ID:82402256 Name:Kei Hirano

Question1

(a) Set V(k(0)) as

$$\begin{aligned} V(k(0)) = \max_{\{c(t)\}} & \int_0^\infty e^{-rt} \frac{c(t)^{1-s}-1}{1-s}\, dt \\ \text{s.t. } & \dot{k}(t) = Ak(t)^\alpha - \delta k(t) - c(t) \\ & k(t) > 0, c(t) \geq 0 \\ & k(0) : given \end{aligned}$$

Consider an small interval of time Δt and we can rewrite V(k(0)) as follows,

$$\begin{aligned} V(k(0)) = \max_{\{c(t)\}} & \left\{ \int_0^{\Delta t} e^{-rt} \frac{c(t)^{1-s}-1}{1-s}\, dt + \int_{\Delta t}^\infty e^{-rt} \frac{c(t)^{1-s}-1}{1-s}\, dt \right\} \\ \text{s.t. } & \dot{k}(t) = Ak(t)^\alpha - \delta k(t) - c(t) \\ & k(t) > 0, c(t) \geq 0 \\ & k(0) : given \end{aligned}$$

Here, from Taylor series around t=0,

$$\begin{aligned} \int_0^{\Delta t} e^{-rt} \frac{c(t)^{1-s}-1}{1-s}\, dt &= \int_0^{\Delta t} (1 - rt + o(t)) \left(\frac{c(0)^{1-s}-1}{1-s} + c(0)^{-s}\dot{c}(0)t + o(t) \right) dt \\ &= \frac{c(0)^{1-s}-1}{1-s}\Delta t + c(0)^{-s}\dot{c}(0)(\Delta t)^2 - \frac{1}{2}\frac{c(0)^{1-s}-1}{1-s} r(\Delta t)^2 \\ &\quad - \frac{1}{3} rc(0)^{-s}\dot{c}(0)(\Delta t)^3 + \int_0^{\Delta t} \left(1 - rt + \frac{c(0)^{1-s}-1}{1-s} + c(0)^{-s}\dot{c}(0)t + o(t)\right) o(t)\, dt \\ &= \frac{c(0)^{1-s}-1}{1-s}\Delta t + o(\Delta t) \quad \left(\because \forall a \in \mathbb{R}, \left| \int_0^{\Delta t} ao(t)dt \right| \leq \int_0^{\Delta t} 2tdt = (\Delta t)^2 = o(\Delta t) \right) \end{aligned}$$

And also,

$$\int_{\Delta t}^\infty e^{-rt} \frac{c(t)^{1-s}-1}{1-s}\, dt = e^{-r\Delta t} \int_{\Delta t}^\infty e^{-r(t-\Delta t)} \frac{c(t)^{1-s}-1}{1-s}\, dt$$

Thus we have

$$\begin{aligned} V(k(0)) = \max_{\{c(t)\}} & \left\{ \frac{c(0)^{1-s}-1}{1-s}\Delta t + o(\Delta t) + e^{-r\Delta t} \int_{\Delta t}^\infty e^{-r(t-\Delta t)} \frac{c(t)^{1-s}-1}{1-s}\, dt \right\} \\ \text{s.t. } & \dot{k}(t) = Ak(t)^\alpha - \delta k(t) - c(t) \\ & k(t) > 0, c(t) \geq 0 \\ & k(0) : given \end{aligned}$$

$$
\begin{aligned}
&= \max_{c(0)} \left\{ \frac{c(0)^{1-s}-1}{1-s}\Delta t + o(\Delta t) + e^{-r\Delta t} V(k(\Delta t)) \right\} \\
&\quad \text{s.t. } \dot{k}(t) = Ak(t)^{\alpha} - \delta k(t) - c(t) \\
&\qquad c(0) \geq 0 \\
&\qquad k(0) : given \\
\Leftrightarrow e^{r\Delta t} V(k(0)) &= \max_{c(0)} \left\{ \frac{c(0)^{1-s}-1}{1-s} e^{r\Delta t}\Delta t + o(\Delta t) + V(k(\Delta t)) \right\} \\
&\quad \text{s.t. } \dot{k}(t) = Ak(t)^{\alpha} - \delta k(t) - c(t) \\
&\qquad c(0) \geq 0 \\
&\qquad k(0) : given \\
\Leftrightarrow (1 + r\Delta t + o(\Delta t))V(k(0)) &= \max_{c(0)} \left\{ \frac{c(0)^{1-s}-1}{1-s} (1 + r\Delta t + o(\Delta t))\Delta t + o(\Delta t) + V(k(\Delta t)) \right\} \\
&\quad \text{s.t. } \dot{k}(t) = Ak(t)^{\alpha} - \delta k(t) - c(t) \\
&\qquad c(0) \geq 0 \\
&\qquad k(0) : given \\
\Leftrightarrow r\Delta t V(k(0)) &= \max_{c(0)} \left\{ \frac{c(0)^{1-s}-1}{1-s}\Delta t + o(\Delta t) + V(k(\Delta t)) - V(k(0)) \right\} \\
&\quad \text{s.t. } \dot{k}(t) = Ak(t)^{\alpha} - \delta k(t) - c(t) \\
&\qquad c(0) \geq 0 \\
&\qquad k(0) : given
\end{aligned}
$$

