"Fourier took a prominent part at his home in promoting the Revolution."

Florian Cajori, A History of Mathematics (1893)

3

Fourier Series Part I: Introductory Concepts

Fourier series are one of the most widely used and powerful tools in all of applied mathematics, with applications that range from signal analysis to compression algorithms for making smaller computer file sizes. Before investigating Fourier series in detail, it will be useful to revisit some basic concepts about infinite series in general.

3.1 Terminology

In the study of Fourier series, a number of new concepts arise. For convenience, some of the more important definitions and vocabulary are summarized here.

- **Fourier series.** A series expressed in terms of a weighted sum of sine or cosine functions (or both). The sine and cosine functions are of increasing frequency, and the weight functions represent the corresponding amplitude for that function. As an example, a general Fourier sine series on $x \in [0,1]$ is given in terms of the amplitudes B_n and frequencies (modulated by the integer n) as follows

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(nPix)$$

- **Amplitude.** The amplitude of a periodic function is the maximum value of its vertical magnitude. For a sine function in the form $B\sin x$, the coefficient B represents its amplitude. For example, if $B = 2$, then the function oscillates between ± 2 with changing x.

- **Period.** A periodic function, f, is said to have a period P_λ if $f(x) = f(x+nP)$ for all positive integers n. In other words, the period is the distance (space or time) along the x-axis that defines the repeating unit

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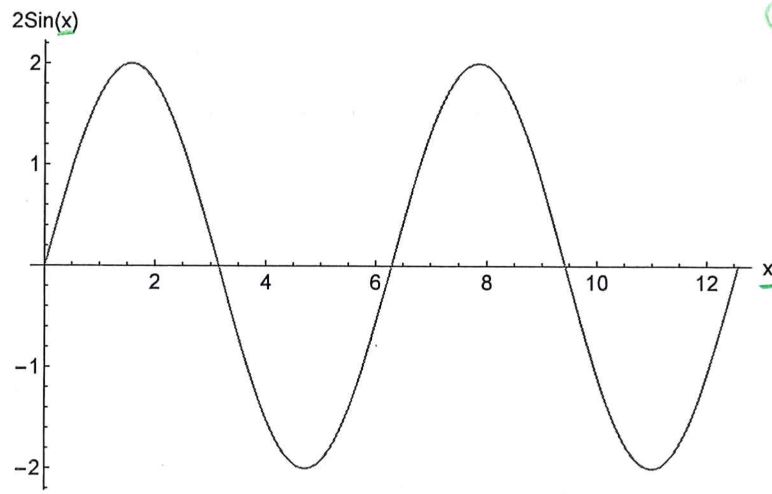

Fig. 3.1 The periodic function $f(x) = \sin(x)$ plotted over the interval $0 \leq x \leq 4\pi$. The amplitude of this function is $B = 2$. The period of this function is $P = 2\pi$; the function repeats itself every multiple of 2π so that $f(x) = f(n2\pi x)$ for all positive integers, n.

of the periodic function. When the independent variable is space, sometimes P_λ is called the *wavelength* and given the symbol λ. The subscript λ is added to the the symbol for the period (P_λ) and the frequency (F_λ, defined below) so that these symbols are defined uniquely (and as a reminder of their connection with the concept of wavelength).

- **Frequency.** If the independent variable is time, the period is sometimes expressed as the *frequency*. Because of its frequent use, it has become common to speak of the frequency even when the independent variable is not time. Frequency is best thought of as the number of complete cycles of the periodic function per unit length or unit time. The relationship between period, P_λ, and frequency, F is

$$F = \frac{1}{P}$$

Therefore, if the Period is $P_\lambda = 2\pi$, the frequency is $F_\lambda = 1/(2\pi)$. This indicates that there is one full cycle (one period) occurring every 2π units of the independent variable. The SI units for frequency are the Hertz (Hz) which measures the number of periods completed per second.

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- **Convergence.** Convergence of series is a complicated topic. We discussed the concept of convergence briefly for Taylor series. For Fourier series, the functions that we examine will all converge in some useful sense. For continuous functions, the series will converge pointwise. This means that for any x chosen in the domain, the partial sums evaluated at x_0, ($S_n(x_0)$), get as close to the value of the function at x ($f(x_0)$) as we like by increasing n sufficiently. For discontinuous functions, the Fourier series converge in a sense that will be discussed later in the chapter.

3.2 Review: Power series

Most of us are familiar with infinite series because they are introduced in introductory calculus courses. There is a very intuitive way to think about Taylor series. To set the stage, consider first the problem of fitting a polynomial to a set of data. For example, we all know that a unique line can be fit through two points; similarly, a quadratic through three points, and cubic through four points, etc. This result is supported in a more general context of fitting $n+1$ points exactly by the following theorem (which we state, but do not prove) and so on

Theorem 3.1. *For any set of $k+1$ points (x_i, y_i) where*

- *i is an index such that $i = 1, \ldots k+1$*
- *$x_i \neq x_j$ for all $i \neq j$*

then there is a unique *polynomial of degree* at most *k such that $f(x_i) = y_i$ for all $i = 1, \ldots, k+1$.*

This theorem raises an interesting question. Suppose we have a continuous function $f(x)$ (let's assume that all orders of derivatives also exist and are bounded, so that $f(x)$ is a C^∞ function) defined on some interval $I = (a,b)$. Now, assume we sample $n+1$ points in the interval I. Apparently, we can get some kind of approximation to the function $f(x)$ by fitting an n^{th}-order polynomial to the $n+1$ sampled points. This can be done, in practice, by a method such as Lagrange polynomial interpolation (which we will not discuss). rom not superscript

Now, by definition, all polynomials are of finite order. The order of the highest power can be as *large* as you like, but it cannot be infinity; thus there are a finite number of terms. The interesting question that might occur after thinking about this problem is whether or not one can consider a kind of "infinite" polynomial that could fit continuous curves exactly. Being somewhat loose, we might wonder of something like an "infinite" polynomial would correspond to fitting an "infinite" number of points on the interval I. We would expect such a polynomial to provide an exact (in some sense) representation of the continuous function $f(x)$. if

While this is not necessarily a rigorous mathematical way of posing the question, it does offer a kind of analogy that provides some basis for working with what are called power series. Recall, a polynomial of order n is given by that

$$f(x) = \sum_{n=0}^{n=N} a_n (x-b)^n$$

where, here a *shift* of the amount b has been included in the polynomial. A power series, then, is the generalization of this idea to the limit $n \to \infty$.

$$f(x) = \sum_{n=0}^{n=\infty} a_n (x-b)^n$$

Note that any finite number of terms from this series is a polynomial; however, a power series extends the sum to infinity. Despite their similarity, power series and polynomials do exhibit some very different kinds of behavior in appropriate limits. Specifically, we note the following.

1. First, all polynomials converge to a *finite value* in every finite interval, $I = (a,b)$, where a and b are real numbers.
2. Conversely, it is not true that every power series converges. While finite polynomials have finite values on any finite interval, "infinite" polynomials might not converge on any interval.
3. It is not difficult to show that any polynomial tends toward plus or minus infinity as x tends toward plus or minus infinity. In other words, for a polynomial $f(x)$ of n^{th} degree, we have rom not superscript

$$\left| \lim_{x \to \infty} f(x) \right| = \infty$$

4. A power series, in contrast to a polynomial of order n, can converge to *any value*, including finite values, as x tends toward infinity. This is actually quite a strange result in some ways. For example, suppose we have the following behavior for the power series representing $f(x)$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sum_{n=0}^{n=\infty} a_n (x-b)^n = 0$$

Certainly, power series with this kind of behavior exist (for example, the power series for $f(x) = \exp(-x^2)$ behaves this way). However, for *any finite approximation to the power series* using only N terms, we have (regardless of how large N is)

$$\lim_{x\to\infty} \left| \sum_{n=0}^{n=N} a_n (x-b)^n \right| = \infty$$

This is very strange behavior because the power series tends to zero as $x \to \infty$, however *every* finite approximation of that series diverges to $\pm\infty$. It is frequently the case though that extensions of concepts from the finite to the infinite lead to non-intuitive behavior!

3.3 Review: Taylor series

The Taylor series should be a familiar concept from introductory calculus. It was named after the English mathematician Brook Taylor who worked on problems of calculus in the early 1700's. Although primarily known for his work on calculus, his interest ranged widely (as was typical of the times), even authoring a treatise under the unusual title *On the Lawfulness of Eating Blood* which was discovered, upon his demise, to be among his unpublished papers.

The Taylor series is actually derivable directly from the definition of the power series. Note, if we take the n^{th} derivative of the power series we find the following

$$\begin{aligned}
&\textit{first derivative} && f'(x) = a_1 + 2a_2(x-b) + 3a_3(x-b)^2 + \ldots \\
&\textit{second derivative} && f''(x) = 2a_2 + 6a_3(x-b) + \ldots \\
&n^{th}\textit{ derivative} && f^{(n)}(x) = n!a_n + (n+1)!a_{n+1}(x-b) + \ldots
\end{aligned}$$

Starting from this result, if we now set $x = b$, we find that all of the terms except the first are identically zero. The coefficients for the power series above are then defined by

$$\begin{aligned}
f'(b) &= a_1 \\
\frac{1}{2!} f''(b) &= a_2 \\
\frac{1}{n!} f^{(n)}(b) &= a_n
\end{aligned}$$

Substituting this result into the power series above, gives us the *Taylor series* around the point $x = b$

$$f(x) = \sum_{n=0}^{n=\infty} (x-b)^n \frac{f^{(n)}(b)}{n!}$$

Sometimes the series written around $b = 0$ is called the *Maclaurin* series. Setting $b = 0$ gives this series

$$f(x) = \sum_{n=0}^{n=\infty} x^n \frac{f^{(n)}(0)}{n!}$$

Any function that has a convergent Taylor series a set of points, $x \in X$ is called *analytic* on X. Many series (such as the exponential function) converge for all possible values of x, and are thus analytic everywhere. Any finite truncation of a Taylor series is called a *Taylor polynomial*. Note that every finite truncation of a Taylor series is actually a polynomial! Also true is that analytic functions converge nicely: as one increase the number of terms in the sum approximating the function f, the result gets uniformly closer to the actual value of f. An immediate consequence of these properties is that, for every function that is *analytic* in some domain, X, there is a polynomial that can represent that function as closely as we like. Essentially, this follows from the definition of an analytic function, and the Taylor polynomial.

Example 3.1 (Taylor/Maclaurin series examples). The function $f(x) = \exp(x)$ is an interesting example to consider for a Taylor series expansion, in part because it converges exactly everywhere in the domain $x \in (-\infty, \infty)$. To start the analysis, note that we first need an infinite number of derivatives of $f(x)$. While this may seem like a daunting task, often derivatives of functions exhibit a pattern of behavior that can be exploited (using the principle of induction). Consider the following

$$\begin{aligned} f(x) &= \exp(x) \\ f'(x) &= \exp(x) \\ f''(x) &= \exp(x) \\ &\dots \end{aligned}$$

Here, we do not need to work too hard to see the pattern. For *any derivative of order n*, the derivative is given by the equation $f^{(n)}(x) = \exp(x)$. Now, suppose that we expand the series around the point $x = 0$ (i.e., we choose the shift parameter, b, to be zero, which is also the definition of a Maclaurin series). Then, each derivative is equal to unity $f^{(n)}(0) = 1$. The resulting Taylor series is

$$\begin{aligned} \exp(x) &= \sum_{n=0}^{n=\infty} x^n \frac{1}{n!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots \end{aligned}$$

As a second case, suppose we would like to find the Taylor series for $\sin(x)$.

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$$
\begin{aligned}
f(x) &= \sin(x) \\
f'(x) &= (-1)^2 \cos(x) \\
f''(x) &= (-1)^3 \sin(x) \\
f'''(x) &= (-1)^4 \cos(x) \\
&\dots
\end{aligned}
$$

In principle, we can define the Taylor series to be expanded around any real number. And there are good reasons to do that in some cases; for example, if you know you need a particular expansion to be accurate in a specific interval, $I = (a_1, a_2)$, it would be good to choose $a_1 < b < a_2$ for improved accuracy in the expansion. In this example, no particular point was stated, so we adopt $x = 0$ for convenience. Under those conditions, we find

$$
\begin{aligned}
f(0) &= 0 \\
f'(0) &= 1 \\
f''(0) &= 0 \\
f'''(0) &= -1 \\
f^{(4)}(0) &= 0 \\
f^{(5)}(0) &= 1 \\
&\dots
\end{aligned}
$$

In other words, we want only the *odd* terms, and those terms alternate sign. A little thought will indicate that the function $\mathrm{odd}(n) = 2n+1$, $n = 0, 1, 2, \dots$ counts by odd numbers only. To switch signs we multiply by powers of -1; we can define a function that alternates sign by $\mathrm{alt}(n) = (-1)^n$. Putting this together, we find the following result valid for any value of n

$$f^{(2n+1)}(0) = \mathrm{alt}(n) = (-1)^n$$

Note the following examples

$$
\begin{array}{lll}
n = 0 & 2n+1 = 1 & f^{(1)}(0) = (-1)^0 = +1 \\
n = 1 & 2n+1 = 3 & f^{(3)}(0) = (-1)^1 = -1 \\
n = 2 & 2n+1 = 5 & f^{(5)}(0) = (-1)^2 = +1
\end{array}
$$

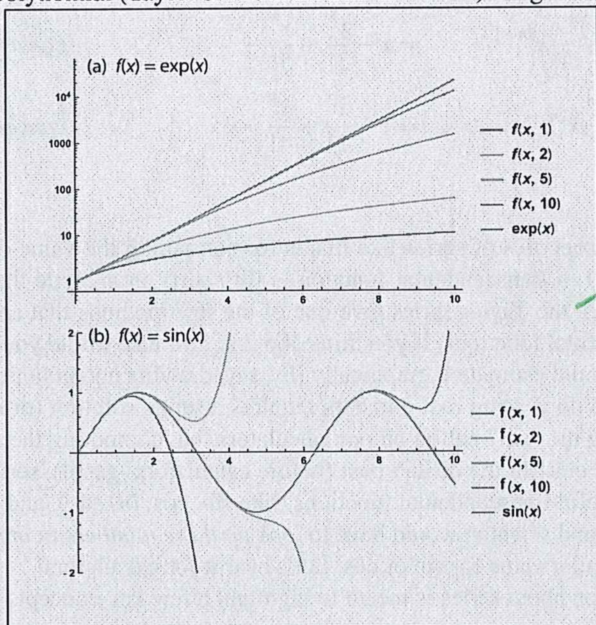
Recalling the definition of the Maclaurin series

$$f(x) = \sum_{n=0}^{n=\infty} x^n \frac{f^{(n)}(0)}{n!}$$

and noting again that only the odd terms are non-zero, the series representation for $\sin(x)$ is

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{n=\infty} x^{2n+1} \frac{f^{(2n+1)}(0)}{(2n+1)!} \\
&= \sum_{n=0}^{n=\infty} x^{2n+1} \frac{(-1)^n}{(2n+1)!} \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\end{aligned}
$$

Plots of finite Taylor polynomial (Taylor series truncated at order n) are given in ~~the figure below~~. Fig. 3.2/



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bf Fig. 3.2: (a) Exponential, and (b) Sine functions plotted for 1, 2, 5, and 10 terms in the approximating Taylor polynomial.

