DSCI 501: Homework 3

Remembering differentiation

1. We will apply some rules we saw in the class to differentiate some functions. Using the power rule, differentiate $f(x) = x^{780}$

   (a) $f^{'}(x) = 779x^{780}$
   (b) $f^{'}(x) = 779x^{779}$
   (c) $f^{'}(x) = 780x^{779}$
   (d) $f^{'}(x) = 781x^{780}$

2. We saw that differentiating sums is the same as differentiating individual terms and summing it up. Differentiate $f(x) = x^3 - 1/x + 4$

   (a) $f^{'}(x) = 3x^2 + 1/x^2 + 4$
   (b) $f^{'}(x) = 3x^2 + 1/x^2$
   (c) $f^{'}(x) = 3x^2 - 1/x^2$
   (d) $f^{'}(x) = 2x^2 - 1/x^2$

3. Find $f^{''}(x)$ for $f(x) = e^x - 3sinx + x^3$.

   (a) $f^{''}(x) = e^x + 3sinx + 6x$
   (b) $f^{''}(x) = e^x + 3cosx + 3x^2$
   (c) $f^{''}(x) = 3sinx + 6x$
   (d) $f^{''}(x) = e^x + 3sinx + 3x^2$

4. Let us now apply the product rule for differentiating functions. Differentiate $f(x) = (x+1)(3x+4)$

   (a) $f^{'}(x) = 3(x+1) + 3x + 4$
   (b) $f^{'}(x) = 6x + 3$
   (c) $f^{'}(x) = 3x + 4$
   (d) $f^{'}(x) = x + 1$

5. Differentiate $f(x) = xe^x cosx$

   (a) $f^{'}(x) = [-xsinx + xcosx + cosx]e^x$
   (b) $f^{'}(x) = [cosx + xcosxe^x - xsinx]e^x$
   (c) $f^{'}(x) = xcosxe^x - xsinxe^x + x$
   (d) $f^{'}(x) = xcosxe^x - xsinxe^x + e^x$

Chain rule

In this section, we will practice the chain rule we discussed in class. Let $g(h)$ and $h(x)$ be two functions. The derivative of g with respect to x is given by $\frac{dg}{dx} = \frac{dg}{dh}\frac{dh}{dx}$.

6. Calculate $f^{'}(x)$ for $f(x) = cos^4 x$. You can either differentiate this directly or can use chain rule.

   (a) $f^{'}(x) = cos^3 x$

   (b) $f^{'}(x) = 4cos^3 x$

   (c) $f^{'}(x) = -4sinxcos^3 x$

   (d) $f^{'}(x) = -4sin^3 x$

7. A little more complex but chain rule should make it straightforward. However, you can do this directly too. $f(x) = e^{sin(x^2)}$

   (a) $f^{'}(x) = 2xe^{sin(x^2)}cos(x^2)$

   (b) $f^{'}(x) = 2xe^{sin(x^2)}$

   (c) $f^{'}(x) = xe^{sin(x^2)}cos(x^2)$

   (d) $f^{'}(x) = e^{sin(x^2)}cos(x^2)$

Differentiation at a given point

You will frequently require to find the differentiation at a given point. You simply need to find $f^{'}(x)$ and evaluate it at the given point. Let us solve some problems.

8. Find $f^{'}(x)$ evaluated at $x = 4$ for the function $f(x) = x^{3/2} + \pi x^2 + 7$

   (a) $3 + 8\pi$

   (b) $3\sqrt{8} + 2\pi$

   (c) $6 + 2\pi$

   (d) $3 + 2\pi + 7$

9. Find $f^{'}(x)$ at the point $x = \pi$ for the function $f(X) = sin(x)e^{cos(x)}$

   (a) $-1/e$

   (b) $1/e^2$

   (c) e

   (d) e^2

   (e) $-e$

Partial differentiation, the Jacobian, and the Hessian

In this section, we will practice some partial differentiation which will then lead to calculating the total derivative. Remember that partial differentiation involves treating every variable that you are not differentiating as constants.

10. Find the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ for $f(x,y,z) = x^2y + y^2z + z^2x$

    (a) $\frac{\partial f}{\partial x} = 2xy + z^2$
    $\frac{\partial f}{\partial y} = x^2 + 2yz$
    $\frac{\partial f}{\partial z} = y^2 + 2zx$

    (b) $\frac{\partial f}{\partial x} = 2y + z$
    $\frac{\partial f}{\partial y} = x + 2yz$
    $\frac{\partial f}{\partial z} = y + 2zx$

    (c) $\frac{\partial f}{\partial x} = 2xy + z^2$
    $\frac{\partial f}{\partial y} = 2x^2 + 2yz$
    $\frac{\partial f}{\partial z} = 2y^2 + 2zx$

