DSCI 501: Homework 1

Remembering simultaneous equations

1. Solving simultaneous equations is the process of finding the values of the variables (here a and b) that satisfy the system of equations. Let's start with the simplest type of simultaneous equation, where we already know all but one of the variables:

$$\begin{aligned} 5a + b &= -8 \\ a &= 2 \end{aligned}$$

   Substitute the value of a into the first equation to find b, then select the correct values of a and b below.

   (a) $a = 2, b = 8$
   (b) $a = 2, b = -10$
   (c) $a = 2, b = 10$
   (d) $a = 2, b = -18$

2. The first goal when solving simple simultaneous equations should be to isolate one of the variables. For example, try taking the second equation away from the first to solve the following pair of equations:

$$\begin{aligned} 6a - 4b &= 14 \\ 4a - 4b &= 4 \end{aligned}$$

   First find a and then substitute a into one of the equations to find b. Select the correct values of a and b below.

   (a) $a = 1, b = 4$
   (b) $a = 5, b = 4$
   (c) $a = 6, b = 6$
   (d) $a = 4, b = 5$

3. Now let us look at elimination method. You can use this method when the coefficients do not match.

   For example, to solve the following problem, multiply both sides of the first equations by 3 and use the earlier method (substitution) to solve for a and b.

$$\begin{aligned} 2a + 3b &= 18 \\ 6a - 4b &= 2 \end{aligned}$$

   (a) $a = 1, b = 4$

(b) $a = 3, b = 4$

(c) $a = 6, b = 6$

(d) $a = 4, b = 5$

4. Systems of simultaneous equations can have more than two unknown variables. Here we have a system with three variables; a, b and c. First try to find one of the variables by elimination or substitution, which will lead to two equations and two unknown variables. Continue the process to find all of the variables. You can generalize this to any number of variables. However, think about how many equations are needed to uniquely determine n variables. We will discuss this when we cover linear independence in later topics.

   Solve for $a, b, andc$

$$3a - 2b + c = 7$$
$$a + b + c = 2$$
$$3a - 2b - c = 3$$

   (a) $a = -1, b = 3, c = 2$

   (b) $a = 1, b = -1, c = 2$

   (c) $a = -1, b = -3, c = 4$

   (d) $a = -1, b = 4, c = 2$

Vector operations

We will explore some basics of vector and vector operations. Figure 1 has five vectors u, v, w, x, y. The grid sizes are of unit length.

5. What is the numerical representation of vector u?

   (a) $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   (c) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