Taylor series for specific functions are widely available, and are also not unreasonably difficult to compute directly from the definition. Note, however, that all Taylor series have a *radius of converge*. In other words, there is a domain ~~that~~ on which a Taylor series will converge (i.e., tend toward a definite number as the number of terms in the sum increase, and that definite number is equal to the value of f at that point). It is generally easy to determine if a series converges or not, but it is much more difficult to determine its radius of convergence. We will not discuss convergence properties of series in this chapter, but we will be careful to list their radius of convergence if it is known. A few well-known Taylor series are given by the following:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (\textit{converges for all } x)$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad (\textit{converges for } |x| < 1)$$

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$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad = x + x^2 + x^3 + \ldots \qquad (\textit{converges for } |x| < 1)$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \qquad = 1 + 2x + 3x^2 + 4x^3 + \ldots \qquad (\textit{converges for } |x| < 1)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad (\textit{converges for all } x)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad (\textit{converges for all } x)$$

the ability to compute/

One of the most useful properties of series like this is ~~for computing~~ the value of *transcendental* functions (Recall from Chapter 1, a transcendental function is (in short) an analytic that cannot be expressed by a finite polynomial). Thus, the Taylor series give one of the few methods that are available to compute approximations to transcendental functions. If you think about it, how else would you compute, for example, the value of $\sin \pi/12$? You could estimate it graphically (by, say, drawing a giant unit circle) like the ancient Greek's did, but a method to do it using real numbers requires a series solution (or some other algorithm). We get accustomed to hitting the "sin" button on our calculators, but computing the actual values of the sin function is actually very difficult! In the distant past (before calculators, gasp!), series solutions were used to compute the values for useful transcendental functions (like sin, cos, *ln*, etc.), and they were published in very large books. Engineers and scientist would have to *look up these numbers in immense tables* whenever they wanted to know a particular value for sin or cos. Let's hear it for calculators!

In summary, this discussion about series is meant to highlight a few key concepts.

1. Power series can be interpreted as an extension of the concept of polynomials to infinite degree.
2. Taylor and Maclauren series are a special cases of power series.
3. Power series may converge only on some finite interval; it is also possible that they do not converge at all.
4. Because the sum is an infinite one, the limiting behavior of a series (e.g., as $|x| \to \infty$) might not be well approximated by a finite truncation of the series.
5. Infinite sums frequently thwart intuition. Infinite sums can create behavior that would otherwise not be expected!

3.4 Trigonometric Series

Power series are hardly the only kind of series that provides useful results. In fact, expansions in *trigonometric series* are probably more widely used than any other kind of series expansion. A trigonometric series is one that contains a trigonometric function that varies with the index of the sum. A few examples illustrate some ~~examples of~~ trigonometric series on the interval (0,1). In each of these examples, n is an integer greater than or equal to zero.

$$(a) \qquad B_n = -50\cos\left(\frac{n\pi}{r} + \frac{\pi}{4}\right) \qquad f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

$$(b) \qquad A_n = -50\cos\left(\frac{n\pi}{r} + \frac{\pi}{4}\right)) \qquad f(x) = \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

$$(c) \qquad B_n = \frac{\sin\left(\frac{\pi n}{2}\right)}{n} \qquad f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

$$(c) \qquad A_n = \frac{\sin\left(\frac{\pi n}{2}\right)}{n} \qquad f(x) = \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

The plots of these series appear in Fig. 3.3 where ~~here~~ we have used A_n to indicate the coefficients of cosine series, and B_n to indicate the coefficient for sine series.

As you can see in these results, trigonometric series can allow the representation of some very peculiar functions. Some of the interesting features about the particular functions plotted above are as follows.

- Although the two functions used to create the new functions are C^∞ (i.e., infinitely smooth analytic functions), the series seem to give functions appear to be, in some cases, non-differentiable at every point.
- Although the two functions used to create the new functions are C^∞ (i.e., infinitely smooth analytic functions), the series seem to give functions appear to be, in some cases, discontinuous.
- The sine and cosine series give very different series solutions even when $A_n = B_n$.

Although all of these examples are interesting, it is not clear that any of them correspond to any classical function that we recognize, either polynomial or transcendental. So, although we certainly can generate many intersting trigonometric series, the questions arises: "if I am given a specific, known (polynomial or transcendental function) on a finite interval, can I determine a trigonometric series for that?" This is a question that occurred to Joseph Fourier, a French engineer, in the early 1800's. The answer to this question, remarkably, is a resounding yes for almost any function that one can imagine. In fact, modern mathematics was dramatically shaped by the quest for the answer to this question. Not only did it cause mathematicians to re-think the notion of what a function is, but it caused them to refine the mathematical methods that have led to modern mathematical analysis.

3.5 Fourier Series

For power series, one can think of the expansions as being in an infinite set of polynomials (although, technically, there is no infinite-order polynomial). For trigonometric series, the series expansions are, not surprisingly, in trigonometric functions.

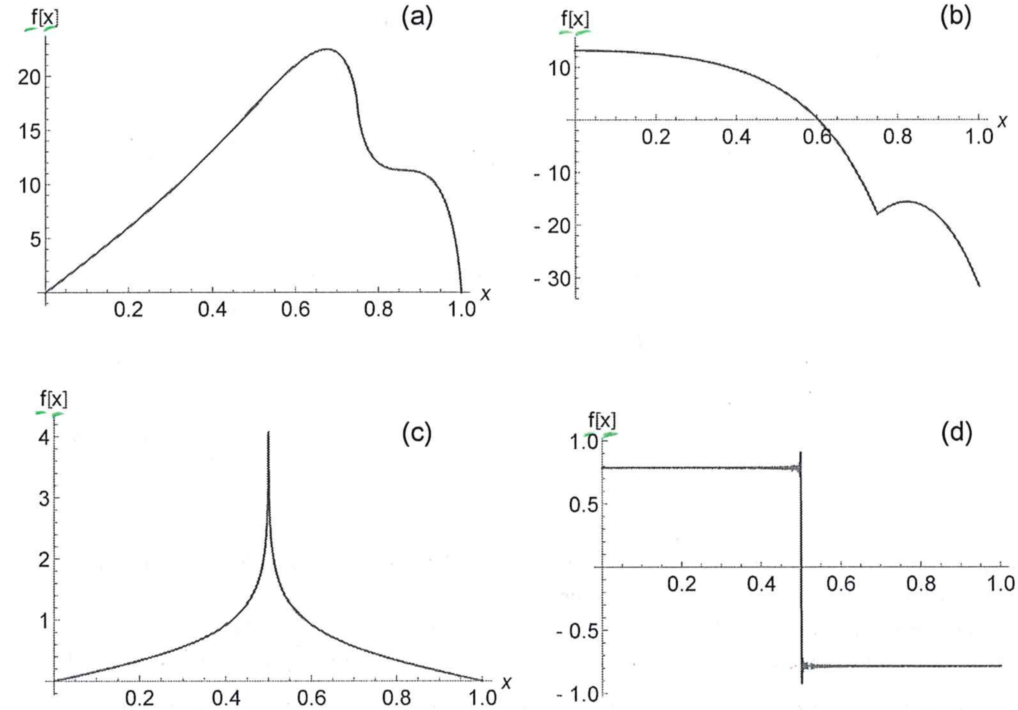

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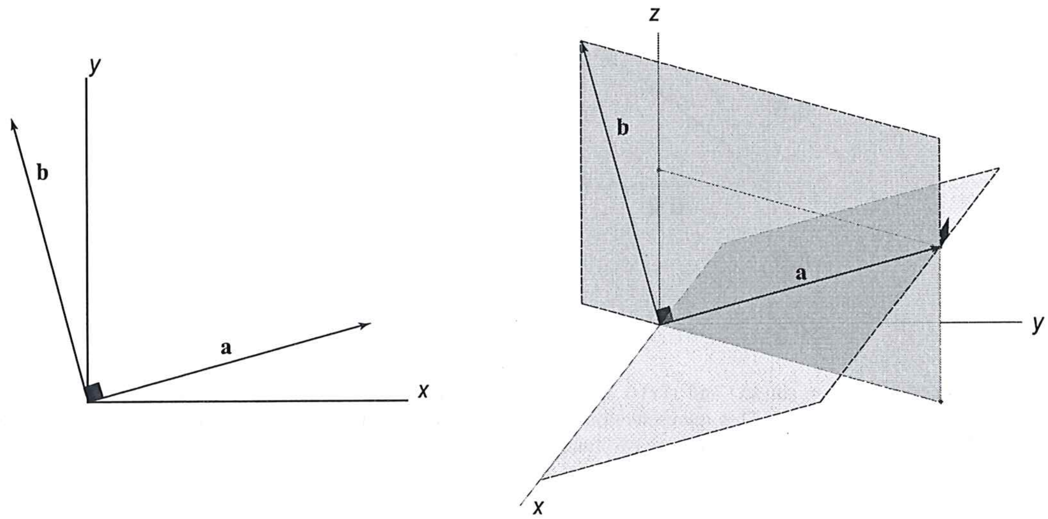
Fig. 3.3: The functions given by (a)-(d) above.

For the development of the Taylor series, we were able to develop a scheme in which we (1) proposed an expansion in an infinite series with an infinite number of (unknown) coefficients, a_n, and (2) determined the infinite number of coefficients by finding repeating patterns in the derivatives needed, so that any derivative of order n could be explicitly computed if n were specified.

For Fourier series, we will do something similar. However, instead of the unknown parameters being a function of an infinite number of derivatives, we will be able to express the unknown parameters as an infinite number of *integrals*. While this sounds on the surface to be dire, like the case for Taylor series, we will find that we can determine repeating patterns in the integrals so that we can derive closed-form expressions for the integrals in terms of the series index n.

A critical component for the development of the Fourier series is the extension of the concept of *orthogonality* to continuous functions. While on the surface this may not immediately make intuitive sense, it can be made intuitive by analogy with the familiar concept of orthogonality for finite vectors.



two/ three/

Fig. 3.4: Perpendicular vectors **a**, and **b** in 2- and 3-dimensions.

3.5.1 *Orthogonality Revisited*

two/ three/

At some point in our mathematical education, most of us have encountered the concept of *orthogonality* for two vectors. Perpendicular vectors in both 2- and 3-dimensions are illustrated in Fig. 3.4. There are two related concepts that help us define and describe perpendicular vectors in 2- and 3-dimensions.

two/ three/

1. First, any two vectors **a** and **b** are said to be perpendicular if the dot product between the two vectors is zero. That is, two vectors are perpendicular if $\mathbf{a} \cdot \mathbf{b} = 0$. Recalling that the dot product of two vectors is equal to their magnitudes times the cosine of the angle between them, we have

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

where θ is the angle between the two vectors in the plane that contains them both. Note that this is zero, exactly when $\theta = \pi/2 = 90°$.

2. Second, there are a set of mutually perpendicular *basis vectors* that can be used to define any arbitrary vector as a linear *weighted sum* of the basis vectors. These vectors are often given the symbols **i**, **i**, and **i** corresponding to unit vectors in the x-, y-, and z- directions, respectively. Thus, in the Cartesian coordinate system, we have

$$\mathbf{i} = (1,0,0) \qquad \mathbf{j} = (0,1,0) \qquad \mathbf{k} = (0,0,1)$$

Using these basis vectors, any vector $\mathbf{a} = (a_1, a_2, a_3)$ can be represented by the following weighted sum of the basis vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

In summary then, every vector can be decomposed into a linear weighted sum of its basis vectors, and each of these basis vectors is, by definition, perpendicular to the others. These ideas are relatively familiar ital x2 /

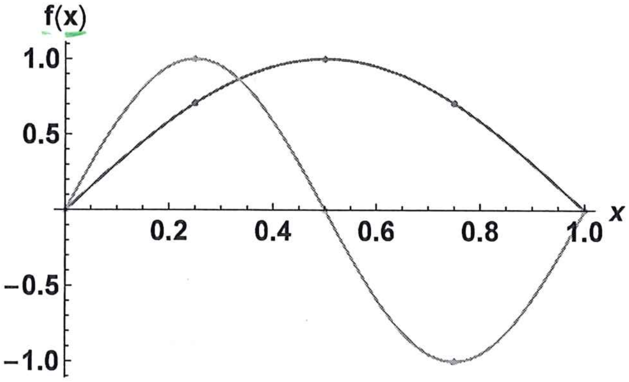


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Fig. 3.5: The functions $f(x) = \sin(\pi x)$ and $g(x) = \sin(2\pi x)$, approximated by four discreet sampling intervals (defined by 5 points, $\mathbf{x} = (0, 1/4, 1/2, 3/4, 1)$). This representation results in 5-dimensional vectors representing the values of the sine functions. The continuous representation of these two functions is shown for comparison.

two / three / 9/

and easy to grasp in 2- and 3-dimensions. Of course, these concepts can be extended to any finite number of dimensions. We cannot visualize such extensions, but mathematically the concepts remain valid. We will discuss in what sense two *functions* can be perpendicular by making an analogy of functions being approximated by finitely-long vectors.

Now, consider two *finite* approximations to the functions $f(x) = \sin(\pi x)$ and $g(x) = \sin(2Pix)$ on the interval $I = [0, 1]$. By finite approximation, in this case we mean

1. Segmenting the interval into N pieces represented by the vector $\mathbf{x}$, where the spacing is given by $\Delta x_i = 1/N$, and the components of the vector are given by the recursive relationship $x_{i+1} = x_i + \Delta x_i,\ i = 0, 1, 2, \ldots, N$.
2. Determining the values of $\sin(\pi x_i)$ and $\sin(2\pi x_i)$, for $i = 0, 1, 2, \ldots, N$.

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As a simple (and crude) example, consider the case of $N = 4$. For this case, the values of $\mathbf{x}$ are given by $\mathbf{x} = (0, 1/4, 1/2, 3/4, 1)$. For notation, define the vector of values for the $\sin(\pi x_i)$ function by ${}_{1\pi}\mathbf{S} = \sin(\pi x_i),\ i = 0, 1, 2, 3, \ldots, N$. So, for this example, the result is ${}_{1\cdot\pi}\mathbf{S} = (0, \sin(\pi/4), 1, \sin(\pi/4), 0)$. We will use a similar definition for ~~the~~ $\sin(2\pi x_i)$, so that ${}_{2\pi}\mathbf{S} = (0, 1, 0, -1, 0)$. The two sine functions, and the points for the discrete approximation of them, are illustrated in Fig. 3.5.

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Although this is a rather crude approximation to the two sine functions, note that for our carefully chosen sampling points, we have the following result

$$
\begin{aligned}
{}_{1\pi}\mathbf{S} \cdot {}_{2\pi}\mathbf{S} &= (0, \sin(\pi/4), 1, \sin(\pi/4), 0) \cdot (0, 1, 0, -1, 0) \\
&= 0 \cdot 0 + \sin(\pi/4) \cdot 1 + \sin(\pi/4) \cdot (-1) + 0 \cdot 0 \\
&= 0
\end{aligned}
$$

⊙/

So, even though this is an approximate representation of the two sine functions, we find that the finite vectors representing the functions themselves are orthogonal. This is a bit more compelling when we use a larger number of points, say $N = 100$. For that case, the approximate discrete functions (each with 101 points) are illustrated in Fig. 3.6.

