Orlicz norms exercises

// Exercise 2.1. Young functions

1° Super-additivity: $\psi(a+b) \geq \psi(a) + \psi(b)$

Proof: $\psi'' \geq 0 \Rightarrow \psi' \nearrow$. Assume w.l.o.g. $a \geq b$.

$$\psi(a+b) - \psi(b) = \int_b^{a+b} \psi'(x)\, dx \geq a\psi'(b)$$

$$\geq a\psi'(a)$$

$$\geq \int_0^a \psi'(t)\, dt = \psi(a) - \psi(0)$$

$$= \psi(a). \quad \blacksquare$$

2° Let $\varphi_x(t) = \dfrac{\psi(tx)}{t}$.

Then $\varphi_x(1) = \psi(x)$

$$\varphi_x'(t) = \frac{x}{t}\psi'(tx) - \frac{1}{t^2}\psi(tx)$$

$$= \frac{1}{t^2}\left(u\psi'(u) - \psi(u)\right)\Big|_{u=tx}$$

$$= \frac{1}{t^2}\Big(\underset{0}{\underset{\shortparallel}{\psi(0)}} - \psi(u) - (0-u)\psi'(u)\Big) \geq 0$$

by convexity.

$\Rightarrow \varphi_x(t) \geq \varphi_x(1)$. $\blacksquare$

// $\boxed{\text{Exercise 2.2}}$ (Orlicz max):

Reduction: $\dfrac{\max_j |X_j|}{\max_j K_j} \leq \max_j \dfrac{|X_j|}{K_j} = \max_j |Z_j|.$

Now:
$$\psi\left(\mathbb{E}\left[\max_j |Z_j|\right]\right) \overset{\text{Jensen}}{\leq} \mathbb{E}\left[\psi\left(\max_j |Z_j|\right)\right]$$
$$\leq \mathbb{E}\left[\max_j \psi(|Z_j|)\right]$$
$$\leq \mathbb{E}\left[\sum_j \psi(|Z_j|)\right] \leq N.$$

+ Use monotonicity of $\psi^{-1}(\cdot)$. (Show it first...) $\square$

(Alternatively, one can use superadditivity & triangle inequality...)

Or using superadditivity

Exercise 3.1

$\alpha < 1$, $x \in \mathbb{R}_+$.

let $\psi_\alpha(x) = \underbrace{\exp(x^\alpha)}_{\psi_+} \mathbb{1}_{x \geq x_\alpha} + \underbrace{\frac{e^{1/\alpha}}{x_\alpha}}_{\psi_-} x \, \mathbb{1}_{x < x_\alpha}$.

where $x_\alpha = \alpha^{-1/\alpha} \iff x_\alpha^\alpha = \frac{1}{\alpha}$.

(1.) ψ_α is $C^1(\mathbb{R}_+)$ and a Young function.

Proof:

$$\psi_+'(x) = \alpha x^{\alpha-1} \exp(x^\alpha) \implies \psi_+'(x_\alpha) = \alpha x_\alpha^{\alpha-1} \exp(x_\alpha^\alpha)$$

$$\psi_-'(x) = \frac{e^{1/\alpha}}{x_\alpha} = (e\alpha)^{1/\alpha} \qquad = \alpha \alpha^{-\frac{\alpha-1}{\alpha}} e^{1/\alpha} = (\alpha e)^{1/\alpha}.$$

So, ψ_α is in $C^1(\mathbb{R}_+)$. For Young, let's check convexity:

$x < x_\alpha$: $\psi_\alpha''(x) = 0$.

$x \geq x_\alpha$: $\psi_\alpha''(x) = \left(\alpha^2 x^{2\alpha-2} + \alpha(\alpha-1) x^{\alpha-2}\right) \exp(x^\alpha)$

$$= \alpha x^{\alpha-2} \big(\underbrace{\alpha x^\alpha - 1}_{\geq 0} + \alpha\big) \exp(x^\alpha)$$

