Efficient and Near-Optimal Online Portfolio Selection

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Outline

Introductory part:

- Online portfolio selection.
- Universal Portfolios.
- Our algorithm: VB-FTRL.

Technical part:

- **VB-FTRL as an approximation of Universal Portfolios**.
- Regret analysis: key ideas.
- Quasi-Newton implementation of VB-FTRL.

. . .

PROFIT (???)

Online portfolio selection

Online portfolio selection (Cover, 1991)

- $\mathrm{Cap}_0 = 1$
- **For** $t \in [T] := \{1, 2, ..., T\}$ **do**
  - Select $w_t \in \Delta_d$ // distribute money over d assets
  - Receive $x_t \in \mathbb{R}_+^d$ // obtain new asset returns

$$\mathrm{Cap}_t := \underbrace{x_t^\top w_t}_{\text{overall return at round } t} \mathrm{Cap}_{t-1}.$$

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- "Money compounds **multiplicatively**." "Market plays **against** you."
- **Goal:** choose $w_1, ..., w_T$ to earn large final capital $\mathrm{Cap}_T = \prod_{t \in [T]} x_t^\top w_t$.

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- Defining the instantaneous **losses**

$$\ell_t(w) := -\log(x_t^\top w),$$

maximizing $\mathrm{Cap}_T \Leftrightarrow$ minimizing the final **cumulative loss**

$$-\log(\mathrm{Cap}_T) = \sum_{t \in [T]} \ell_t(w_t).$$

Online optimization formulation

- **Baseline:** the best **offline, static** selection
  - also called the best "constantly-rebalanced portfolio" (CRP).

Regret of $w_{1:T}$ given $x_{1:T}$:

$$\mathscr{R}_T(w_{1:T}|x_{1:T}) := \sum_{t\in[T]} \ell_t(w_t) - \min_{w\in\Delta_d} \sum_{t\in[T]} \ell_t(w).$$

- **Goal:** guarantee small regret **uniformly** over all possible markets:

$$\sup_{x_1,\dots,x_T\in\mathbb{R}^d_+} \mathscr{R}_T(w_{1:T}|x_{1:T}) = o_d(T).$$

Cover's algorithm

Follow-The-Leader (FTL)

$$w_{t+1} \in \operatorname*{Argmin}_{w \in \Delta_d} \sum_{\tau \in [t-1]} \ell_\tau(w),$$

i.e. select the best portfolio for the observed market.

- **Fails:** for $(x_1, x_2, x_3, x_4, \dots) = (e_1, e_2, e_1, e_2, \dots)$ in $\mathbb{R}^2$ the regret is $\Omega(T)$.

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- **Key trick:** robustify the procedure by exponential weighting:

Universal Portfolios (a.k.a. Exponentially Weighted Average Forecaster)

Play $w_t = \mathbb{E}_{\phi_t} w$ where averaging is over the **distribution** with density

$$\phi_t(w) \propto \exp\left(- \sum_{\tau \in [t-1]} \ell_\tau(w) \right), \quad w \in \Delta_d.$$

English: "to each $w \in \Delta_d$ assign the weight proportional to the amount of money it would have earned on the observed market, and take the average."

Cover's algorithm (cont.)

Theorem (Cover, 1991)

For any realization $x_{1:T}$ of the market, Cover's algorithm has regret at most

$$(d-1)\log(T+1).$$

Regret guarantee: not only sublinear in T, but near-constant.

- Doesn't depend on the magnitudes of x_t's and is linear in d.
- Unimprovable for $T \gtrsim d$ (Cesa-Bianchi and Lugosi, 2006).

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Computationally prohibitive: amounts to integration over the density ϕ_t.

- ϕ_t is **log-concave** $\Rightarrow$ can sample from it to approximate $\mathbb{E}_{\phi_t}[w]$.
- Kalai and Vempala (2002): runtime $O(d^4(T+d)^{14})$ per round.