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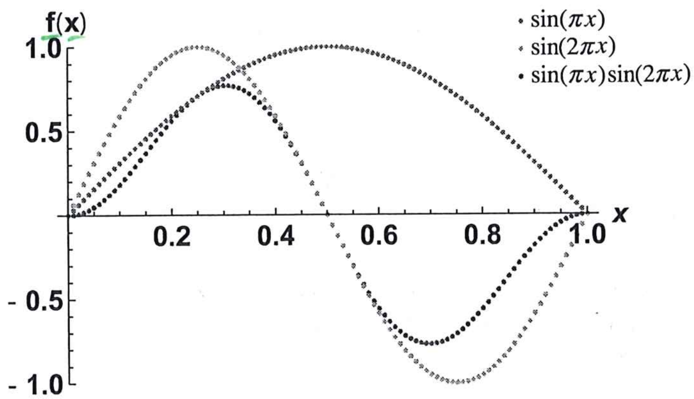


Fig. 3.6: The functions $f(x) = \sin(\pi x)$ and $g(x) = \sin(2\pi x)$, approximated by four discreet sampling intervals (defined by 101 points). This representation results in 101-dimensional vectors representing the values of the sine functions.

For this discrete representation, we can see that the number of points used is high enough that we really are capturing the features of the two sine functions. We have a large list of numbers for $\mathbf{x}$, ${}_{1\pi}\mathbf{S}$, and ${}_{2\pi}\mathbf{S}$, as follows

$$\mathbf{x} = (0, 1/100, 2/100, 3/100, \ldots, 1)$$
$${}_{1\pi}\mathbf{S} = (0, \sin(\pi/100), \sin(2\pi/100), \ldots, 0)$$
$${}_{2\pi}\mathbf{S} = (0, \sin(2\pi/100), \sin(4\pi/100), \ldots, 0)$$

Au: As meant?

While we can verify that ${}_{1\pi}\mathbf{S} \cdot {}_{2\pi}\mathbf{S}$ by direct computation, a look at the graph in Fig. 3.6 makes this somewhat unnecessary. The red points plotted in this graph represent the product of the two sine functions. A little though will indicate that the symmetry of the problem guarantees that the sum of the red points (which represents the dot product ${}_{1\pi}\mathbf{S} \cdot {}_{2\pi}\mathbf{S}$) will sum to zero.

While this is an approximate method to understand how two continuous functions can be though of as being orthogonal, it also is to some extent rigorous. For example, consider the quantity

$$\sum_{i=0}^{i=N+1} {}_{1\pi}\mathbf{S}\Delta x_i = \sum_{i=0}^{i=N+1} \sin(\pi x_i)\Delta x_i$$

In the appropriate limit, this gives

$$\lim_{N\to\infty} \left(\sum_{i=0}^{i=N+1} \sin(\pi x_i)\Delta x_i \right) = \int_0^1 \sin(\pi x)\, dx$$

And, in a similar fashion, we have

$$\sum_{i=0}^{i=N+1} {}_{1\pi}\mathbf{S}\cdot{}_{2\pi}\mathbf{S}\Delta x_i = \sum_{i=0}^{i=N+1} \sin(\pi x_i)\sin(2\pi x)\Delta x_i$$

In the appropriate limit, this gives

$$\lim_{N\to\infty}\left(\sum_{i=0}^{i=N+1} \sin(\pi x_i)\sin(2\pi x)\Delta x_i\right) = \int_0^1 \sin(\pi x_i)\sin(2\pi x)\,dx = 0$$

In particular, it is easy to validate that this last integral is identically zero. Thus, the concept of two continuous functions being *orthogonal* to one another really does have a direct, demonstrable connection to the case of finite-sized vectors being orthogonal.

3.5.2 Fourier Sine Series

Fourier series are a special kind of trigonometric series that have somewhat astounding properties. Fourier series use only the sine and cosine functions to expand a function $f(x)$ as a series. In starting the investigation of Fourier series, we will consider first only the interval $I = \{x : x \in [0,1]\}$; this interval is sometimes called the *unit interval*. We can expand the definition of the Fourier series to other intervals once we understand how they work on the unit interval.

To start, consider the most basic question that we can ask. If we have a simple, analytic function on an interval $x \in [0,1]$, can we determine the Fourier series for it? To be concrete, let's suppose we have a specific function, say $f(x) = e^{-x}$, and we decide we would like to find a sine series for that function. The question is, can we find a series of the form following

$$f(x) = \sum_{n=0}^{\infty} B_n \sin(n\pi x) \tag{3.1}$$

and

Or, being specific to the example given, $f(x) = e^{-x}$

$$e^{-x} = \sum_{n=0}^{\infty} B_n \sin(n\pi x) \tag{3.2}$$

where for both expressions, n is an integer, and B_n is an infinite sequence of constants (i.e., the sequence $B_n = (B_1, B_2, B_3, B_4, \ldots)$. There are two primary questions that we need to address about such a proposed series. ~~These are~~

1. Is there a method (an algorithm or constructive proof) that allows us to determine the infinite sequence of constants B_n?
2. If we can find the constants B_n, can we show that the resulting series converges to the function $f(x)$?

Before we proceed, we need to make a few notes. First, we note that because $\sin(0\pi x) = 0$, technically we do not need to start the series at $x = 0$; we could start at $x = 1$. By convention, sine series are usually written as

$$e^{-x} = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

It is not wrong to start the series at $n = 0$ though! The function $\sin(0\pi x)$ is one of the basis functions for the sine series; it just does not add anything if it is maintained. Secondly, note the following identity (which we discussed in the context of orthogonality in the previous section)

$$\int_{x=0}^{x=1} \sin(n\pi x)\sin(m\pi x)dx = \begin{cases} 0 \, if \, m \neq n \\ \frac{1}{2} \, if \, m = n \end{cases} \tag{3.3}$$

In these expressions, both n and m are integers, not necessarily equal. Because of the direct analogy with the dot product in the case of finite-length vectors, we call two functions *orthogonal* when they meet a condition like Eq. (3.3). With this information in hand, we now note the following reasonably amazing result. If we multiply both sides of Eq. (3.2) by $\sin(m\pi x)$ and integrate, we find

$$\int_{x=0}^{x=1} e^{-x}\sin(m\pi x)dx = \sum_{n=1}^{\infty} B_n \int_{x=0}^{x=1} \sin(n\pi x)\sin(m\pi x)dx$$

where we have taken the integral inside the sum (and this is always allowable for our purposes; the same is not necessarily true for differentiation!) The reason that this is interesting, is because of what it does to the right-hand side of the equation. When we started, we had an infinite number of values of B_n to contend with. But, because the integral on the right-hand side is only non-zero when $m = n$, then we must have the following

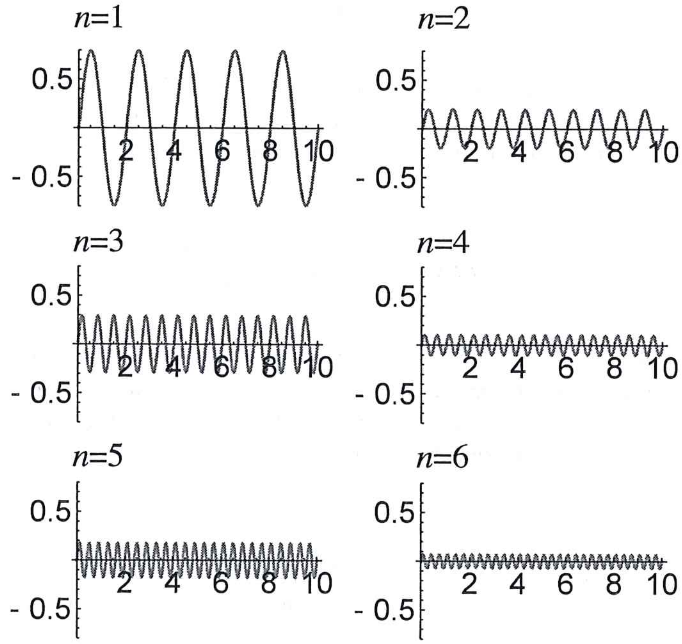
$$\int_{x=0}^{x=1} e^{-x}\sin(n\pi x)dx = B_n \int_{x=0}^{x=1} \sin^2(n\pi x)dx$$

In other words, all of the terms in the series, after integration, are zero, except for the *one term* where n and m are equal. For that one term, the integral becomes the integral of $\sin^2(n\pi x)$ over the interval $x \in [0,1]$. By the identities above, this is exactly $\frac{1}{2}$. Thus, we end up with the result (rearranging a little) that allows us to compute B_n:

$$B_n = 2\int_{x=0}^{x=1} e^{-x}\sin(n\pi x)dx$$

This integral can be done by parts (and then using the fact that it generates a repeating function that can be collected on one side), from a table of integrals, or from software like Mathematica. The result is

$$B_n = \frac{\pi n - e\pi n\cos(\pi n)}{\pi^2 n^2 + 1}$$



Au: Fig. 3.7 is not called out in the text.

Fig. 3.7: The functions $B_n \sin(n\pi x)$ for $n = 1, 2, 3, 4, 5, 6$.

And, while this result looks a bit strange, we have nonetheless apparently computed the value of B_n for all possible values of n. In other words, we have exactly the coefficients we need to compute the result for our series. Our series apparently takes the form

$$e^{-x} = \sum_{n=1}^{\infty} \frac{\pi n - e\pi n \cos(\pi n)}{\pi^2 n^2 + 1} \sin(n\pi x)$$

Can this possibly be true? Is there some way that a bunch of sine curves, properly weighted, and added together can somehow form an exponential? Well, let's find out. Below, I have computed the first 10 values of B_n

$$B_n = \{0.790704, 0.196239, 0.287042, 0.099972, 0.173461, 0.0668818, \\ 0.124146, 0.050223, 0.0966368, 0.0402013\}$$

These values represent the *amplitude* or the height of each of the sine functions up to $n = 6$. To see the functions themselves, we need only compute the values of $B_n \sin(n\pi x)$, $n = 1, 2, 3, 4, 5, 6$. These are plotted on Fig. 3.8. As a matter of terminology, note that the functions $\sin(n\pi x)$ are called *basis* functions. For each such basis function, the integer n controls the *frequency* of the basis function, and the associated coefficient B_n controls the *amplitude* of the basis function.

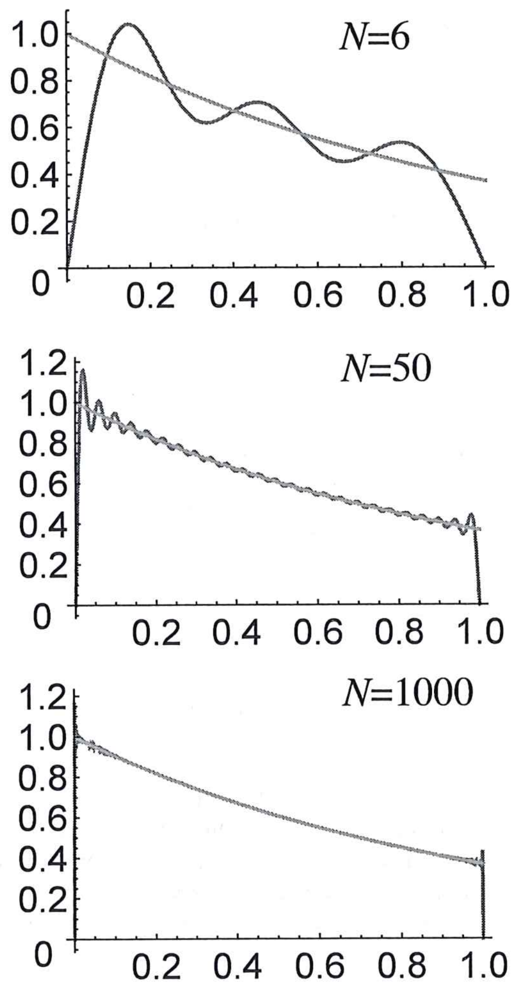


Fig. 3.8: The partial sum of the series, for the total number of terms, N, equal to $N = 6, 50,$ and 1000.

It is clear from the plot in Fig. 3.8 that each of the terms in the series is a sine function, but each with a different amplitude, and each with a frequency that increases with *n*. The most remarkable thing, however, is what happens when we add these components together. Adding the first 6 terms (as we have above) gives us an approximation to the function $f(x) = e^{-x}$ that is not necessarily good (although, it is not necessarily bad either!). If we compute the first 50 terms, things begin to look much nicer. Finally, with the first 100 terms, the function and the series are almost indistinguishable at most points.

The set of functions $E = \{0,\ \sin(\pi x),\ \sin(2\pi x),\ \sin(3\pi x), \ldots\}$ can be thought of as a set of *orthogonal basis functions* from which new functions can be built. There is an analogy with *orthogonal basis vectors* here that was introduced in §3.5.1.

For infinite-dimensional vectors (functions), the equivalent of the dot-product is an integral that is called the *inner product* or sometimes the L^2 (pronounced "ell-two") inner product. Although we will not make extensive use of it, there is even a specific symbol used in mathematics to indicate the L^2 inner product. On the interval $x \in [0,1]$ we would have

$$\langle \sin(n\pi x), \sin(m\pi x) \rangle = \int_{x=0}^{x=1} \sin(n\pi x)\sin(m\pi x)\,dx$$

To summarize the main results for the sine series, we have the following. For any smooth function $f(x)$, on $x \in [0,1]$, the Fourier sine series is given by

$$f(x) = \sum_{n=0}^{\infty} B_n \sin(n\pi x) \tag{3.4}$$

$$B_n = 2\int_{x=0}^{x=1} f(x)\sin(n\pi x)dx, \quad n = 0,1,2,3,\ldots \tag{3.5}$$

or, equivalently, because $\sin(0\pi x) = 0 \Rightarrow B_0 = 0$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) \tag{3.6}$$

$$B_n = 2\int_{x=0}^{x=1} f(x)\sin(n\pi x)dx, \quad n = 1,2,3,\ldots \tag{3.7}$$

We can think of the set of functions $E_{sin} = \{0, \sin(\pi x), \sin(2\pi x), \sin(3\pi x), \ldots\}$ as being the basis functions from which any function on the unit interval $I = [0,1]$ can be reconstructed. The reconstruction requires an infinite sum to be made, and the sum is a weighted one, where the weights (amplitudes) are given by the values of B_n. Note that there is an almost exact correspondence here to the case of reconstructing an arbitrary finite vector $\mathbf{a}$ by computing the sum of a weighted set of basis functions: $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

3.6 Fourier Cosine Series

In the previous section, we (somewhat remarkably!) showed that for nice smooth functions, we can use nothing other than calculus to develop the representation for the Fourier sine series. As you might imagine, the same thing is possible for the cosine, and the procedure is roughly the same.

For the sine series, we found that the set of basis functions $E_{\sin} = \{\sin 0\pi x,\ \sin(\pi x),\ \sin(2\pi x),\ \sin(3\pi x), \ldots\}$ allowed us to construct a sine series on $x \in [0,1]$ of the form

$$f(x) = \sum_{n=1}^{n\to\infty} B_n \sin(n\pi x)$$

And, using the *orthogonality* of the functions $\sin(n\pi x)$ and $\sin(m\pi x)$ $(m,n \in \mathbb{N})$, we were able to determine an integral equation for *all* of the amplitude coefficients, B_n. Note that, although we excluded it from the sum, the zero function is, technically, one of the functions in this list of basis functions (given that it is the function associated with $\sin(0\pi x)$). Making this analogy, we might guess that the set of basis functions for the cosine series might be $E_{\cos} = \{\cos 0\pi x,\ \cos(\pi x),\ \cos(2\pi x),\ \cos(3\pi x), \ldots\}$. This points out one important distinction between the sine and cosine series: the basis function associated with $n = 0$ is nonzero for the cosine series! To be specific, note that $\cos(0\pi x) = 1$. This is a non-symmetry that creates a few headaches, mostly because we are always having to keep track of this fact. However, we can continue forward as before. To start, we suggest the following form for the cosine series

$$f(x) = \sum_{n=0}^{n\to\infty} A_n \cos(n\pi x) \tag{3.8}$$

where now the series *must* start at zero. Again, by convention, this series is often written in a form that allows the sum to start at $n = 1$ (just as is done for the sine series). Somewhat ridiculously, I am even going to keep the $\cos(0\pi x)$ function as such in the series

$$f(x) = A_0' \cos(0\pi x) + \sum_{n=1}^{n\to\infty} A_n \cos(n\pi x) \tag{3.9}$$

Here, I have adopted the unusual notation of A_0' for the first term in the Fourier cosine series. The reason for this is that in this text, we will represent this term in a way that is not conventional, but it is useful and leads to fewer mistakes. This will be discussed further below.