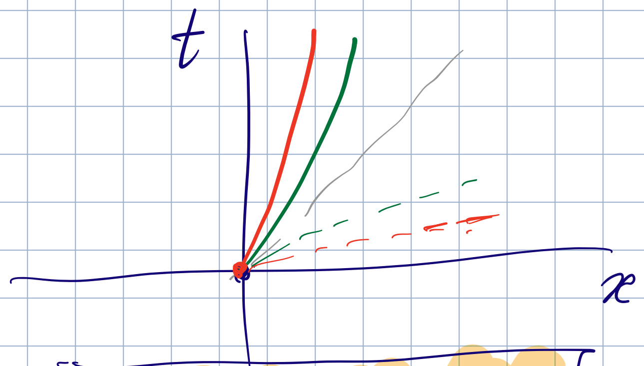
Other checks are trivial. One can instead take $\tilde{x}_\alpha = \left(\frac{1-\alpha}{\alpha}\right)^{1/\alpha}$.

(2.) Straightforward calculation.

(3.) $\exp(x^\alpha) - C_\alpha \leq \psi_\alpha(x) \leq \exp(x^\alpha)$.

$$\log^{1/\alpha}(t) \leq \psi_\alpha^{-1}(t) \leq \log^{1/\alpha}(t + C_\alpha)$$

<u>Proof</u>: Second line follows from the first, picture:



$\psi \geq \varphi \iff \psi^{-1} \leq \varphi^{-1}$.

⊙ $\boxed{\psi_\alpha(x) \leq \exp(x^\alpha)}$:

Note that this holds for $x \geq x_\alpha$, and for $x \leq x_\alpha$ one has

$$\frac{\psi_\alpha(x)}{\exp(x^\alpha)} = (e\alpha)^{1/\alpha} \underbrace{x \exp(-x^\alpha)}_{r(x)}.$$

$$r'(x) = \exp(-x^\alpha)\left[1 - \alpha x^\alpha\right] \geq 0$$

so $r(x) \leq r(x_\alpha) = 1$.

⊙ $\boxed{\psi_\alpha(x) \geq \exp(x^\alpha) - 1}$:

Again trivial for $x \geq x_\alpha$, and for $x \leq x_\alpha$ one has

$$\varphi(x) := \psi_\alpha(x) - \exp(x^\alpha) = (e\alpha)^{1/\alpha} x - \exp(x^\alpha)$$

$$\varphi(0) = -1$$

$$\varphi'(x) = (e\alpha)^{1/\alpha} - \alpha x^{\alpha-1} \exp(x^\alpha) \in [0, (e\alpha)^{1/\alpha}]$$

since $0 \le \alpha x^{\alpha-1} \exp(x^\alpha) \le \alpha x_\alpha^{\alpha-1} \exp(x^\alpha) = (e\alpha)^{1/\alpha}$.

$$\Rightarrow \varphi(x) \ge \varphi(0) = -1.$$

Exercise 3.2

1. Assume X is (K, α)-sub-Weibull, that is

$$\boxed{\mathbb{E} \exp\left(\left(\frac{|X|}{K}\right)^\alpha\right) \le 1}$$

Then $\|X\|_{\psi_\alpha} \le 1$. Follows from $\psi_\alpha(x) \le \exp(x^\alpha)$.

2. By (1), $\|X_j\|_{\psi_\alpha} \le K_j$. Then by Prop. 2.1,

$$\left\| \max_{j \in [N]} |X_j| \right\|_{\psi_\alpha} \le \psi_\alpha^{-1}(N) \max_{j \in [N]} K_j$$

$$\le \log^{1/\alpha}(N+1) \max_{j \in [N]} K_j.$$

(3.) Obvious for $\alpha \geq 1$. For $\alpha < 1$, one has

$$\mathbb{E}\left[\psi_{\alpha/p}\left(\frac{|X|^p}{2K^p}\right)\right] \underset{\substack{\uparrow \\ \text{super-additivity}}}{\leq} \frac{1}{2}\,\mathbb{E}\left[\psi_{\alpha/p}\left(\frac{|X|^p}{K^p}\right)\right]$$

$$[\text{upper bound}] \leq \frac{1}{2}\,\mathbb{E}\left[\exp\left(\frac{|X|}{K}\right)^{\alpha}\right]$$

$$[\text{lower bound}] \leq \frac{2}{2}\,\mathbb{E}\left[\psi_{\alpha}\left(\frac{|X|}{K}\right)\right].$$

$\boxtimes$