \frac{1}{\sqrt{2\pi}} \int_{t \geq u} -\frac{1}{t} de^{-\frac{t^2}{2}}.$$

(int. by parts) $$= \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{t} e^{-\frac{t^2}{2}} \Big|_u^{\infty} + \int_{t \geq u} e^{-\frac{t^2}{2}} d\frac{1}{t} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{u} e^{-\frac{u^2}{2}} + \frac{1}{\sqrt{2\pi}} \int_{t \geq u} -\frac{1}{t^2} e^{-\frac{t^2}{2}} dt$$

$$= \frac{1}{u} \phi(u) + \frac{1}{\sqrt{2\pi}} \int_{t \geq u} -\frac{1}{2t^3} e^{-\frac{t^2}{2}} dt^2$$

$$\geq \frac{1}{u} \phi(u) + \frac{1}{\sqrt{2\pi}} \int_{t \geq u} -\frac{1}{2u^3} e^{-\frac{t^2}{2}} dt^2$$

$$= \frac{1}{u} \phi(u) + \frac{1}{\sqrt{2\pi}} \frac{1}{u^3} e^{-\frac{u^2}{2}}.$$

+

$$= \left(\frac{1}{u} - \frac{1}{u^3} \right) \phi(u).$$

3. (b) As shown in 3(a), we have

$$\int_u^\infty \phi(t)dt = \frac{1}{u}\phi(u) + \frac{1}{\sqrt{2\pi}}\int_{t\geq u} -\frac{1}{2t^3} e^{-\frac{t^2}{2}} dt^2$$

$$= \frac{1}{u}\phi(u) + \frac{1}{\sqrt{2\pi}} \int_{t\geq u} \frac{1}{\cancel{2}t^3} d e^{-\frac{t^2}{2}}$$

$$\text{(int. by parts)} = \frac{1}{u}\phi(u) + \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{t^2}{2}} \frac{1}{t^3}\Big|_u^\infty - \int_{t\geq u} e^{-\frac{t^2}{2}} d\frac{1}{t^3}\right)$$

$$= \left(\frac{1}{u} - \frac{1}{u^3}\right)\phi(u) + \frac{1}{\sqrt{2\pi}} \int_{t\geq u} \frac{3}{t^4} e^{-\frac{t^2}{2}}dt$$

$$= \left(\frac{1}{u} - \frac{1}{u^3}\right)\phi(u) + \frac{1}{\sqrt{2\pi}} \int_{t\geq u} \frac{3}{t^5} d e^{-\frac{t^2}{2}}$$

$$\leq \left(\frac{1}{u} - \frac{1}{u^3}\right)\phi(u) + \frac{1}{\sqrt{2\pi}} \int_{t\geq u} \frac{3}{u^5} d e^{-\frac{t^2}{2}}$$

$\oplus$ $$= \left(\frac{1}{u} - \frac{1}{u^3} + \frac{1}{u^5}\right)\phi(u). \qquad \square$$

3.2 $$\frac{1}{2} - \Phi(u) = \int_0^u \phi(t)dt \qquad (\text{Assume } u \geq 0)$$

$$= \int_0^1 u\,\phi(tu)dt$$

$$= \int_0^1 \frac{u}{\sqrt{2\pi}} \sum_{i=0}^\infty \frac{\left(-\frac{(ut)^2}{2}\right)^i}{i!} dt$$

$$= \int_0^1 \frac{1}{\sqrt{2\pi}} \sum_{i=0}^\infty \frac{(-1)^i u^{2i+1} t^{2i}}{2^i\, i!} dt$$

since $\sum_{i=0}^\infty \left|\frac{(-1)^i u^{2i+1} t^{2i}}{2^i\, i!}\right| \leq u\exp\left(\frac{u^2}{2}t^2\right) \leq u\exp\left(\frac{u^2}{2}\right)$ on $t \in [0,1)$

By dominated convergence theorem,

$\oplus$ $$\frac{1}{2} - \Phi(u) = \frac{1}{\sqrt{2\pi}} \sum_{i=0}^\infty \frac{(-1)^i u^{2i+1}}{2^i\, i!} \int_0^1 t^{2i} dt = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \frac{(-1)^k u^{2k+1}}{2^k k!(2k+1)}.$$

Well done!