In exactly an analogous way as for the sine series, we also have an orthogonality condition for the inner product for the cosine functions. ~~Note~~

$$\int_{x=0}^{x=1} \cos(n\pi x)\cos(m\pi x)dx = \begin{cases} 0 & \textit{if}\ m \neq n \\ \frac{1}{2} & \textit{if}\ m = n \neq 0 \\ 1 & \textit{if}\ m = n = 0 \end{cases} \tag{3.10}$$

~~(3.11)~~

Au: Equations will need to be renumbered from this point forward.

Note that the case $n = m = 0$ leads to a result that is different from the rest of the cases $(n, m > 0)$. As we did for the sine series, we are now going to work out a method to determine the values of A_n (and A_0') for the cosine series. To start, we can multiply both sides of Eq. (3.9) by $\cos(m\pi x)$ and integrate. This gives

Au: Correct?

$$\int_{x=0}^{x=1} f(x)\cos(m\pi x)\,dx = A_0' \int_{x=0}^{x=1} \cos(0\pi x)\cos(m\pi x)\,dx + \sum_{n=1}^{n\to\infty} \int_{x=0}^{x=1} A_n \cos(n\pi x)\cos(m\pi x)\,dx.$$

Using the results from Eq. (3.10.2), we have (and, specifically, recalling that every term in this integrated version of the series is *zero* except for the one term where $n = m$) the following two cases.

- Case 1: $n = m = 0$

$$\int_{x=0}^{x=1} f(x)\cos(0\pi x)\,dx = A_0'$$

so that

$$A'_0 = \int_{x=0}^{x=1} f(x)\,dx$$

- Case 2. $n = m$, and $n, m > 0$

$$\int_{x=0}^{x=1} f(x)\cos(n\pi x)\,dx = \frac{1}{2}A_n$$

so that

$$A_n = 2\int_{x=0}^{x=1} f(x)\cos(n\pi x)\,dx$$

Here, we have used $m = n$ to express the final result in terms of n (again, this is the conventional notation). Now, if we look at the results for A'_0 and A_n we notice something that is a little bit annoying. The general form for A_n actually works for A'_0 also, except the result would be two times too large. So, there are two equally reasonable ways to proceed.

1. We just remember the formula for A'_0 and for A_n ($n > 0$) separately, and go along on our way.
2. We remember only the formula for A_n, with the idea that it also works for $n = 0$; but then we also have to redefine A'_0 as follows

$$A'_0 = \frac{A_0}{2}$$

The reality is that both are fine, and both require that you remember two different things. The use of A_0 is one adopted by most texts by convention. However, in practice, this is not the best approach. Instead, I prefer to think of the first term of the Fourier cosine series as being the *average value* of the function over the interval. For the unit interval, the interval length is $L = 1$. Thus we have the corresponding average

$$A'_0 = \frac{1}{1}\int_{x=0}^{x=1} f(x)\,dx$$

Later, when we allow the intervals to be of arbitrary length, $I = [0, L]$, the first term in the Fourier cosine series will *still be* the average value, computed for this more general case by

$$A'_0 = \frac{1}{L}\int_{x=0}^{x=L} f(x)\,dx$$

In summary, we have

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{n \to \infty} A_n \cos(n\pi x) \tag{3.12}$$

$$A_n = 2 \int_{x=0}^{x=1} f(x) \cos(n\pi x)\, dx, \;\; n = 0, 1, 2, \ldots \tag{3.13}$$

or, preferably

$$f(x) = A_0' + \sum_{n=1}^{n \to \infty} A_n \cos(n\pi x) \tag{3.14}$$

$$A_0' = \int_{x=0}^{x=1} f(x)\, dx \tag{3.15}$$

$$A_n = 2 \int_{x=0}^{x=1} f(x) \cos(n\pi x)\, dx, \;\; n = 1, 2, \ldots \tag{3.16}$$

where we *always* compute the first term in any cosine series as the average value over the appropriate interval.

These results are essentially identical to those for the sine series, except that we always will have to deal with the $n = 0$ term of the cosine series (which is identically zero for the sine series) whenever we use the cosine series. This all makes cosine series just a little less fun, but ultimately it will be worth it. We will discuss ~~later~~ reasons that we might prefer one series over another. Later

Example 3.2 (Fourier Sine and Cosine Series Compared). The function $f(x) = x$ is a frequent example for sine and cosine series, mostly because the expressions for A_n and B_n are integrable. To start, let's compute the sine series. This is given by

$$B_n = 2 \int_{x=0}^{x=1} x \sin(n\pi x)\, dx$$

Integrating by parts gives

$$B_n = -2\frac{\cos(n\pi)}{n\pi} = -\frac{2(-1)^n}{n\pi}$$

So, the result is

$$\begin{aligned} f(x) &= \sum_{n=1}^{n \to \infty} B_n \sin(n\pi x) \\ x &= \sum_{n=1}^{n \to \infty} -\frac{2(-1)^n}{n\pi} \sin(n\pi x) \end{aligned}$$

Now, for the cosine series we have

$$A_n = 2 \int_{x=0}^{x=1} x\cos(n\pi x)\,dx$$

Integrating by parts gives

$$A_n = \frac{2(-1+(-1)^n)}{n^2\pi^2}$$

Note that for $n = 0$ this gives us 0/0, which is undefined. Rather than deal with that problem, we go back to the original definition for the first term in the series

$$A_0' = \int_{x=0}^{x=1} x\,dx = \frac{1}{2}$$

So, in conclusion, we have

$$\begin{aligned} f(x) &= A_0' + \sum_{n=1}^{n\to\infty} A_n \cos(n\pi x) \\ x &= \frac{1}{2} + \sum_{n=1}^{n\to\infty} \frac{2(-1+(-1)^n)}{n^2\pi^2}\cos(n\pi x) \end{aligned}$$

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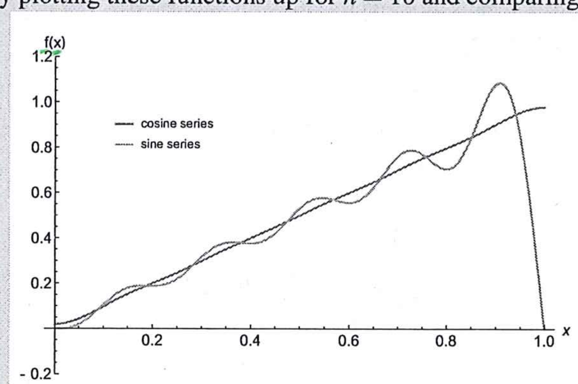
Thus, we have two *different* series for the same function. Is there any practical difference between them? We can check by plotting these functions up for $n = 10$ and comparing.



Au: Fig. 3.9 not called out in the text.

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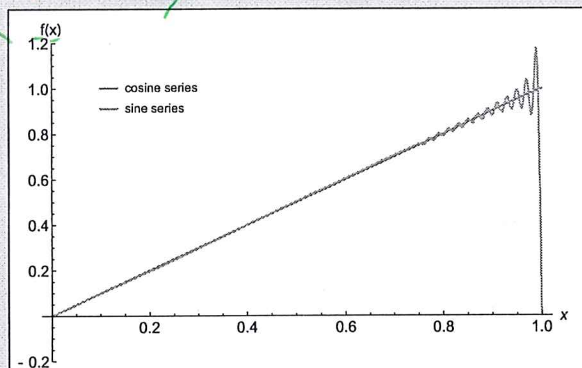
Fig. 3.9: Cosine and sine series for the function $f(x) = x$, each with 10 terms.

We have two different series for this function, and it is clear from observation that the behavior of the two series is quite different. First of all, the series using the cosine basis functions seems to converge much better than the one using the sine series. Second, the sine series is zero at the location delete lines / ital x2

$x = 1$, which is not what we want. As we increase the number of terms, we can see that this behavior never really goes away. ~~Here is~~ a plot of the same two series with a total number of terms equal to 100 for each series Fig. 3.10 shows



(bf)

Fig. 3.10: Cosine and sine series for the function $f(x) = x$, each with 100 terms.

(lc) one

As you can see, something slightly odd happens near $x = 1$ with the sine series. The explanation is, at least in part, this: The sine series is forced to be zero at all multiples of π; thus each term in $\sin(n\pi x)$ is *identically* zero when $x = 1$. The best that the sine series can do is to try to create a discontinuity that drops precipitously from 1 to zero as you approach $x = 1$ from the left. Which is what it, in fact, does do. There are a few more problems that arise (~~for instance~~, one can observe some oscillations near the point $x = 1$ with the sine series) that will be discussed later. e.g.:

3.7 Comparison of Basis Functions

Both the Fourier sine and cosine series have a similar features. For each, we can think of any function f on a finite interval $D = [a, b]$ as being decomposed into its constituent amplitudes and frequencies, as represented by the weighted sine or cosine functions. These functions form a *basis*, as mentioned above, although it is a basis with an infinite number of basis "vectors" (in this case, the basis vectors are the sine and cosine functions).

It is helpful to see some of the parallels between the sine and cosine series. In Table 3.1, the basis functions, as a function of n, are listed. There are relatively clear parallels between the two kinds of basis functions, with the exception of the $n = 0$ case. For that case, there is a lack of parallelism. For the sine function, we have that $\sin(0) = 0$, whereas for the cosine function, $\cos(0) = 1$. So, technically, there is a basis function for the sine functions for $n = 0$; however, because that function is just the zero constant function, it adds nothing to the corresponding series. For the cosine series, we have a different constant function given by $\cos(0) = 1$. This function does contribute to the series by allowing the series to be translated by a constant. This actually can make it easier for some problems; as an example, the cosine series for $f(x) = 1$ is just the $n = 0$ term (all other terms are zero). For the sine series, the expansion for $f(x) = 1$ is substantially more complicated. This distinction is made explicit by Eqs. (3.17)-(3.18).

$$\text{sine series:} \qquad 1 = \sum_{n=1}^{\infty} B_n \sin(n\pi x) = \sum_{n=1}^{\infty} \frac{1-(-1)^n)}{n\pi} \sin(n\pi x) \tag{3.17}$$

$$\text{cosine series:} \qquad 1 = A_0' + \sum_{n=1}^{\infty} A_n \cos(n\pi x) = 1,$$

$$(\text{i.e., } A_0' = 1, A_n = 0 \text{ for } n > 0) \tag{3.18}$$

Table 3.1: Basis functions for sine and cosine Fourier series.

n	$\sin(n\pi x)$	$\cos(n\pi x)$
0	0	1
1	$\sin(\pi x)$	$\cos(\pi x)$
2	$\sin(2\pi x)$	$\cos(2\pi x)$
3	$\sin(3\pi x)$	$\cos(3\pi x)$
4	$\sin(4\pi x)$	$\cos(4\pi x)$
$\vdots$	$\vdots$	$\vdots$
n	$\sin(n\pi x)$	$\cos(n\pi x)$

3.8 The Spectrum

This is somewhat tangential to our primary focus on Fourier series, but it is also interesting, and helps better explain what is going on in Fourier series. Suppose we look more at both the sine and cosine series for the function

$$f(x) = x, \quad x \in [0,1] \tag{3.19}$$

that we examined above. Recall, we had the following list of amplitudes as a function of n, where $n \in \mathbb{N}$ (i.e., $n = 1,2,3,\ldots$) for the sine and cosine series for this function

$$\text{cosine} \quad \begin{cases} A_0' = \dfrac{1}{2} \\[2ex] A_n = \dfrac{2(-1+(-1)^n)}{n^2\pi^2} \qquad n = 1,2,3,\ldots \end{cases}$$

$$\text{sine} \qquad B_n = \frac{-2(-1)^n}{n\pi} \qquad n = 1,2,3,\ldots$$

There is an interesting fact about these two series. Although the functions that they represent are infinite dimensional (in the *uncountable* infinity sense of the word infinite), the series representations are *countably* infinite. In other words, if you know all of the amplitudes, you can reconstruct the function. In a sense, this represents a form of compression of the information embedded in a function. In fact, Fourier series are sometimes used for exactly that purpose (e.g., in the compression of images in the JPEG format!)

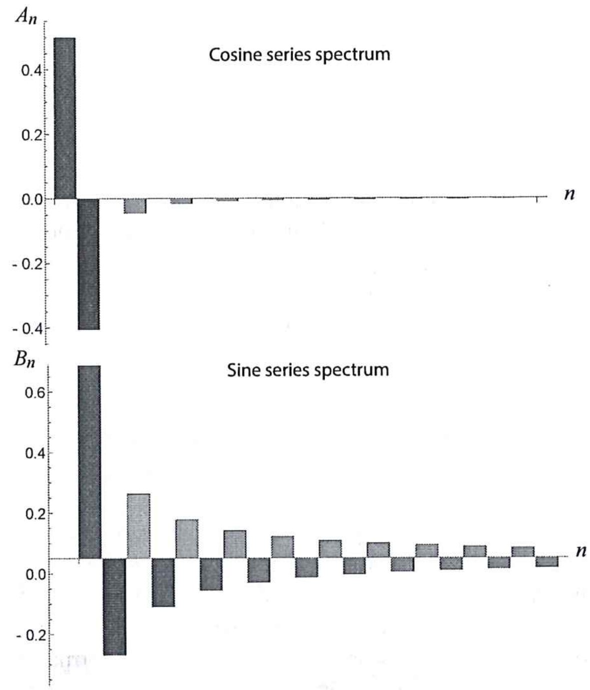


Fig. 3.11 The spectra for the cosine series (top) and sine series (bottom). The first 20 amplitudes are shown.

The amplitudes of the sine and cosine functions are useful for a number of reasons. The main one is that they can give you some sense for how quickly a function converges. The graph of A_n or B_n versus n represents the graph of the amplitudes, and it is called the *spectrum* of a sine or cosine series. As an example, let's look at the two spectrum plots for the sine and cosine series for $f(x) = x$ (Fig. 3.11).

Looking at these two spectra, one can see several important differences. The primary one is that the cosine series for this function is dominated by the first two or three (nonzero) amplitudes; the remaining amplitudes are quite small in comparison, and they decay (become smaller) very quickly. For the sine series, although it is true that the first two amplitudes are the largest (in magnitude), they don't exactly dominate the others, and they do not seem to decay as quickly. In fact, it is not clear that any of the first 20 amplitudes for the sine series might be neglected compared to the first two. This gives us some guidance as to how well the two series converge. By examining these plots, one might expect that the cosine series would give good results even if it included only the amplitudes A_0, A_1 and A_3 (A_2 and all other even amplitudes being zero). Whereas the same cannot be said for the sine series. To check this, the plot of the cosine and sine series constructed from the first three (nonzero) amplitudes are given in the plot shown in Fig. 3.12. Examining the results of Fig. 3.12 suggests that our assertion about the relative sizes and the rate of decay of sizes of the amplitudes was largely correct.