Dividing both sides by Δt,

$$
\begin{aligned}
rV(k(0)) &= \max_{c(0)} \left\{ \frac{c(0)^{1-s}-1}{1-s} + \frac{o(\Delta t)}{\Delta t} + \frac{V(k(\Delta t)) - V(k(0))}{\Delta t} \right\} \\
&\quad \text{s.t. } \dot{k}(t) = Ak(t)^{\alpha} - \delta k(t) - c(t) \\
&\qquad c(0) \geq 0 \\
&\qquad k(0) : given
\end{aligned}
$$

Taking the limit Δt to zero,

$$
\begin{aligned}
rV(k(0)) &= \max_{c(0)} \left\{ \frac{c(0)^{1-s}-1}{1-s} + V'(k(0))\dot{k}(0) \right\} \\
&\quad \text{s.t. } \dot{k}(t) = Ak(t)^{\alpha} - \delta k(t) - c(t) \\
&\qquad c(0) \geq 0 \\
&\qquad k(0) : given \\
\Leftrightarrow rV(k(0)) &= \max_{c(0)} \left\{ \frac{c(0)^{1-s}-1}{1-s} + V'(k(0))(Ak(0)^{\alpha} - \delta k(0) - c(0)) \right\} \\
&\quad \text{s.t. } c(0) \geq 0 \\
&\qquad k(0) : given
\end{aligned}
$$

Therefore,

$$rV(k) = \max_c \left\{ \frac{c^{1-s}-1}{1-s} + V'(k)(Ak^\alpha - \delta k - c) \right\} \quad \square$$

(b) From the first-order condition,

$$c^{-s} = V'(k)$$

Thus, optimal consumption is

$$c = V'(k)^{-\frac{1}{s}} \quad \square$$

Question2

(a) We compute the *forward difference* as

$$dV_{i,f} = \frac{V(k_{i+1}) - V(k_i)}{k_{i+1} - k_i}$$

and the *backward difference* as

$$dV_{i,b} = \frac{V(k_i) - V(k_{i-1})}{k_i - k_{i-1}}.$$

Then we can calculate $c_{i,f}$ and $c_{i,b}$ from the first-order condition, $c = (V'(k))^{-\frac{1}{s}}$. And from $c_{i,f}$ and $c_{i,b}$, we can also calculate $\mu_{i,f}$ and $\mu_{i,b}$, from the constraint, as

$$\mu_{i,f} = Ak_i^\alpha - \delta k_i - c_{i,f}, \quad \mu_{i,b} = Ak_i^\alpha - \delta k_i - c_{i,b}.$$

Because $\dfrac{c(t)^{1-s}-1}{1-s}$ is increasing and concave, we have

$$dV_{i,f} \leq dV_{i,b} \Leftrightarrow c_{i,b} \leq c_{i,f} \Leftrightarrow \mu_{i,f} \leq \mu_{i,b}.$$

