1. (a)

Total: 65/100

~~inf $M_X(\lambda)e^{-\lambda u}$~~

By Talyor expansion

$$\frac{E[e^{\lambda X}]}{e^{\lambda u}} = \frac{\sum_n \frac{\lambda^n E[X^n]}{n!}}{\sum_n \frac{\lambda^n u^n}{n!}} = \lim_{h\to\infty} \frac{1+\lambda E[X]+\cdots+\frac{\lambda^n E[X^n]}{n!}}{1+\lambda u+\cdots+\frac{\lambda^n u^n}{n!}}$$

$\boxed{\text{Cauchy's third inequality}}$ — this is equivalent to the whole problem, so I expect you to prove it. (It's a simple proof, but nonetheless...)

$$\geq \lim_{h\to\infty} \min_{k\leq n} \frac{\lambda^k E[X^k]}{\lambda^k u^k} = \inf_{k\in\mathbb{Z}^+} \frac{E[X^k]}{u^k}$$

+1

(b) If X is symmetric, then

$$\inf_{\lambda>0} M_X(\lambda)e^{-\lambda u} = \frac{1}{2}\left(M_X(\lambda)e^{-\lambda u} + M_{-X}(\lambda)e^{-\lambda u}\right)$$

By the same process as in (a), we have

$$\frac{E[e^{\lambda X}]}{e^{\lambda u}} + \frac{E[e^{-\lambda X}]}{e^{\lambda u}}$$

$$= \frac{\sum_n \frac{\lambda^n E[X^n]}{n!} + \sum_n \frac{\lambda^n (-1)^n E[X^n]}{n!}}{\sum_n \frac{\lambda^n u^n}{n!}} = \lim_{h\to\infty} \frac{2+\cdots+2\lambda^{2n}\frac{E[X^{2n}]}{(2n)!}}{1+\lambda u+\cdots+\frac{\lambda^n u^n}{n!}}$$

$$\geq \lim_{k\to\infty} \min_{n\leq k} \frac{E[X^{2n}]}{u^{2n}} = \frac{1}{2}\inf_{k\in\mathbb{Z}^+} E[X^{2k}]u^{-2k}$$

+.

2. Since

$E(UV) \le (E|U|^p)^{1/p} (E|V|^q)^{\frac{1}{q}}$ with $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$

Let $U = e^{(1-\lambda)t_1X}$, $V = e^{\lambda t_2 X}$, $p = \frac{1}{1-\lambda}$ $q = \frac{1}{\lambda}$, then we have

$$E[e^{(1-\lambda)t_1X} \cdot e^{\lambda t_2 X}] \le E[e^{t_1X}]^{1-\lambda} E[e^{t_2X}]^{\lambda}$$

Take log of both sides, we have

$$K_X(\lambda t_1 + (1-\lambda)t_2) \le (1-\lambda) K_X(t_1) + \lambda K_X(t_2)$$ + Correct!

3.1

3. (a) It's well known that $\phi'(u) + u\phi(u) = 0$

Then we have $\Phi(u) = \int_u^\infty \phi(t)\,dt$

$$= \int_u^\infty -\frac{\phi'(t)}{t}\,dt$$

$$\cancel{\le \frac{1}{u}\int_u^\infty \phi'(t)\,dt = \frac{1}{u}\phi(u)}$$

~~thus the lower upper bound is proved.~~

$$= -\int_u^\infty \frac{\phi(t)}{t^2}\,dt + \left[-\frac{\phi(t)}{t}\right]_u^\infty$$

$$= \frac{\phi(u)}{u} - \int_u^\infty \frac{\phi(t)}{t^2}\,dt$$

$$\le \frac{\phi(u)}{u}$$ +

And $\Phi(u) = \frac{\phi(u)}{u} - \int_u^\infty \frac{\phi(t)}{t^2}\,dt$

$$= \frac{\phi(u)}{u} + \left[\frac{\phi(t)}{t^3}\right]_u^\infty - \int_u^\infty \frac{-3\phi(t)}{t^4}\,dt$$

$$= \frac{\phi(u)}{u} - \frac{\phi(u)}{u^3} + \int_u^\infty \frac{3\phi(t)}{t^4}\,dt \ge \frac{\phi(u)}{u} - \frac{\phi(u)}{u^3}$$