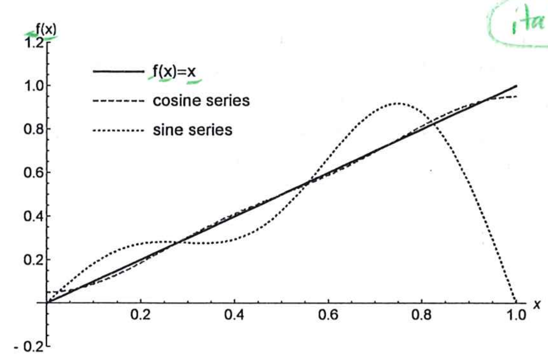


Fig. 3.12 Plots of the cosine and sine series, using only the first three nonzero amplitudes.

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The spectrum also has significant practical applications. Whenever one is tackling problems that contain time-series that have periodic components (e.g., tide heights over several weeks, atmospheric pressure over several days, hourly temperatures collected over a month), the spectrum can tell you much about what frequencies (frequency $=n/2$) are important. Similarly, one can use a spectral analysis to help analyze data for *noise* and even build filters to remover noise. This is explored in the next example.

Example 3.3 (Noise filtering of a DC voltage). Suppose you are working in a lab, and a piece of equipment is turned off and on using a direct current (DC) signal. When the equipment ~~gets~~ senses a large voltage spike (say, $20 < V < 50$ volts) it turns on. When it senses a small voltage spike ($2 < V < 10$) it turns off. Voltage changes smaller than about $2V$ are ignored. To make this piece of equipment work, you need a nice clean source of DC voltage that can send voltage spikes of various sizes. Suppose that you have a DC source, and you send out low- and high-voltage spikes about every 1 second. You measure its voltage output for (what you expect to be) a sequence of $25V$ and $5V$ voltage spikes occurring every 1 s or so. You record this information (an oscilloscope is a device that can do this), and you find that your DC signal is actually ~~really~~ not very clean at all. In fact, it is contaminated with all kinds of noise. Suppose it looks like the plot in Fig. 3.13. This is kind of a disaster, because there is so much noise in the signal that some of the small pulses sometimes generate voltage that is over $10V$ (thus, they would not turn off the device!)

You show your results to the local person-who-can-build-anything, and they say that you just need a DC filter to remove the high-frequency contamination coming from the rectifier (which is a device that converts AC current from an outlet into DC current). They can build one for you as long as you can suggest what frequencies to filter out.

Au: Since V is a unit of measure, not a variable, shouldn't it appear in Roman (not italics), and with a space between the numeric value and the unit?

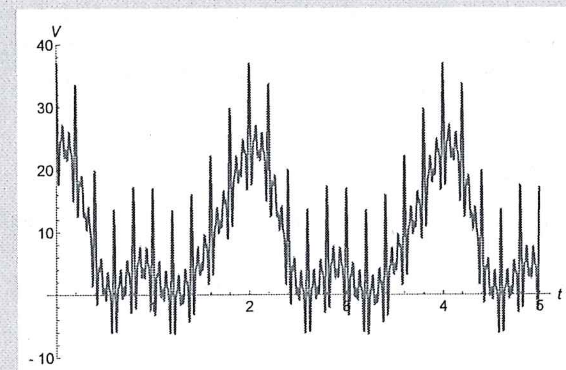

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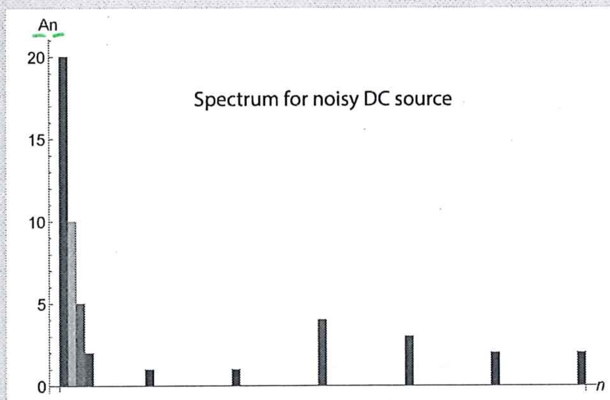
Fig. 3.13: A noisy voltage-versus-time signal.



n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A_n	20	10	5	2	0	0	0	0	0	0	1	0	0	0	0	0

n		21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
A_n		0	0	0	0	0	0	0	0	0	4	0	0	0	0	0

n		41	42	43	44	45	46	47	48	49	50	51	52	53	54	55
A_n		0	0	0	0	0	0	0	0	0	2	0	0	0	0	0

Fig. 3.14: A DCT spectrum for your noisy current data.

Knowing what you know about Fourier series, you decide that what you really need is the spectrum for your DC current. There are many ways of computing spectra directly from raw data; the most popular one (which will not be described in detail!) is call the *Fast Fourier Transform* or FFT. Actually, you will use a fast version of what is called the discrete cosine transform (DCT) so that you only get cosine amplitude-frequency information. Being a Mathematica whiz, you compute the spectrum, and you find the spectrum shown in Fig. 3.14. Looking at this, you immediately see the problem! You expect the signals to have a period around 1 s (i.e., $n \approx 2$, but you see that there are a number of components with an n much greater than this value. The solution seems clear: you just need to develop

second

a filter to “cut off” all of the high frequencies. Looking at the plot, you decide it is safe to cut off all of the values where $n > 10$ (which means that you decided to keep the first little bump in the spectrum plot). To check to see if it this is sufficient, you even compute the new signal based on this filtering. You can basically reconstruct the correct cosine signal by taking the first four non-zero values of A_n. You find the result

$$V(t) = 9 + 10\cos(\pi x) + 5\cos(2\pi x) + 2\sin(3\pi x) + \cos(10\pi x)$$

Plotting this will allow you to assess how well your filter will work. The plot appears as Fig. 3.15. Looking at these results, you decide that your cut off filter suggestion will work. You get ”on” and ”off” peaks of the right magnitude, and the voltage fluctuations are small enough that they will be ignored.

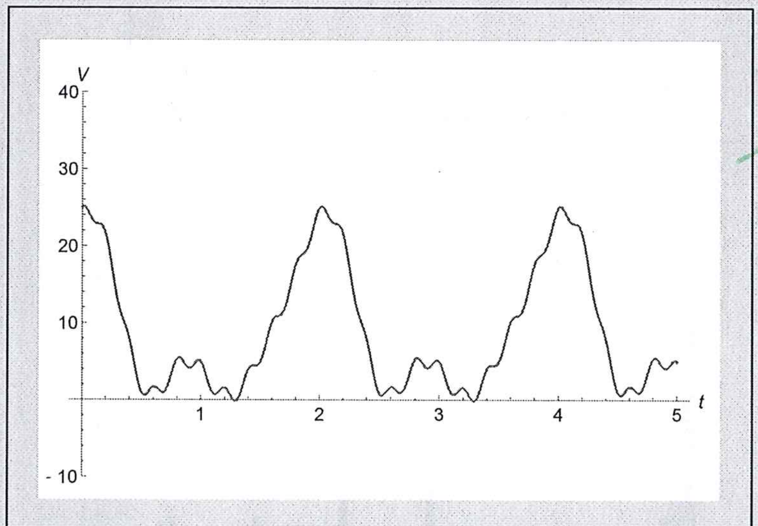


Fig. 3.15: A nicely filtered voltage signal!

3.9 Change of Interval

In the examples above, we have examined the Fourier series exclusively on the interval $x \in [0,1]$. Of course, we might want to define the Fourier series on some more general interval, say $x \in [0,L]$. This change is actually not all that difficult. When we think about it, changing to the new interval basically means that the function that occurs on $x \in [0,1]$ is either stretched or compressed along the x-axis so that it now occurs on $x \in [0,L]$; the vertical behavior of the function remains unchanged (except that it is mapped to these stretched or compressed coordinates). Fig. 3.16 gives a rough idea of the process.

Making this change in the domain is not as difficult as it might seem. To start, think about the following variable

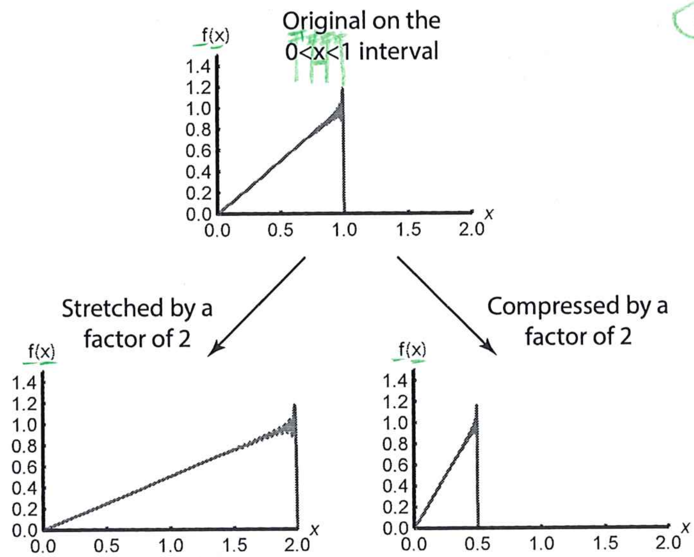


Fig. 3.16 Rescaling the domain of the Fourier series.

$$x = \frac{z}{L}$$

where $z \in [0, L]$. While z on the right-hand side goes from 0 to L, the variable x on the left-hand side only goes from 0 to 1. We have essentially *mapped* the $[0, L]$ onto the $[0, 1]$ interval with this transformation. Why is this useful? Well, let's consider the Fourier sine series for the function $f(x) = x$ that we derived above for $x \in [0, 1]$. The result was

$$f(x) = \sum_{n=1}^{n\to\infty} -\frac{2(-1)^n}{n\pi} \sin(n\pi x), \quad x \in [0, 1]$$

Now, consider the series

$$f\left(\frac{z}{L}\right) = \sum_{n=1}^{n\to\infty} -\frac{2(-1)^n}{n\pi} \sin\left(n\pi\frac{z}{2}\right), \quad x \in [0, 2]$$

This is basically the same series, except the interval is now twice as long. In fact, recognizing that the variable z is just a symbol for the independent variable, we could re-label the variable in this new function back to x if we like. Also, we technically do not need to put parameters like L inside our notation for the function, just the list of independent variables. To make this clear, we can write

$$f\left(\frac{x}{L}\right) = g(x)$$

So, our longer-interval series can be equivalently written (in terms of the independent variable x) as follows (ital)//

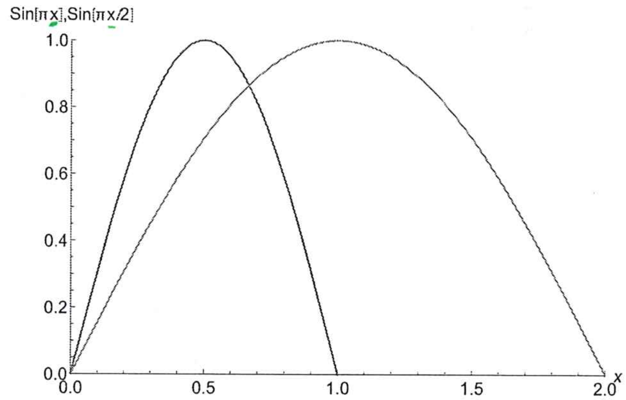


Fig. 3.17 Two sine functions with $n = 1$. One spans twice the distance of the other, but they both correspond to the first *frequency* of the sine series.

Au: Fig. 3.17 not called out in the text.

$$g(x) = \sum_{n=1}^{n\to\infty} -\frac{2(-1)^n}{n\pi}\sin\left(n\pi\frac{x}{2}\right), \quad x \in [0,2]$$

Au: Okay?

that/ Note, this is no longer the Fourier series for $f(x) = x$; it is the Fourier series for ~~we have defined the Fourier series for~~ a *new* function ‡/

$$g(x) = \frac{1}{2}x$$

⊙/

If you think about what happens here, the value of $g(1)$ is the same as the value of $f(1/2)$; similarly $g(2)$ is the same as the value of $f(1)$. So, the net result is the original function f is now stretched out over an interval that is twice as long. This actually is the case that is plotted in Fig. 3.16 (upper plot and lower left plot). So, that is really all that there is to the process of changing the interval. ↑/

If you think about what has happened here, it is really not all that complicated. In short, we have replaced the set of orthogonal basis functions $E_{\sin} = \{0,\ \sin(1\pi x),\ \sin(2\pi x), \cdots, x \in [0,1]\}$ with the new set of basis functions $E_{\sin} = \{0,\ \sin(1\pi x/L),\ \sin(2\pi x?), \cdots, x \in [0,L]\}$.

one/ you// ⊙/

Now, we have on little detail to clean up. Suppose you are given a series on an interval that is not $x \in [0,1]$ to begin with. You could (a) convert the function to an interval that covered $x \in [0,1]$ by an appropriate transformation of variables, and and carry on as we did above, or (b) we could just re-derive the expressions for B_n for the larger interval. Generally, the second of these two options is going to be much more convenient. To do this, we just re-create the process that we did with the unit interval. We end up with the following integrals we need to evaluate

$$\int_{x=0}^{x=L} \sin(n\pi x/L)\sin(m\pi x/L) = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \end{cases} \tag{3.20}$$

‡/

And, we ultimately find that for $x \in [0,L]$

$$B_n = \frac{2}{L}\int_{x=0}^{x=L} f(x)\sin\left(n\pi\frac{x}{L}\right)\,dx$$

Note that this reduces to the same expression for the unit interval when $L = 1$!

3.10 Fourier series on symmetric intervals around zero

Although we have been examining Fourier sine and cosine series on positive intervals such as $I = [0, L]$, in actuality the natural domain for Fourier sine and cosine series are symmetric intervals around zero, $I = [-L, L]$. To understand this, we need primarily just two concepts

1. The concept of *odd* versus *even* functions.
2. The concept that the functions $sin(n\pi x/L)$ and $\cos(n\pi x/L)$ each repeat with a period of $2L$ (N.B., for $n = 1$, the functions repeat exactly once in an interval of $2L$; for $n > 1$ the functions repeat multiple times in an interval of $2L$). note that

We start the discussion with definitions of odd and even functions.

3.10.1 Even and Odd Functions

So far, we have discussed only Fourier series on the interval $[0, L]$, for some positive value for L. In general, there is absolutely nothing preventing a Fourier series to be defined on *any* finite interval $I = \{x : a < x < b\}$ where a and b are any real numbers. However, including the origin in the domain has the potential to create some technical problems for finding Fourier sine and cosine series. Before starting the discussion on Fourier series on more general intervals, we will discuss the concept of *odd* and *even* functions, and their relationship to Fourier series.

Definition 3.1. Suppose a function f is defined on some *symmetric interval* around origin ($x = 0$), i.e., that is $I = [-L, L]$, where a and is some positive real number. Then the function may be *even*, *odd*, or *neither even nor odd* as follows.

- The function f is said to be *odd* on I if $f(x) = -f(-x)$ for all $x \in I$.
- The function f is said to be *even* on I if $f(x) = f(-x)$ for all $x \in I$.
- If neither of these two conditions is true, then f is neither even nor odd on the interval I.

The concepts of even and odd functions are more than just a obscure mathematical property. It turns out that, for every single function defined on a symmetric interval around zero can be decomposed into an even part and and odd part whose sum gives the original function. The proof of this is intuitive and clever, so it will be explained in a few lines.