Thus, if $\mu_{i,f} \leq \mu_{i,b} < 0$, we set $V'(k_i) = dV_{i,b}$ because k will decrease. If $\mu_{i,f} \leq 0 \leq \mu_{i,b}$, we set $V'(k_i) = (Ak_i^\alpha - \delta k_i)^{-s}$ because in this case, we can think k_i is almost at the steady state, or $\dot{k}_i = 0$. If $0 < \mu_{i,f} \leq \mu_{i,b}$, we set $V'(k_i) = dV_{i,f}$ because k will increase. Thus, we can get V'(k) for all $k \in [k_{\min}, k_{\max}]$ and this is so-called upwind scheme. $\square$
(b) The implicit scheme starts with

$$\frac{V^{n+1}(k_i) - V^n(k_i)}{\Delta} + rV^{n+1}(k_i) = \frac{(c_i^n)^{1-s}-1}{1-s} + (V^{n+1}(k_i))'(Ak_i^\alpha - \delta k_i - c_i^n)$$

where n is the value after n iteration and $c_i^n = (V^n(k_i))'^{-\frac{1}{s}}$.
From the upwind scheme, we can rewrite this equation as

$$\frac{V^{n+1}(k_i) - V^n(k_i)}{\Delta} + rV^{n+1}(k_i) = \frac{(c_i^n)^{1-s}-1}{1-s} + dV_{i,f}^{n+1}(Ak_i^\alpha - \delta k_i - c_{i,f}^n)^+ + dV_{i,b}^{n+1}(Ak_i^\alpha - \delta k_i - c_{i,b}^n)^-$$

$$
\begin{aligned}
&= \frac{(c_i^n)^{1-s}-1}{1-s} + \frac{V^{n+1}(k_{i+1}) - V^{n+1}(k_i)}{\Delta k}(\mu_{i,f}^n)^+ \\
&+ \frac{V^{n+1}(k_i) - V^{n+1}(k_{i-1})}{\Delta k}(\mu_{i,b}^n)^- \\
&= \frac{(c_i^n)^{1-s}-1}{1-s} + x_i^n V^{n+1}(k_{i-1}) + y_i^n V^{n+1}(k_i) + z_i^n V^{n+1}(k_{i+1}) \qquad (1)
\end{aligned}
$$

where

$$
\begin{aligned}
x_i^n &= -\frac{\min(\mu_{i,b}^n, 0)}{\Delta k} \\
y_i^n &= -\frac{\max(\mu_{i,f}^n, 0)}{\Delta k} + \frac{\min(\mu_b^n, 0)}{\Delta k} \\
z_i^n &= \frac{\max(\mu_{i,f}^n, 0)}{\Delta k}
\end{aligned}
$$

Since $V^{n+1}(k)$ is a I dimensional vector, we can rewrite equation (1) into a matrix form,

$$
\left[\left(\frac{1}{\Delta} + r\right) I - P^n\right] V^{n+1}(k) = U^n + \frac{V^n(k)}{\Delta}
$$

where

$$
P^n = \begin{pmatrix}
y_1^n & z_1^n & 0 & \cdots & 0 \\
x_2^n & y_2^n & z_2^n & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & x_{I-1}^n & y_{I-1}^n & z_{I-1}^n \\
0 & \cdots & 0 & x_I^n & y_I^n
\end{pmatrix}, \quad U^n = \begin{pmatrix}
\frac{(c_1^n)^{1-s}-1}{1-s} \\
\vdots \\
\frac{(c_I^n)^{1-s}-1}{1-s}
\end{pmatrix}
$$

Then we can get

$$
V^{n+1}(k) = \left[\left(\frac{1}{\Delta} + r\right) I - P^n\right]^{-1} \left(U^n + \frac{V^n(k)}{\Delta}\right)
$$

QuestionB

(1) Set V(a(0)) as

$$
\begin{aligned}
V(a(0)) = \max_{\{c(t)\}} &\int_0^\infty e^{-rt} \frac{c(t)^{1-s}-1}{1-s}\, dt \\
\text{s.t. } &\dot{a}(t) = \omega + Ra(t) - c(t) \\
&a(0) : given
\end{aligned}
$$

Consider an small interval of time Δt and we can rewrite $V(a(0))$ as follows,

$$
V(a(0)) = \max_{\{c(t)\}} \left\{ \int_0^{\Delta t} e^{-rt} \frac{c(t)^{1-s}-1}{1-s}\, dt + \int_{\Delta t}^\infty e^{-rt} \frac{c(t)^{1-s}-1}{1-s}\, dt \right\}
$$

$$\text{s.t. } \dot{a}(t) = \omega + Ra(t) - c(t)$$
$$a(0) : given$$

The following results can be obtained by doing the same thing as Question1-(a).

$$
\begin{aligned}
V(a(0)) = \max_{c(0)} &\left\{ \frac{c(t)^{1-s}-1}{1-s}\Delta t + o(\Delta t) + e^{-r\Delta t}V(a(\Delta t)) \right\} \\
\text{s.t. } &\dot{a}(t) = \omega + Ra(t) - c(t) \\
&a(0) : given
\end{aligned}
$$