Open problem

Propose a **regret-optimal** and **computationally feasible** algorithm.

Progress timeline

Open problem

Propose a **regret-optimal** and **computationally feasible** algorithm.

Algorithm	Regret	Runtime	Source
Universal Portfolios	$d\log(T)$	$\exp(d, T)$	Cover (1991)
		$d^4 T^{14}$	Kalai and Vempala (2002)
Online Grad. Descent	$G\sqrt{Td}$	d	Zinkevich (2003)
Exponentiated Grad.	$G\sqrt{T\log(d)}$	d	Helmbold et al. (1998)
Online Newton Step	$Gd\log(T)$	d^3	Hazan et al. (2007)
Soft-Bayes	$\sqrt{Td\log(d)}$	d	Orseau et al. (2017)
Ada-BARRONS	$d^2\log^p(T)$	$d^{2.5}(T+d)$	Luo et al. (2018)
BISONS	$d^2\log^p(T)$	$\text{poly}(d)$	Zimmert et al. (2022)
AdaMix+DONS	$d^2\log^p(T)$	$d^3\log^p(T+d)$	Mhammedi and Rakhlin (2022)
VB-FTRL	$d\log(T)$	$d^2 T$	**Our result**

Log-barrier regularization

- Regularize the observed losses with the **log-barrier** of Δ_d, i.e. consider

$$L_t(w) := \sum_{\tau \in [t]} \ell_t(w) \; - \; \lambda \sum_{i \in [d]} \log(w[i]) \quad \text{with } \lambda > 0.$$

LB-FTRL – Log-Barrier Follow-The-Regularized-Leader

$$w_t = \operatorname*{argmin}_{w \in \Delta_d} L_{t-1}(w).$$

- **Self-concordant** (SC) $\Rightarrow$ Newton's method, in $O(d^2 T)$ per step.
- Van Erven et al. (2020): $O(\sqrt{dT \log(T)})$ regret for $\lambda \approx \sqrt{\frac{T+d}{d}} \gg 1$
- **Conjectured** $O(d \log(T))$ regret for $\lambda = O(1)$, but it was **disproved** by Zimmert et al. (2022):

$$\mathscr{R}_T(w_{1:T} | x_{1:T}) \gtrsim \mathbf{2^d} \log(T) \log\log(T) \;\; \text{when} \;\; T \gtrsim \text{poly}(2^d).$$

Key ingredient: volumetric barrier

Key insight: it suffices to add to $L_t(w)$ a **volumetric** regularizer:

$$V_t(w) := \tfrac{1}{2} \log\det[\nabla^2 L_t(w)]$$

—**volumetric barrier** for the set of observed linear constraints:

$$w \in \Delta_d : \quad x_\tau^\top w > 0, \quad \forall \tau \in [t]\}.$$

- First studied by Vaidya (1989) as a **self-concordant barrier** for a polytope, improved compared to the logarithmic barrier, i.e. $L_t(w)$.

VB-FTRL – Follow-The-Regularized-Leader with the Volumetric Barrier:

$$w_t = \operatorname*{argmin}_{w \in \Delta_d} \underbrace{L_{t-1}(w) + \mu V_{t-1}(w)}_{P_{t-1}(w)}.$$

- Vaidya (1989) proved that P_{t-1} is strictly convex & **self-concordant**. (Informally, its 3rd derivatives are bounded in terms of the Hessian.)

Main result

In terms of the "hybrid" barrier $P_{t-1}(w) := L_{t-1}(w) + \mu V_{t-1}(w)$ we have

$$w_t = \operatorname*{argmin}_{w \in \Delta_d} P_{t-1}(w).$$

- VB-FTRL is **near-optimal** in terms of regret:

Theorem

For any $x_{1:T}$, VB-FTRL run with $\lambda = 16$, $\mu = 7$ produces $w_{1:T}$ such that

$$\mathscr{R}_T(w_{1:T}|x_{1:T}) \leqslant 30\,(d-1)\log(T+16d).$$

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- VB-FTRL is **computationally cheap**:

Theorem

A quasi-Newton method run on P_{t-1} from w_{t-1} converges **linearly** to w_t.