Theorem 3.2. *Every function, f, that is neither even nor odd on a symmetric interval I around zero can be decomposed into a sum of two functions on that interval, one of which is even, and the other odd.*

Proof. Define two new functions as follows

$$f_{even}(x) = (f(x) + f(-x))/2 \qquad f_{odd}(x) = (f(x) - f(-x))/2$$

It is easy to verify that $f_{even}(-x) = f_{even}(x)$ and $f_{odd}(-x) = -f_{odd}(x)$, and thus these two functions are even and odd, respectively. It is also easy to verify that $f_{even}(x) + f_{odd}(x) = f(x)$. ■

It turns out that this decomposition into even and odd components is also *unique*; that is, there is no other such decomposition, only the one that we constructed above. Examples of the sine function and the cosine function on the interval $-1 < x < 1$ is given in Fig. 3.18 ~~((b) and (d))~~. This graphical presentation allows us to note that the functional definitions given above lead to geometrical properties of even and odd functions that can be observed in their graphs. In particular, note the following

- The cosine function is an *even* function ($\cos(-\pi x) = \cos(\pi x)$). Graphically, we can generate the cosine function on $[-1,0]$ by reflecting the cosine function on $[0,1]$ about the vertical axis.
- The sine function is an *odd* function ($\sin(-\pi x) = -\sin(\pi x)$). Graphically, we can generate the sine function on $[-1,0]$ by reflecting the sine function on $[0,1]$ about the vertical axis, and then a second time about the horizontal axis.

There are a few additional features about even and odd functions that should be mentioned. The first is an assessment of what happens when one multiplies even and odd functions. As an example of even and odd functions to keep in mind, think of the even function $f_{even}(x) = x^2$ and the odd function $f_{odd}(x) = x^3$ on $x = [-1,1]$. Now we state the following theorems (without proof)

Theorem 3.3 (Products of even and odd functions). *Suppose we have and even function $f_{even}(x)$ and an odd function $f_{odd}(x)$ defined on the interval $[-L,L]$. Then, the products of these functions have the following characteristics*

1. *$f_{even} \times f_{even}$ is an even function.*
2. *$f_{even} \times f_{odd}$ is an odd function.*
3. *$f_{odd} \times f_{odd}$ is an even function.*

For easy visualization, these results are summarized graphically in Fig. 3.19.

Theorem 3.4 (The integral of an even and odd functions). *Suppose we have and even function $g_{even}(x)$ and an odd function $g_{odd}(x)$ defined on the interval $[-L,L]$. Then the integral of g_{even} over the interval $[-L,L]$ is non-zero. The integral of g_{odd} over the interval $[-L,L]$ is zero.*

The reason that these two theorems are important, is because they explain what we can and cannot do with Fourier sine and cosine series. In particular, recalling that the sine series is odd, and the cosine series is even, then we have the following results. Suppose $f(x)$ function on the interval $I = [-L,L]$. Then, the following things must be true

1. If $f(x)$ is even, then it can be expanded only as a cosine series. A sine series will not work because each integral would involve an even times an odd function, which is odd; the resulting integrals defining B_n would be zero.
2. If $f(x)$ is odd, then it can be expanded only as a sine series. A cosine series will not work because each integral would involve an odd times an even function, which is odd; the resulting integrals defining A_n would be zero.

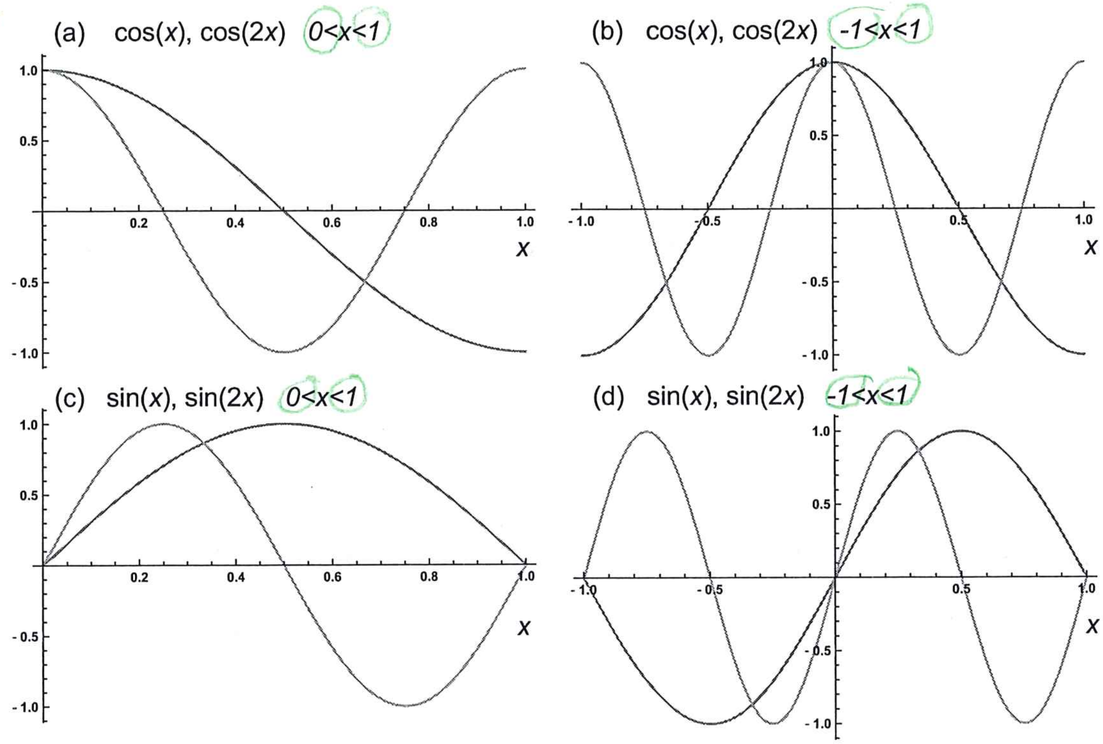


Fig. 3.18: Sine and cosine functions on the intervals $0 < x < 1$ and $-1 < x < 1$. From the definitions in the text (and by examining parts (b) and (d) of this figure), it should be clear that cosine is an even function and sine is and odd function on the interval $-1 < x < 1$.

3. If $f(x)$ is neither even nor odd, then it can be thought of as being the sum of two functions $f(x) = f_{even}(x) + f_{odd}(x)$. The Fourier series must have both sine and cosine components to represent the entire function.

Now that we have an understanding of even and odd functions, we can consider finding the Fourier transform of general functions on symmetric intervals around zero.

3.10.2 Fourier series on the interval $I = [-L, L]$

Now consider the symmetric interval $I = [-1, 1]$. It turns out that the relevant basis functions for this are $\sin(n\pi x)$ and $\cos(n\pi x)$, just as they were for the unit interval $I = [0, 1]$. To see this more clearly, examine the behavior of both the sine and cosine functions on the interval $I = [-1, 1]$ as shown in Fig. 3.18. It is easy to see that the interval $I = [-1, 1]$ allows the sine and cosine functions to go through an integer number of periods; in fact, the functions on the interval $[-1, 0]$ are just reflections of the functions on the interval $[0, 1]$ as discussed above. The basis functions on the interval $I = [-1, 1]$ are exactly the same as those on $I = [0, 1]$; specifically, the functions $\sin(n\pi x)$ and $\cos(n\pi x)$.

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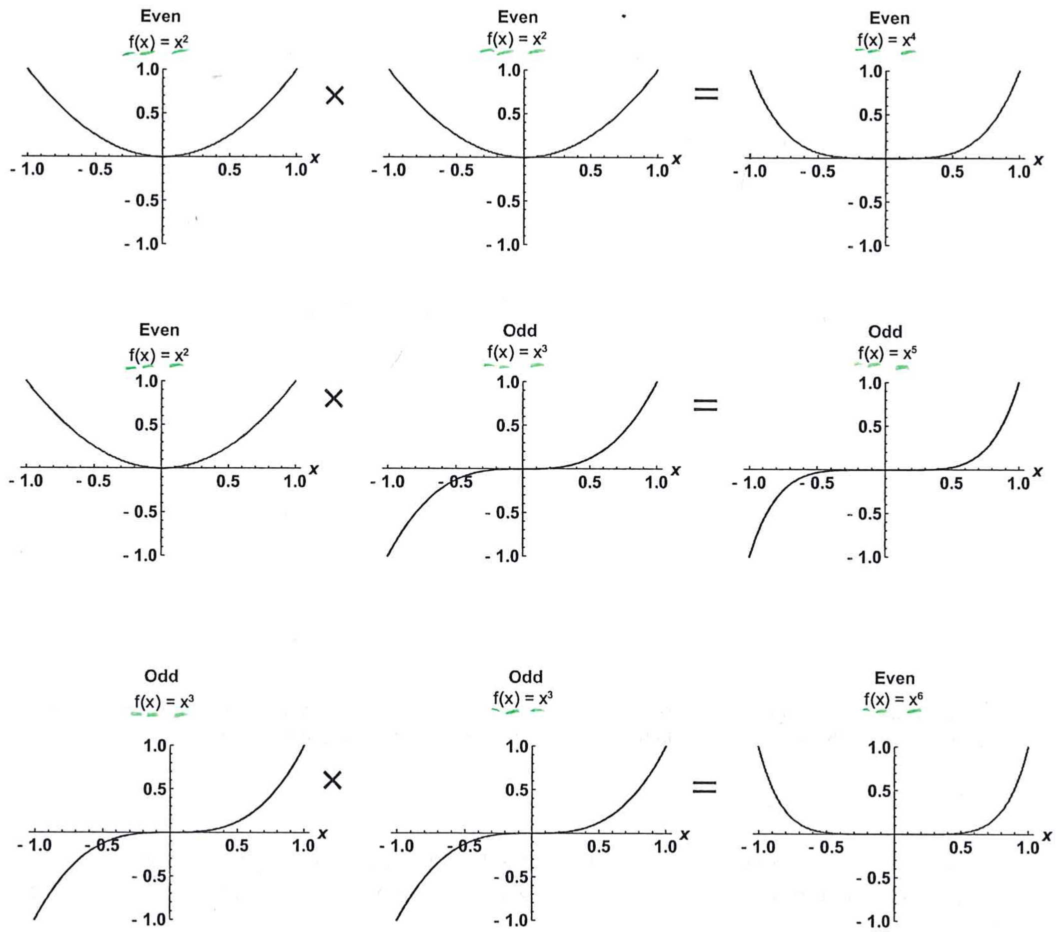

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Fig. 3.19: The result of multiplying even and odd functions. In this figure, the multiplication happens from left to right, across rows.

There is one slightly tricky point about this change of interval. Note that for a symmetric interval $I = [-L, L]$, we can map this interval to the interval $I = [-1, 1]$ simply by dividing independent variable by L. In other words, the new basis functions become $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$. This is worth noting, because we are still dividing by L, which corresponds to only one-half of the full interval width (the interval width is $2L$). This is unlike the case of the change of interval on $I = [0, L]$, where we divided by the entire width of the interval, L.

The best way to think about the relationship between Fourier series on $I = [0, 1]$ versus those on $I = [-1, 1]$ is to think of them as the *same* series, where we are simply ignoring the negative part of the interval. In fact, this is exactly what we are doing when we compute a Fourier series for a function $f(x)$ on an interval $I=[0,1]$. If we expand $f(x)$ as a sine series, then we are ignoring an *odd* extension of the solution on the interval $[-1,0]$. If we expand $f(x)$ as a sine series, then we are ignoring an *even* extension of the solution on the interval $[-1,0]$. This can be made a bit more clear through an example. Before proceeding, however, we note the following integrals on the general interval $I=[-L,L]$

$$\int_{x=-L}^{x=L} \sin(n\pi x/L)\sin(m\pi x/L) = \begin{cases} 0 & n \neq m \\ L & n=m \end{cases} \tag{3.21}$$

$$\int_{x=-L}^{x=L} \cos(n\pi x)\cos(m\pi x)dx = \begin{cases} 0 & if\ m \neq n \\ L & if\ m=n\neq 0 \\ 2L & if\ m=n=0 \end{cases} \tag{3.22}$$

Note that the interval is now $2L$, but the bounds of the integral go from $-L$ to L. This sometimes creates some confusion, so be aware of the details here.

These two integrals are still *orthogonal*, but the case for $n=m$ now evaluates to twice the value we had for the unit interval. This makes sense, since the total domain of integration is twice that of the unit interval. Following this process through to the evaluation of A_n and B_n leads to the (hopefully unsurprising) results

$$\begin{aligned} A_0' &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x)dx, \quad n=1,2,3,\ldots \\ B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x)dx \quad n=1,2,3,\ldots \end{aligned}$$

with

$$f(x) = A_0' + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

or

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

Recall, it is easy to produce the formula for the first term of the cosine series (A_0') if you remember that it is just the *average* of the function.

Example 3.4 (Sine series for $f(x)=x$ on $I=[0,1]$ versus on $I=[-1,1]$). Lets return to the familiar case of the sine series expanded fo the function $f(x)=x$. To begin, lets compute the sine series for this function on the interval $I = [0,1]$. Following the details above, we have

$$B_n = 2\int_0^1 x\cos(n\pi x)dx = \frac{-2(-1)^n}{n\pi}$$

Which gives us the series

Au: Okay as simplified?

$$x = \sum_{n=0}^{n=\infty} \frac{-2(-1)^n}{n\pi}\sin(n\pi x), \quad \text{for } 0 < x < 1$$

This should be familiar; we have computed this series before. Now, let's try the steps that we did previously for the unit interval ~~for the interval~~ $I = [-1,1]$. We start with the definition of the series expansion on $I = [-1,1]$

$$x = \sum_{n=0}^{n=\infty} B_n \sin(n\pi x), \quad \text{for } -1 < x < 1$$

As done previously, we use orthogonality to determine the B_n. Multiplying both sides of the last equation by $sin(m\pi x)$ and integrating gives us

$$\int_{-1}^{1} x\sin(m\pi x)\,dx = \sum_{n=0}^{n=\infty} B_n \int_{-1}^{1} \sin(n\pi x)\sin(m\pi x)\,dx$$

and, using the orthogonality of the sine functions, only one term in the sum on the right hand side is non-zero; this corresponds to the case where $n = m$

$$B_n = \int_{-1}^{1} x\sin(m\pi x)\,dx$$

Finally, computing this integral gives

$$B_n = \frac{-2(-1)^n}{n\pi}$$

which gives us the series

$$x = \sum_{n=0}^{n=\infty} \frac{-2(-1)^n}{n\pi}\sin(n\pi x), \quad \text{for } -1 < x < 1$$

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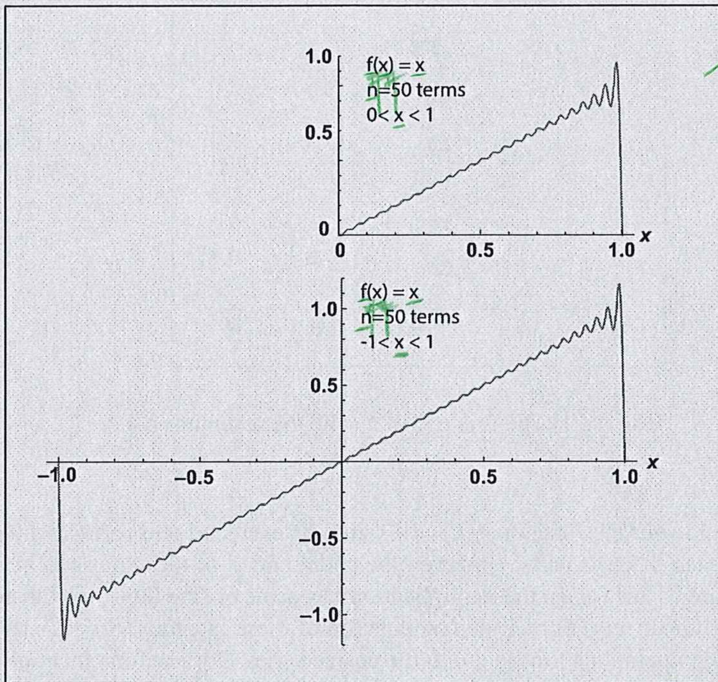

Fig. 3.20: Comparison of the Fourier sine series for the function $f(x) = x$ on the domains $I_1 = [0,1]$ and $I_2 = [-1,1]$. Note that sine is and odd function, thus the function on the symmetric domain $I_2 = [-1,1]$ is also odd.

