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“Science is a partial differential equation. Religion is a boundary condition.”

Alan Turing, genius developer of modern computing theory.

# 5

## Introduction to Partial Differential Equations, Conservation Laws, and Constitutive Equations

### 5.1 Introduction

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Partial differential equations (PDEs) are the mathematical workhorses of continuum systems. The term *partial* differential equation indicates that the dependent variable depends on two or more independent variables. This is in contrast to an ordinary differential equation (Chapter 2), where there is always one dependent and one independent variable. For this chapter, we will use the symbol  $u$  to indicate the independent variable, and the conventional symbols  $x$ ,  $y$ ,  $z$ , and  $t$  to represent the independent variables corresponding to three spatial dimensions and one time dimension. Most of the equations in this introductory chapter on PDEs will be linear and 2-dimensional, meaning one time and one space dimension, or two space dimensions.

### 5.2 Terminology

- **Extensive property.** An extensive property is one that depends upon the size of the domain of interest. As an example, the mass of material in a domain is an extensive property of that domain.
- **Intensive property.** An intensive property is one that is independent of the size of the domain of interest. While mass is an *extensive* property of a domain, density is an *intensive* property, because it depends in no way upon the total size of the domain.
- **Flux.** A flux is the amount of some extensive property passing through a unit area per unit time. Technically, a flux may be more generally defined in terms of vector quantities; under these circumstances the flux is defined as the amount of the extensive property passing through a unit area per unit time, where

the unit area is defined with respect to a specific unit vector indicating the direction normal to the surface; both normal and tangential fluxes can be defined in multiple dimensions. In one space dimension these additional distinctions are not material.

- **Partial derivative.** Partial derivatives are defined whenever there is more than one independent variable defined for function. To be a little more concrete, consider derivatives of a function of one space dimension,  $x$ , and one time dimension,  $t$ . The partial derivative in space would be formulated as for an ordinary derivative in space, where the time component of the function were simply held constant.
- **Partial differential equation (PDE).** A differential equation that contains more than one independent variable. The equations themselves are composed of algebraic combinations of functions multiplied by partial derivatives. Often, such equations arise from physical considerations of continuum systems, although they may arise in any number of ways from any number of disciplines. For such equations, partial derivatives do not necessarily have to appear for each independent variable.
- **Order of a PDE.** The order of a PDE is given by the highest order of derivative that appears in the equation.
- **Transient PDE.** Assume we have a partial differential equation with one space and one time variable. A transient PDE is one where an initial condition evolves to new configurations in the spatial dimension as time changes.
- **Steady state conditions.** The steady state for a PDE is the condition where the time partial derivative is identically zero for all other combinations of the remaining independent variables. For example, in one space and one time dimension with domain  $x \in [a, b]$ , the steady state is defined by  $\partial u(x, t) / \partial t = 0$  for all points in  $[a, b]$ ; note that if this occurs at some finite time,  $t_1$ , then it must also be true that  $\partial u(x, t) / \partial t = 0$  for all  $t > t_1$ . In other words, once the system achieves steady state, it remains in that state (unless the system is perturbed by external forces).
- **Ancillary conditions.** Like for ODEs, ancillary conditions are the additional information required to determine a *particular* solution from the set of all possible solutions to a PDE. Unlike ODEs, because more than one independent variable is involved, the ancillary conditions technically specify *whole functions* rather than just constants (although constant functions are very common!).
- **Parabolic PDE.** In one time and one space dimension, a parabolic PDE is one characterized by an initial configuration that evolves smoothly over time. The classical example of a parabolic PDE is the heat or diffusion equation. For that equation, an initial distribution spreads out over time, becoming smoother and more uniform over time.
- **Hyperbolic PDE.** In one time and one space dimension, a hyperbolic PDE is one that represents wave-like behavior. An initial perturbation is translated through the domain either without changing shape, or with shape changes (which can result either from self-sharpening behavior in nonlinear wave equations, to dissipating behavior in equations that represent energy loss from waveforms as they translate).
- **Elliptic PDE.** There is no exact analogue for an elliptic PDE in one time and one space dimension. However, in one space dimension alone, an elliptic PDE can be considered to be the *steady state* version of

a corresponding parabolic PDE. As an example, a steady-state diffusion-reaction problem can generate a solution in which the solution is not uniform in space, but at each spatial point the solution does not change in time. Physically, this would indicate that the processes of diffusion and reaction balanced one another.

- **Well-posed PDE problem.** A well-posed problem in the context of PDEs means the following.

1. The solution exists. In other words, there is sufficient information provided by the ancillary conditions such that the problem can be solved.
2. The solution is unique. This requirement indicates that not only must solutions be found, but one must have enough information to generate a *particular* solution to the problem. Another way of stating this is that the solution should not represent a *class* of functions, but, rather, a specific function that is computable given all of the parameters involved.
3. The solution must depend continuously upon the initial data. This concept is a bit less obvious than the other two, but, in short, it means only that the solution to the problem cannot behave chaotically. Here, not behaving *chaotically* means that very small changes (or *perturbations*) to the initial conditions do not lead to dramatically different results. This is a feature of well-posed linear equations: small perturbations to the system lead only to small changes in the solution.

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## 5.3 Partial Derivatives and the Types of Partial Differential Equations

Many processes in science and engineering are governed by partial differential equations. Essentially, most systems that can be approximated as being a *continuum* and where one needs to understand the time-space behavior of the system (as opposed to the integrated qualities of the system as appears in material balances) are best described by partial differential equations. Before continuing with examples, however, it is useful to recall the definition of a *partial derivative*.

7/1

### 5.3.1 Partial derivatives

A partial derivative describes the slope of a curve that is a function of more than one independent variable. For example, suppose a function  $u$  depends upon the spatial variables  $x$  and  $y$ . Then, the derivative in the  $x$ - and  $y$ -directions are given by

$$\lim_{\Delta x \rightarrow 0} \frac{\partial u(x, y)}{\partial x} \equiv \frac{[u(x + \Delta x, y) - u(x, y)]}{\Delta x} \quad (5.1)$$

$$\lim_{\Delta y \rightarrow 0} \frac{\partial u(x, y)}{\partial y} \equiv \frac{[u(x, y + \Delta y) - u(x, y)]}{\Delta y} \quad (5.2)$$

(5.3)

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*Au:* Equations will need to be renumbered from this point forward.

Here it is understood that  $\Delta x$  is a positive value. Similarly, the second derivative with respect to  $x$  and  $y$  can be specified by

$$\lim_{\Delta x \rightarrow 0} \frac{\partial^2 u}{\partial x^2} \equiv \frac{[\frac{\partial u}{\partial x}(x + \Delta x, y) - u(x, y)]}{\Delta x} \quad (5.4)$$

$$\lim_{\Delta y \rightarrow 0} \frac{\partial u(x, y)}{\partial y} \equiv \frac{[u(x, y + \Delta y) - u(x, y)]}{\Delta y} \quad (5.5)$$

These expressions can be iterated to derive even higher-order derivatives. As a final note, there are a few common methods for indicating partial derivatives in PDEs. The other conventional method is to use subscripts to indicate the independent variable of differentiation. The subscript is repeated  $n$  times to indicate an  $n^{th}$  order derivative. Thus, the following are equivalent

$$\frac{\partial u}{\partial x} \Leftrightarrow u_x \quad \frac{\partial u}{\partial t} \Leftrightarrow u_t \quad (5.6)$$

$$\frac{\partial^2 u}{\partial x^2} \Leftrightarrow u_{xx} \quad \frac{\partial^2 u}{\partial t^2} \Leftrightarrow u_{tt} \quad (5.7)$$

### 5.3.2 Characterization of linear PDE types

Partial differential equations can be characterized primarily by three major features.

- Equation order.** Partial differential equations are characterized in part by the order of the highest derivative (in any independent variable). This is known as the *order* of the PDE. The vast majority of PDEs seen in science and engineering are of first or second order, although equations up to fourth order are sometimes encountered. Equations above fourth order are seldom encountered, but they occasionally do arise as the description of some physical process. All first order equations in  $2$  independent variables describe wave-like behavior; these are listed in Table 5.1. Some examples of second-order equations are given in Table 5.2. Relevant examples of linear equations of order higher than two are given in Table 5.3.
- Equation type.** There are several well-studied behaviors seen in linear partial differential equations. In particular, for linear second-order partial differential equations *in two independent variables* of the form

$$a \frac{\partial^2 u}{\partial t^2} + 2b \frac{\partial^2 u}{\partial t \partial x} + c \frac{\partial^2 u}{\partial x^2} + d \frac{\partial u}{\partial x} + e \frac{\partial u}{\partial t} + f u + g = 0 \quad (5.8)$$

every such can be described as exhibiting three types of behavior. Note that more than one type of behavior may be exhibited by the same equation if the coefficients are functions; that is, the equation type can change as a function of time or space. For equations with constant coefficients, only one equation type is defined, and it does not depend on any of the independent variables.

With the appropriate change of variables, every equation of the form of Eq. (5.8) can be put in its *canonical form*. While determining the equation type is a useful exercise, it does not necessarily tell us all that we need to know about a particular problem. For example, the fact that the general equation given by Eq. (5.8) can always be put into its canonical form by a coordinate transformation completely ignores the fact that such a transformation may make any associated boundary and initial conditions

Q: Why are the three equations below displayed within boxes?

so complex as to render the problem nearly unsolvable. Thus, the idea of the canonical form must be taken with the proverbial grain of salt. While it is useful information, in more practical problems this information may be of minor importance.

- a. **Parabolic.** The canonical form for the parabolic equation with one time and one space variable is

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

These equations have generally smooth, dissipative solutions. The heat and diffusion conservation equations are examples.

- b. **Hyperbolic.** The canonical form for the parabolic equation with one time and one space variable is

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

Hyperbolic equations are characterized by representing wave-like behavior. These waves can be smooth and periodic (like ocean waves) or sharp, like the wave front of a compressible gas that creates a sonic boom.

- c. **Elliptic.** The canonical form for the parabolic equation with two space variables is

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

While mathematically, there can be time-dependent elliptic equations, most physical applications come from steady-state descriptions of problems that are determined by their boundary conditions. The steady, 2-dimensional distribution of heat in a plate would be an example. Steady groundwater flow is also an example of an elliptic equation.