Multiplying both sides by $e^{r\Delta t}$,

$$
\begin{aligned}
e^{r\Delta t}V(a(0)) = \max_{c(0)} &\left\{ \frac{c(t)^{1-s}-1}{1-s}\Delta t e^{r\Delta t} + o(\Delta t) + V(a(\Delta t)) \right\} \\
\text{s.t. } &\dot{a}(t) = \omega + Ra(t) - c(t) \\
&a(0) : given \\
\Leftrightarrow (1 + r\Delta t + o(\Delta t))V(a(0)) = \max_{c(0)} &\left\{ \frac{c(t)^{1-s}-1}{1-s}\Delta t(1 + r\Delta t + o(\Delta t)) + o(\Delta t) + V(a(\Delta t)) \right\} \\
\text{s.t. } &\dot{a}(t) = \omega + Ra(t) - c(t) \\
&a(0) : given \\
\Leftrightarrow r\Delta t V(a(0)) = \max_{c(0)} &\left\{ \frac{c(t)^{1-s}-1}{1-s}\Delta t + o(\Delta t) + V(a(\Delta t)) - V(a(0)) \right\} \\
\text{s.t. } &\dot{a}(t) = \omega + Ra(t) - c(t) \\
&a(0) : given \\
\Leftrightarrow rV(a(0)) = \max_{c(0)} &\left\{ \frac{c(t)^{1-s}-1}{1-s} + \frac{o(\Delta t)}{\Delta t} + \frac{V(a(\Delta t)) - V(a(0))}{\Delta t} \right\} \\
\text{s.t. } &\dot{a}(t) = \omega + Ra(t) - c(t) \\
&a(0) : given
\end{aligned}
$$

Taking the limit Δt to zero,

$$
\begin{aligned}
rV(a(0)) = \max_{c(0)} &\left\{ \frac{c(t)^{1-s}-1}{1-s} + V'(a(0))\dot{a}(0) \right\} \\
\text{s.t. } &\dot{a}(t) = \omega + Ra(t) - c(t) \\
&a(0) : given \\
\Leftrightarrow rV(a(0)) = \max_{c(0)} &\left\{ \frac{c(t)^{1-s}-1}{1-s} + V'(a(0))(\omega + Ra(0) - c(0)) \right\} \\
\text{s.t. } &a(0) : given
\end{aligned}
$$

Therefore,

$$rV(a) = \max_{c} \left\{ \frac{c^{1-s}-1}{1-s} + V'(a)(\omega + Ra - c) \right\} \quad \square$$

(2) From the first-order condition

$$c^{-s} = V'(a)$$

Thus, optimal consumption is

$$c = V'(a)^{-\frac{1}{s}} \quad \square$$

(3)
We compute the *forward difference* as

$$dV_{i,f} = \frac{V(a_{i+1}) - V(a_i)}{a_{i+1} - a_i}$$

and the *backward difference* as

$$dV_{i,b} = \frac{V(a_i) - V(a_{i-1})}{a_i - a_{i-1}}.$$

Then we can calculate $c_{i,f}$ and $c_{i,b}$ from the first-order condition, $c = (V'(a))^{-\frac{1}{s}}$. And from $c_{i,f}$ and $c_{i,b}$, we can also calculate μ_f and μ_b, from the constraint, as

$$\mu_{i,f} = \omega + Ra_i - c_{i,f}, \quad \mu_{i,b} = \omega + Ra_i - c_{i,b}.$$

Because $\dfrac{c(t)^{1-s} - 1}{1-s}$ is increasing and concave, we have

$$dV_{i,f} \leq dV_{i,b} \Leftrightarrow c_{i,b} \leq c_{i,f} \Leftrightarrow \mu_{i,f} \leq \mu_{i,b}.$$

Thus, if $\mu_{i,f} \leq \mu_{i,b} < 0$, we set $V'(a_i) = dV_{i,b}$ because a will decrease. If $\mu_{i,f} \leq 0 \leq \mu_{i,b}$, we set $V'(a_i) = (\omega + Ra_i)^{-s}$ because in this case, we can think a_i is almost at the steady state, or $\dot{a}_i = 0$. If $0 < \mu_{i,f} \leq \mu_{i,b}$, we set $V'(a_i) = dV_{i,f}$ because a will increase. Thus, we can get $V'(a)$ for all $a \in [a_{\min}, a_{\max}]$ and this is so-called upwind scheme. $\square$