Comparing this with the result for the unit interval, we find that the two series are identical! Thus, when we expand a function as a sine series on the interval $I = [0,1]$, we really are generating the same result as for the interval $I = [-1,1]$, it is just that we are ignoring the negative portion of the solution (because, by definition of the domain being $I = [0,1]$, it is not in the domain that is of interest to us). A plot can help compare the solution over the two domains; such a plot given in fig. 3.20.

Au: This sentence does not make sense grammatically. Please reword.

The case where a function is neither even nor odd is an interesting one. To start, we will *attempt* to expand the function using only a sine series and then only a cosine series. We will find that *neither* series is able to expand the function by itself. However, the *sum* of the two series will reproduce the function. In short, the sine series reproduces the *odd* part of the function, and the cosine series reproduces the *even* part of the function. An example is really useful here.

Example 3.5 (Cosine and Sine series for $f(x) = e^{-x}$ on $I = [-1,1]$).
The function $f(x) = e^{-x}$ is neither even nor odd on $I = [-1,1]$; this is easy to see by looking at a plot of the function on this interval.

Au: Fig. 3.21 not called out in the text.

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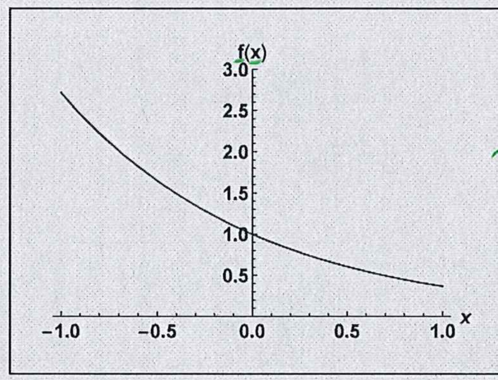

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Fig. 3.21: The function e^{-x} is neither even nor odd.

In principle, we could *split* the function $f(x) = e^{-x}$ into its even and odd parts, and then compute the Fourier series for each of these parts. However, the computation of the series coefficients A_n and B_n will actually take care of this for us; the multiplication by cosine or sine filters out the even and odd parts of the function (respectively) during the computation of these integrals. So, let's try computing the two parts as separate operations, starting with the cosine series. For reinforcement of the ideas of orthogonality, this example will be shown with all steps developed (rather than simply relying on the formulas derived earlier).

$$e^{-x} = \sum_{n=0}^{n=\infty} A_n \cos(n\pi x), \quad \text{where for this problem } L = 1.$$

Multiplying both sides by $\cos(m\pi x)$ and integrating gives

$$\int_{-1}^{1} e^{-x}\cos(m\pi x)dx = \sum_{n=0}^{n=\infty} A_n \int_{-1}^{1} \cos(n\pi x)\cos(m\pi x)dx$$

For the integral on the right, there are three cases that can occur: (1) $m \neq n$, in which case the integral is zero (2) $n = m \neq 0$, in which case the integral is equal to 1 and (3) $n = m = 0$, in which case the integral is equal to 2 (the length of the interval!) Regardless of case, for each value of m chosen, there is only *one* nonzero value of the sum as n goes from zero to infinity. Thus, we can rewrite this equation as (and, remembering that because $n = m$, we are free to set the index to n everywhere)

$$\int_{-1}^{1} e^{-x}\cos(n\pi x)dx = A_m \int_{-1}^{1} \cos(n\pi x)\cos(n\pi x)dx$$

We treat the $n = 0$ case independently. For that case, we find that the integral on the right gives the result of 2, and, solving for A_0', we find

$$A_0' = \frac{1}{2}\int_{-1}^{1} e^{-x}\cos(n\pi x)dx = \sinh(1)$$

For the remaining values of n, we find

$$A_n = \int_{-1}^{1} e^{-x}\cos(n\pi x)dx = \frac{(-1)^n(-1+e^2)}{e(1+n^2\pi^2)}$$

Together, these yield the series

$$f_{even}(x) = \sinh(1) + \sum_{n=1}^{n=\infty} \frac{(-1)^n(-1+e^2)}{e(1+n^2\pi^2)}\cos(n\pi x)$$

Here, the function is labeled "$f_{even}(x)$ because it is not in fact the function $f(x) = e^{-x}$; this can be seen in Fig. 3.22. The associated Fourier cosine series *must be* an even function, and we can verify that it is. Conceptually, this funtion represents the "even" component of the function $f(x) = e^{-x}$.

For the sine series, we will not reproduce the entire sequence of operations leading to the resulting series. Instead, we perform operations analogous to those above, and find the result (presented previously)

$$B_n = \int_{-1}^{1} e^{-x}\sin(n\pi x)dx = \frac{(-1)^n(-1+e^2)n\pi}{e(1+n^2\pi^2)} \tag{3.23}$$

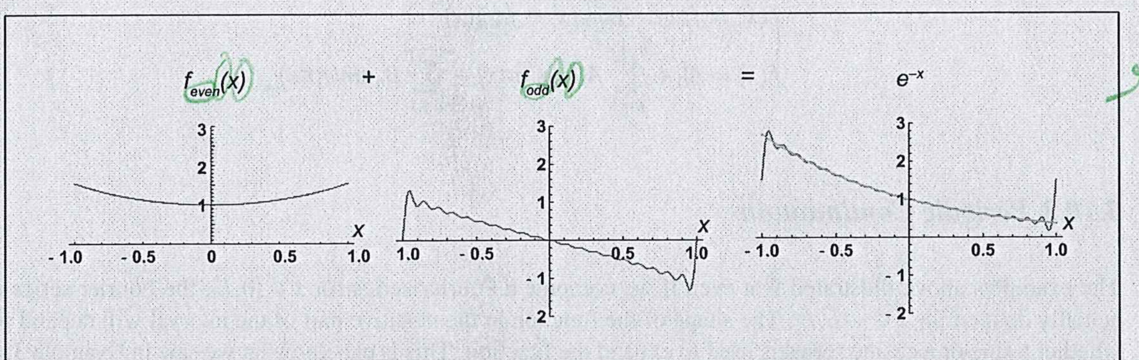

Fig. 3.22: The function e^{-x} is the sum of an even (cosine) Fourier series plus and odd (sine) Fourier series

And, the corresponding series is

$$f_{odd}(x) = \sum_{n=1}^{n=\infty} \frac{(-1)^n(-1+e^2)n\pi}{e(1+n^2\pi^2)}\sin(n\pi x)$$

This function is also plotted in Fig. 3.22. By inspection, it is clear that the sine series is in fact an odd function.

Finally, note that the sum of these two series actually reproduces the function that we seek

$$e^{-x} = f_{odd}(x) + f_{even}(x)$$
$$= \sinh(1) + \sum_{n=1}^{n=\infty} \frac{(-1)^n(-1+e^2)}{e(1+n^2\pi^2)} \cos(n\pi x) + \sum_{n=1}^{n=\infty} \frac{(-1)^n(-1+e^2)n\pi}{e(1+n^2\pi^2)} \sin(n\pi x)$$

Figure 3.22 illustrates the sum of these two series, showing that the sum of the sine and cosine series expansions for the function results in a total series that reproduces the function.

From the example above, we can draw some conclusions.

1. Functions on symmetric intervals around zero are either even, odd, or neither even nor odd. For the latter case, it is *always* possible to decompose the function into unique even and odd parts, i.e., $f(x) = f_{even}(x) + f_{odd}(x)$.
2. Even functions are represented exclusively by Fourier cosine series.
3. Odd functions are represented exclusively by Fourier sine series.
4. For functions that are neither even nor odd, we need to sum both the Fourier cosine and sine series so that both the even and odd components of the function are represented. This expression takes the general form

$$f(x) = f(x) + f_{even}(x) + f_{odd}(x)$$
$$f(x) = A_0' + \sum_{n=1}^{n\to\infty} A_n \cos(n\pi x) + \sum_{n=1}^{n\to\infty} B_n \sin(n\pi x)$$

3.10.3 Periodic Continuation

The examples above illustrated that even if we compute a Fourier series for $x \in [0, L]$, the Fourier series is actually defined for $x \in [-L, L]$. The shape of the function in the negative part of the interval will depend on whether a sine or a cosine series is used to expand the function. This is exactly what we saw in Example 3.4.

It turns out that the Fourier series on an interval $[-L, L]$ is actually a *periodic* function. That is, outside of the interval $[-L, L]$, the function simply repeats by generating copies of the function that is defined for $[-L, L]$.

As an example, consider our cosine and sine series developed for $x \in [0, 1]$ in Example 3.2. Recall the two functions that we found for the unit interval were

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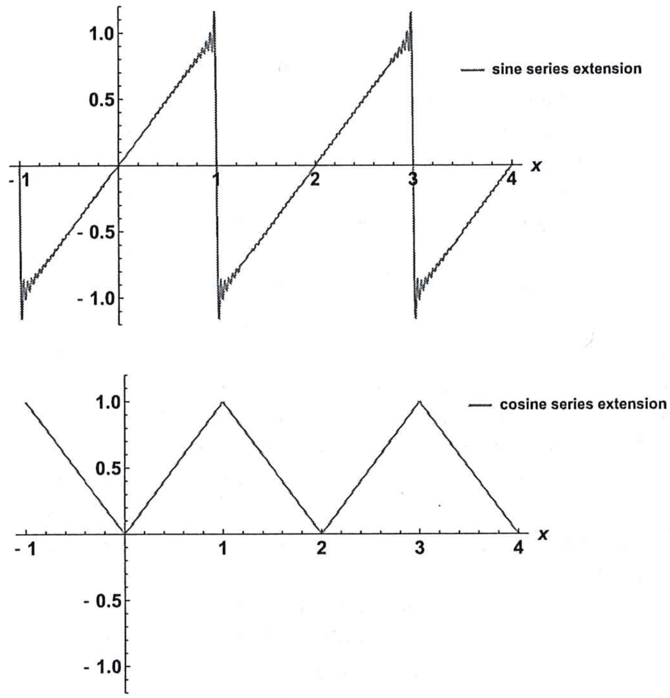

Fig. 3.23 The extensions of the Fourier sine and cosine series for the function $f(x) = x$, x to the interval $x \in [-1,4]$.

cosine function

$$x = \frac{1}{2} + \sum_{n=1}^{n\to\infty} \frac{2(-1+(-1)^n)}{n^2\pi^2}\cos(n\pi x) \tag{3.24}$$

sine function

$$x = \sum_{n=1}^{n\to\infty} \frac{-2(-1)^n)}{n\pi}\sin(n\pi x) \tag{3.25}$$

Now, when we plot them over $x \in [-1,4]$ (Fig. 3.23), we find that each of the functions repeats what is represented on $x \in [-1,1]$. This makes sense each of the basis functions $(\sin(\pi x),\ \sin(2\pi x),\ldots)$ are defined for any value of x, and by construction, each of the trigonometric functions have a period of $2/n$. Thus the lowest frequency functions involved are $\sin(\pi x)$ and $\cos(\pi x)$, and both of these repeat exactly with a period equal to 2. All other functions repeat even more frequently, but each repeats n times over an interval of 2. Therefore the function constructed from the sum of these has no choice but to also repeat with a period of 2. Comparing the sine and cosine series for $f(x) = x$ in Fig. 3.23, it is clear that this is in fact the case. This example also underscores again the differences between the odd (sine) series expansions and the even (cosine) series expansions.

3.11 Introduction to Convergence of Fourier Series

In calculus, the issue of the convergence arises in the study of infinite series. The determination of whether or not a particular series will converge in some sense is a central problem in applied mathematics.

It would not be an understatement to say that the study of the convergence properties of Fourier series actually ushered in a new era of analysis in mathematics. When Fourier series were first introduced, Fourier claimed without proof (starting around 1807, when his initial results were presented) that the method would work for *any* function. It should be noted, however, that even the notion of what properly constituted a *function* and the definition of *integration* were not well-defined at the time. In that context, one can better understand Fourier's overly-enthusiastic proclamations.

Over the ensuing time since Fourier proposed his theory, it has gradually been realized that the theory is much more robust than might have initially been envisioned. While the details of convergence can involve very technical mathematical concepts, it is not incorrect to state that many of the functions that arise naturally in physics and engineering converge in some useful and intuitive sense.

The earliest serious study of the convergence of Fourier series was done by P. Dirichlet, who in 1829 proposed the first theorem regarding the convergence of Fourier series. While by modern standards this theorem is substantially more restrictive than technically necessary, it is a useful touchpoint for understanding the convergence of Fourier series. The conditions proposed by Dirichlet are *sufficient* to gurantee convergence, but not necessary. In other words, there are series for functions that do not meet these conditions, but the series still converge. Regardless, the Dirichlet conditions cover many of the important functions that would arise in more applied problems. While a more detailed discussion of convergence properties of Fourier series will be delayed, we state the Dirichlet conditions as follows.

Theorem 3.5 (Convergence of Fourier Series: Dirichlet conditions.). *The Fourier series for a function f exists on an interval I if the following conditions are met.*

1. *The function is* bounded *over the domain I. This is sometimes stated more rigorously by the idea that the function f is* absolutely integrable, *which means*

$$\int |f(x)|\,dx < \infty$$

2. *The function is piecewise C^1 continuous. This means, by definition, that there exist a* finite number *of points where the right and left derivatives are not equal (i.e., a finite number of discontinuities in the derivative).*

3. *The function f has only a* finite *number of maxima or minima in the interval. In other words, the function can not oscillate infinitely fast anywhere in the interval.*

As a note, the finite number of maxima or minima prohibits functions such as $f(x) = \frac{1}{\sin(1/x)}$ from meeting the conditions because such functions oscillate infinitely fast as $x \to 0$. If the the three conditions above are met, then the function f can be said to have a Fourier series that converges everywhere in the domain, I. The series converges pointwise to its actual value $f(x)$ except at points of discontinuity. At points of discontinuity x_d the series converges to the average value on the two sides of the discontinuity, i.e.,

$$f(x_d) = \frac{1}{2}[f(x_d^+) + f(x_d^-)]$$

Here, $f(x_d^+)$ represents the value of the function at x_d approaching from the right, and $f(x_d^-)$ represents the value of the function at x_d approaching from the left.

Problems

Practice Problems

For the following problems, find the *Fourier sine series* on the interval $x \in [0,1]$ for the function indicated. You do not have to re-derive the formula for the coefficients B_n (although you are certainly welcome to if that works better for you). Plot the function using 50 terms. Also plot the associated spectrum for the first 50 terms.