There are some problems with formal classification systems for PDEs that prevent them from being universally useful. For example, there is a classification system for all linear second-order PDEs with two (or more) independent variables. While such classifications can provide some insight, they still have weaknesses. For example, in these classification systems, the role of derivatives less than order 2 are not considered at all. This can create some difficulties, especially when these first-order terms dominate the problem. As an example, the classical convection-diffusion equation in one spatial dimension can be given by

$$\frac{\partial u}{\partial t} = -v_0 \frac{\partial u}{\partial x} + D \frac{\partial^2 u}{\partial x^2}$$

While the traditional classification schemes presented in most texts on partial differential equations would qualify this as a *parabolic* equation, this is not an entirely satisfactory answer. In particular, when the convective term (the first-order term) is much larger than the diffusive term (the second-order term), then this equation behaves as a *hyperbolic* equation. This is of some significance, because, for example, numerical methods for solving parabolic (smooth) and hyperbolic (sometimes abrupt) equations can be quite different. As a second example, consider the following elliptic equation in two spatial variables

$$K_x \frac{\partial^2 u}{\partial x^2} + K_y \frac{\partial^2 u}{\partial y^2} = 0$$

while this is a perfectly acceptable elliptic PDE, we encounter a problem in the limit  $K_y \rightarrow 0$ . In that limit, our equation is suddenly no longer elliptic by the conventional criterion. While it is true that it can be then considered to be an ordinary differential equation (since there is only one independent variable), it does raise significant problems.

Therefore, in this text while we will use the terms *parabolic*, *hyperbolic*, and *elliptic* to be descriptive, there will be no special effort made to cover the formal methods of characterizing linear PDEs by invoking a transformation of the coordinates. For higher-order PDEs, technically there is no conventional classification system; however, occasionally one of the categorizations listed above (parabolic, hyperbolic, elliptic) will be used if the solutions to the higher-order equation share similar features as the behavior for a second order equation.

**3. Linear or nonlinear.** Linear equations describe many phenomena in engineering and science, often as an approximating behavior of a more general nonlinear equation. The linearity of a PDE can be determined using the same techniques that are used to establish linearity for ODEs. For some phenomena (e.g., turbulence in the Navier-Stokes equations), nonlinearity is essential to phenomenon being represented. A few examples of nonlinear problems are given in Table 5.4. While nonlinear problems are both relevant and significant, their solutions generally require specialized methods that are beyond the scope of this introductory text.

Table 5.1 Examples of first-order linear PDEs with two independent variables. All first order PDEs represent waves of one form or another.

Type	Independent Variables in 2 Dimensions	Name	Example Application	Example Equation
One-dimensional waves	space, $x$ time, $t$	convection equation	Solute transport	$\frac{\partial u}{\partial x} = -v_0 \frac{\partial u}{\partial t}, \quad v_0 > 0$
One-dimensional waves	space, $x$ time, $t$	The Maxwell equations	Electrodynamics	$\frac{\partial E}{\partial x} = \frac{1}{c} \frac{\partial B}{\partial t}$ $\frac{\partial B}{\partial x} = \frac{1}{c} \frac{\partial E}{\partial t} + J$

## ✓ 5.4 The Origins of Partial Differential Equations in Science and Engineering: The Axioms of Conservation

There is a nearly inexhaustible supply of applications of PDEs in engineering and science. Essentially, any systems that are subject to the laws of continuum mechanics are usually expressed as partial differential equations. Usually, these partial differential equations arise by considering conservation principles. Conservation principles are *axiomatic* statements (or *laws*) that cannot be proven from more fundamental ideas, but they are consistent with all other information and measurements that exist. The primary axiomatic statements regarding conservation principles in continuum mechanics can be listed as follows.

Table 5.2: Examples of second-order linear PDEs with two independent variables

Type	Independent Variables in 2 Dimensions	Name	Example Application	Example Equation
Parabolic	space, $x$ time, $t$	The heat equation	Heat or mass transport	$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$
Elliptic	space, $x$ space, $y$	The Laplace equation	Steady groundwater flow	$K \frac{\partial^2 u}{\partial x^2} + K \frac{\partial^2 u}{\partial y^2} = 0$
Hyperbolic	space, $x$ time, $t$	The wave equation	Movement of deep water waves	$\frac{\partial^2 u}{\partial t^2} = K \frac{\partial^2 u}{\partial x^2}$
(none)	space, $x$ time, $t$	The linear Korteweg-de Vries equation	Movement of shallow water waves	$\frac{\partial u}{\partial t} = K \frac{\partial^3 u}{\partial x^3}$
(none)	space, $x$ time, $t$	The vibration equation	Vibrations in beams or thin plates	$\frac{\partial^2 u}{\partial t^2} = K \frac{\partial^4 u}{\partial x^4}$

Table 5.3: Examples of a few linear, higher-order PDEs with two independent variables

Name	Independent Variables in 2 Dimensions	Application	Equation
The linear Korteweg-de Vries equation	space, $x$ time, $t$	Movement of shallow water waves	$\frac{\partial u}{\partial t} = K \frac{\partial^3 u}{\partial x^3}$
The vibration equation	space, $x$ time, $t$	Vibrations in beams or thin plates	$\frac{\partial^2 u}{\partial t^2} = K \frac{\partial^4 u}{\partial x^4}$

Table 5.4: Examples of some important nonlinear PDEs in two independent variables.

Name	Independent Variables in 2 Dimensions	Application	Equation
Nonlinear diffusion	space, $x$ time, $t$	Solute transport at high concentrations	$\frac{\partial u}{\partial t} = D(u) \frac{\partial^2 u}{\partial x^2}$
Burgers' equation	space, $x$ time, $t$	one-dimensional analogue to Navier-Stokes	$\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} + v \frac{\partial^2 u}{\partial x^2}$
The nonlinear Korteweg-de Vries equation	space, $x$ time, $t$	shallow water waves with nonlinear restoring function	$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} - 6u \frac{\partial u}{\partial x}$

1. **Conservation of mass.** This is encapsulated in the (approximate) idea that matter can be neither created nor destroyed. In reality, it is possible to convert some matter to energy (and vice versa), but this is not a usual situation in continuum mechanics, so the conservation of mass can be assumed to be valid. An example of a conservation of mass expression would be the continuity equation that is encountered in fluid mechanics.
2. **Conservation of momentum.** In continuum systems, the conservation of momentum is analogous to Newton's laws for discrete bodies. In short, conservation of momentum for continuum systems states that Newton's laws apply not only to discrete bodies, but to any portion one can imagine being cut out of a discrete body. Conservation of momentum is where we get the important equations describing the flow of fluids (the Navier-Stokes equations) or the forces distributed in a solid container under pressure. Technically, there are two parts of the axiom of conservation of momentum: Conservation of linear momentum, and conservation of angular momentum. Most of the material in this text will focus only on problems of linear momentum.
3. **Conservation of energy.** Conservation of energy is a complementary axiom to of the conservation of mass, and is usually stated that energy can be neither created nor destroyed. As mentioned above, however, we know that mass and energy can be interchanged under somewhat extreme conditions (e.g., when objects are moving very fast, are very small, or quantum mechanical interactions are relevant). Generally, continuum mechanics does not consider such systems, so the conservation of energy can safely be taken to be a truthful axiomatic statement.

As a related note, each of the quantities above represents a thermodynamically *extensive* quantity; that is, the value of the mass, momentum, or energy computed depends upon the physical size of the system. Conversely, one may consider any of these three quantities on a per volume basis (i.e., the mass density, the momentum density, or the energy density), which makes these densities intensive quantities. For the most part, in continuum mechanics it is best to discuss the dependent variables in terms of densities rather than extensive quantities. For later use, we will use lower case letters (in particular, the letter  $u$ ) to indicate intensive variables, and upper case letters (e.g.,  $U$ ) to indicate extensive variables.

Usually, the development of partial differential equations come from application of these conservation laws to systems that can be considered to be a *continuum*. A continuum is simply any region of space filled with a material in which can be assumed to behave as if it varied smoothly from point to point. We know that, fundamentally, this cannot be true because all matter is composed of atoms, which are distinctly not continuous at the molecular scale. However, when we consider large numbers of atoms together, their behavior can often be well approximated as if it were a continuous system. In fact, this is the basic idea behind the field of continuum mechanics. In continuum mechanics, a material is represented by *representative volumes* of atoms that are numerous enough that they behave as if they were a continuum. A coherent example of a representative volume can be given by the application of the ideal gas law to noble gases. Noble gases tend to obey the ideal gas law rather closely at near standard temperature and pressures. Clearly, a gas is made up of a collection of individual atoms or molecules, but if there are a large enough number of them, then they behave as if they can be described by as a single bulk medium. Hence, a large enough molar concentration of a noble gas at nearly earth-surface temperatures and pressures will be well described by the ideal gas law.

However, if one could (at least conceptually) maintain a small number of gas atoms (say,  $n = 10$  atoms) at room temperature in, say, a 1 liter volume, the ideal gas law would fail to predict the pressure very accurately. In fact, the very definition of pressure (force per unit volume) would be challenging for a such a small number of atoms. The problem is that the amount of momentum transfer between a small number of atoms and the walls of the container would be a rapidly varying function of time. Because we think about the molecules as having more-or-less random thermal motion (i.e., they have kinetic energy that is proportional

to the temperature), with only 10 atoms in a 1 liter container, the atoms collide with the walls in a way that appears somewhat random.