(4) The implicit scheme starts with

$$\frac{V^{n+1}(a_i) - V^n(a_i)}{\Delta} + rV^{n+1}(a_i) = \frac{(c_i^n)^{1-s} - 1}{1-s} + (V^{n+1}(a_i))'(\omega + Ra_i - c_i^n)$$

where n is the value after n iteration and $c_i^n = (V^n(a_i))'^{-\frac{1}{s}}$.
From the upwind scheme, we can rewrite this equation as

$$\begin{aligned}
\frac{V^{n+1}(a_i) - V^n(a_i)}{\Delta} + rV^{n+1}(a_i) =& \frac{(c_i^n)^{1-s} - 1}{1-s} + dV_{i,f}^{n+1}(\omega + Ra_i - c_{i,f}^n)^+ + dV_{i,b}^{n+1}(\omega + Ra_i - c_{i,b}^n)^- \\
=& \frac{(c_i^n)^{1-s} - 1}{1-s} + \frac{V^{n+1}(a_{i+1}) - V^{n+1}(a_i)}{\Delta a}(\mu_{i,f}^n)^+ \\
&+ \frac{V^{n+1}(a_i) - V^{n+1}(a_{i-1})}{\Delta a}(\mu_{i,b}^n)^-
\end{aligned}$$

$$=\frac{(c_i^n)^{1-s}-1}{1-s}+x_i^n V^{n+1}(a_{i-1})+y_i^n V^{n+1}(a_i)+z_i^n V^{n+1}(a_{i+1}) \tag{2}$$

where

$$\begin{aligned}
x_i^n &= -\frac{\min(\mu_{i,b}^n,0)}{\Delta a}\\
y_i^n &= -\frac{\max(\mu_{i,f}^n,0)}{\Delta a}+\frac{\min(\mu_{i,b}^n,0)}{\Delta a}\\
z_i^n &= \frac{\max(\mu_{i,f}^n,0)}{\Delta a}
\end{aligned}$$

Since $V^{n+1}(a)$ is a I dimensional vector, we can rewrite equation (2) into a matrix form,

$$\left[\left(\frac{1}{\Delta}+r\right)I-P^n\right]V^{n+1}(a)=U^n+\frac{V^n(a)}{\Delta}$$

where

$$P^n=\begin{pmatrix} y_1^n & z_1^n & 0 & \cdots & 0\\ x_2^n & y_2^n & z_2^n & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & x_{I-1}^n & y_{I-1}^n & z_{I-1}^n\\ 0 & \cdots & 0 & x_I^n & y_I^n \end{pmatrix}, \quad U^n=\begin{pmatrix} \frac{(c_1^n)^{1-s}-1}{1-s}\\ \vdots \\ \frac{(c_I^n)^{1-s}-1}{1-s}\end{pmatrix}$$

Then we can get

$$V^{n+1}(a)=\left[\left(\frac{1}{\Delta}+r\right)I-P^n\right]^{-1}\left(U^n+\frac{V^n(a)}{\Delta}\right)$$

QuestionC

(1) Set $V(K_0)$ as

$$\begin{aligned}
V(K_0) = \max_{\{I(t)\}} &\int_0^\infty e^{-rt}[F(K(t))-\Psi(I(t),K(t))]\,dt\\
\text{s.t. } &\dot{K}(t)=I(t)-\delta K(t)\\
&K(0)=K_0
\end{aligned}$$

Consider an small interval of time Δt and we can rewrite $V(K_0)$ as follows,

$$\begin{aligned}
V(K_0)=\max_{\{I(t)\}} &\left\{\int_0^{\Delta t} e^{-rt}[F(K(t))-\Psi(I(t),K(t))]\,dt+\int_{\Delta t}^\infty e^{-rt}[F(K(t))-\Psi(I(t),K(t))]\,dt\right\}\\
\text{s.t. } &\dot{K}(t)=I(t)-\delta K(t)
\end{aligned}$$

$$K(0) = K_0$$

Here, from Taylor series around t=0,

$$\begin{aligned}
&\int_0^{\Delta t} e^{-rt}[F(K(t)) - \Psi(I(t), K(t))]\, dt \\
=&\int_0^{\Delta t} (1 - rt + o(t)) \left(F(K_0) + F'(K_0)t - \Psi(I(0), K_0) - \frac{\partial \Psi(I(0), K_0)}{\partial I} \dot{I}(0)t - \frac{\partial \Psi(I(0), K_0)}{\partial K} \dot{K}(0)t + o(t) \right) dt \\
=&(F'(K_0) - \Psi(I(0), K_0))\Delta t + o(\Delta t)
\end{aligned}$$

