"It was a sort of act of faith with us that any equations which describe fundamental laws of Nature must have great mathematical beauty in them. It was a very profitable religion to hold and can be considered as the basis of much of our success."

Paul Dirac on the laws of quantum mechanics, where the delta function arises as an important feature

4

The Step and Delta Functions

The concept of what constitutes a *function* was briefly covered in Chapter 1. While the concept itself seems simple enough, this is in part because ~~generally~~ we have been generally been exposed to the concept throughout our education. However, the word *function* itself did not even exist until the late 1600's, when the mathematicians Gottfried Leibniz (of calculus fame) and Johann Bernoulli (a Swiss mathematician, who's son Daniel was famous for the Bernoulli principle of fluid mechanics, and for the gamma function, discussed below) began to develop the concept of function more formally. Even into the early 1800's, the concept of *function* was still thought by many to apply only to *analytic* functions (see Chapter 1).

Many mathematicians contributed to the generalization of the concept of what defined a function. However, it is not an overstatement to say that it was the theory of Joseph Fourier who prompted modern efforts for defining functions. Through the use of Fourier series (see Chapter 3), Joseph Fourier was able to *construct* functions that behaved very much unlike the functions that mathematicians were used to contemplating. For example, Fourier series existed ~~which~~ defined functions with discontinuities in the value (i.e., jumps) within the function's domain. More alarmingly, these discontinuous functions were constructed entirely from infinite series of *continuous* functions, which seemed to present a conceptual paradox.

While we will not study function theory as a separate topic, we will in this section cover two important functions that come up frequently in applications: (1) The step function, (2) and the delta function. The second of these, the delta function, is technically not a function; is more correctly called a *generalized function* or a *distribution*. However, the terminology *delta function* is so thoroughly ingrained in science and engineering that we will adopt this (slightly incorrect) terminology.

You may have encountered the concept of the delta function previously. The delta function is a good example of a mathematical concept whose justification was very much inspired by the fields of engineering and physics. In fact, the function is often referred to as the Dirac delta function in honor of the physicist Paul Dirac (8 August 1902 20 October 1984), who used the delta function as an important component of his description of quantum mechanics. Another common application of the delta function is to represent point charges in electrostatics. In engineering mechanics, the delta function is used to represent point loads, as described above for the case of the force acting on a beam. As mentioned above, the delta function was routinely in used applications long before it was understood mathematically. In the 1950's, the more general theory of generalized functions (which includes the delta function) was finally established by a mathematician named Laurent Schwartz. While the theory extended the notion of what constituted a function to new mathematical constructs, we will not pursue that course here. Instead, we will focus specifically on the delta function and the related step function. The ideas will be developed using primarily the tools of calculus.

4.1 Terminology

- **Generalized function.** A notion that expanded the definition of functions. In particular, a generalized function is a mathematical object that may not meet the definition of a regular (classical) function, but can be described by its action on other functions via integration. All regular functions are generalized functions, but some generalized functions are not regular functions. The delta function is the ~~most~~ well known example of a generalized function.

- **Delta function.** A generalized function that physically represents a concentrated source over a very small (relative to other scales of the problem) time or space interval. Mathematically, the delta function posed many difficulties; ultimately, these unusual mathematical objects were given a sound mathematical framework generally known as the theory of *generalized functions*. The word generalized was chosen because these new objects had function-like utility, but were not functions in any mathematically conventional sense. While the delta function does not behave like any known classical function (e.g., it is non-zero at only one point, but its integral is nonetheless equal to 1), it does arise from applied, physically-based considerations. As an example, point charges in the theory of electrostatics are representable by delta functions. Similarly, point forces in the mechanics of materials can be represented by delta functions.

- **Delta sequence.** A function, f that is indexed by an integer, n, such that the sequence of functions $(f_1, f_2, f_3, \ldots)$ becomes increasingly peaked and narrow as $n \to \infty$. An example is the conventional Gaussian function, written in the form

$$G(x;a,n) = \frac{1}{\sqrt{\pi (a/n)^2}} \exp\left(-\frac{x^2}{(a/n)^2}\right)$$

This function is illustrated for several values of n in Fig. 4.1. It is easy to see in this figure that as n increases, the function becomes more peaked and narrower. In the limit, this sequence of functions becomes a *delta function* (hence its name).