To make this clearer, suppose we fix a time interval  $\Delta t$  such that, on average, 5 of the molecules contact the walls of the container over the time period. Of course, with so few molecules involved, sometimes this number would be higher (say, 7 or 8 contact the wall in  $\Delta t$ ) and sometimes lower (say 2 or 3 contact the wall in  $\Delta t$ ). Viewed this way, the number of atoms having collided with the walls of the container would be a random variable. Thus, the pressure that one might measure would vary wildly from near zero (when no molecules contact the container walls) to a maximum value when all ten molecules contact the surface over the time interval  $\Delta t$ . Clearly, this situation would lead to a pressure that varied in time, proportional to the number of molecules that contacted the surface over the time interval. Therefore the ideal gas law would only be met in an average sense. If the molecules are assumed to behave independently, then the variance of the number of collisions with the walls would be inversely proportional to the number of molecules involved, consistent with the Law of Large Numbers that you may have studied in statistics. Thus, one can always drive down the variance of the behavior of our example gas by increasing the number of molecules. With a very large number of molecules, the variance rapidly becomes incredibly small. Under these conditions, the gas can be treated as a continuum, and the ideal gas law would hold with very high fidelity.

## 5.5 Derivation of Differential Conservation Equations

Assuming that a system can be treated as a continuum, then one can develop conservation equations on the basis of differential balances for the quantity of interest. In this section, we will focus on generating a general balance equation for mass, linear momentum, or energy, regardless of which quantity is of interest. Each of these quantities subscribe to the same fundamental axiom for conservation. Following the discussion above, this axiom can be stated as follows.

**Axiom 1** (Axiom of Conservation for Mass, Momentum, or Energy—Form 1). *For any isolated system, the amount of mass, momentum, or energy contained in that system is a constant.*

While this axiom is probably familiar and is absolutely correct, it is not necessarily the most useful form of the axiomatic statement for mass, momentum, and energy that we can make for the purposes of making computations. The problem is that we are often interested in sub-volumes of an isolated system. Fortunately, there are other statements of the axiom of conservation of mass, momentum, and energy that are equivalent to the version given above, but are more useful for applications to sub-volumes of isolated systems.

**Axiom 2** (Axiom of Conservation for Mass, Momentum, or Energy—Form 2). *For any sub-volume cut from an isolated system, the following balance is maintained in accordance with the laws of conservation of mass, momentum, and energy.*

The alternate form of the axiom can be stated as follows. For any volume  $V$  cut out of a continuum, the following balance law is valid for the extensive variable  $U$  (where  $U$  represents the mass, momentum, or energy in the volume  $V$ ).

$$\left\{ \begin{array}{l} \text{rate of} \\ \text{accumulation} \\ \text{of } U \text{ in } V \end{array} \right\} = \left\{ \begin{array}{l} \text{rate of } U; \\ \text{entering the} \\ \text{volume } V \end{array} \right\} - \left\{ \begin{array}{l} \text{rate of } U \\ \text{leaving the} \\ \text{volume } V \end{array} \right\} + \left\{ \begin{array}{l} \text{rate that sources or reactions} \\ \text{change the amount} \\ \text{of } U \text{ in the volume } V \end{array} \right\} \quad (5.9)$$

This form of the axiom of conservation is equivalent to the first, but it is more useful for developing a computable balance because we often want to do this for some fixed sub-volume of a system rather than for the system as a whole. The axiom gives us a direct way to setting up the balance; we need only convert our statement given in words into an equivalent statement using the language of mathematics.

To begin, we define a the sub-volume of the isolated system as illustrated in Fig. 5.1. In this introduction to computing balances, we will impose some constraints so that the problem can be considered to vary only in one spatial direction (the  $x$ -direction). Thus, we make the following, rather strong assumption.

**Assumptions.** All intensive properties of interest change only in the  $x$ -direction; thus, the intensive property of interest is constant over any cross-sectional area perpendicular to the  $x$ -direction. This necessarily requires that the cross-sectional area,  $A$  be constant.

We will consider more general balances in fully 3-dimensional systems in later (optional) material.

To start determining the mathematical statement of the balance, we note the following.

1. The intensive variable of interest (mass density (concentration), momentum density, or energy density) is denoted by the dependent variable  $u(x, t)$ . The variable  $u$  depends only on the independent variables  $x$  and  $t$ .
2. The flux,  $j$  is defined as the amount of mass, momentum, or energy passing through a unit area in a unit time. Thus, the fluxes described here must have units of mass per unit area per unit time, momentum per unit area per unit time, or energy per unit area per unit time. Computing the total mass, momentum, or energy crossing an area  $A$  per unit time would be given by

$$\left\{ \begin{array}{c} \text{rate of} \\ U \\ \text{crossing } A \end{array} \right\} = \int_{y, z \in A(x)} j(x, y, z, t) d\eta dz$$

$$= A(x) j(x)$$

$$= A j(x) \quad (5.10)$$

where the second line of this expression results because we have assumed explicitly that all properties over the area perpendicular to the  $x$ -direction are constant (thus,  $j(x, y) = \text{const}$  on  $A(x)$ ), and the last line occurs because the cross-sectional area is assumed to be constant. Note that fluxes are signed quantities. In the notation here, fluxes pointed in the positive  $x$  direction have a positive sign; fluxes pointed in the negative  $x$  direction would have a negative sign. This sign convention is inherent in our statement of the balance axiom given by (2). Although we have drawn the fluxes in Fig. 5.1 as though they were positive, our mass balance axiom is correct regardless of the actual sign of the fluxes at the locations  $x$  and  $x + \Delta x$ .

3. The total amount of the extensive property within the volume  $V$  is given by the integral of its density over the volume. In other words,

$$\left\{ \begin{array}{l} \text{total} \\ \text{amount of} \\ U \\ \text{in } V \end{array} \right\} = \int_{\xi, y, z \in V(x)} u(x, y, z, t) d\xi dy dz$$

$$= A \int_x^{x+\Delta x} u(\xi, t) d\xi \quad (5.11)$$

With this definition in place, note that the *rate of accumulation* (which could be positive or negative) of  $U$  in the volume  $V$  is given by

$$\left\{ \begin{array}{l} \text{rate of} \\ \text{accumulation} \\ \text{of } U \text{ in } V \end{array} \right\} = \frac{\partial}{\partial t} \int_{\xi, y, z \in V(x)} u(x, y, z, t) d\xi dy dz$$

$$= A \frac{\partial}{\partial t} \int_x^{x+\Delta x} u(\xi, t) d\xi \quad (5.12)$$

4. Finally, we consider the amount of internal reaction or other source or sink within the volume to be given by a function  $s$ , so that the total amount of reaction or source or sink is given by

$$\left\{ \begin{array}{l} \text{rate that sources or reactions} \\ \text{change the amount} \\ \text{of } u \text{ in the volume } V \end{array} \right\} = \int_{\xi, y, z \in V(x)} s(\xi, y, z, t) d\xi dy dz$$

$$= A \int_x^{x+\Delta x} s(\xi, t) d\xi \quad (5.13)$$

Now we are in a position to consider converting our word statement given above into a mathematical statement. Substituting the mathematical statements of the word quantities developed above, the axiomatic statement given by Eq. 2 is given in mathematical terms by

$$A \frac{\partial}{\partial t} \int_x^{x+\Delta x} u(\xi, t) d\xi = A j(x) - A j(x + \Delta x) + A \int_x^{x+\Delta x} s(\xi, t) d\xi \quad (5.14)$$

Clearly, the constant  $A$  plays no role in this expression, so we can simplify this to

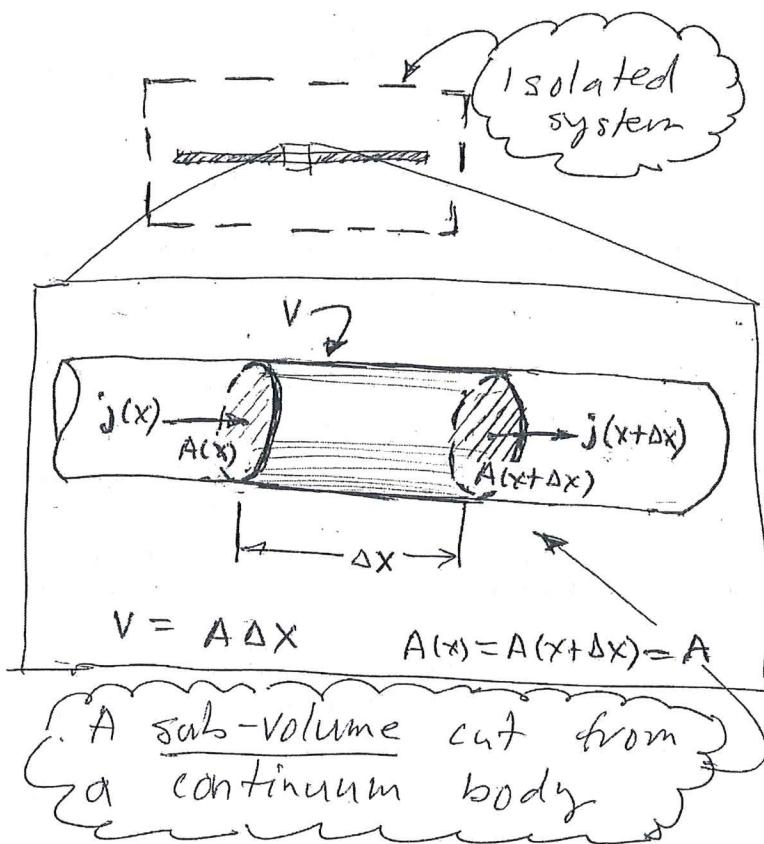
$$\frac{\partial}{\partial t} \int_x^{x+\Delta x} u(\xi, t) d\xi = j(x) - j(x + \Delta x) + \int_x^{x+\Delta x} s(\xi, t) d\xi \quad (5.15)$$

At this juncture, this does not quite look like a differential balance equation. However, there is one important piece of information that we have not yet imposed. Because we are considering our system to be a *continuum*, we are free to let the volume  $V$  be as small as we like. More specifically, for the one-dimensional representation, we can let  $\Delta x \rightarrow 0$ . This will simplify our expression significantly. To start, note that the variable of integration  $\xi$  is such that  $x < \xi < x + \Delta x$ . Thus, as  $\Delta x \rightarrow 0$ , we must also have  $|\xi - x| \rightarrow 0$ . Now consider the following two Taylor series expansions for  $u(\xi, t)$  and  $s(\xi, t)$  around the point  $x$ . These are given by

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**Fig. 5.1** A representative volume extracted from a continuum body. For our purposes, we will assume various symmetries that allow us to consider only one spatial dimension. In particular, the system shown has a constant cross-sectional area,  $A$ , and variables of the system change only in the  $x$ -direction (thus they are constant on every cross-sectional area.)