And also,

$$\int_{\Delta t}^{\infty} e^{-rt}[F(K(t)) - \Psi(I(t), K(t))]\, dt = e^{-r\Delta t} \int_{\Delta t}^{\infty} e^{-r(t+\Delta t)}[F(K(t)) - \Psi(I(t), K(t))]\, dt$$

Thus we have

$$\begin{aligned}
V(K_0) &= \max_{\{I(t)\}} \Bigg\{ (F(K_0) - \Psi(I(0), K_0))\Delta t + o(\Delta t) \\
&\qquad + e^{-r\Delta t} \int_{\Delta t}^{\infty} e^{-r(t+\Delta t)}[F(K(t)) - \Psi(I(t), K(t))]\, dt \Bigg\} \\
&\quad \text{s.t. } \dot{K}(t) = I(t) - \delta K(t) \\
&\quad K(0) = K_0 \\
&= \max_{I(0)} \left\{ (F(K_0) - \Psi(I(0), K_0))\Delta t + o(\Delta t) + e^{-r\Delta t} V(K(\Delta t)) \right\} \\
&\quad \text{s.t. } \dot{K}(t) = I(t) - \delta K(t) \\
&\quad K(0) = K_0 \\
\Leftrightarrow e^{r\Delta t} V(K_0) &= \max_{I(0)} \left\{ (F(K_0) - \Psi(I(0), K_0)) e^{r\Delta t} \Delta t + o(\Delta t) + V(K(\Delta t)) \right\} \\
&\quad \text{s.t. } \dot{K}(t) = I(t) - \delta K(t) \\
&\quad K(0) = K_0 \\
\Leftrightarrow (1 + r\Delta t + o(\Delta t)) V(K_0) &= \max_{I(0)} \{ (F(K_0) - \Psi(I(0), K_0))\Delta t + o(\Delta t) + V(K(\Delta t)) \} \\
&\quad \text{s.t. } \dot{K}(t) = I(t) - \delta K(t) \\
&\quad K(0) = K_0 \\
\Leftrightarrow rV(K_0) &= \max_{I(0)} \left\{ (F(K_0) - \Psi(I(0), K_0)) + \frac{o(\Delta t)}{\Delta t} + \frac{V(K(\Delta t)) - V(K_0)}{\Delta t} \right\} \\
&\quad \text{s.t. } \dot{K}(t) = I(t) - \delta K(t) \\
&\quad K(0) = K_0
\end{aligned}$$

Taking the limit Δt to 0,

$$rV(K_0) = \max_{I(0)} \left\{ (F(K_0) - \Psi(I(0), K_0)) + V'(K_0) \dot{K}(0) \right\}$$

$$
\begin{aligned}
&\text{s.t. } \dot{K}(t) = I(t) - \delta K(t) \\
&\quad K(0) = K_0 \\
= &\max_{I(0)} \{F(K_0) - \Psi(I(0), K_0) + V'(K_0)(I(0) - \delta K_0)\} \\
&\text{s.t. } K(0) = K_0
\end{aligned}
$$

Therefore,

$$
rV(K) = \max_I \{F(K) - \Psi(I, K) + V'(K)(I - \delta K)\} \quad \square
$$

(2) From the first-order condition

$$
\frac{\partial \Psi(I, K)}{\partial I} = V'(K) \quad \square
$$

Economic interpletation: This means the addition of initial capital K requires the addition of cost adjustment via investment I.