(b) $\Phi(u) = \frac{\phi(u)}{u} - \frac{\phi(u)}{u^3} + \int_u^\infty \frac{3\phi(t)}{t^4}\,dt = \frac{\phi(u)}{u} - \frac{\phi(u)}{u^3} + \frac{3\phi(u)}{u^5} - \int_u^\infty \frac{15\phi(t)}{t^6}$ +

$$\le \frac{\phi(u)}{u} - \frac{\phi(u)}{u^3} + \frac{3\phi(u)}{u^5}$$ +

☑

3.(a)

3.2 Note that

$$\int_0^u e^{-\frac{t^2}{2}} dt = \frac{u}{1\cdot 1} - \frac{u^3}{2\cdot 3} + \frac{u^5}{8\cdot 5} \cdots$$

is that obvious ?!

$$= \sum_{k=0}^{\infty} (-1)^k \frac{u^{2k+1}}{(2k+1) 2^k k!}$$

Thus $\frac{1}{2} - \Phi(u) = \frac{1}{2} - (1 - \int_{-\infty}^{u} \phi(t) dt)$

$$= -\frac{1}{2} + \int_{-\infty}^{0} \phi(t) + \int_0^u \phi(t) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^u e^{-\frac{t^2}{2}} dt$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k+1}}{2^k k! (2k+1)}$$

+/-

4.(i) $E[X] = \cancel{E[X \geq (1-t)E[X]]}\ E[X \mathbb{1}\{X \geq (1-t)E[X]\}] + E[X \mathbb{1}\{X \leq (1-t) E[X]\}]$

$$\leq (1-t)E[X] + E[X^2]^{\frac{1}{2}} P(X \geq (1-t)E[X])^{\frac{1}{2}}$$

$$\Rightarrow P(X \geq (1-t)EX) \geq \frac{t^2 E[X]^2}{E[X^2]}$$

+

~~(ii) Let $Y = X - E[X]$~~

~~Then $E[Y] = 0$ $Var(Y) = E[Y^2] = \sigma^2$ $Var(X)$~~

~~Then $P(Y \geq \theta) = P(Y+u \geq \theta + u)$~~

~~$\leq P((Y+u)^2 \geq (\theta+u)^2)$~~

~~$\leq \frac{E[(Y+u)^2]}{(\theta+u)^2} = \frac{Var(X)+u^2}{(\theta+u)^2}$~~

~~Let $u = \sigma^2/\theta$, we have~~

~~$P(Y \geq \theta) \leq \frac{\sigma^2}{\sigma^2+\theta^2}$~~ ~~$P(X \geq (1-t)E[X])$~~

~~Let $\theta = -tE[X]$, we have $P(Y \geq -tE[X])$~~

1 (a)

(ii) We first prove $P(X-E[X]\geq\theta)\leq\frac{\sigma^2}{\theta^2+\sigma^2}$

let $Y=X-E[X]$, then $E[Y]=0$, $Var(Y)=E[Y^2]=\sigma^2$

So $P(X-E[X]\geq\theta)=P(Y\geq\theta)=P(Y+u\geq\theta+u)$

$$\leq\frac{E[(Y+u)^2]}{(\theta+u)^2}=\frac{\sigma^2+u^2}{(\theta+u)^2}$$

Minimizing over u, we have when $u=\frac{\sigma^2}{\theta}$, we have

$$\boxed{P(X-E[X]\geq\theta)\leq\frac{\sigma^2}{\sigma^2+\theta^2}}$$

this is an upper bound not a lower bound.

~~$\Rightarrow P(X-E[X]\leq\theta)\geq1-\frac{\sigma^2}{\sigma^2+\theta^2}=\frac{\theta^2}{\sigma^2+\theta^2}$~~

~~let $\theta=-E[X]$, we have~~

~~$P(X\leq E[X]+E[X])\geq$~~

which implies the desired inequality.