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$$u(\xi, t) = u(x, t) + (\xi - x) \frac{\partial u}{\partial x} \Big|_{(x,t)} + \dots \quad (5.16)$$

$$s(\xi, t) = s(x, t) + (\xi - x) \frac{\partial s}{\partial x} \Big|_{(x,t)} + \dots \quad (5.17)$$

Now, because we are considering  $\Delta x \rightarrow 0$ , we also have  $|\xi - x| \rightarrow 0$ . In Chapter 1, we noted that all derivatives of analytic functions must remain bounded within their domain. This means that we can *always* find a small enough value for  $\Delta$  such that the second term in each of the Taylor series above is as small as we like. In the limit as  $\Delta x$  tends toward zero (but is not zero), the Taylor series results in the following approximations

$$u(\xi, t) = [u(x, t) + \alpha_1 \Delta x] \quad (5.18)$$

$$s(\xi, t) = [s(x, t) + \alpha_2 \Delta x] \quad (5.19)$$

where the final term represents a measure of the error involved, with  $\alpha_1$  and  $\alpha_2$  being two finite constants. This means the two integrals can be simplified as follows.

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$$\frac{\partial}{\partial t} \int_x^{x+\Delta x} [u(x,t) + \alpha_1 \Delta x] d\xi = j(x,t) - j(x+\Delta x, t) + \int_x^{x+\Delta x} [s(x,t) + \alpha_2 \Delta x] d\xi$$

$$\frac{\partial}{\partial t} [u(x,t) + \alpha_1 \Delta x] \int_x^{x+\Delta x} d\xi = j(x,t) - j(x+\Delta x, t) + [s(x,t) + \alpha_2 \Delta x] \int_x^{x+\Delta x} d\xi$$

$$\frac{\partial}{\partial t} [u(x,t) + \alpha_1 \Delta x] \Delta x = j(x,t) - j(x+\Delta x, t) + s(x,t) + \alpha_2 \Delta x \Delta x$$

$$\frac{\partial}{\partial t} [u(x,t) + \alpha_1 \Delta x] = -\frac{[j(x+\Delta x, t) - j(x,t)]}{\Delta x} + [s(x,t) + \alpha_2 \Delta x] \quad (5.20)$$

Note in the final line of this expression, we have rearranged the two flux terms so that the resulting expression is in the conventional form for the definition of the derivative. Now, if we take the limit as  $\Delta x \rightarrow 0$ , we find the result known as the general conservation equation.

$$\boxed{\frac{\partial u(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x} + s(x,t)} \quad (5.21)$$

Au:Box?

## 5.6 Constitutive Equations and Flux Laws

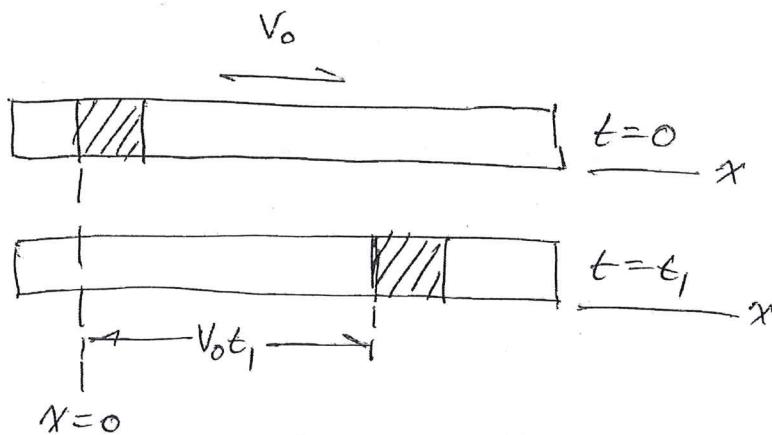
It is not an understatement to say that Eq. (5.21) is the most important equation in continuum mechanics. In one very simple equation, a relatively universal statement about the conservation of mass, momentum, and energy is expressed. Plus the statement is mathematically interpretable in terms of physical intuition: accumulation in a control volume is equal to the difference between mass in and mass out (the flux term) plus any generation or reaction within the volume.

One thing that the conservation equation *does not* do is express conservation of  $u$  in terms of only the dependent variable  $u$ . Instead, we are faced with needing to know both the fluxes,  $j$ , and the source terms,  $s$ , in order to solve the conservation equation. The source terms are generally straightforward: they are either specified by somewhat familiar terms representing kinetic reactions (or equivalent processes for energy or momentum), or they are nonhomogeneous terms (rendering the equation itself nonhomogeneous). The flux terms, however, are somewhat different. We know on the basis of the physics of the problem that the flux of  $u$  should somehow involve the variable  $u$  itself. The expressions that relate the flux  $j$  to a function of the dependent variable,  $u$  are called *constitutive equations*. They play a central role in both the thermodynamics of nonequilibrium systems and in continuum mechanics (in fact, these two fields overlap significantly). There are some deep results in the theory of constitutive equations, and they continue to be an active area of research. In the material that follows, we will list some of the more important constitutive relationships from the perspective of mathematical modeling of continuum systems, but aside from a few side notes, we will not dwell too heavily on the details of their genesis. While many constitutive equations were initially determined empirically (and were often called *laws* given that they were otherwise axiomatic statements at the time of their discovery), many of them have been given more fundamental interpretations through their formulation in statistical mechanics.

(cont'd)

**Fig. 5.2 Convective flux.**

The effect of a purely convective flux in an ideal plug flow reactor with a uniform velocity profile.



For each of the examples that appear below, for now we restrict the discussion to one space dimension. This will be relaxed in material that appears later on in the text.

### 5.6.1 Convective Fluxes

The most fundamental kind of flux is referred to by either the term *convective flux* or *advective flux*. There is some disagreement in the literature (and, occasionally, even debates) about the terminology, usually with strong statements that one or the other of these two terms is the “correct” terminology followed by some historical fact presented as proof. Regardless of these details, meaning of language is created via usage, and both terms are used to describe the same phenomenon widely enough that one should not attempt to distinguish between the two words unless the author has carefully defined the meaning. From here forward, only the term *convective flux* will be used.

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A convective flux is one that transports the intensive quantity,  $u$  by virtue of motion of the whole medium. As an explicit example, if one were to release 1,000 rubber ducks on the surface of a river, the motion of these ducks could be said to define the convective flux of ducks (assuming they were normalized by the appropriate definition of cross sectional area). Another familiar example is the motion of a pulse of a chemical species in an ideal plug flow reactor, which is illustrated in Fig. 5.2. The idea is that the motion of  $u$  follows the bulk velocity of the medium in which  $u$  is embedded; obviously this makes the most sense if that medium is a fluid (gas or liquid), although there are forms of matter that are not clearly one or the other or are mixtures of matter types, but still exhibit bulk motion. Magma, sand flows, and plasmas are examples of matter that can exhibit liquid-like convective fluxes. Mathematically, a convective flux is stated by the relationship

$$j_c(x, t) = u(x, t)v(x, t) \quad (5.22)$$

where  $u$  is the dependent variable (concentration, momentum per unit volume, or specific energy), and  $v$  is the velocity field that describes the motion of the material in which  $u$  is embedded.

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*Example 5.1 (Mass conservation: The continuity equation, A hyperbolic equation).*

A simple and important example of convective fluxes combined with the conservation equation is the *continuity equation*. For now, assume that we interpret  $u$  as the density of a gas,  $u \leftrightarrow \rho$  in a steady flow field in one dimension, with no source or sink terms ( $s = 0$ ). This system is composed of only one chemical species—the fluid. The convective flux is then  $j_c(x, t) = u(x, t)v(x)$ , and the conservation equation gives

$$\frac{\partial u(x, t)}{\partial t} = -\frac{\partial(uv)}{\partial x} \quad (5.23)$$

*Example 5.2 (Mass conservation: The first order wave equation, A hyperbolic equation).*

A first order wave equation occurs under conditions such as the ideal plug flow reactor illustrated in Fig. 5.2. For this example, we can think of  $u$  as representing the concentration of a species dissolved in a fluid (liquid or gas) in steady flow in a uniform flow field. Thus, the system is composed of at least two chemical species: one is the fluid, and the other the chemical species that is dissolved in the fluid. It is assumed that there are no reactions or other sources, so that  $s = 0$ . The flux is given by the same relation as above:  $j_c(x, t) = u(x, t)v(x)$ . The conservation equation gives

$$\frac{\partial u(x, t)}{\partial t} = -\frac{\partial(uv)}{\partial x} \quad (5.24)$$

For many liquids, the concept of incompressibility is applicable. While all matter is *compressible* to some extent, liquids are often well approximated as being incompressible. In short, this means simply that no more liquid can enter a control volume than that which exits the volume (i.e., there can be no liquid accumulation in a volume). Another way of stating this is that the density of the liquid is a constant. From the continuity equation above, we immediately find that  $\partial v / \partial x = 0$  for an incompressible fluid. This also implies, then, that the fluid velocity must be a constant,  $v(x) = v_0$ . Under these conditions, the conservation equation can be written in the form

$$\frac{\partial u(x, t)}{\partial t} = -v_0 \frac{\partial u}{\partial x} \quad (5.25)$$

Which is a first-order wave equation. Although we are not technically *solving* PDEs at this juncture, a little thought will indicate that in 1D, if the initial condition is simply being translated through space at a constant velocity,  $v_0$ . If the initial condition is given by the function  $f(x)$ , then the solution to the PDE is given by the translated version of the initial condition for time  $t$ , i.e.,

$$u(x, t) = f(x - v_0 t) \quad (5.26)$$

### 5.6.2 Gradient Laws that Define Fluxes

A gradient law is any constitutive expression that describes the flux as being proportional to the spatial gradient of  $u$ . In one dimension, the gradient is represented simply by the derivative,  $\partial u / \partial x$ . As mentioned above, gradient laws have a long history in the field of continuum mechanics. A few are given in the following.

$$\text{Fick's Law (mass):} \quad j_g = -D \frac{\partial u}{\partial x} \quad u = \text{concentration} \quad (5.27)$$

$$\text{Fourier's Law (energy):} \quad j_g = -K \frac{\partial u}{\partial x} \quad u = \text{temperature} \quad (5.28)$$

$$\text{Newton's Law of Viscosity (momentum):} \quad j_g = -\mu \frac{\partial u}{\partial x} \quad u = \text{velocity} \quad (5.29)$$

While gradient laws were in fact determined experimentally originally, in 1905 Albert Einstein illustrated one of the first *derivations* of a gradient law from what was to become known as statistical mechanics. Einstein was able to show that if large molecules could be conceived of as being represented by a very large number of non-interacting particles moving in random directions that changed direction over every fixed increment of time, then such motion naturally led to Fick's law as a continuum approximation to the motion. While his derivations might be viewed as being somewhat rough by modern standards, one also has to remember that the very notion of the existence of molecules was not entirely accepted by the scientific community in 1905. Not only did Einstein develop one of the first derivations of a constitutive equation from a more fundamental theory (statistical mechanics), but his work was one of the final indications that the molecular theory of matter must in fact be valid. Einstein's theory was tested just a few years later (1908) by a brilliant French experimentalist named Jean-Baptiste Perrin. After Perrin's work was published, there were virtually no longer any valid arguments that could be effectively waged against the molecular nature of matter.