   (d) $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

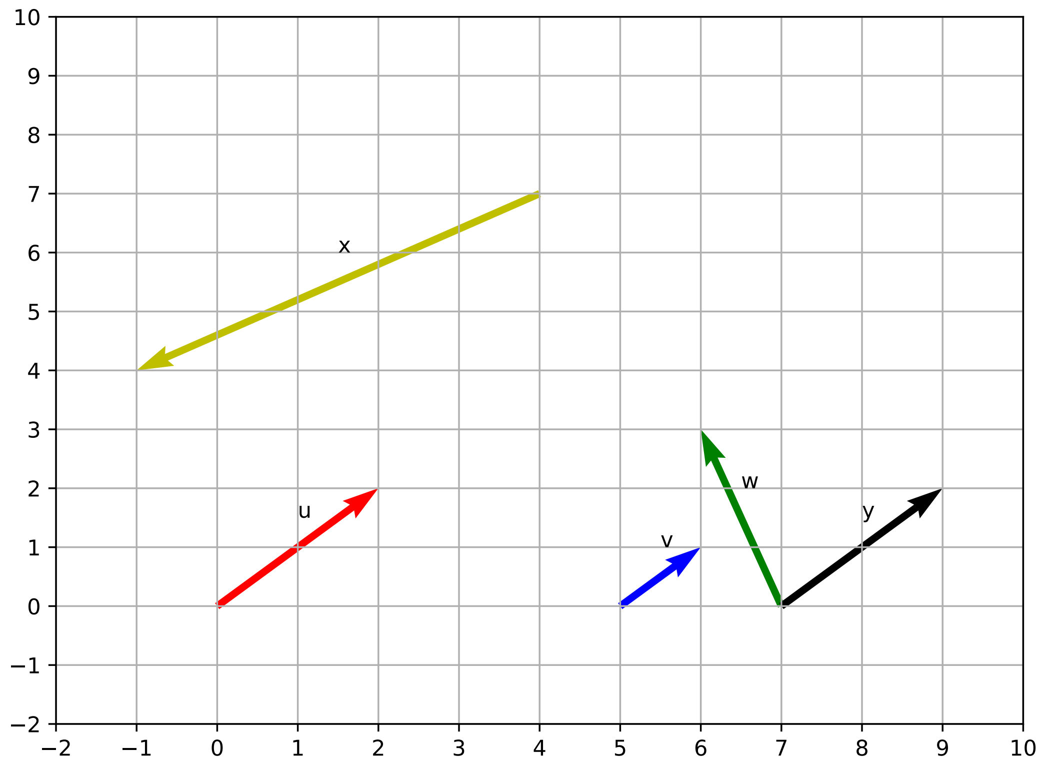

Figure 1: Vectors

6. Select all the vectors in Figure 1 that represent $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

   (a) u
   (b) v
   (c) w
   (d) x
   (e) y

7. Select the vector in Figure 1 that represent $\begin{bmatrix} -5 \\ -3 \end{bmatrix}$

   (a) u
   (b) v

(c) w

(d) x

(e) y

8. Select all the vectors in Figure 1 that are equal to $2v$.

(a) u

(b) v

(c) w

(d) x

(e) y

9. What is the vector $y + w$?

(a) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(d) $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$

10. What is the vector $y - w$?

(a) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$

Vector Dot products

Dot product is one of the most important and useful concept in linear algebra. In the following questions, we will test our understanding of dot product.

The size or the magnitude of a vector with two components is calculated using Pythagoras's theorem. If vector $r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$, then the magnitude or the size of the vector is simply $\sqrt{r_1^2 + r_2^2}$. This can easily be generalized to any size vector.

11. Using the above, calculate the magnitude of vector $s = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \\ \sqrt{6} \end{bmatrix}$

   (a) 6

   (b) 36

   (c) $\sqrt{35}$

   (d) 7

12. Dot product of two vectors with n components is defined as $a.b = a_1b_1 + a_2b_2 + \ldots a_nb_n$. Find the dot product of the vector $r = \begin{bmatrix} 3 \\ 2 \\ 8 \\ -5 \end{bmatrix}$ and $s = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

   (a) $r.s = -1$

   (b) $r.s = 0$

   (c) $r.s = \begin{bmatrix} 6 \\ -2 \\ 0 \\ -5 \end{bmatrix}$

   (d) $r.s = \begin{bmatrix} 5 \\ -1 \\ 8 \\ -4 \end{bmatrix}$

13. Now remember the projections we discussed in class. What is the scalar projection of v on u where $u = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} -6 \\ 5 \\ 10 \end{bmatrix}$

   (a) 1

   (b) 10

   (c) 2

   (d) 3

14. What is the vector projection of v onto u where $u = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} -6 \\ 5 \\ 10 \end{bmatrix}$

   (a) $\begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix}$

   (c) $\begin{bmatrix} 0 \\ -8 \\ 6 \end{bmatrix}$

   (d) $\begin{bmatrix} 0 \\ -8/5 \\ 6/5 \end{bmatrix}$

15. Consider any two vectors u and v. Which of the following is always correct? Verify the result with an example.

   (a) $||u + v|| \leq ||u|| + ||v||$

   (b) $||u + v|| > ||u|| + ||v||$

   (c) $||u + v|| = ||u|| + ||v||$

   (d) $||u + v|| \geq ||u|| + ||v||$

Change of basis

16. In class, we discussed standard basis and in such a standard basis, we have the following vectors: $v = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$, $b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, and $b_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$. Find vector v in the basis of $b_1, b_2, b_3, and b_4$. Hint: b vectors are pairwise orthogonal to each other.

(a) $v_b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(b) $v_b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(c) $v_b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

(d) $v_b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Linear independence

The set of vectors, $\{u_1, u_2, \ldots, u_n\}$ are linearly independent if for any real values $c_1, c_2, \ldots, c_n$, the equation

$$c_1 u_1 + c_2 u_2 + \ldots + c_n u_n = 0$$

has only the solution $c_1 = c_2 = \ldots = c_n = 0$. In other words you cannot write any vector u_i in terms of other vectors with nonzero coefficients.

17. Which set of the following sets of vectors are linearly independent?

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$