Problem 4

(i) First, by Cauchy inequality,

$$E[X 1_{\{X \geq (1-t)EX\}}]^2 \leq E[X^2]\, E[1_{\{X \geq (1-t)EX\}}^2]$$

$$= E[X^2]\, P(X \geq (1-t)EX) \quad (*)$$

Also, notice that

$$(1-t)EX \geq E[X 1_{\{X \leq (1-t)EX\}}]$$ + (Note that $X \geq 0$ is used here)

which is $\quad EX - tEX \geq EX - E[X 1_{\{X \geq (1-t)EX\}}]$

therefore $\quad tEX \leq E[X 1_{\{X \geq (1-t)EX\}}]$. +

since $X \geq 0$, $\quad t^2 (EX)^2 \leq E[X 1_{\{X \geq (1-t)EX\}}]^2$ +

$$(\text{by } *) \leq E[X^2]\, P(X \geq (1-t)EX)$$ +

$\square$.

(ii) To show $\quad P(X \geq (1-t)EX) \geq \dfrac{t^2 (EX)^2}{t^2 (EX)^2 + Var(X)}$.

it is the same to show $\quad P(X < (1-t)EX) \leq \dfrac{Var(X)}{Var(X) + t^2 (EX)^2}$.

Notice that $\quad P\left(X - EX - \dfrac{Var(X)}{tEX} < -tEX - \dfrac{Var(X)}{tEX}\right)$ ≤ 0

Agreed

$$\leq P\left(\left(X - EX - \frac{Var(X)}{tEX}\right)^2 > \left(tEX + \frac{Var(X)}{tEX}\right)^2\right)$$

$$(\text{Markov ineq.}) \leq \frac{E\left[\left(X - EX - \frac{Var}{tEX}\right)^2\right]}{\left(tEX + \frac{Var(X)}{tEX}\right)^2} \overset{+}{=} \frac{Var(X) + \frac{Var(X)^2}{(tEX)^2}}{\left(tEX + \frac{Var(X)}{tEX}\right)^2}$$

Smart (shorter than the one I've seen before)

$$\overset{+}{=} \frac{Var(X)}{t^2 (EX)^2 + Var(X)}$$

So the Cantelli inequality is proved. Bravo! +

4(iii) Use Hölder Ineq. Here q satisfies $\frac{1}{p}+\frac{1}{q}=1$,

$$E[|X \mathbb{1}_{\{X \geq (1-t)EX\}}|] \leq E[|X|^p]^{\frac{1}{p}} E[\mathbb{1}_{\{X \geq (1-t)EX\}}^q]^{\frac{1}{q}}$$

since $X \geqslant 0$, $E[X \mathbb{1}_{\{X \geq (1-t)EX\}}]^p \leq E[|X|^p] \, P(X \geq (1-t)EX)^{p-1}$

Again notice that $E[X \mathbb{1}_{\{X \geq (1-t)EX\}}] \geq tEX$

(argument made in 4(i))

we have $(tEX)^p \leq E[|X|^p] \, P(X \geq (1-t)EX)^{p-1}$

which is exactly $P(X \geq (1-t)EX) \geq \left(\frac{t^p (EX)^p}{E[|X|^p]}\right)^{\frac{1}{p-1}}$ +

Problem 5

(a) let $X \sim \chi_2^2$, then $X = Z_1^2 + Z_2^2$ where Z_1, Z_2 independent std normal

$$M_2(t) = E[e^{tX}] = E[e^{tZ_1^2 + tZ_2^2}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t(x^2+y^2)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dx\,dy$$

$$= \int_0^{\infty}\int_0^{2\pi} \frac{1}{2\pi} e^{tr^2} e^{-\frac{1}{2}r^2} \cdot r \, d\theta \, dr$$

$$= \int_0^{\infty} r \exp\left(\left(t-\tfrac{1}{2}\right)r^2\right) dr = \frac{1}{2t-1} \cdot (-1) = \frac{1}{1-2t}$$

Assume $t < \frac{1}{2}$.

+

if $t \geq \frac{1}{2}$, then the integral $\int_0^{\infty} r e^{(t-\frac{1}{2})r^2} dr$ diverges.

Therefore, $M_2(t) = \frac{1}{1-2t}$, $t < \frac{1}{2}$.

Also $M_{2d}(t) = E[e^{t\sum_{i=1}^{2d} Z_i^2}] \overset{iid}{=} E[e^{tZ_1^2 + tZ_2^2}]^d = \frac{1}{(1-2t)^d}$

(b) $P(X > x) \overset{X>0 \text{ a.s.}}{=} P(e^{tX} > e^{tx}) \leq \frac{E[e^{tX}]}{e^{tx}} = \frac{e^{-tx}}{(1-2t)^d}$.

Now improve the bound by picking t. for $0 < t < \frac{1}{2}$.