#### Example 5.3 (Mass conservation: Fick's second law, A parabolic equation).

Sometimes Fick's gradient law combined with the conservation equation is called "Fick's second law". Given the information above, the result is straightforward to develop. Substituting the gradient law for mass for the flux,  $j$ , the conservation equation gives (again, assuming that  $s = 0$ , so there are no reactions or sources in this formulation)

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} \left( -D \frac{\partial u}{\partial x} \right)$$

or

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \quad (5.30)$$

Clearly, the use of Fourier's law in the conservation equation for energy will yield analogous results. Thus, Eq. (5.38) is known both as the *diffusion* equation and the *heat* equation.

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Example 5.4 (Momentum conservation: Burgers' Equation Mixed parabolic and hyperbolic).

Burgers' equation is a one-dimensional analogue to the well-known Navier-Stokes equations describing fluid motion. While Burgers' equation was initially derived as a tool for studying the Navier-Stokes equations (and, thus, one without particular physical relevance), it was later understood to also describe certain kinds of waves in fluids. The flux for Burgers' equation consists of the combination of a convective flux for momentum, and Newton's gradient law for viscosity. For conservation of momentum, the variable of interest is  $q = \rho u$ ; in other words, it is the momentum per unit volume. Thus, the flux takes the form

$$j = q - \mu \frac{\partial q}{\partial x} \quad (5.31)$$

Substituting this result into the general conservation equation (and assuming that  $s = 0$ ) yields

$$\frac{\partial q}{\partial t} = -\frac{\partial}{\partial x} \left( q - \mu \frac{\partial q}{\partial x} \right)$$

or, upon substituting  $q = \rho u$  and assuming constant density

$$\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} + \frac{\mu}{\rho} \frac{\partial^2 u}{\partial x^2} \quad (5.32)$$

This is the first example of a conservation equation that contains both *convective* and *gradient* fluxes.

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Example 5.5 (Mass conservation: Convection-diffusion-reaction. Mixed parabolic and hyperbolic).

The convection-diffusion-reaction equation is a familiar one science and engineering, and it has been used to describe everything from the motion of dissolved contaminants in groundwater to the spread of infectious diseases. Technically, systems that have convective motion experience *dispersion* which is the combination of spreading due to molecular diffusion plus spreading due to fluctuations (or velocity gradients) in the velocity field. Regardless, both diffusion and dispersion have the same mathematical form (assuming that neither is a strong function of space), and mathematically they are identical problems. The reaction term is expressed in the source or sink term,  $s$ , appearing in the conservation equation. This is the first example to contain a non-zero source term. For this effort, consider  $u$  to be the concentration of a reacting chemical species,  $v_0$  to be a spatially constant (uniform) one-dimensional velocity field, and the reaction rate to be first order in  $u$ , that is  $s = -ku$ . The resulting expression is very similar to the previous result for Burgers' equation.

$$j = uv_0 - D \frac{\partial u}{\partial x} \quad (5.33)$$

$$s = -ku \quad (5.34)$$

Substituting this result into the general conservation equation yields

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} \left( uv_0 - D \frac{\partial u}{\partial x} \right) - ku$$

or, upon using the fact that  $v_0$  is a constant

$$\frac{\partial u}{\partial t} = -v_0 \frac{\partial u}{\partial x} + D \frac{\partial^2 u}{\partial x^2} - ku \quad (5.35)$$

This is the first example of a conservation equation that contains both *convective* and *gradient* fluxes and a source term.

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Example 5.6 (Mass conservation: The Groundwater Flow Equation in 1-dimension, Elliptic).

Examples of *elliptic* problems with two spatial variables present the challenges noted above (specifically, the fluxes become vectors, which is a topic that will be covered in a separate chapter). Presenting examples of elliptic equations with one space dimension is also somewhat challenging. However, the following is a reasonable example. To start, recall the one-dimensional conservation equation on  $x \in [0, L]$  of the form

$$\frac{\partial u(x, t)}{\partial t} = -\frac{\partial j(x, t)}{\partial x} + s(x, t)$$

In groundwater flow, Darcy's law is the gradient constitutive equation that relates hydraulic head,  $h$ , to the rate of groundwater flow per unit area  $j$ . In one-dimension, the expression takes the form

$$j = -K \frac{\partial u}{\partial x}$$

where here  $u$  = hydraulic head in meters,  $K$  is the hydraulic conductivity in  $m/s$ . We are ignoring a amplification term known as the storativity for ease in the presentation. Substituting the flux into the general conservation equation yields

$$\frac{\partial u(x, t)}{\partial t} = K \frac{\partial^2 h(x, t)}{\partial x^2} + s(x, t)$$

While this result is technically a parabolic equation, we consider specifically the case where the groundwater flow is at steady state. Steady state is defined by the state in which, for all locations  $x$  in the domain,  $\partial u / \partial t = 0$ . Thus, the balance equation is given by

$$0 = K \frac{\partial^2 h(x)}{\partial x^2} + s(x)$$

Technically, this equation is *elliptic*. More conventionally, this equation would appear with two spatial variables in the form of *Poisson's* equation.

$$K_x \frac{\partial^2 u}{\partial x^2} + K_y \frac{\partial^2 u}{\partial y^2} = -s(x)$$

If we had conditions of symmetry on the boundaries of the  $x$ -y domain, we can imagine conditions in which the flow in the  $y$ -direction is zero. Under those conditions, we would recover an equation that is identical to the one-dimensional form that was derived above (even though the problem applied to a 2-dimensional domain).

Hopefully at this juncture, you can see that the simple derivation of the conservation equation is indeed a powerful tool. Combined with the constitutive expressions for fluxes and representations for the source term, a variety of meaningful continuum mechanical expressions can be derived.

## ✓ 5.7 Ancillary Conditions

The problem of determining how many ancillary conditions are needed for a PDE is very much like the analogous problems for ODEs. Here, for concreteness, we will continue to discuss problems with at most two independent variables, one for space and one for time. The case of two independent variables in space adds a complication; it introduces fluxes as vector quantities. Thus, a fuller analysis of the elliptic problems discussed earlier in this chapter will be delayed until the chapter on multidimensional conservation equations.

### ✓ 5.7.1 Order of the PDE and determining necessary ancillary conditions

For problems in one space dimension, it is relatively straightforward to determine the necessary number of ancillary conditions. For clarity, we consider the diffusion / heat equation in the form

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad (5.36)$$

where  $\alpha^2$  is a parameter that is real and (necessarily) positive, and represents either the diffusion coefficient or the thermal conductivity. The way to think about these problems is to consider how many *integrations* are needed to eliminate the derivatives that appear. Here we are using the word “integrations” somewhat informally (although, when we explore separation of variables, we will find that the solutions are indeed found by integration in the conventional sense that we have used the term in the study of ODEs).

In this example, we have the following derivatives to resolve:

1. One time derivative of first order.
2. One space derivative of second order.

Thus, to resolve the problem, we must conduct one “integration” in time, and two “integrations” in space. This generates a set of three unknown constants, one from each integration. In reality, the unknowns may be whole *functions* because we are dealing with two dimensions; while the unknown values are fixed in one of

the two dimensions, they remain potentially free in the other dimension. This will be more clear when we consider an example.

### 5.7.2 Types of ancillary conditions

There is an additional detail that needs some attention regarding the ancillary conditions. Generally, we should think of ancillary conditions as a hierarchy of conditions which usually start with specifying the value of the function, and may (depending on the order of the equation) include specifying derivatives of the function. In other words, suppose we consider the two ancillary conditions in space needed for the diffusion equation given above. Before continuing on, we will make note of the three fundamental types of ancillary conditions for second-order PDEs composed of one space and one time variable, which we assume to be represented by the dependent variable  $u$ , and independent variables  $x$  and  $t$  (thus, the function being specified by the ancillary conditions is  $u(x, t)$ ).