- **Step function or Heaviside function.** Step functions are functions that are discontinuous at a single point, x_0. The function is zero for $x < x_0$. At the point of discontinuity, the step function jumps from 0 to the value 1, and remains at that value for $x > x_0$. At $x = x_0$, the function is technically not defined; however, a better way of thinking about this is that the function can take on any value, c between 0 and 1 at x_0. As such, the step function is actually always an entire *equivalence class* (see §1.2.3) of functions, where the value chosen for the function at x_0 defines the particular function in the class.

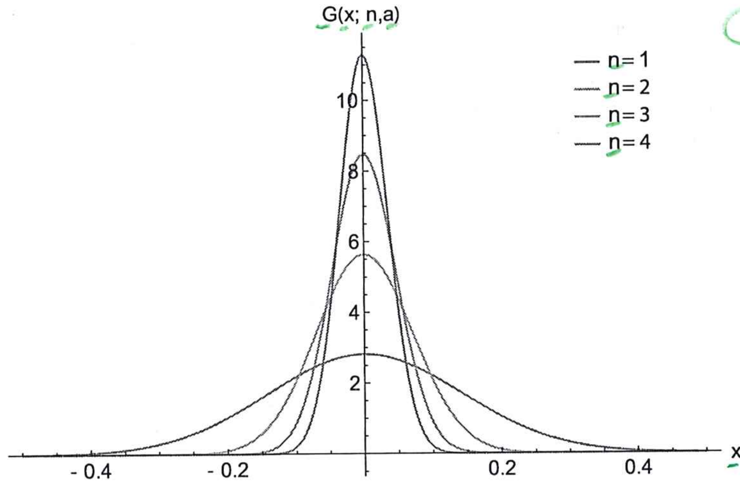


Fig. 4.1 **A delta sequence.** The Gaussian $G(x;\frac{1}{5},n)$ with standard deviation $\sigma = \frac{1}{5}n$; $n \in \mathbb{N}$ is an example of a *delta sequence*. As n increases, the function becomes increasingly peaked and narrow, even though the area stays constant at 1.

- **Generalized derivative.** The derivative of a generalized function. This concept will be defined by appealing to integration by parts. The ideas is to give concrete *meaning* to what it means to, for example, take the derivative of the step function. Classically, the step function has no derivative defined at the point of discontinuity (i.e., the function jumps a finite distance over an interval of size zero, so its classical derivative is infinite.) The theory of generalized functions gives meaning to the "derivative" of the step function. While we will not explore the theory of generalized functions *per se*, we will nonetheless adopt an approach that gives an intuitive (but mathematically rigorous) notion of the derivative of the step function.

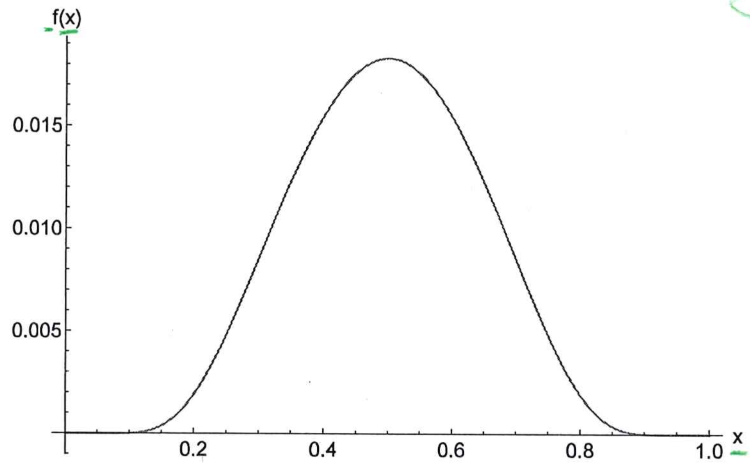
- **Function compact support.** A *function of compact support* is (for a single independent variable) a function that is nonzero only on some closed interval. While it is hard to conjure up images of such function, there are many examples that are even C^∞ (differentiable any number of times). One routinely adopted example is

$$f(x) = \exp(1/x)\exp[1/(1-x)]$$

This function is plotted in Fig. 4.2. Functions of compact support that are also C^∞ are sometimes called *bump functions* or *mollifying functions*.