$\log \frac{e^{-tx}}{(1-2t)^d} = -tx - d \cdot \log(1-2t)$, take derivative w.r.t. t.

the derivative is $-x + \frac{2d}{1-2t}$, 2nd order derivative is $\frac{4d}{(1-2t)^2} > 0$

Stationary point $t^* = \frac{1}{2} - \frac{d}{x} \in (0, \frac{1}{2})$

so $P(X > x) \leq \frac{e^{-t^* x}}{(1-2t^*)^d} = \left(\frac{x}{2d}\right)^d \exp\left(d - \frac{x}{2}\right) = \exp\left(d \log\left(\frac{x}{2d}\right) - \frac{x-2d}{2}\right)$

+

Problem 5. (c)*

(i) Claim: $\log(1+u) \le u - \frac{1}{4}\min\{u, u^2\}$, $\forall u \ge 0$

indeed, when $0 \le u \le 1$, $\min\{u, u^2\} = u^2$

$$g(u) = \log(1+u) - u + \frac{1}{4}u^2$$

$$g'(u) = \frac{1}{1+u} - 1 + \frac{u}{2} = \frac{u^2+u+\cancel{2}-2u-\cancel{2}}{2(1+u)}$$

so $g(u) \le g(0) = 0$ on $u \in [0,1]$. +

$$= \frac{u^2-u}{2(1+u)} \le 0$$

when $u \ge 1$, $u \le u^2$, then

$$h(u) = \log(1+u) - u + \frac{1}{4}u = \log(1+u) - \frac{3}{4}u.$$

$h'(u) = \frac{1}{1+u} - \frac{3}{4} < 0$, so $g(u) \le g(1) < 0$, $\forall u \ge 1$

So the claim holds. +

$$P(X - 2d > z) \le \exp\left(d \log\left(1+\frac{z}{2d}\right) - \frac{z}{2}\right)$$

$$\le \exp\left(d \cdot \left(\frac{z}{2d} - \frac{1}{4}\min\left\{\left(\frac{z}{2d}\right), \left(\frac{z}{2d}\right)^2\right\}\right) - \frac{z}{2}\right)$$

$$= \exp\left(-\frac{d}{4}\min\left\{\left(\frac{z}{2d}\right)^2, \frac{z}{2d}\right\}\right)$$

$$= \begin{cases} \exp\left(-\frac{z^2}{16d}\right), & \text{if } 0 \le \frac{z}{2d} \le 1 \\ \exp\left(-\frac{1}{8}z\right), & \text{if } \frac{z}{2d} > 1. \end{cases}$$

+

Problem 6

(a) $\text{Risk}(sX) = E[\|sX - \mu\|^2]$

$$= s^2 E[\|X - \cancel{\#}\mu\|^2 + \|\frac{\mu}{s} - \mu\|^2].$$

$$= s^2 \cdot d + s^2 \cdot (\frac{1}{s} - 1)^2 \|\mu\|^2$$

$$= s^2 d + (s-1)^2 \|\mu\|^2.$$

For $s < 0$, $\text{Risk}(-sX) \le \text{Risk}(sX)$, so $-s$ is always better.

~~For~~ So we can restrict discussion to $s \in [0, \infty)$.

For $s > 1$, $s-1$ is always better, so we can restrict discussion to $s \in [0, 1]$. +

(b) $\text{Risk}(sX) = (d + \|\mu\|^2) s^2 - 2\|\mu\|^2 s + \|\mu\|^2$

The stationary point of this quadratic form is

$$s^* = \frac{\|\mu\|^2}{d + \|\mu\|^2} = 1 - \frac{d}{d + \|\mu\|^2}.$$ +

(c) Because s^* is not known, it involves true value μ.

For $(1 - \frac{d}{\|X\|^2})X = \hat{\mu}$, it approximates $\|\mu\|^2$ using the observation we have, which is $\|X\|^2$. +

Question 6(d)*.

$$\hat{\mu}^{\delta} = \left(1-\frac{\delta}{\|X\|^2}\right)X$$

denote $g_i(X) = \left(1-\frac{\delta}{\|X\|^2}\right)X_i$, then

$$\frac{\partial g_i(X)}{\partial x_j} = \begin{cases} 1-\frac{\delta}{\|X\|^2} + \frac{2\delta X_i^2}{\|X\|^4}, & \text{if } i=j \\ \frac{2X_i\,\delta\, X_j}{\|X\|^4}, & \text{otherwise} \end{cases}$$

①

$$E\left[\left|\frac{\partial g_i(X)}{\partial x_j}\right|\right] \le \max\left\{E\left[\left|1-\frac{\delta}{\|X\|^2}+\frac{2\delta X_i^2}{\|X\|^4}\right|\right],\ |2\delta|\,E\left[\left|\frac{X_iX_j}{\|X\|^4}\right|\right]\right\}$$

$$\le \max\left\{1+3|\delta|\,E\left[\frac{1}{\|X\|^2}\right],\ 2|\delta|\,E\left[\frac{1}{\|X\|^2}\right]\right\}$$

$$= 1+|3\delta|\,E\left[\frac{1}{\|X\|^2}\right].$$

For $d \geq 3$. $\|X\|^2$ is a shifted chi-squared random variable with degree of freedom >2,

therefore the integral exists, $E\left[\frac{1}{\|X\|^2}\right] < \infty$

For $d=1,2$, $\frac{\delta}{\|X\|^2}$ is not integrable.