1. **Specified value conditions.** In PDEs, specifying the value of the solution at its time-space boundaries is the most fundamental kind of ancillary condition. As will be discussed below, for a problem to be well-posed it often must have some portion of its boundary for each independent variable identified by a specified value condition. The convention in the study of PDEs is that space and time variables are treated somewhat differently (and, in fact, they are physically quite different—we can think of traveling backwards or forwards in even one space dimension, but only forwards in time!) Thus, there are separate names given to the two different kinds of specified value conditions.
  - a. **Specified value in the space dimension.** These conditions are called *Dirichlet* or *first-type* boundary conditions. The term “boundary” is used to emphasize that they are spatial conditions. Thus, these conditions specify the value of the dependent variable,  $u$ , at one or more spatial locations for all values of the independent variable,  $t$ , defining the time domain. This boundary condition is named after the German mathematician Johann Peter Gustav Lejeune Dirichlet (1805–1859).
  - b. **Specified value in the time dimension.** When applied to the time dimension, the specified value condition is called an *initial condition*. Thus, an initial condition specifies the value of the dependent variable,  $u$ , at  $t = 0$  (or some other time considered to be the start of the clock associated with the problem) for all values of the independent variable,  $x$ , defining the spatial domain.
2. **Specified derivative conditions.** Specified derivative conditions are, in a sense, “weaker” than specifying the value of the dependent variable. Thus, specifying the derivative of the dependent variable gives the problem more freedom to adapt to the requirements imposed by the PDE itself. In other words, one can meet a specified derivative condition at a boundary, but the value of the function at that boundary is otherwise unconstrained by the boundary condition itself. Because of this, some care has to be taken when applying specified derivative conditions. It is possible to generate problems that have no unique solution if only specified derivative conditions are imposed. Like specified value conditions, the convention in the study of PDEs is that different terminologies are used for space and time conditions of this type.
  - a. **Specified derivative in the space dimension.** These conditions are known as *Neumann* or *second-type* boundary conditions. These conditions specify the derivative of the dependent variable,  $u$ , at one or more spatial locations for all values of the independent variable,  $t$ , defining the time domain. The condition is named after the German mathematician Carl Gottfried Neumann (1832–1925).

- b. **Specified derivative in the time dimension.** When applied to the time dimension, the specified derivative condition is still referred to as an *initial condition*; there is no special reason for this other than convention. Thus, this initial condition specifies the derivative of the dependent variable,  $u$ , at  $t = 0$  (or some other time considered to be the start of the clock associated with the problem) for all values of the independent variable,  $x$ , defining the spatial domain. In more mathematical contexts, the specified value plus specified derivative initial condition is sometimes also called a *Cauchy condition* in honor of the French mathematician Baron Augustin-Louis Cauchy (1789–1857).

3. **Linear combinations of specified value and specified derivative conditions.** There are instances where a linear combination of the two, more fundamental, types of ancillary conditions is useful and contains and often necessary physical content. These kinds of conditions will not be brought up in any detail for the remainder of this chapter, but an example of a useful linear combination of ancillary conditions as a model for a physical boundary condition will be discussed in the material related to *solving PDEs*. When applied to spatial boundaries, the linear combination condition is called a *Robin condition* after the French mathematician Victor Gustave Robin (1855–1897). rom
4. **Periodic conditions.** There are some applications where the boundary conditions are also required to behave periodically on the domain. This is not actually a separate boundary condition *per se*, so much as a constraint on the kinds of solutions that can be attained. The use of conditions to amend or constrain the boundary conditions is a topic that is more advanced than the material being covered in this text, so we will not discuss ~~the~~ in detail.

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### 5.7.3 Applications of ancillary conditions with one space and one time dimension

Let's return to the diffusion problem introduced above.

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad (5.37)$$

We know that we need three ancillary conditions total to account for the undetermined information (functions) created by integrating the equation. For the diffusion problem, the integrations involved necessarily require two *boundary conditions* and one *initial condition* to be specified.

We could specify these conditions by any of the following options.

Option 1 (two Dirichlet conditions, one initial condition)

$$u(0, t) = u_0(t) \quad u(L, t) = u_L(t) \quad u(x, 0) = f(x)$$

Option 2 (one Dirichlet, one Neumann condition, one initial condition)

$$\frac{\partial u(0, t)}{\partial x} = g_0(t) \quad u(L, t) = u_L(t) \quad u(x, 0) = f(x)$$

Option 3 (one Dirichlet, one Neumann condition, one initial condition)

$$u(0,t) = u_0(t) \quad \frac{\partial u(L,t)}{\partial x} = g_L(t) \quad u(x,0) = f(x)$$

In other words, we should generally work up through the sequence of functions in the order  $u, u_x, u_{xx}, \dots$  when specifying the ancillary conditions. Another way of stating this is that, for problems in one space and one time dimension, both space and time should have at least one Dirichlet (specified value) condition specified at some point in the problem.

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Missing from the list above is the possible combination

Option 4 (two Neumann conditions)

$$\frac{\partial u(0,t)}{\partial x} = g_0(t) \quad \frac{\partial u(L,t)}{\partial x} = g_L(t) \quad u(x,0) = f(x)$$

The reason that this problem is not listed is that it requires special handling. This is best shown by using a direct example.

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*Example 5.7 (The problem of two Neumann conditions).*

Consider the elliptic problem above for groundwater flow. Suppose we have the following PDE and two boundary conditions on the interval  $x \in [0, L]$

$$\begin{aligned} K_x \frac{\partial^2 u}{\partial x^2} &= 0 \\ \frac{\partial u(0,t)}{\partial x} &= g_0 \\ \frac{\partial u(L,t)}{\partial x} &= g_L \end{aligned}$$

This problem is actually an easy one to solve. It requires just two direct integrations of the first equation. An outline of the process is as follows

$$\begin{aligned} \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) &= 0 \quad \text{Divide by } K_x \neq 0; \text{ rewrite derivatives in equivalent form} \\ \frac{\partial u}{\partial x} &= C_1 \quad \text{Integrate once, as indefinite integration} \\ u(x) &= C_1 x + C_2 \quad \text{Integrate a second time, as indefinite integration} \end{aligned}$$

Now, note that the derivative of the solution implies

$$\frac{\partial u}{\partial x} = C_1$$

Indicating that the derivative is, everywhere, a constant. Thus, for the original statement to make sense, we must have  $g_0 = g_L = G$ , for  $G$  a constant. In other words, the solution does not allow the flux at the two ends of the domain to be different from one another. The solution at this juncture is

$$u(x) = Gx + C_2$$

Unfortunately, we have run out of unique boundary conditions with which to solve for  $C_2$ ! This situation may be repairable if one can generate some other constraint on the system (usually considering some physical conservation principle) to replace the lack of a second, unique boundary condition. For example, if the average velocity were known, i.e.,

$$\frac{1}{L} \int_0^L u(x) dx = U_0$$

Then, this would be sufficient to replace the lack of information that we have by having a repeated Neumann condition. Specifically, note

$$\frac{1}{L} \int_0^L u(x) dx = U_0$$

$$\frac{1}{L} \int_0^L (Gx + C_2) dx = U_0$$

$$\frac{1}{L} \left( \frac{GL^2}{2} + C_2 L \right) = U_0$$

$$\implies C_2 = U_0 - \frac{GL}{2}$$

This is an indication that the application of two Neumann conditions must be considered very carefully in general to assure that the following are met:

1. The problem must make *physical* sense, including in the limits as time becomes arbitrarily large.
  2. The problem must have sufficient content so that the problem is *well posed*. This just means that the solution exists, the solution is unique, and the solution does not exhibit “chaotic” behavior.

An problem combining all of the ideas in this section is given in the following example illustrating a set of boundary and initial conditions for a reaction-diffusion equation that is well posed.

*Example 5.8 (A well-posed reaction-diffusion equation.).*

A well-posed problem is one in which the following conditions are met.

1. The problem has a solution. This means that the PDE itself is constructed so as to be internally consistent (i.e., it makes mathematical or physical sense), and that it has sufficient ancillary information so that solutions may be found.

2. The problem is unique. This means that not only can we find solutions, but we can find particular solutions that represent specific conditions for which we would like solutions. This means that we do not find an equivalence class of solutions (where each solution is valid, but no two solutions are linear combinations of the others), but rather a single solution that is fully specified by appropriate ancillary conditions.
3. The problem does not behave chaotically. This is usually stated as requiring that the problem depend *continuously upon the initial data*, although that is not the most intuitive statement that can be made. In short, this requirement indicated that if the ancillary conditions of the problem are perturbed by a small amount, the problem solution is changed in a small amount. While this statement can be made mathematically more rigorous, the essential idea is sufficient for our purposes.

Consider the parabolic problem for a diffusion-reaction system on the interval  $x \in [0, L]$ .

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - ku$$

This problem has one spatial derivative of order 2, and one time derivative of order 1. Thus, we need two ancillary conditions for the spatial derivatives, and one for the time derivative. Now, consider the following statement.

$$\begin{aligned} & \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - ku \\ B.C. 1 & \quad u(0, t) = u_0 \\ B.C. 2 & \quad \frac{\partial u(L, t)}{\partial x} = 0 \\ I.C. 1 & \quad u(x, 0) = f(x) \end{aligned}$$

According to the guidelines above, the boundary conditions include one Dirichlet condition, and one other condition (in this case a Neumann condition, although a second Dirichlet condition would be acceptable also if it represented the physical context of the problem). Thus, the spatial ancillary conditions appear to meet our requirements. Note also that the ancillary conditions are technically functions. Even though they are constant functions, both boundary conditions assign a particular value of  $u(0, t)$  or  $\partial u(L, t)/\partial x$  to each boundary for every possible value of  $t$ ; thus, these two statements represent functions of time. A little thought indicates why this must be true; if the functions were not defined for all times of relevance, then there would exist some times for which we did not have the appropriate ancillary conditions, and the problem would no longer be well posed. An analogous statement can be made about the initial condition. The initial condition indicates what the state of the system is at  $t = 0$  (or, in principle, any time  $t_0$  that was assigned to define the "start" of the problem). Of course, this initial condition must state what happens for every location  $x$  in the domain, or else there would exist locations for which no ancillary space data existed, and the solution in time could not be found (the problem, again, would not be well posed).

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## 5.8 Transformations in the solutions to PDEs

Transformations of PDEs are an application of the idea that “if you have a problem you do not know how to solve, try to convert it into a problem you do know how to solve.” While, *in principle* this is a great idea, one of the problems with the general use of transformations in PDEs is difficult to state definite rules to find such transformations, especially for nonlinear problems.

cap

### 5.8.1 Symmetries and transformations

Some transformations for linear and nonlinear problems can be learned by understanding what are called the differential symmetries of the problem (i.e., the symmetries associated with the infinitesimal geometry of the system). While that topic is well beyond the scope of this text, the basic results for the approach were found by a German Amalie Emmy Noether (1882–1935). Not only is it notable that Emmy Noether was a female mathematician at a time where there were very few, but her work led to some deep theories and even entire disciplines in mathematics that did not exist before her work. In short, Noether's theorem proclaims that every conservation law has a continuous *symmetry*, and vice versa for every continuous symmetry, there's an associated conservation law.