- **Gamma function, Γ.** The gamma function is an analytic function on the real line. One interpretation of the gamma function is that it extends the notion of the factorial to the real numbers. The history of the gamma function reads like a "who's who" of classical mathematics, with important contributions regarding the applications and properties of the function being established by Leonhard Euler (1707–1783), Adrien-Marie Legendre (1752–1833), Carl Friedrich Gauss (1777–1855), Joseph Liouville (1809–1882), Karl Weierstrass (1815–1897), and Charles Hermite (1822–1901), among others.

Fig. 4.2 A compact function. This function is non-zero only for $0 \leq x \leq 1$, and is C^∞. Functions like this that are both compact and C^∞ form an important class of functions called *bump* functions.



4.2 The Step and Delta Functions: The Basic Idea

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In this section, the purpose is to build the basic intuition about the delta function and the step function. The intuition about the delta function is actually fairly simple to visualize. Physically, the delta function represents some quantity (e.g., a force, a concentration, heat energy) that is *idealized* as if it were concentrated at a single point. There is a purpose for this model of a physical system. For example, consider the force of a sharp wedge bearing down on a rigid beam. In Fig. 4.3, a force acting on a beam is illustrated at two levels of resolution. From afar, the idea of representing the force as if it were applied at a single point seems reasonable. However, we have to consider what we actually mean by applying a force to a single point. Physically, this is impossible; all physical contacts must occur through an interaction of finite areas. Physically, a *point* does not exist; a point is a strictly mathematical abstraction. Thus, if we zoom in on the region where the force is applied, it is clearly not occurring at a single point; in fact, the application of the force happens over a small area, and, in general, such interactions will occur in a complex way that we will not be able to know in detail.

4.2.1 Mathematical modeling of a point source

While it is true that *physically* we cannot apply a force to a single point, there are instances where *mathematically* we would like to model a quantity that is applied to a small area (relative to some other dimension; in this case, the length of the beam) as if it were applied to a point. The reason that we might want to make such a model are as follows.

1. Generally, we would not know the *actual* distribution of forces concentrated over a small region. Thus, we would not know how to represent the distribution of forces exactly.
2. Even if we could represent the distribution of forces exactly, there is the sense that, because of the difference in scales, $h \ll L$, it seems unlikely that the particular distribution of forces over h will be relevant.