One quasi-Newton step for P_{t-1} can be performed in $O(d^2(T+d))$.

Connection of VB-FTRL with Universal Portfolios

Universal Portfolios: play $w_t = \mathbb{E}_{w \sim \phi_t} w$ with probability density

$$\phi_t(w) \propto \exp\left(-\frac{1}{\mu} \sum_{\tau \in [t-1]} \ell_\tau(w)\right), \quad w \in \Delta_d.$$

Variational characterization (Gibbs' duality)

Let $\mathrm{Supp}(\Delta_d)$ be the set of all distributions supported on Δ_d, and define

$$\mathcal{F}_t[\phi] := \mathcal{L}_t[\phi] - \mu\mathcal{H}[\phi]$$

where $\mathcal{L}_t[\phi] = \mathop{\mathbb{E}}_{w \sim \phi} L_t(w)$, and $\mathcal{H}[\phi]$ is the **differential entropy** of ϕ, i.e. $\mathcal{H}[\phi] := \mathop{\mathbb{E}}_{w \sim \phi} [-\log \phi(w)]$. Then

$$\phi_t = \mathop{\mathrm{argmin}}_{\phi \in \mathrm{Supp}(\Delta_d)} \mathcal{F}_{t-1}[\phi].$$

- μ is "temperature:" from Dirac on the leader to the uniform measure.
- λ (contained inside L_t) biases the leader towards the Dirac on $\frac{1}{d}\mathbb{1}_d$.

Optimization viewpoint

$$\min_{\phi \in \mathrm{Supp}(\Delta_d)} \{\mathcal{F}_t[\phi] \; := \; \mathcal{L}_t[\phi] - \mu \mathcal{H}[\phi]\}$$

is a **convex** optimization problem—but infinite-dimensional one, thus hard.

- Even **evaluating** $\mathcal{F}_t[\phi]$ is already **challenging** (expectation over ϕ).

Minimize approximately over a smaller class of distributions.

Outline

1º. Replace $L_t(w)$ with its quadratic approximation near $\hat{w} = \mathbb{E}_\phi[w]$.

2º. By self-concordance, this approximation is valid in a Dikin ellipsoid of L_t around $\hat{w} \Longrightarrow$ focus on the densitites **supported** in this ellipsoid.

3º. The ellipsoid constraint suggests to focus on Gaussian distributions.

Approximation of L_t via self-concordance

Definition

Let $\mathbb{A}_d = \{w \in \mathbb{R}^d : \mathbb{1}^\top w = 1\}$. The r-Dikin ellipsoid of L_t at $\hat{w}$, $r > 0$, is

$$\mathcal{E}_{t,r}(\hat{w}) := \left\{ w \in \mathbb{A}_d : \ \|w - \hat{w}\|_{\nabla^2 L_t(\hat{w})} < r \right\}.$$

- L_t is SC with domain $\mathrm{ri}(\Delta_d)$, so $\mathcal{E}_{t,1}(\hat{w}) \in \Delta_d$ for any $\hat{w} \in \Delta_d$.[1]

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- Controlled quadratic approximation for all $\hat{w} \in \Delta_d$ and $w \in \mathcal{E}_{t,\,1/2}(\hat{w})$:

$$\frac{1}{5}\|w - \hat{w}\|^2_{\nabla^2 L_t(\hat{w})} \leqslant L_t(w) - L_t(\hat{w}) - \nabla L_t(\hat{w})^\top (w - \hat{w}) \leqslant \frac{4}{5}\|w - \hat{w}\|^2_{\nabla^2 L_t(\hat{w})}.$$