What is meant by a symmetry here can be roughly thought about in terms of our conventional notion of the word. Take, for example, a square drawn of a piece of paper. If the square is rotated by any integer multiple of  $\frac{\pi}{2}$  radians, then it appears to be unchanged. Thus, it has a rotational symmetry with respect to rotations of  $\frac{\pi}{2}$  radians. Similarly, a square is unchanged by reflections across its horizontal or vertical axes; thus, it has two kinds of reflection symmetry. Many of the familiar and unfamiliar kinds of symmetries are discussed in a text by the famous physicist Herman Weyl (Weyl, 1952). While most familiar examples represent discrete symmetries (i.e., the rotational symmetry for a square happens for only certain angles of rotation), continuous symmetries are a bit more complex. While the details are not important here, one can think of continuous symmetries as symmetries that exist for a sequence of infinitesimal actions (e.g., rotations, translations).

Regardless, the importance of symmetries and of Emmy Noether's theorem have been critical to the study of PDEs and to the study of various physical phenomena (often expressed through PDEs). A few deep results in quantum mechanics have even been discovered by looking for certain kinds of physical symmetries in the mathematical laws that describe it.

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### 5.8.2 An example transformation

For our purposes, we might best understand how transformations can lead to a simplified problem by an explicit example. In the following example, we propose a continuous symmetry in time; that is, we find that if we exponentially rescale the dependent variable in time, we capitalize on a particular kind of symmetry (a dilation-of-time symmetry) that puts the problem in a new, much simpler form.

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Example 5.9 (Transformation of a reaction-diffusion equation).

Consider a parabolic problem similar to the one described in the previous example for a diffusion-reaction system on the interval  $x \in [0, 1]$ . This problem meets our (somewhat limited) definition of being well-posed.

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - ku$$

B.C. 1

$$u(0, t) = 0$$

B.C. 2

$$u(1, t) = 0$$

I.C. 1

$$u(x, 0) = f(x)$$

*An: This is the last chapter of the book.*

In the next chapter, we will examine finding solutions to this problem using separation of variables. Actually, the solution to the pure diffusion problem is well known, and is reasonably straightforward. While the solution to the reaction-diffusion problem is possible using separation of variables, it does contain a number of complications that make the solution unwieldy.

Thus, we consider the following well-known transformation by Danckwerts (1951) (who was a chemical engineer) for transforming this problem into a pure diffusion problem. The transformation is accomplished by defining the following new dependent variable  $w$

$$u(x, t) = w(x, t) \exp(-kt)$$

To adopt this transformation, we simply substitute the right-hand side for  $u$  in both the PDE and ancillary conditions. The following results

$$\frac{\partial}{\partial t}(w(x, t)e^{-kt}) = D \frac{\partial^2}{\partial x^2}(w(x, t)e^{-kt}) - k(w e^{-kt})$$

B.C. 1

$$w(0, t)e^{-kt} = 0$$

B.C. 2

$$w(1, t)e^{-kt} = 0$$

I.C. 1

$$w(x, 0)e^0 = f(x)$$

Applying the product rule for derivatives and extracting the exponential from the space derivative yields

$$\frac{\partial w(x, t)}{\partial t} e^{-kt} - k w(x, t) e^{-kt} = e^{-kt} D \frac{\partial^2}{\partial x^2} w(x, t) - k w(x, t) e^{-kt}$$

B.C. 1

$$w(0, t)e^{-kt} = 0$$

B.C. 2

$$w(1, t)e^{-kt} = 0$$

I.C. 1

$$w(x, 0)e^0 = f(x)$$

Noting that the exponentials are not zero, and that for the initial condition the exponential is identically 1, we have the final result

$$\frac{\partial w(x,t)}{\partial t} = D \frac{\partial^2 w(x,t)}{\partial x^2}$$

B.C. 1       $w(0,t) = 0$   
 B.C. 2       $w(1,t) = 0$   
 I.C. 1       $w(x,0) = f(x)$

31

101

Which is a diffusion-like equation.

cap II

## 5.9 Steady state problems

The steady state for a transient problem (i.e., one that is a function of time, and the solution evolves over time from an initial configuration) is simply the state where the partial derivatives in time approach zero. The word *approach* is used here because for transient second-order PDEs, the steady state usually occurs as time becomes arbitrarily large (i.e., as  $t \rightarrow \infty$ ); thus, it represents a *limiting* behavior.

When transient problems are composed of one space and one time dimension, then the steady state solutions are identical in form to ordinary differential equations. There are some philosophically delicate points here to be made about the use of partial versus total derivatives for such problems (technically, as  $t \rightarrow \infty$ , they are no longer functions of time, thus there is only one independent variable, so one can certainly argue that they are formally, in fact, ODEs in the limiting case), but we will not dwell on these. Instead, we will treat such problems as being *de facto* ODEs, while continuing to keep in mind that the equation itself came from a more general form. An example of handling the determination of a steady state solution is given below.

Example 5.10 (A steady-state reaction-diffusion equation.). Let's start with the reaction-diffusion problem given above. Specifically, we have

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - ku$$

B.C. 1       $u(0,t) = u_0$   
 B.C. 2       $\frac{\partial u(L,t)}{\partial x} = 0$   
 I.C. 1       $u(x,0) = f(x)$

101

Now, as time becomes arbitrarily large, we expect this equation to reach some steady state. To determine if this is, in fact, true, we can examine what happens if the time derivative is set to zero. If a solution exists, then the problem has a steady state. Specifically, we have the problem (after dividing the balance equation through by  $D$ ).

following /

$$\frac{d^2u}{dx^2} - \frac{k}{D}u = 0$$

B.C. 1                       $u(0,t) = u_0$   
 B.C. 2                       $\frac{du(L,t)}{dx} = 0$

Note that we no longer have need for the initial condition  $u(0,t)$  because there is no integration in time required; the partial derivatives have also been replaced by ordinary derivatives, since time is no longer an independent variable in this equation.

Solving this equation is the same as described in Chapter 2 on ODEs. It is a homogeneous equation, and solving it requires only that we first determine which case it represents by computing the roots of the characteristic equation. Noting  $a = 1$ ,  $b = 0$ , and  $c = -k/D$ , we have the roots given by

$$s_1 = +\sqrt{k/D} \quad s_2 = -\sqrt{k/D}$$

For ease in notation, set  $\beta = |\sqrt{k/D}|$ . The solution is now

$$u(x) = C_1 \exp(\beta x) + C_2 \exp(-\beta x)$$

$$du(x)/dx = C_1 \beta \exp(\beta x) - C_2 \beta \exp(-\beta x)$$

The second boundary condition gives us

$$0 = C_1 \beta \exp(\beta L) - C_2 \beta \exp(-\beta L)$$

and the first boundary condition

$$u_0 = C_1 + C_2 \implies -C_2 = C_1 - u_0$$

Substituting this last relation into the second boundary condition expression gives

$$C_1 [\exp(\beta L) + \exp(-\beta L)] = u_0 \exp(-\beta L)$$

or, recalling the definition of the hyperbolic cosine  $\cosh(x) = 1/2(e^x + e^{-x})$ , we have

$$C_1 = \frac{u_0 \exp(-\beta L)}{2 \cosh(\beta L)}$$

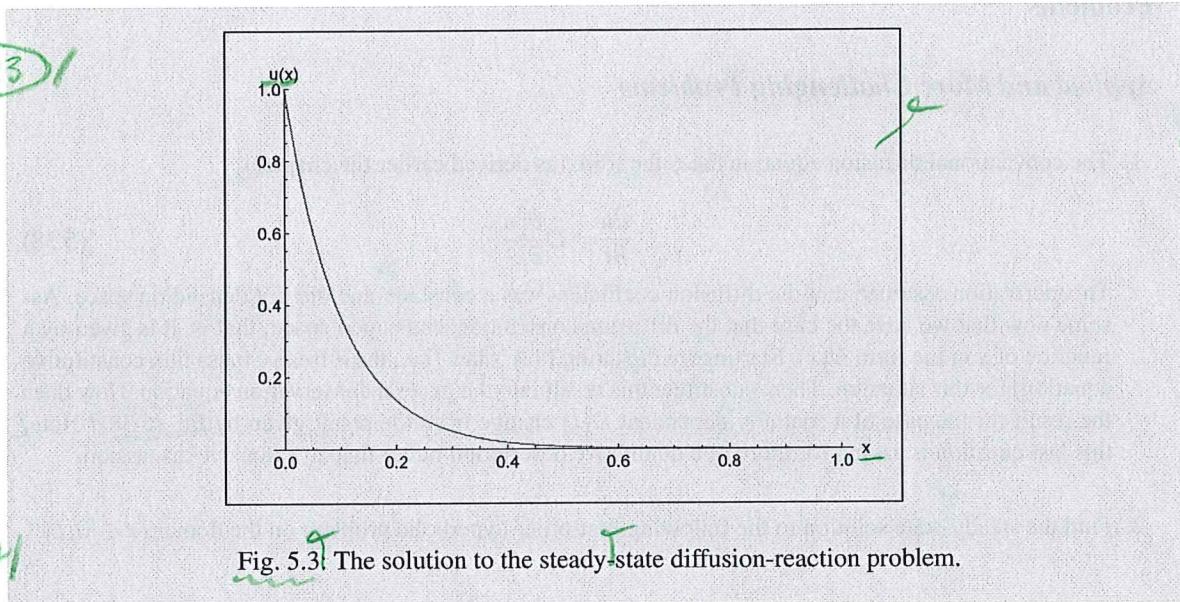
$$C_2 = u_0 - \frac{u_0 \exp(-\beta L)}{2 \cosh(\beta L)}$$

The final solution is

$$u(x) = \frac{u_0 \exp(-\beta L)}{2 \cosh(\beta L)} \exp(\beta x) + \left( u_0 - \frac{u_0 \exp(-\beta L)}{2 \cosh(\beta L)} \right) \exp(-\beta x)$$

For reference, the solution to this problem is given in Fig. ??.

An: Correct?