    (d) $\frac{\partial f}{\partial x} = 2xy + z^2$
    $\frac{\partial f}{\partial y} = yx^2 + 2yz$
    $\frac{\partial f}{\partial z} = y^2 + 2zx$

11. Total derivative for $f(x,y,z)$ where $x = x(t), y = y(t), z = z(t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

Find the total derivative for

$$f(x,y,z) = cos(x)sin(y)e^{2z}, x(t) = t+1, y(t) = t-1, z(t) = t^2.$$

(a) $\frac{df}{dt} = e^{2t^2}[-sin(t+1)sin(t-1) + cos(t+1)cos(t-1) + 4tcos(t+1)sin(t-1)]$

(b) $\frac{df}{dt} = e^{2t^2}[sin(t+1)sin(t-1) + cos(t+1)cos(t-1) + 4tcos(t+1)sin(t-1)]$

(c) $\frac{df}{dt} = e^{2t^2}[-sin(t+1)sin(t-1) - cos(t+1)cos(t-1) + 4tcos(t+1)sin(t-1)]$

(d) $\frac{df}{dt} = -sin(t+1)sin(t-1) + cos(t+1)cos(t-1) + 4tcos(t+1)sin(t-1)$

12. Calculate the Jacobian row vector for $f(x, y) = 2x^2y + xy + 5$.

    (a) $J = [4x + xy + 5, 2x^2 + y]$
    (b) $J = [4xy + y, 2x^2 + x]$
    (c) $J = [4x + y, 2x^2 + x]$
    (d) $J = [xy + y, x^2 + x]$

13. Calculate the Jacobian row vector for $f(x, y) = x^2 + e^y e^z + cos(x)sin(z)$ at the point $(0, 0, 0)$.

    (a) $J(0, 0, 0) = [2, 2, 1]$
    (b) $J(0, 0, 0) = [0, 1, 2]$
    (c) $J(0, 0, 0) = [2, 1, 0]$
    (d) $J(0, 0, 0) = [1, 2, 1]$

14. Find the Jacobian of the following vector functions at the point $(0, \pi)$

$$u(x, y) = x^2y - cos(x)sin(y)$$

$$v(x, y) = e^{x+y}.$$

    (a) $\begin{bmatrix} e^\pi & 1 \\ 0 & 1 \end{bmatrix}$
    (b) $\begin{bmatrix} e^\pi & 1 \\ 0 & e^\pi \end{bmatrix}$
    (c) $\begin{bmatrix} 0 & 1 \\ e^\pi & e^\pi \end{bmatrix}$
    (d) $\begin{bmatrix} 1 & 0 \\ e^\pi & e^\pi \end{bmatrix}$

15. Find the Hessian for the function $f(x, y, z) = xycos(z) - sin(x)e^y z^3$ and evaluate at $(0, 0, 0)$

    (a) $H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$
    (b) $H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $H = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

16. Given,

$$\begin{aligned} f(x_1, x_2) &= cos(x_1)sin(x_2) \\ x_1(u_1, u_2) &= 2u_1^2 + 3u_2^2 - u_2 \\ x_2(u_1, u_2) &= 2u_1 - 5u_2^3 \\ u_1(t) &= e^{t/2} \\ u_2(t) &= e^{-2t} \end{aligned}$$

Calculate $\frac{df}{dt}$ in matrix form. We can write $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}\frac{\partial u}{\partial t}$

(a) $\begin{bmatrix} -sin(x_1)sin(x_2) & cos(x_1)cos(x_2) \end{bmatrix} \begin{bmatrix} 4u_1 & 6u_2 - 1 \\ 2 & -15u_2^2 \end{bmatrix} \begin{bmatrix} \frac{e^{t/2}}{2} \\ -2e^{-2t} \end{bmatrix}$

(b) $\begin{bmatrix} cos(x_1)cos(x_2) & sin(x_1)sin(x_2) \end{bmatrix} \begin{bmatrix} 4u_1 & 6u_2 - 1 \\ 2 & -15u_2^2 \end{bmatrix} \begin{bmatrix} \frac{e^{t/2}}{2} \\ -2e^{-2t} \end{bmatrix}$

(c) $\begin{bmatrix} cos(x_1)cos(x_2) & sin(x_1)sin(x_2) \end{bmatrix} \begin{bmatrix} 6u_2 - 1 & 4u_1 \\ 2 & -15u_2^2 \end{bmatrix} \begin{bmatrix} \frac{e^{t/2}}{4} \\ -2e^{-2t} \end{bmatrix}$

(d) $\begin{bmatrix} -sin(x_1)sin(x_2) & cos(x_1)cos(x_2) \end{bmatrix} \begin{bmatrix} 6u_2 - 1 & 4u_1 \\ 2 & -15u_2^2 \end{bmatrix} \begin{bmatrix} \frac{e^{t/2}}{2} \\ -2e^{-2t} \end{bmatrix}$