$$P\{Y\geq-t\mu\}\geq\frac{t^2\mu^2}{t^2\mu^2+\sigma^2}$$

$$P\{Y\geq-\varepsilon\}\geq\frac{\varepsilon^2}{\varepsilon^2+\sigma^2},$$

$$\Downarrow$$

$$P\{Y\leq-\varepsilon\}\leq\frac{\sigma^2}{\varepsilon^2+\sigma^2}$$

(iii) $E[X]=E[X\mathbb{1}\{X\geq(1-t)E[X]\}]+E[X\mathbb{1}\{X\leq(1-t)E[X]\}]$

$$\leq(1-t)E[X]+E[X^p]^{\frac{1}{p}}P(X\geq(1-t)E[X])^{\frac{p-1}{p}}$$

$$\Rightarrow P(X\geq1-t)\geq\left(\frac{tE[X]}{E[|X|^p]^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}=\left(t^p\frac{(E[X])^p}{E[|X|^p]}\right)^{\frac{1}{1-p}}$$

+

5.(a)

$M_2(t) = \mathbb{E}[e^{t(z_1^2+z_2^2)}]$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t(z_1^2+z_2^2)} e^{-\frac{z_1^2}{2}} e^{-\frac{z_2^2}{2}} dz_1 dz_2$

$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{(t-\frac{1}{2})r^2} r dr d\theta$

$= \int_0^{\infty} e^{(t-\frac{1}{2})r^2} r dr d\theta$

$= \begin{cases} \frac{1}{1-2t} & t<\frac{1}{2} \\ \infty & t \geq \frac{1}{2} \end{cases}$

So $M_{2d}(t) = \mathbb{E}[e^{t(z_1^2+\cdots+z_{2d}^2)}]$

$= \mathbb{E}[e^{t(z_1^2+z_2^2)}]^d$

$= \frac{1}{(1-2t)^d}.$

This is unreadable, sorry ...

t.

(b) $\mathbb{P}(X>x)$

$= \mathbb{P}(e^{tX} > e^{tx})$

$\leq \frac{E[e^{tX}]}{e^{tx}}$

$= \frac{e^{-tx}}{(1-2t)^d}$

By taking inf over $t<\frac{1}{2}$

$\mathbb{P}(X>x) \leq \inf_{t<\frac{1}{2}} \frac{e^{-tx}}{(1-2t)^d}$

~~let~~ The optimal $t = \frac{1}{2} + \frac{d}{x}$

We have $\mathbb{P}(X>x) \leq e^{(d\log(\frac{x}{2d}) - \frac{x-2d}{2})}$

+1

6. (a)

~~When $s>1$~~

When $s>1$

$Risk_\mu[\hat{\mu}] = s^2 d > d = Risk_\mu[X]$

So $\hat{\mu}$ is dominated by X.

what about the case $s<0$?

+/-

(b) Consider the following problem:

$$\min_{s} \mathbb{E}_\mu[\|sX-\mu\|^2]$$

By optimality condition, we have

$s^* \mathbb{E}[X^T X] = \mathbb{E}[X^T \mu]$

$s^*(d - \|\mu\|^2 + 2\|\mu\|^2) = \|\mu\|^2$? - am I supposed to read this?

$$\Rightarrow s^* = \frac{\|\mu\|^2}{d+\|\mu\|^2} = 1 - \frac{d}{d+\|\mu\|^2}$$

+.

(c) Note that $\mathbb{E}[\|X\|^2] = d + \|\mu\|^2$

+

So you can replace $d+\|\mu\|^2$ by $\|X\|^2$.

$\hat{\mu}^* = s^* X$ is not proper because μ is unknown.

1. Assume such planar Venn diagram exists.

Note that

$$V_n = V_{n-1} + 2(n-1)$$

$$\begin{cases} E_n = E_{n-1} + \cancel{2(n+1)} \; \cancel{2(n-1)} \; (2n-3)(n-1) \\ F_n = F_{n-1} + 1 + 3\binom{n}{2} \end{cases}$$

??, (wrong)

??, (wrong)

~~clearly $V_n - E_n + F_n + 2$ for $n \geq 1$. if~~

clearly $V_5 - E_5 + F_5 \neq 2$ even if $V_4 - E_4 + F_4 = 2$

The actual relationships are:

$$\begin{cases} V_n = V_{n-1} + 2(n-1) & (1) \\ E_n = E_{n-1} + 4(n-1) & (2) \\ V_n - E_n + F_n = 2 \quad (\text{Euler}) & (3) \end{cases}$$

Here, (2) is because the n-th circle must intersect each of $(n-1)$ previous ones, adding $V_n - V_{n-1} = 2(n-1)$ arcs that arise from the "old" arcs (those on the previous circles) and the same number of arcs on the added circle.