- As a result, for any ϕ with $\mathbb{E}_\phi w = \hat{w}$ and supported on $\mathcal{E}_{t,\,1/2}(\hat{w})$,

$$\underbrace{L_t(\hat{w}) + \frac{1}{5}\left\langle \mathrm{Cov}[\phi], \nabla^2 L_t(\hat{w}) \right\rangle}_{\underline{\mathcal{L}}_t[\phi]} \leqslant \mathcal{L}_t[\phi] \leqslant \underbrace{L_t(\hat{w}) + \frac{4}{5}\left\langle \mathrm{Cov}[\phi], \nabla^2 L_t(\hat{w}) \right\rangle}_{\bar{\mathcal{L}}_t[\phi]}.$$

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- This suggests to replace $\min\limits_{\phi \in \mathrm{Supp}(\Delta_d)} \mathcal{L}_t[\phi] - \mu \mathcal{H}[\phi]$ with the problem

$$\min_{\substack{\hat{w} \in \Delta_d, \; \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathrm{Supp}(\mathcal{E}_{t,1/4}(\hat{w}))}} \bar{\mathcal{L}}_t[\phi] - \mu \mathcal{H}[\phi]$$

Dikin approximation accuracy

Suppressing the subscript t, let

$$\phi^\star := \underset{\phi \in \mathsf{Supp}(\Delta_d)}{\operatorname{argmin}} \underbrace{\mathcal{L}[\phi] - \mu\mathcal{H}[\phi]}_{\mathcal{F}[\phi]} \quad \text{and} \quad \bar{\phi} \in \underset{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/4}(\hat{w}))}}{\operatorname{Argmin}} \underbrace{\bar{\mathcal{L}}[\phi] - \mu\mathcal{H}[\phi]}_{\bar{\mathcal{F}}[\phi]},$$

i.e. $\phi^\star$ is Cover's distribution, and $\bar{\phi}$ is its approximation.

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i.e. $\phi^\star$ is Cover's distribution, and $\bar{\phi}$ is its approximation.

- As it turns out, $\bar{\phi}$ is not much worse than ϕ^* in terms of $\mathcal{F}[\cdot]$, namely:

Proposition 1

For any $\lambda \geqslant 1$ and $\mu \geqslant 0$,

$$\mathcal{F}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\bar{\phi}] \leqslant \min_{\phi \in \mathsf{Supp}(\Delta_d)} \mathcal{F}[\phi] + 1.5\mu(d-1)\log(T + \lambda d) + C\mu + c.$$

- The first inequality is trivial. I shall outline the proof of the second one.

Dikin approximation accuracy: proof sketch (I)

$$\bar{\phi} \in \operatorname*{Argmin}_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w},\\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/4}(\hat{w}))}} \underbrace{\bar{\mathcal{L}}[\phi] - \mu \mathcal{H}[\phi]}_{\bar{\mathcal{F}}[\phi]}. \qquad (\bar{\mathsf{P}})$$

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- Recall that for $\phi \in \mathsf{Supp}(\mathcal{E}_{1/2}(\hat{w}))$ we also have $\mathcal{F}[\phi] \geqslant \underline{\mathcal{F}}[\phi]$, and let

$$\underline{\phi} \in \operatorname*{Argmin}_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w},\\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/2}(\hat{w}))}} \underbrace{\underline{\mathcal{L}}[\phi] - \mu\mathcal{H}[\phi]}_{\underline{\mathcal{F}}[\phi]}.$$