An: Are you able to  
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## Problems

### Applied and More Challenging Problems

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1. The conventional diffusion equation takes the form (as derived earlier the chapter)

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \quad (5.38)$$

This derivation assumed that the diffusion coefficient was a constant, and did not depend on space. Assume now that we have the case that the diffusion constant depends upon space; that is, it is given by a function of  $x$  in the form  $D(x)$ . Starting by re-stating Fick's law (i.e., the diffusive mass flux constitutive equation) for this situation. Then, substitute this result into the general conservation equation. How does the result for the case of a spatially-dependent  $D(x)$  change from the result given by Eq. (5.38)? Note that this last question is asking for more of a qualitative description rather than an extensive discussion.

9/91

77  
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77

2. Find the steady-state solution to the following first-order hyperbolic problem on the domain  $x \in [0, L]$ .

$$\begin{aligned} \frac{\partial u}{\partial t} &= -v_0 \frac{\partial u}{\partial x} - ku \\ \text{B.C. 1} &\quad u(0, t) = u_0 \\ \text{I.C. 1} &\quad u(x, 0) = 0 \end{aligned}$$

Does the solution to this problem match your intuition? This last question is a qualitative one, so a qualitative answer is all that is expected.

- that?  
it is possible to  
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3. A theme in this text is one in which, when faced with a problem that one does not know how to solve, change the problem into one that can be solved. While this is easy to state, it is not always easy to know how to transform such problems, however!

One interesting transformation that was invented by the famous chemical engineer Peter Danckwerts (1916–1984) is one that transforms a diffusion-reaction problem into one of only diffusion, greatly simplifying the solution process. To start, consider the following problem with Neumann boundary conditions.

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - ku \quad (5.39)$$

$$\text{B.C. 1} \quad \left. \frac{\partial u}{\partial x} \right|_{(0,t)} = 0 \quad (5.40)$$

$$\text{B.C. 2} \quad \left. \frac{\partial u}{\partial x} \right|_{(L,t)} = 0 \quad (5.41)$$

$$\text{I.C.} \quad u(x, 0) = f(x) \quad (5.42)$$

Now, consider the following transformation of variables

101

41

that!

$$u(x,t) = \exp(-kt)w(x,t)$$

Show that substituting this transformation into the diffusion-reaction problem given by Eqs. (5.39)-(5.39) will transform that set of equations into the following equations in terms of the variable  $w$

$$\frac{\partial w}{\partial t} = D \frac{\partial^2 w}{\partial x^2} \quad (5.43)$$

$$B.C.1 \quad \left. \frac{\partial w}{\partial x} \right|_{(0,t)} = 0 \quad (5.44)$$

$$B.C.2 \quad \left. \frac{\partial w}{\partial x} \right|_{(L,t)} = 0 \quad (5.45)$$

$$I.C. \quad w(x,0) = f(x) \quad (5.46)$$

- 31 one 191
4. Burgers' equation is a nonlinear equation that is used as a prototype for the Navier-Stokes equation in 1-dimension, and in applications to dissipative wave transport. The equation is nonlinear, taking the form

$$u_t + uu_x - vu_{xx} = 0 \quad (5.47)$$

Nonlinear equations are generally difficult to solve analytically (if they can be solved at all!) For this problem, there is a famous two-step transformation that makes the equation linear. This transformation, called the Cole-Hopf transformation begins as follows. First, we define the following relationship among variables.

$$u = w_x$$

Now, substituting this into the original equation, we find an equation in  $w$  of the form

$$w_{xt} + w_x w_{xx} - vw_{xxx} = 0$$

It is not clear at this juncture that anything has been made better, but we will press on. First note the following identity

$$\frac{\partial}{\partial x} \left( \frac{1}{2} w_x^2 \right) = w_x w_{xx}$$

Note that we have mixed notation for derivatives here, which is a perfectly acceptable thing to do as long as you keep everything straight. This result is easy to check by a simple application of the chain rule. It is also useful to recall that compositions of derivatives combine as follows

$$\frac{\partial}{\partial x} w_x = \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial x} \right) = \frac{\partial^2 w}{\partial x^2} = w_{xx}$$

Now, this means that the transformed Burgers' equation can be rewritten as follows

$$w_{xt} + \frac{\partial}{\partial x} \left( \frac{1}{2} w_x^2 \right) - v w_{xxx} = 0$$

All of this work has paid off a bit, because now we can extract one partial derivative with respect to  $x$  outside the expression as a whole, giving

$$\frac{\partial}{\partial x} \left( w_t + \frac{1}{2} w_x^2 - v w_{xx} \right) = 0$$

Both sides of this can be integrated once with respect to  $x$  using an indefinite integration. The result is

$$w_t + \frac{1}{2} w_x^2 - v w_{xx} = C_1$$

While this equation seems perhaps *better* than the original one, it is still nonlinear because of the term  $1/2w_x^2$ . Now, for the second transformation, which is the basis for this problem. Show that the transformation of variables given by

$$w(x, t) = -2v \ln[z(x, t)]$$

Will lead to a *linear* second-order PDE of the diffusion equation form, i.e.,

$$z_t = v z_{xx}$$

Where we have assumed that  $C_1 = 0$ .

5. The Cahn–Hilliard (CH) equation is a balance equation that predicts how two phases in a fluid will separate starting from a highly mixed initial configuration. In this case,  $u$  is actually a dependent variable that encodes the internal *configuration* of the geometry of the two phases. The variable  $u$  is initially viewed as a variable that takes on the values  $u \in [0, 1]$ . The value of  $u$ , then, indicates one of the two phases. A value of 1 indicates the first phase, and a value of 0 indicates the second phase. Variables that take on values of 1 or 0 to represent phases are sometimes called *phase indicator* variables in the study of continuum mechanics.

The nonlinear balance equation for the phase indicator variable  $u$  takes the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} \left( D u^3 - D u - \gamma D \frac{\partial^2 u}{\partial x^2} \right) \quad (5.48)$$

If you are interested in the topic, an example of the application of the CH equation in one dimension is reported in the following paper. Argentina et al. (2005)

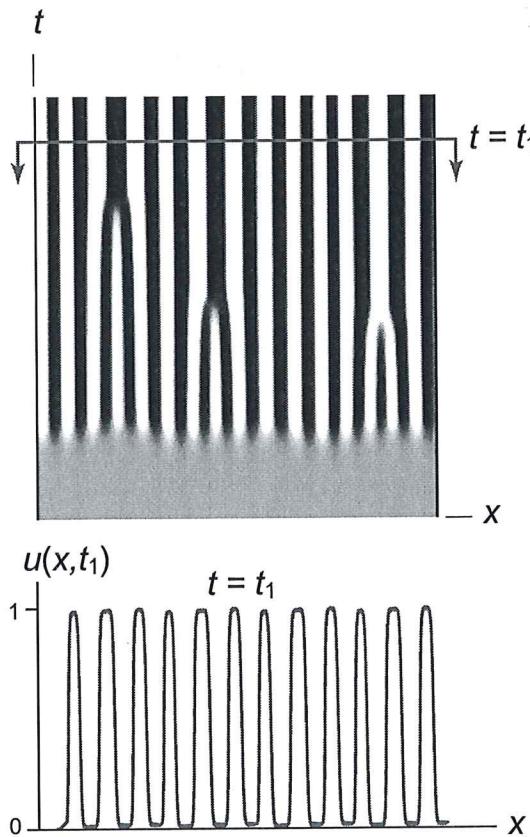
Argentina, M., Clerc, M., Rojas, R., Tirapegui, E.: Coarsening dynamics of the one-dimensional Cahn-Hilliard model. *Physical Review E* 71(4), 046210 (2005)

The behavior of the equation is very interesting. Starting from a (relatively) uniform initial condition representing a fully mixed fluid, the CH equation predicts how the fluid will separate. As an analogue, think about how an oil-water emulsion can form, over time, large oil droplets as the small droplets coalesce (this may be an observation you have had in something as mundane as vinegar-and-oil salad dressing). While the applications of the CH are most interesting in 2- and 3-dimensional systems, there

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**Fig. 5.4 Solution to the Cahn-Hilliard Equation.**

This equation transforms an initial mixture of phases (denoted by the gray region near the time origin) to one in which the two phases separate. The upper figure shows the value of  $u(x, t)$  as a function of both  $x$  and  $t$ ; the values in the domain represent an intensity plot (white=0, black=1, with gray representing values in between) for each time space location. The bottom figure illustrates the spatial variation of  $u(x, t_1)$  for a fixed value of time; it corresponds to the horizontal line labeled " $t = t_1$ " in the upper figure. It is remarkable that this nonlinear equation generates such extreme behavior. Figures based on data presented by Argentina et al. (2005).



are 1-dimensional solutions also. As an example, Fig. 5.4 provides the time-space value of the indicator function for one such solution.

Recall, the general conservation equation (with no source) for 1 dimension takes the form

$$\frac{\partial u(x, t)}{\partial t} = -\frac{\partial}{\partial x} j(x, t) \quad (5.49)$$

Comparing Eq. (5.49) with the expression given by Eq. (5.48), determine what the flux,  $j(x, t)$  must be as a function of  $u$ . To be clear, you need only compare these two equations to determine what the flux must be interpreted as in Eq. (5.48); this involves a bit of algebra and application of the derivative operators, but nothing else. To start, think about how you can pull out the operator  $-\partial/\partial x$  from the right-hand side of Eq. (5.48). Once you do that, what this operator acts upon will then be the flux,  $j(x, t)$ .

6. Consider a parabolic problem, similar to the one described in example 5.9, on the interval  $x \in [0, 1]$ . This problem is slightly different from the example because the reaction rate coefficient is a function of time. The balance equation, boundary, and initial conditions are

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - k(t)u$$

*B.C. 1*       $u(0, t) = 0$   
*B.C. 2*       $u(1, t) = 0$   
*I.C.* 1       $u(x, 0) = f(x)$

31

For this problem, show that the transformation

$$u(x, t) = \exp \left( - \int_{\tau=0}^{\tau=t} k(\tau) d\tau \right)$$

can be used to transform the reaction-diffusion problem into a purely diffusion problem.

## References

- (Cap II)
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