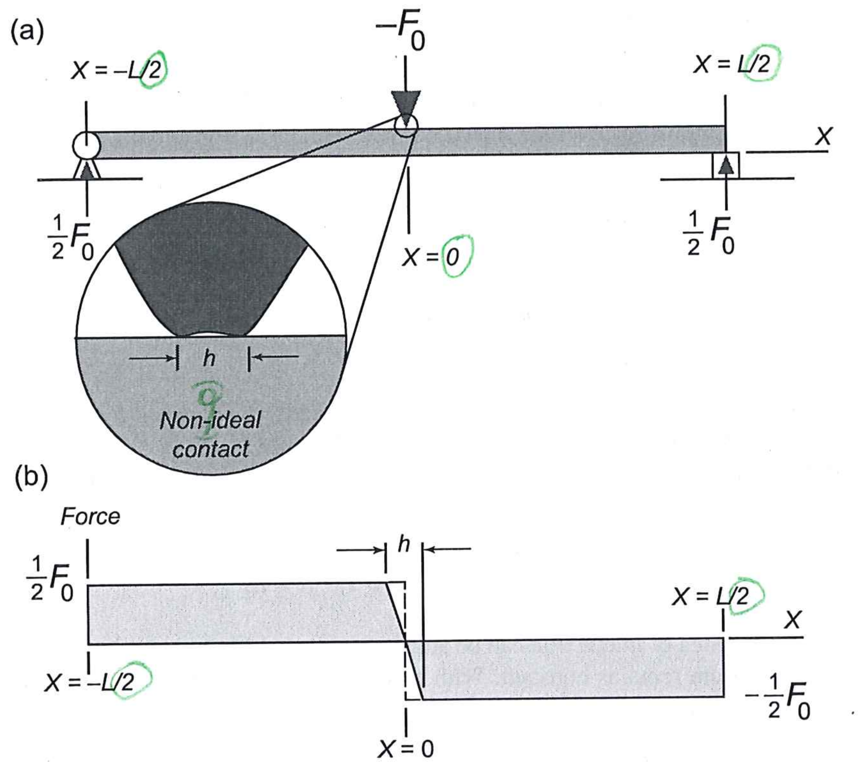


Fig. 4.3: **Idealization of non-ideal system.** Mathematical modeling always requires a level of abstraction from "reality". The trick in modeling is to make abstractions that are good approximations to the real problem, and also lead to mathematically tractable results. The use of the delta function is an example of an abstraction (i.e., an representation that is not exact, but generates the correct behavior anyway) for many physical quantities of interest. In the example in part (a) of this illustration, the actual force applied to the beam is not ideal. This means that it does not occur at a single point, but is a complex function of x that, in general, we would not know explicitly (see expanded view). In part (b) of this figure, we show what the force diagram (also known as a *shear diagram*) would look like for a force that is distributed over a small area of size h. Here, we have approximated the force as being uniformly distributed over the distance h, even though the actual distribution may be more complex.

Au: Okay? Seems like a word is missing.

better

3. There is some hope that a mathematical model that represents the force as if it were applied to a single point would offer mathematical simplifications than would a more detailed model.

At this juncture, then, we need to address the question "*is it possible to generate a sensible model of the force as if it applied to a single point?*" This is our goal for this chapter.

To start our thinking about the problem, consider the set of functions illustrated in Fig. 4.4. Each of the rectangles shown has the same area—an area of unity (i.e., *area* = 1). Clearly, for each integer increase in n, the width of the rectangle decreases by a factor of 2, and the height increases by a factor of 2. Thus, as n increases, the rectangle becomes increasingly peaked and narrow. This function is given explicitly by

$$F_0\delta_n(x) = \begin{cases} F_0/(h/n) & \text{for } -\frac{h}{2n} < x < \frac{h}{2n} \\ 0 & \textit{otherwise} \end{cases} \tag{4.1}$$

Some additional interpretation is helpful here. Consider the case $n = 1$ and $n = 2$. These case represents the force of the beam, were it to be spread uniformly over the interval of size h and $h/2$ respectively.

$$F_0\delta_1(x) = \begin{cases} F_0\frac{1}{h} & \text{for } -\frac{h}{2} < x < \frac{h}{2} \\ 0 & \textit{otherwise} \end{cases} \tag{4.2}$$

$$F_0\delta_2(x) = \begin{cases} F_0\frac{2}{h} & \text{for } -\frac{h}{4} < x < \frac{h}{4} \\ 0 & \textit{otherwise} \end{cases} \tag{4.3}$$

Increasing values of n affect the force density in obvious ways. Note additionally, the constant F_0 is distinct from the delta function, δ_n; its role is to modify the *magnitude* of the delta function.

As a concrete example of these functions, suppose we were to integrate, $F_0\delta_1$, over the interval $-h/2 \leq x \leq h/2$. We would find

$$\int_{\xi=-h/2}^{\xi=h/2} F_0\frac{1}{h}\delta_1(\xi)\,d\xi = F_0\int_{\xi=-h/2}^{\xi=h/2}\frac{1}{h}\delta_1(\xi)\,d\xi = F_0\frac{1}{h}h = F_0 \tag{4.4}$$

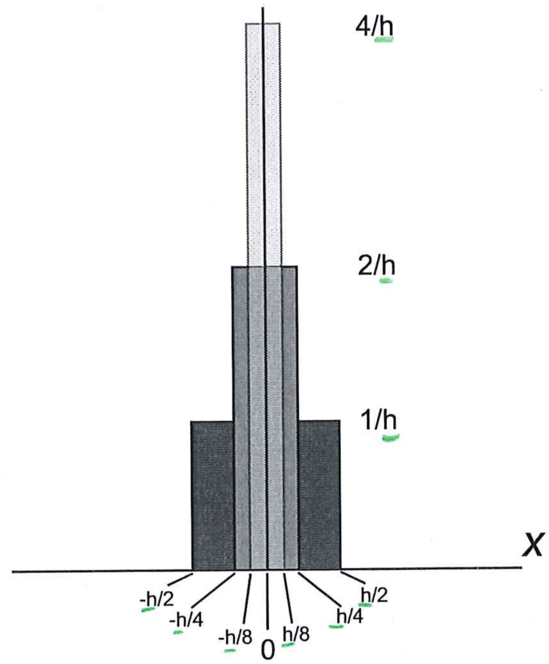
We now have a model in which the area of interaction can be adjusted to be as larger or small as we like, while the total force applied to the beam remains constant. With this representation, we can compute the force on the beam at any location x as follows