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- Moreover, let $\underline{\phi}^+(\cdot) = 2\underline{\phi}(\underline{w} + 2(\cdot - \underline{w}))$ be the "squeezing-by-factor-2" of $\underline{\phi}$ towards $\underline{w} = \mathbb{E}_{\underline{\phi}}[w]$, i.e. the distribution of $\underline{w} + \frac{1}{2}(w - \underline{w})$ for $w \sim \underline{\phi}$; then

$$\mathbb{E}_{\underline{\phi}^+}[w] = \mathbb{E}_{\underline{\phi}}[w] = \underline{w} \quad \text{and} \quad \underline{\phi}^+ \in \mathsf{Supp}(\mathcal{E}_{1/4}(\underline{w})).$$

$$\bar{\phi} \in \operatorname*{Argmin}_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/4}(\hat{w}))}} \underbrace{\bar{\mathcal{L}}[\phi] - \mu\mathcal{H}[\phi]}_{\bar{\mathcal{F}}[\phi]}. \qquad (\bar{\mathsf{P}})$$

$$\mathcal{F}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\bar{\phi}] \leqslant \min_{\phi \in \mathsf{Supp}(\Delta_d)} \mathcal{F}[\phi] + 1.5\mu(d-1)\log(T+\lambda d) + C\mu + c.$$

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$$\mathbb{E}_{\underline{\phi}^+}[w] = \mathbb{E}_{\underline{\phi}}[w] = \underline{w} \quad \text{and} \quad \underline{\phi}^+ \in \mathsf{Supp}(\mathcal{E}_{1/4}(\underline{w})).$$

- In other words, $(\underline{w}, \underline{\phi}^+)$ is **feasible** in $(\bar{\mathsf{P}})$, and therefore $\bar{\mathcal{F}}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\underline{\phi}^+]$
- But we also have $\mathrm{Cov}[\underline{\phi}^+] = \frac{1}{4}\mathrm{Cov}[\underline{\phi}]$ and $\mathcal{H}[\underline{\phi}^+] = \mathcal{H}[\underline{\phi}] - (d-1)\log(2)$, so

$$\bar{\mathcal{F}}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\underline{\phi}^+] \leqslant \underline{\mathcal{F}}[\underline{\phi}] + \mu(d-1)\log(2).$$

Dikin approximation accuracy: proof sketch (II)

So far, for

$$\bar{\phi} \in \underset{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/4}(\hat{w}))}}{\operatorname{Argmin}} \bar{\mathcal{F}}[\phi] \quad \text{and} \quad \underline{\phi} \in \underset{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/2}(\hat{w}))}}{\operatorname{Argmin}} \underline{\mathcal{F}}[\phi]$$

we have the following:

$$\mathcal{F}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\bar{\phi}] \leqslant \underline{\mathcal{F}}[\underline{\phi}] + \mu(d-1)\log(2).$$

- But since $\underline{\mathcal{F}}[\phi] \leqslant \mathcal{F}[\phi]$ for all $\phi \in \mathsf{Supp}(\mathcal{E}_{1/2}(\hat{w}))$, we conclude that

$$\mathcal{F}[\bar{\phi}] \leqslant \min_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/2}(\hat{w}))}} \mathcal{F}[\phi] \;+\; \mu(d-1)\log(2).$$

- We've gotten rid of the **objective approximation.** It remains to prove that

$$\min_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/2}(\hat{w}))}} \mathcal{F}[\phi] \leqslant \min_{\phi \in \mathsf{Supp}(\Delta_d)} \mathcal{F}[\phi] + O(\,\mu(d-1)\log(T+\lambda d) + \mu + 1\,).$$

Dikin approximation accuracy: proof sketch (III)

It remains to prove that

$$\min_{\substack{\hat{w}\in\Delta_d,\ \mathbb{E}_\phi w = \hat{w},\\ \phi\in\mathrm{Supp}(\mathcal{E}_{1/2}(\hat{w}))}} \mathcal{F}[\phi] \leqslant \min_{\phi\in\mathrm{Supp}(\Delta_d)} \mathcal{F}[\phi] + O(\,\mu(d-1)\log(T+\lambda d) + \mu + 1\,).$$

