DSCI 501: Homework 2

Matrix transformation

1. Consider a problem of rotating a digital figure by 30 degree anticlockwise. Which of the following matrix would do the transformation?

   (a) $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$

   (b) $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

   (c) $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$

   (d) $\begin{bmatrix} \sqrt{3}/2 & \sqrt{3}/2 \\ 1/2 & 1/2 \end{bmatrix}$

Solving linear equations with matrices

2. Lets say you go shopping every Friday to buy three items (A, B, and C). In week 1, you buy 6 A, 4 B, and 2 C, all for \$56. In week 2, you buy 4 A, 2 B, and 4 C, all for \$46. In week 3, you buy 3 A, 3 B, and 3 C for 45. Assuming a, b, c as the unit cost of A, B, C respectively, construct a matrix and vector for this linear system i.e.,

   $$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = T$$

   (a) $A = \begin{bmatrix} 3 & 3 & 3 \\ 4 & 2 & 4 \\ 6 & 4 & 2 \end{bmatrix}, T = \begin{bmatrix} 56 \\ 46 \\ 45 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 6 & 4 & 2 \\ 4 & 2 & 4 \\ 3 & 3 & 3 \end{bmatrix}, T = \begin{bmatrix} 56 \\ 46 \\ 45 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 4 & 2 \\ 3 & 3 & 3 \end{bmatrix}, T = \begin{bmatrix} 45 \\ 46 \\ 56 \end{bmatrix}$

(d) $A = \begin{bmatrix} 4 & 2 & 4 \\ 6 & 4 & 2 \\ 3 & 3 & 3 \end{bmatrix}, T = \begin{bmatrix} 56 \\ 46 \\ 45 \end{bmatrix}$

3. The above system, $A * r = T$ could be solved using row echelon technique we discussed in class. Do not interchange the rows. What is the echelon form of the system?

(a) $\begin{bmatrix} 2 & 2 & -2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 26 \\ 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ -13 \\ 5 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ -13 \\ 5 \end{bmatrix}$

4. Using back substitution, solve the system of equations. Select the right solution.

(a) $\begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$

5. Now assume that you go and buy the same amount of A, B and C week in and week out and every week you get a new list of totals. Now to solve the system in

general, you will need to find the inverse of matrix A. Use the Gaussian elimination method to find the inverse and verify your solution using numpy in Python. Select the inverse of A from the following:

(a) $\begin{bmatrix} 0.5 & 0.25 & -0.5 \\ 0.5 & -0.5 & 2/3 \\ -0.25 & 0.25 & 1/3 \end{bmatrix}$

(b) $\begin{bmatrix} 0.25 & 0.25 & -0.5 \\ 0 & -0.5 & 2/3 \\ -0.25 & 0.25 & 1/6 \end{bmatrix}$

(c) $\begin{bmatrix} 0.25 & 0.25 & -0.5 \\ 0 & 1 & 2/3 \\ -0.25 & 0.25 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0.25 & 0.25 & -0.5 \\ 0 & -0.5 & 2/3 \\ 1 & 0.25 & 1 \end{bmatrix}$

Matrix multiplication

6. If A ($m \times n$), B ($n \times p$)and C ($n \times p$)are matrices and $AB = AC$. Is it always true that $B = C$.

   (a) Yes

   (b) No

7. The matrix product $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ is equal to

   (a) $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}$

   (c) $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$

   (d) $\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$

8. Let A and B be $n \times n$ matrices with $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. If A and B are upper triangular matrices, then $a_{ik} = 0$ or $b_{kj} = 0$ when

   (a) $k < i$ and $k < j$ only

   (b) $k < i$ and $k > j$ only

   (c) $k > i$ and $k < j$ only

   (d) $k > i$ and $k > j$ only

9. $(ABC)^T$ is equal to

   (a) $A^T B^T C^T$

   (b) $A^T C^T B^T$

   (c) $C^T A^T B^T$

   (d) $C^T B^T A^T$

Orthogonal matrices

A square matrix Q that satisfies

$$Q^{-1} = Q^T$$

is called an orthogonal matrix. Also it follows that $QQ^T = Q^T Q = I$. Rows and columns of orthogonal matrix are an orthonormal basis i.e., they are perpendicular to each other and magnitude of 1. Nice thing about orthogonal matrices are its ease of calculating its inverse. The inverse is just its transpose i.e., $Q^T Q = I$.

10. Which of the following is not orthogonal?

   (a) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

11. Which matrix rotates a 3×1 column vector an angle θ counterclockwise around the $x-$axis?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$

(b) $\begin{bmatrix} sin\theta & 0 & cos\theta \\ 0 & 0 & 0 \\ cos\theta & 1 & -sin\theta \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 1 & sin\theta & cos\theta \end{bmatrix}$

(d) $\begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12. Which of the following transformation matrix will move row one to row two, row two to row three, and row three to row one?

(a) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

13. Which of the following is the LU decomposition of $\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 2 & -10 & 6 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

Determinants

Explore numpy library in python to find determinant of any square matrix.

14. Assume A and B are invertible matrices. Which of the following is false?

    (a) $\det A^{-1} = 1/\det A$
    (b) $\det A^T = \det A$
    (c) $\det (A + B) = \det A + \det B$
    (d) $\det (AB) = \det A * \det B$

Eigen value decomposition

Let A be a square matrix, x is a vector, and λ a scalar. The eigenvalue for A solves

$$Ax = \lambda x$$

for eigenvalues λ_i with corresponding eigenvectors x_i. Eigenvalues can be determined by solving the characteristic equation $\det(A - \lambda I) = 0$ or simply $\lambda^2 - \mathrm{Tr}\, A\lambda + \det A = 0$, where $\mathrm{Tr}\, A$ is the trace which is the sum of the diagonal elements of A.

15. Which of the following is an eigenvector of $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$?

    (a) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} \sqrt{2} \\ 1 \\ \sqrt{2} \end{bmatrix}$

Using the eigen properties, we can do matrix diagonalization. If matrix S represents the matrix with eigen vectors and D represents the Diagonal matrix consisting of eigenvalues of matrix A, we can show that

$$A = SDS^{-1}.$$

Diagonalization of matrices makes computation easy and efficient.

16. Given that $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, Find A^5 using the properties of diagonalization.

(a) $\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$