$$F(x) = \frac{F_0}{2} - \int_{\xi=-L/2}^{\xi=x} F_0\delta_n(\xi)\,d\xi \tag{4.5}$$

This result generates a distribution of shear forces in the beam as illustrated in Fig. 4.3. We can interpret the Eq. (4.5) describing this distribution of shear forces as follows. On the left-hand side, the support provides an upward force of $F_0/2$. As we approach the region near $x = 0$, the force per unit width is spread uniformly (by our model) over the small distance h. The force expression given by Eq. 4.5 represents this by an increase in the total downward force as x increases in this region, such that integrating the force density across the width $-h/(2n) < x < h/(2n)$, one recovers the downward force F_0. Finally, as x becomes greater than $h/(2n)$, then the total downward force applied is the amount F_0. A little thought on the physics of the system will hopefully match your intuition.

4.2.2 The path forward: Limiting behavior

Now, although our model for the force distribution is much simpler than what the actual distribution might be, it is still has not achieved the notion of being applied at a single point. In fact, for those who have studied beams in physics or a course in statics, the force diagram illustrated in Fig. 4.3 don't look like those that you have seen before. Usually, the transition at $x = 0$ is treated *as if the force were in fact applied at a single point*. In that case, the force diagram near $x = 0$ would resemble the curve given by the dashed line rather than the solid line, which contains an observable finite interval over which the force increases. Interestingly, we can make our representation for the force as close to the idealized dashed line as we like, simply by



rom/ **Fig. 4.4 Approximations to the delta function.** A *sequence* of rectangle functions, indexed by n. The area under the rectangle is fixed at a value of one. As n increases, the rectangle gets thinner and taller, but the area always remains one.

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making the value of h smaller (or, looking at Eq. (4.1), making n larger). This begins to give us some insight as to the nature of the delta function. It appears that we might think of this function as being a *limit* of the function described above as the width of that function tends toward zero.

The problem now is making mathematical sense out of an idea that is motivated by a physical model. Quite frequently, mathematics and science have interacted to reinforce one another. This was certainly the which/ case for the delta function, ~~where~~ physicists and engineers used ~~the delta function~~ long before it had a solid mathematical theory underpinning it. Because the delta function had such obvious intuition as the limit of a process that had a sensible interpretation, it was widely adopted by scientists and engineers. Correspondingly, mathematicians realized that the delta function was not a function in any conventional sense of the word. There was no theory that described the mathematics of such a function, and its use lead to quite a number of mathematical conundrums. However, as mentioned previously, a full understanding of the mathematics of this problem was finally developed in the 1950s. Now, such functions are used with confidence in both the physical sciences and in mathematics.

4.3 A Construction for the Step and the Delta Functions

The description given above was a relatively intuitive presentation of the delta function. Now, we will firm things up a bit by illustrating how to *construct* delta functions from appropriate sequences. Once we see how this is done and under what conditions it is possible to do so, we can simply adopt the delta function as a "shorthand" notation for a more complex process. Fortunately, the complexities of the process do not