- Note: the leader $w^\star = \operatorname*{argmin}_{w\in\Delta_d} L(w)$ is the **mode** of $\phi^\star = \operatorname*{argmin}_{\phi\in\mathrm{Supp}(\Delta_d)} \mathcal{F}[\phi]$.
- Now, let ϕ^{trc} be the truncation of $\phi^\star$ to $\mathcal{E}_{1/8}(w^\star)$, that is

$$\phi^{\mathrm{trc}}(w) = \frac{\exp\left(-\frac{1}{\mu}L(w)\right)}{\int_{\mathcal{E}_{1/8}(w^\star)}\exp\left(-\frac{1}{\mu}L(w')\right)dw'}, \quad w\in\mathcal{E}_{1/8}(w^\star).$$

- By self-concordance, $\phi^{\mathrm{trc}} \in \mathcal{E}_{\frac{1}{2}}(w^{\mathrm{trc}})$ with $w^{\mathrm{trc}} = \mathbb{E}_{\phi^{\mathrm{trc}}}[w]$. Indeed, for $w\in\mathcal{E}_{1/8}(w^\star)$ $\|w-w^{\mathrm{trc}}\|_{\nabla^2 L(w^{\mathrm{trc}})} \leqslant 2\|w-w^{\mathrm{trc}}\|_{\nabla^2 L(w^\star)} \leqslant 2\|w-w^\star\|_{\nabla^2 L(w^\star)} + 2\|w^{\mathrm{trc}}-w^\star\|_{\nabla^2 L(w^\star)} \leqslant \frac{1}{2}$.
  - In other words, ϕ^{trc} is feasible in the restricted minimization problem, and

$$\min_{\substack{\hat{w}\in\Delta_d,\ \mathbb{E}_\phi w = \hat{w},\\ \phi\in\mathrm{Supp}(\mathcal{E}_{1/2}(\hat{w}))}} \mathcal{F}[\phi] \leqslant \mathcal{F}[\phi^{\mathrm{trc}}].$$

  - It remains to show that $\mathcal{F}[\phi^{\mathrm{trc}}]$ is not much larger than $\mathcal{F}[\phi^\star]$.

Dikin approximation accuracy: proof sketch (IV)

Truncation lemma

As long as $\lambda \geqslant 1$, for

$$\phi^\star(w) = \frac{\exp\left(-\frac{1}{\mu}L(w)\right)}{\int_{\Delta_d} \exp\left(-\frac{1}{\mu}L(w')\right) dw'} \quad \text{and} \quad \phi^{\text{trc}}(w) = \frac{\exp\left(-\frac{1}{\mu}L(w)\right)}{\int_{\mathcal{E}_{1/8}(w^\star)} \exp\left(-\frac{1}{\mu}L(w')\right) dw'}$$

we have that

$$\mathcal{F}[\phi^{\text{trc}}] \leqslant \mathcal{F}[\phi^\star] + 1.5\mu(d-1)\log(T+\lambda d) + 3.2\mu(d+1) + 0.1.$$

Proof outline:

- By Gibbs' duality, $\mathcal{F}[\phi^\star]$ and $\mathcal{F}[\phi^{\text{trc}}]$ are the negative log-partition functions:

$$\mathcal{F}[\phi^\star] = -\mu \log\left(\int_{\Delta_d} \exp\left(-\frac{1}{\mu}L(w')\right) dw'\right),$$
$$\mathcal{F}[\phi^{\text{trc}}] = -\mu \log\left(\int_{\mathcal{E}_{1/8}(w^\star)} \exp\left(-\frac{1}{\mu}L(w')\right) dw'\right).$$

- Compare the volumes of Δ_d and $\mathcal{E}_{1/8}(w^\star)$ using the barrier property of $L(\cdot)$.

Open questions and perspectives

- Further acceleration from $O(d^2(T+d))$ to $O(d^3)$?
- Quantum state estimation?
- Other online learning problems with **intractable near-optimal strategies**, e.g. online linear optimization with bandit feedback?
- Applications beyond online learning?

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