1. $f(x) = C_1$, where C_1 is a constant
2. $f(x) = x$
   (you will have to use integration by parts)
3. $f(x) = x^2$
   (you will have to use integration by parts twice here)
4. $f(x) = \sqrt{x}$
   (A symbolic mathematics program can be used to compute the integral defining the coefficients B_n)
5. $f(x) = \sin(x)$
   (A symbolic mathematics program can be used to compute the integral defining the coefficients B_n)
6. $f(x) = \cos(x)$
   (A symbolic mathematics program can be used to compute the integral defining the coefficients B_n)
7. $f(x) = \begin{cases} 1 \text{ for } x \le \frac{1}{2} \\ 0 \text{ for } \frac{1}{2} < x \le 1 \end{cases}$
8. $f(x) = \begin{cases} 0 \text{ for } x \le \frac{1}{2} \\ 1 \text{ for } \frac{1}{2} < x \le 1 \end{cases}$
9. $f(x) = \begin{cases} x \text{ for } x \le \frac{1}{2} \\ 0 \text{ for } \frac{1}{2} < x \le 1 \end{cases}$
10. $f(x) = \begin{cases} 1-2x \text{ for } x \le \frac{1}{2} \\ 0 \text{ for } \frac{1}{2} < x \le 1 \end{cases}$
11. $f(x) = \begin{cases} 0 \text{ for } x < \frac{1}{3} \\ 1 \text{ for } \frac{1}{3} \le x \le \frac{2}{3} \\ 0 \text{ for } \frac{2}{3} < x \le 1 \end{cases}$

For the following problems, find the *Fourier cosine series* on the interval $x \in [0,1]$ for the function indicated. You do not have to re-derive the formula for the coefficients B_n (although you are certainly welcome to if that works better for you). Plot the function using 50 terms. Also plot the associated spectrum for the first 50 terms.

12. $f(x) = C_1$, where C_1 is a constant
13. $f(x) = x$
    (you will have to use integration by parts)
14. $f(x) = x^2$
    (you will have to use integration by parts twice here)
15. $f(x) = \sqrt{x}$
    (A symbolic mathematics program can be used to compute the integral defining the coefficients A_n)
16. $f(x) = \sin(x)$
    (A symbolic mathematics program can be used to compute the integral defining the coefficients A_n)
17. $f(x) = \cos(x)$
    (A symbolic mathematics program can be used to compute the integral defining the coefficients A_n)

18. $f(x) = \begin{cases} 1 \text{ for } x \leq \frac{1}{2} \\ 0 \text{ for } \frac{1}{2} < x \leq 1 \end{cases}$

19. $f(x) = \begin{cases} 0 \text{ for } x \leq \frac{1}{2} \\ 1 \text{ for } \frac{1}{2} < x \leq 1 \end{cases}$

20. $f(x) = \begin{cases} x \text{ for } x \leq \frac{1}{2} \\ 0 \text{ for } \frac{1}{2} < x \leq 1 \end{cases}$

21. $f(x) = \begin{cases} 1-2x \text{ for } x \leq \frac{1}{2} \\ 0 \text{ for } \frac{1}{2} < x \leq 1 \end{cases}$

22. $f(x) = \begin{cases} 0 \text{ for } x < \frac{1}{3} \\ 1 \text{ for } \frac{1}{3} \leq x \leq \frac{2}{3} \\ 0 \text{ for } \frac{2}{3} < x \leq 1 \end{cases}$

For the following problems, find the *Fourier series* for the function on the interval indicated. Use the full series expansion (involving both sine and cosine components) unless otherwise noted.

23. $f(x) = C_1$, where C_1 is a constant, $x \in [-1,1]$
24. $f(x) = x$, $x \in [-1,1]$
25. $f(x) = |x|$, $x \in [-1,1]$
26. $f(x) = \exp(2x)$, $x \in [-2,2]$.
27. $f(x) = \exp(2x)$, $x \in [0,2]$; do this as a sine series.
28. $f(x) = \exp(2x)$, $x \in [0,2]$; do this as a cosine series.
29. $f(x) = \frac{1}{1+x}$

Applied and More Challenging Problems

Au: In chapters 2 and 4, Applied and More Challenging Problems start with 1. Please check and possibly correct throughout.

23. **Even and odd extensions.** The interval $x \in [0,L]$ is sometimes called the *half interval,* because both the Fourier sine and cosine series on this interval are also defined on the interval $[-L,L]$. In fact, both the Fourier sine and cosine series are defined on $[-L,L]$, and they are periodic with period $2L$. For the sine series, the extension to $x \in [-L,L]$ should result in an odd function since sine is an odd function on this interval. For the cosine series, the extension to $x \in [-L,L]$ should result in an even function since sine is an even function on this interval.

For the following problems, develop *both the Fourier sine and cosine series* for the interval $x \in [0,L]$ indicated. Then, show that both series are defined on $[-L,L]$ by using the the result you have obtained, but plotting it over $x \in [-L,L]$ using $N = 50$ terms. You should find that the sine series is defined by an *odd* extension of the function that appeared for $x \in [0,L]$, and the cosine series is defined by an *even* extension of the function that appeared for $x \in [0,L]$.

a. $f(x) = x, \;\; x \in [0,1]$
b. $f(x) = x^2 \;\; x \in [0,2]$
c. $f(x) = \begin{cases} 1 \text{ for } x \leq \frac{1}{2} \\ 0 \text{ for } \frac{1}{2} < x \leq 1 \end{cases}$

d. $f(x) = \begin{cases} 0 & 0 < x < \frac{1}{4} \\ 8x - 2 & \frac{1}{4} \le x \le \frac{1}{2} \\ 6 - 8x & \frac{1}{2} < x \le \frac{3}{4} \\ 0 & x > \frac{3}{4} \end{cases}$

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(Note: This implicitly indicates that the function is defined on the interval $x \in [0,1]$.)

24. Periodic extensions. This extension of Fourier sine and cosine series on $x \in [-L, L]$ for subsequent periodic intervals is called *periodic extension*. Because both the sine and cosine function are periodic, with period equal to $2L$, when the series developed for $x \in [-L, L]$ is extended in either the positive or negative direction, the solution repeats because of the periodicity of the underlying trigonometric functions.

For the following problems, develop *both the Fourier sine and cosine series* for the interval $x \in [0, L]$ indicated. Then, show that the that the series is actually a periodic one that repeats every $2L$ by plotting the series for $x \in [-5L, 5L]$. Use $N = 50$ terms.

a. $f(x) = x, \;\; x \in [0,1]$

b. $f(x) = x^2 \;\; x \in [0,2]$

c. $f(x) = \begin{cases} 1 \text{ for } x \le \frac{1}{2} \\ 0 \text{ for } \frac{1}{2} < x \le 1 \end{cases}$

d. $f(x) = \begin{cases} 0 & 0 < x < \frac{1}{4} \\ 8x - 2 & \frac{1}{4} \le x \le \frac{1}{2} \\ 6 - 8x & \frac{1}{2} < x \le \frac{3}{4} \\ 0 & x > \frac{3}{4} \end{cases}$

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(Note: This implicitly indicates that the function is defined on the interval $x \in [0,1]$.)

25. Consider the following function, defined piecewise over the interval $0 \le x \le 1$ (Fig. 3.24).

$$f(x) = \begin{cases} 0 & 0 < x < \frac{1}{4} \\ 16x - 4 & \frac{1}{4} \le x \le \frac{1}{2} \\ 12 - 16x & \frac{1}{2} < x \le \frac{3}{4} \\ 0 & x > \frac{3}{4} \end{cases}$$

Find the Fourier sine series for this function. Plot the function using the first 100 terms in the series.

26. Consider the following function, defined piecewise over the interval $0 \le x \le 1$ (Fig. 3.24). Note that this function is the derivative of the piecewise function given in problem 25.

$$f(x) = \begin{cases} 0 & 0 < x < \frac{1}{4} \\ 16 & \frac{1}{4} \le x \le \frac{1}{2} \\ -16 & \frac{1}{2} < x \le \frac{3}{4} \\ 0 & x > \frac{3}{4} \end{cases}$$

Find the Fourier sine series for this function. Plot the function using the first 100 terms in the series.

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27. There are many requirements as to when a Fourier series can be differentiated term-by-term. While this question is a complex one in general, what we can say is that on the interval $x \in [0,1]$, if $f(0) = f(1) = f'(0) = f'(1) = 0$ *and* the function f is piecewise C^2, then both the Fourier sine and cosine series for $f(x)$ can be differentiated to obtain the appropriate series for $f'(x)$. Show that this is the case by doing the following.

   a. Compute the termwise derivative of the Fourier series for the function given in problem 25.
   b. Plot the function using the first 100 terms in the resulting series.
   c. Compare this sum to the exact solution given by the piecewise function given in 26.

Au: Change to 32 + 33? Problem

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28. For some functions, the evaluation of the sine or cosine series requires extra care and thought because the function $f(x)$ itself is one of the series terms. For instance, consider the following piecewise function

$$f(x) = \begin{cases} \sin(2\pi x) & 0 < x \leq \frac{1}{2} \\ 0 & \frac{1}{2} < x \leq 1 \end{cases}$$

For this function, plotted in Fig. 3.25, the set of coefficients for the Fourier sine series, B_n, contain an apparent singularity (the denominator goes to zero). Thus, the problem requires some additional handling. To see one resolution to this problem, complete the following steps.

   a. Compute the Fourier sine series for this problem. Verify that the result for the coefficients is

$$B_n = \frac{4\sin\left(\frac{1}{2}n\pi\right)}{(-4+n^2)\pi}$$

   Note that for $n = 2$, this result presents a problem. The numerator is zero, and the denominator is zero, leading to a $0/0$ indefinite form.

   b. There are a few ways in which this problem can be handled. While the result for B_n is correct, the indeterminate form arises because we have computed the integral for the general case. The indeterminate form is avoided if we consider the integral for the $n = 2$ case directly, because we can make obvious simplifications that are not obvious if we consider the more general case. In particular, note that the integration for the $n = 2$ case is

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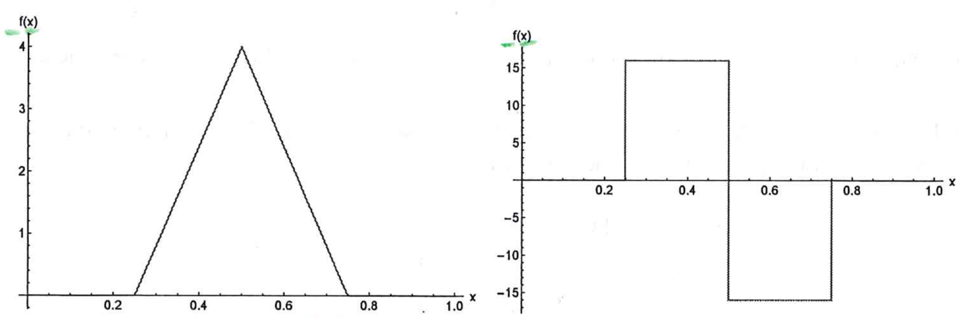

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Fig. 3.24: The piecewise function defined for problems 25 (left) and 26 (right).

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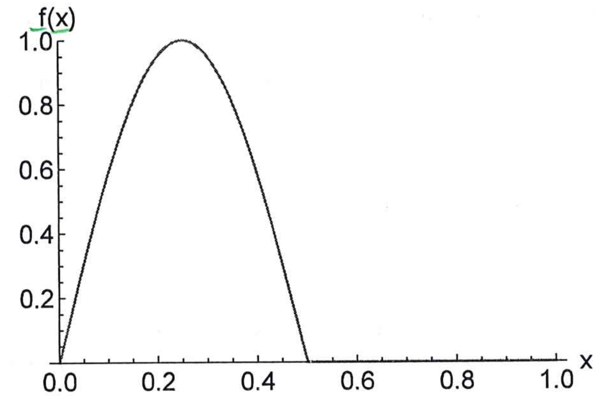

Fig. 3.25 The piecewise function defined for problems 28.

Au: Change to 35?

$$\int_0^{\frac{1}{2}} \sin(2\pi x)\sin(2\pi x)\,dx = \int_0^{\frac{1}{2}} \sin^2(2\pi x)\,dx$$

Once one combines the two functions into $\sin^2(2\pi x)$, the indefinite form no longer occurs. Show that the integration above leads to the result $B_2 = \frac{1}{2}$.

c. There is a second, equivalent way to handle this problem. Recall from your study of calculus L'Hôpital's rule, which states that for an $0/0$ indefinite form

$$\lim_{n\to n_0} \frac{f(n)}{g(n)} = \lim_{n\to n_0} \frac{f'(n)}{g'(n)}$$

Even though the function B_n is not a continuous one, we can certainly treat it as though it were. Show that, treating n as a continuous variable, you can use L'Hôpital's rule on B_n in the form

$$B_2 = \lim_{n\to 2} \frac{4\sin\left(\frac{1}{2}n\pi\right)}{(-4+n^2)\pi}$$

to show that $B_2 = \frac{1}{2}$.

d. Plot the Fourier sine series using $N = 10$ terms, and the piecewise function to compare them.

36 29. The function $f(x) = \exp(-(ax)^2)$ (this is the classical *Gaussian* function used in statistics and physics) has a curious property. Its spectrum has very close to the same shape as the function itself (assuming that a is large enough such that $f(x) << 1$ as $x \to L$, $x) \in [-L, L]$. For this problem, assume that the interval of interest is $x \in [-1, 1]$.

a. To see that the spectrum looks like the original function, compute the Fourier cosine series for this function for the case. To do so, you will need to note the following integral

$$\int_0^1 \exp(-a^2x^2)\cos(n\pi x)\,dx = \begin{cases} \frac{1}{2a}\sqrt{\pi}\,\mathrm{erf}(a), & n=0 \\ \frac{1}{a}\sqrt{\pi}\exp\left(-\frac{\pi^2 n^2}{4a^2}\right)\mathrm{erf}(a), & n>0 \end{cases}$$

b. Assume that $a = 3$, so that $f(x) = \exp(-(3x)^2)$. On two separate plots, plot (i) The original function $f(x)$ on $x \in [0, 1]$, and (ii) $B_n = B(n)$. For the latter function, we can using the following plotting

use/

trick to extend the function to an interpolated version that is smoother looking. The trick is as follows: instead of plotting $B(n)$, plot the coefficient $B(n/100)$. For this example, plot $B(n/100)$ over the interval $n \in [0, 600]$. This should give an interpolated plot that looks smooth (although the values of B_n appear at every division of 100 units on the horizontal axis).

In Mathematica, one command to plot this latter function is given by the following:

```
DiscretePlot[B[n/100], {n, 1, 600}, ExtentSize -> Full,
ColorFunction -> "Rainbow", PlotRange -> All, AxesLabel -> {"n", "B(n)"},
ImageSize -> Large, PlotRange -> All]
```

30. Proving that that the sine and cosine functions are *orthogonal* is something that was not shown in the text, but it is not difficult to illustrate. To do so, one needs to use the following trigonometric identities:

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$
$$\sin(A)\sin(B) = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

Using these identities, show that the following are true.

$$\int_{x=0}^{x=1} \cos(n\pi x)\cos(m\pi x)\,dx = \begin{cases} \frac{1}{2} \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases}$$
$$n = 1, 2, 3, \ldots$$
$$\int_{x=0}^{x=1} \sin(n\pi x)\sin(m\pi x)\,dx = \begin{cases} \frac{1}{2} \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases}$$
$$n = 1, 2, 3, \ldots$$

NOTE: Please work this out carefully by hand.