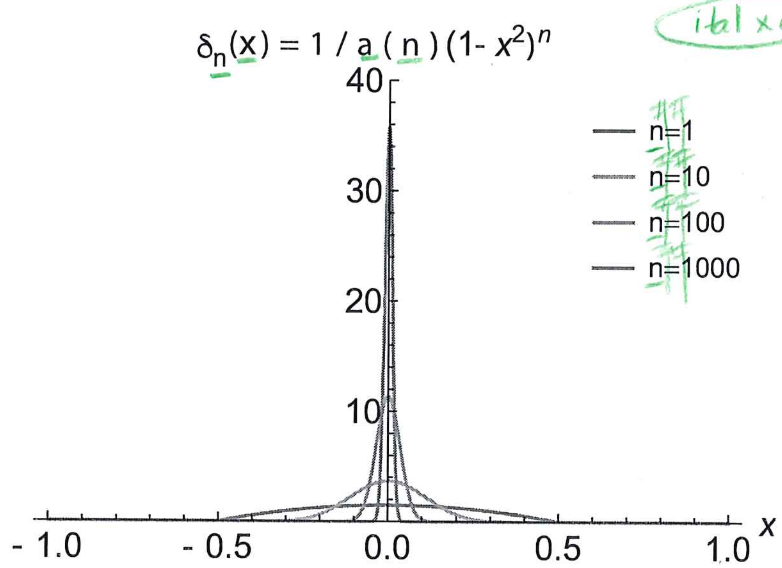

Fig. 4.5 Approximations to the delta function. A *sequence* of functions that approaches the delta function in the limit. These sequences are constructed using the function $\delta_n(x) = 1/a(n)(1-x^2)^n$, which is normalized by $a(n)$ to always have unit area. As the value of n increases, the function rapidly becomes narrower and more peaked.

need to be repeated for every problem; once the delta function is understood, it can be used by adopting an intuitively appealing set of rules.

To start the exploration of the delta and step functions, we are going to revise the previous definition to use functions that are compact and differentiable on an interval. Consider the following function and its integral. The function δ_n is technically compact on $x \in (-1/2, 1/2)$. The exponent n is an integer greater than zero. The function below is normalized by its area, $a(n)$, so that the integral of the function is always unity (i.e., area of the normalized function is $area = 1$) regardless of the value of n. The function is also differentiable $n+1$ times.

$$\delta_n(x) = \begin{cases} 0, & x < -\frac{1}{2} \\ \frac{1}{a(n)}(1-x^2)^n, & -\frac{1}{2} \le x \le \frac{1}{2} \\ 0, & x > \frac{1}{2} \end{cases} \tag{4.6}$$

$$H_n(x) = \int_{\xi=-1/2}^{\xi=x} (1-\xi^2)^n \, d\xi \tag{4.7}$$

$$a(n) = H_n(1/2) = \int_{\xi=-1/2}^{\xi=1/2} (1-\xi^2)^n \, d\xi \tag{4.8}$$

The expression for $H_n(x)$ is the integral of $\delta_n(x)$ to the value x; note that this means that $H_n(1/2)$ is the area under the curve of the function $\delta_n(x)$. This means that $\delta_n(x)$ is *normalized* so that the area under the curve is always one. The mathematical expression for H_n is quite complicated, and not technically necessary at this point for the discussion. To help with the visualization of this function and its integral, these are plotted in Figs. 4.5 and 4.6 for various values of n.

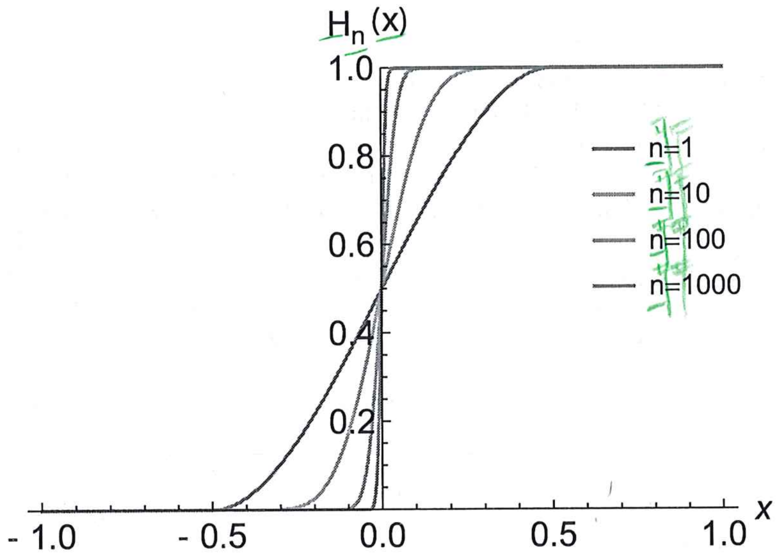

Fig. 4.6 Approximations to the step function. A *sequence* of functions that approaches the *integral* of the delta function in the limit. These sequences are constructed using the function $H_n(x) = 1/a(n)\int_{-1/2}^{x}(1-\xi^2)^n d\xi$. As the value of n increases, the function rapidly becomes abrupt and step-like. The step function is also called the Heaviside function in honor of the electrical engineer Oliver Heaviside who popularized its use.