+

② Also, $|g_i(X)(X_j-\mu_j)|$ is integrable. $\forall i,j$.

By Stein's lemma, $E[(X_i-\mu_i)\,g_i(X)] = E\left[\frac{\partial}{\partial x_i} g_i(X)\right]$.

$$= 1-\delta E\left[\frac{1}{\|X\|^2}\right] + 2\delta E\left[\frac{X_i^2}{\|X\|^4}\right].$$

+

Also expand the left hand side, we get

$$E\left[\sum_{i=1}^{d}\left(1-\frac{\delta}{\|X\|^2}\right)X_i^2-\mu_i X_i\left(1-\frac{\delta}{\|X\|^2}\right)\right]=d-d\delta E\left[\frac{1}{\|X\|^2}\right]+2\delta E\left[\frac{\sum_{i=1}^{d}X_i^2}{\|X\|^4}\right].$$

that is,

$$E[\|X\|^2]-\delta-\sum_{i=1}^{d}\mu_i^2+\cancel{d}\delta E\left[\frac{\sum \mu_i X_i}{\|X\|^2}\right]=d-d\delta E\left[\frac{1}{\|X\|^2}\right]+2\delta E\left[\frac{1}{\|X\|^2}\right]$$

Expand $E[\|X\|^2]=\sum_{i=1}^{d}\mu_i^2+d$, we have

$$-\delta+\cancel{d}\delta E\left[\frac{\sum \mu_i X_i}{\|X\|^2}\right]=(2-d)\delta E\left[\frac{1}{\|X\|^2}\right].$$

$$\Longrightarrow \boxed{-1+\cancel{d}E\left[\frac{\mu^T X}{\|X\|^2}\right]=(2-d)E\left[\frac{1}{\|X\|^2}\right].} \quad + \quad (*)$$

Now we look at the risk function of $\hat{\mu}^\delta$.

$$\mathrm{Risk}(\hat{\mu}^\delta)=E[\|\hat{\mu}^\delta-\mu\|^2]$$

$$=E\left[\left(1-\frac{\delta}{\|X\|^2}\right)^2X^2-2X^T\mu\left(1-\frac{\delta}{\|X\|^2}\right)+\|\mu\|^2\right].$$

$$=E\left[\frac{\delta^2}{\|X\|^2}-2\delta+\frac{2X^T\mu\delta}{\|X\|^2}+X^2-2X^T\mu+\mu^2\right]$$

$$=\delta^2E\left[\frac{1}{\|X\|^2}\right]+\delta\left(-2+2E\left[\frac{\mu^TX}{\|X\|^2}\right]\right)+E[X^2]-2E[X^T\mu]+\mu^2.$$

View the risk function as a quadratic form of δ, then the minimizer is $\delta^*=\dfrac{-2+2E\left[\frac{\mu^TX}{\|X\|^2}\right]}{-2E\left[\frac{1}{\|X\|^2}\right]}$

Using $(*)$, we have $\underline{\delta^*=d-2}$. +

Now $\mathrm{Risk}(\hat{\mu}^{\delta^*})$ is strictly cvx w.r.t. μ.

Problem 7

Lol, OK :)

Reference : Math.StackExchange.com

"Why can a Venn digram for 4+ sets not be constructed using circles?"

① $F_n = |\text{power}(\{1,2\cdots,n\})| = 2^n$ because each $I \subseteq \{1,2\cdots,n\}$ represents a face, in particular, ϕ is the outer region. +

② $V_n = 2\cdot\binom{n}{2}$ because for any two i, j circle, they must intersect on two distinct points, otherwise they ~~overlap~~ cannot create a face for $A_i \cap A_j$. +

③ Every Vertex has degree 4. otherwise, there should be 3 circles sharing one intersection point.

Not the way I did it brief an...

Then not all intersections $\{A_1\cap A_2, A_1\cap A_3, A_2\cap A_3, A_1\cap A_2\cap A_3\}$ are present as faces.

$\sum_{i\in V_n} \deg(i) = 2|E_n|$ for simple graph, so $E_n = 2V_n$.

For $n=4$, $F_4 + V_4 - E_4 = 16 - 12 = 4 \neq 2$ contradicts Euler form.

therefore cannot draw Venn diagram for $n=4$.

Then, of course, cannot draw for $n \geq 4$, since removing one circle from n-Venn diagram results in a $(n-1)$-Venn diagram.

+