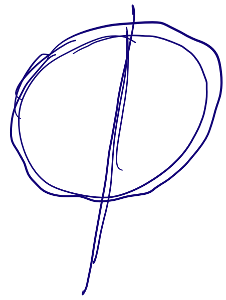
2°

(a) $X \sim \chi^2_2$

pdf: $f(x) = \frac{1}{2} e^{-\frac{x}{2}}$



$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \sim N(0, I_2)$.

$P(W_1^2 + W_2^2 \geq R^2) =$

$$= \frac{1}{2\pi} \int_{\sqrt{w_1^2+w_2^2} \geq R} e^{-\frac{w_1^2+w_2^2}{2}} = \frac{1}{\cancel{2\pi}} \int_R^{+\infty} e^{-\frac{r^2}{2}} \cancel{2\pi}\, r\, dr$$

$$= e^{-\frac{r^2}{2}} \Big|_{+\infty}^{R} = e^{-\frac{R^2}{2}}$$

$$\Rightarrow \boxed{f_2(x) = \frac{1}{2} e^{-\frac{x}{2}}} \quad \boxed{F_2(x) = 1 - e^{-\frac{x}{2}}}$$ for $x \geq 0$.

(b)

$$\boxed{M_2(t) = \begin{cases} \frac{1}{1-2t}, & t < \frac{1}{2} \\ +\infty, & t \geq \frac{1}{2} \end{cases}}$$

Indeed:

$$E[e^{tX}] = \frac{1}{2} \int_0^{+\infty} e^{tx - \frac{x}{2}} dx = \frac{1}{2} \int_0^{+\infty} e^{-(1-2t)\frac{x}{2}} dx$$

$$= \int_{0}^{+\infty} e^{-(1-2t)z}\, dz = \begin{cases} \frac{1}{1-2t} & \text{if } t<\frac{1}{2} \\ +\infty & \text{o.w.} \end{cases}$$

$z = \frac{x}{2}$

(c) $M_{2d}(t) = \mathbb{E}\left[e^{t(X_1+\dots+X_d)}\right] = \left(M_2(t)\right)^d$

$= \frac{1}{(1-2t)^d}$ for $t<\frac{1}{2}$.

(d) Chernoff: then $M_{2d}(t) = \frac{1}{(1-2t)^d}$

$$\mathbb{P}(X \geq x) = \inf_{t \in \mathbb{R}} \mathbb{P}(e^{tX} \geq e^{tx}) \leq$$

$$\inf_{t \in \mathbb{R}} M_{2d}(x) e^{-tx} = \inf_{t<\frac{1}{2}} \frac{e^{-tx}}{(1-2t)^d}$$

$$\log \mathbb{P}(X \geq x) \leq \inf_{t<\frac{1}{2}} \left\{ -d\log(1-2t) - tx \right\}.$$

$$= -\sup_{t<\frac{1}{2}} \Big\{ \underbrace{d\log(1-2t) + tx}_{\text{concave in } t} \Big\}.$$

Unconstrained

$$x = \frac{2d}{1-2t} \qquad 1-2t = \frac{2d}{x}$$

$$\bar{t} = \frac{1}{2} - \frac{d}{x} < \frac{1}{2}.$$

$$\boxed{\bar{t} \geq 0 \text{ for } x \geq 2d}$$

$\Downarrow$ for $x \geq 2d$

$$\log \mathbb{P}(X \geq x) = -\bar{t}x + d\log(1-2\bar{t})$$

$$= -\left(\frac{1}{2} - \frac{d}{x}\right)x + d\log\left(\frac{2d}{x}\right)$$

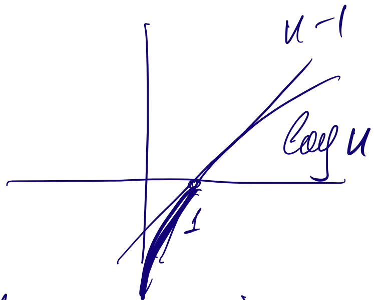
$$= \boxed{d\left(1+\log\left(\frac{2d}{x}\right)\right) - \frac{x}{2}}$$

That is: $\mathbb{P}(X \geq x) = e^{-\frac{x}{2}} e^{d[1+\log(\frac{2d}{x})]}$.

(e) Now: let $z = x - 2d \geq 0$, then $x = 2d + z$.

$$\mathbb{P}(X \geq x) = \mathbb{P}(X \geq 2d+z) = e^{-\left(\frac{2d+z}{2}\right)} e^{d[\cancel{1}+\log(\frac{2d}{2d+z})]}.$$

$$= e^{-\frac{z}{2} + d\log\left(\frac{1}{1+\frac{z}{2d}}\right)}.$$



$\log u \le u-1$, so

$$d\log\left(\frac{1}{1+\frac{z}{2d}}\right) \le d\left(\frac{1}{1+\frac{z}{2d}}-1\right) = -\frac{z}{2+\frac{z}{d}} = -\frac{dz}{2d+z}$$

$$\Rightarrow \mathbb{P}(X \ge 2d+z) \le e^{-\frac{z}{2}-\frac{dz}{2d+z}}$$

$$= e^{-dz\left(\frac{1}{2d}+\frac{1}{2d+z}\right)} = \boxed{e^{-z}}$$

On the other hand: $$e^{-\frac{dz}{2d+z}} \le \begin{cases} e^{-\frac{\min(2d,z)}{4}} \\ e^{-cz} & \text{when } z \le d. \end{cases}$$

$$\frac{dz}{2d+z} \ge \frac{2dz}{4\max(2d,z)} = \frac{\min(2d,z)}{4}$$

$$\log\left(\frac{1}{1+u}\right) \le \frac{1}{1+u}-1 = \frac{-u}{1+u}$$ for all $u \ge 0$.

$$\log\left(\frac{1}{1+u}\right)$$

$$e^{-dz\left(\frac{4d+z}{2d(2d+z)}\right)} = e^{-\frac{z^2}{2(2d+z)}} \cdot e^{-\frac{2dz}{2d+z}}$$

$$\leq e^{-\frac{z^2}{2(2d+z)}} \cdot e^{-\frac{\min\{2d,z\}}{2}}$$

$$\leq e^{-\frac{z^2}{4\max(2d,z)} - \frac{\min\{2d,z\}}{2}}$$

$$= \boxed{\begin{cases} e^{-\frac{z^2}{8d}-\frac{z}{2}} \leq e^{-\frac{z^2}{8d}} & \text{if } z \leq 2d \\ e^{-\frac{z}{4}-d} \leq e^{-\frac{z}{4}} & \text{if } z > 2d \end{cases}}$$

$$e^{-\frac{z^2}{8d}} e^{-\frac{z}{2}} \qquad e^{-\frac{z^2}{8d}} \geq e^{-\frac{z}{4}}$$

$$e^{-\frac{z}{4}} = \delta \iff z = 4\log\frac{1}{\delta} \quad \text{if } z > 2d.$$

$$e^{-\frac{z^2}{8d}} = \delta \iff z = \sqrt{8d\log\frac{1}{\delta}} \quad \text{if } z \leq 2d.$$

$$\begin{cases} e^{-\frac{z}{4}\left(\frac{z}{2d}\right)} & \text{if } z \leq 2d. \\ e^{-\frac{z}{4}} & \text{if } z > 2d. \end{cases}$$

$$\frac{z}{4} \cdot \frac{\min(z, 2d)}{2d} = \log\frac{1}{\delta}$$

$$\bar{z} = \max\left\{4\log\frac{1}{\delta},\ \sqrt{8d\log\frac{1}{\delta}}\right\}.$$