A few things are apparent from these plots. First, the plot of the function $\delta_n(x)$ shows that $\delta_n(x)$ is a compact function (i.e., its domain is a finite one, $x \in [-1/2, 1/2]$) that becomes increasingly narrow and tall as n becomes large enough. It appears that, as we increase n, the maximum of $\delta_n(x)$ increases in value, and the values of the function not immediately near the origin become arbitrarily small, comparatively. In fact, we can determine the *second spatial moment about the vertical axis* (the variance) of the function $\delta_n(x)$ by integration by parts. This is a very messy computation, so the details will not be shown. However, the result is the following.

$$\begin{aligned}\sigma(n) &= \frac{1}{F_\Phi(1/2,n)}\int_{x=-1/2}^{x=1/2} x^2(1-x^2)^n\,dx \\ &= \frac{1}{4(2n+3)}\end{aligned} \tag{4.9}$$

Knowing the characteristic width of this function, we can see that as n increases without bound, we have the somewhat strange result

$$\lim_{n\to\infty}\sigma(n) = 0 \tag{4.10}$$

With a little bit more work, we can find the area under the curve (also known as the *zeroth spatial moment*) for $\delta_n(x)$

$$a(n) = \int_{\xi=-1/2}^{\xi=1/2}(1-\xi^2)^n\,d\xi = \frac{\sqrt{\pi}\,\Gamma(n+1)}{2\Gamma\left(n+\frac{3}{2}\right)} \tag{4.11}$$

The function defined by the symbol Γ is known as the *gamma function*; it is described in more detail in ~~the gray text box for~~ Example 4.1 ~~(The gamma function).~~

Example 4.1 (The gamma function).

We are all familiar with the idea of the factorial n that applies to any integer. This function is plotted for the first few integers in Fig. 4.7. One might wonder if there is concept similar to the factorial but for any real number. It turns out that there is, and this function is called the gamma function. The gamma function is one of the most interesting and important functions in applied mathematics, and it was derived originally by Daniel Bernoulli. However, for our purposes, we can think of it an the idea of extending the factorial. We will just state the definition of the gamma function and begin using it. Late on, we may return to this definition to extend its use to other kinds of problems. The gamma function is defined by

$$\Gamma(n) = \int_{x=0}^{x=\infty} x^{n-1} e^{-x}\, dx\,, \text{ where} n \in \mathbb{R} \tag{4.12}$$

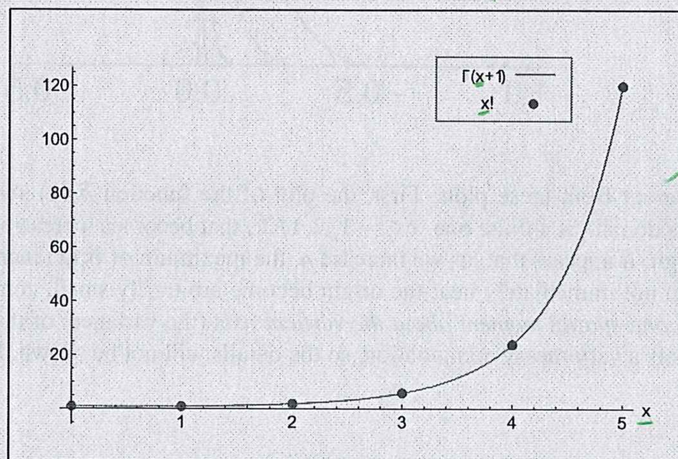

Fig. 4.7: The gamma function plotted over continuous x; for comparison, the factorial computed at the integer values of x is shown in red.

Here, we are using n to indicate any positive real number, rather than only an integer. While it is not conventional to use n to indicate a real number, it helps us remember that when n is equal to an integer value, it is equal to the factorial.