(3) We compute the *forward difference* as

$$
dV_{i,f} = \frac{V(K_{i+1}) - V(K_i)}{K_{i+1} - K_i}
$$

and the *backward difference* as

$$
dV_{i,b} = \frac{V(K_i) - V(K_{i-1})}{K_i - K_{i-1}}.
$$

Here, we assume that we can calculate $I_{i,f}$ and $I_{i,b}$ from the first-order condition, $\frac{\partial \Psi(I, K)}{\partial I} = V'(K)$. (I mean, if $\Psi(I, K) = I + K$, we cannot calculate $I_{i,f}$ and $I_{i,b}$ and solution doesn't exist.) And from $I_{i,f}$ and $I_{i,b}$, we can also calculate μ_f and μ_b, from the constraint, as

$$
\mu_{i,f} = I_{i,f} - \delta K_i, \quad \mu_{i,b} = I_{i,b} - \delta K_i.
$$

Here, we also assume that $F(K) - \Psi(I, K)$ is increasing and concave function for I and K. Then we have

$$
dV_{i,f} \leq dV_{i,b} \Leftrightarrow I_{i,b} \leq I_{i,f} \Leftrightarrow \mu_{i,f} \leq \mu_{i,b}.
$$

Thus, if $\mu_{i,f} \leq \mu_{i,b} < 0$, we set $V'(K_i) = dV_{i,b}$ because K will decrease. If $\mu_{i,f} \leq 0 \leq \mu_{i,b}$, we set $V'(K_i) = \frac{\partial \Psi(\delta K_i, K_i)}{\partial I}$ because in this case, we can think K_i is almost at the steady state, or $\dot{K}_i = 0$. If $0 < \mu_{i,f} \leq \mu_{i,b}$, we set $V'(K_i) = dV_{i,f}$ because K will increase. Thus, we can get $V'(K)$ for all $a \in [a_{\min}, a_{\max}]$ and this is so-called upwind scheme. $\square$

(4) The implicit scheme starts with

$$
\frac{V^{n+1}(K_i) - V^n(K_i)}{\Delta} + rV^{n+1}(K_i) = F(K_i) - \Psi(I_i^n, K_i) + (V^{n+1}(K_i))'(I_i^n - \delta K_i)
$$

where n is the value after n iteration and I_i^n is the solution of $\dfrac{\partial \Psi(I_i^n, K_i)}{\partial I} = (V^n(K_i))'$.
From the upwind scheme, we can rewrite this equation as

$$\begin{aligned}\frac{V^{n+1}(K_i) - V^n(K_i)}{\Delta} + rV^{n+1}(K_i) =& F(K_i) - \Psi(I_i^n, K_i) + dV_{i,f}^{n+1}(I_{i,f}^n - \delta K_i)^+ + dV_{i,b}^{n+1}(I_{b,f}^n - \delta K_i)^- \\ =& F(K_i) - \Psi(I_i^n, K_i) + \frac{V^{n+1}(K_{i+1}) - V^{n+1}(K_i)}{\Delta K}(\mu_{i,f}^n)^+ \\ &+ \frac{V^{n+1}(K_i) - V^{n+1}(K_{i-1})}{\Delta K}(\mu_{i,b}^n)^- \\ =& F(K_i) - \Psi(I_i^n, K_i) + x_i^n V^{n+1}(K_{i-1}) + y_i^n V^{n+1}(K_i) + z_i^n V^{n+1}(K_{i+1})\end{aligned}$$

where

$$\begin{aligned}x_i^n &= -\frac{\min(\mu_{i,b}^n, 0)}{\Delta K} \\ y_i^n &= -\frac{\max(\mu_{i,f}^n, 0)}{\Delta K} + \frac{\min(\mu_{i,b}^n, 0)}{\Delta K} \\ z_i^n &= \frac{\max(\mu_{i,f}^n, 0)}{\Delta K}\end{aligned}$$

Since $V^{n+1}(K)$ is a L dimensional vector, we can rewrite equation (3) into a matrix form,

$$\left[\left(\frac{1}{\Delta} + r\right) I - P^n\right] V^{n+1}(K) = U^n + \frac{V^n(K)}{\Delta}$$

where

$$P^n = \begin{pmatrix} y_1^n & z_1^n & 0 & \cdots & 0 \\ x_2^n & y_2^n & z_2^n & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & x_{L-1}^n & y_{L-1}^n & z_{L-1}^n \\ 0 & \cdots & 0 & x_L^n & y_L^n \end{pmatrix}, \quad U^n = \begin{pmatrix} F(K_L) - \Psi(I_L^n, K_L) \\ \vdots \\ F(K_L) - \Psi(I_L^n, K_L) \end{pmatrix}$$

Then we can get

$$V^{n+1}(a) = \left[\left(\frac{1}{\Delta} + r\right) I - P^n\right]^{-1} \left(U^n + \frac{V^n(K)}{\Delta}\right)$$