Now that we have defined the gamma function, we can turn to the area under the curve for the function $f(x) = (1-x^2)^n$. It turns out that

$$a(n) = \int_{x=-\frac{1}{2}}^{x=\frac{1}{2}} (1-x^2)^n\, dx = \frac{\sqrt{\pi}\,\Gamma(n+1)}{2\Gamma\left(n+\frac{3}{2}\right)} \tag{4.13}$$

which gives us the necessary result for the area under the curve $a(n)$.

Some additional features of the gamma function

We have the following identity (which will not be proven here)

$$\Gamma(n) = (n-1)! \tag{4.14}$$

There are many interesting properties for the gamma function. Possibly the most relevant one is as follows

$$\gamma(1) = 1$$
$$\gamma(n+1) = n\gamma(n)$$
$$n! = \gamma(n+1)$$

The last of these is the statement that, for integer values, the gamma function is equivalent to the factorial. One can derive this by using the identity above, and building up starting from $n = 1$.

So now, let's look at what happens to this function as n becomes large. In Fig. 4.5, the function is plotted for various values of n. What starts out as a very flat function for $n = 1$ rapidly becomes very highly peaked as the exponent n increases. To make this easier to discuss, define $\delta_n(x) = \frac{1}{a(n)} f_1(x)$ so that

$$f_1(x) = (1-x^2)^n \tag{4.15}$$

Note also the following

$$\lim_{n\to\infty} a(n) = \lim_{n\to\infty} \frac{\sqrt{\pi}\Gamma(n+1)}{2\Gamma\left(n+\frac{3}{2}\right)} = 0 \tag{4.16}$$

Thus, as n increases, the area used to normalize the curve also tends toward zero.

Now, note the following. The maximum of the function $f_1(x) = (1-x^2)^n$ is at $x = 0$, and its value is always $f_1(0) = 1$. However, every other point in the function has a value that is less than 1. Recall, for a positive number, m_0 less than 1, we have

$$m_0 > m_0^2 > m_0^3 > m_0^4 \ldots \tag{4.17}$$

This means that as n grows, all of the values of the function become smaller, except the value at $x = 0$, which remains 1 regardless of how large n is. Thus, once normalized by $a(n)$, the only option for the function is to become more and more peaked as n becomes arbitrarily large, with the value at $x = 0$ being equal to $1/a(n)$.

Recalling that $a(n)$ is tending toward zero as n becomes arbitrarily large, ~~and~~ we find that ~~in the limit~~ our function f_δ behaves quite unusually in the limit. It apparently has zero width (as measured by the variance), and infinite height. And, because of its construction, it also has a total area of 1; in other words

$$\lim_{n\to\infty} \left(\int_{-\infty}^{\infty} f_\delta(x,n)\, dx \right) = 1$$

While it may seem like a peaked function with zero width could not somehow also have a finite area, note that the width (variance) times height (roughly, the area) of this function is, in the limit, a $0 \cdot \infty$ form. Recall from Chapter 1, we had the example of the function $g(x) = \frac{1}{x}\sin(x)$. In the limit, as $x \to 0$, we can show using L'Hôpital's rule that

$$\lim_{n\to\infty} \frac{\sin(x)}{x} = \lim_{n\to\infty} \frac{\cos(x)}{1} = 1 \tag{4.18}$$

Therefore, it is not that unusual that a $0 \cdot \infty$ form leads to a result that is neither 0 nor ∞. This is essentially the case that we have for our function f_δ.

It is also interesting to look at the *integral* of the function $\delta_n(x)$ as n increases. Although that function is given mathematically by the expression given by Eq. (4.7), it may be more helpful just to look at Fig. 4.6 to see how this function behaves. When the values of n are very small, the function $\delta_n(x)$ is somewhat wide and smooth. Thus, as we integrate up to values of x in the range $-1/2 < x < 1/2$, the result is a smooth "S" shaped function with a maximum of 1. The function F_Φ has a maximum of 1 because the function f_δ is normalized to always have an area of 1.

For F_Φ, the function becomes more like a *step* as the value of n becomes arbitrarily large. This is consistent with what we know for the function f_δ; we know that as n becomes arbitrarily large, the function f_δ becomes increasingly narrow. Thus, the parts that contribute most to the integral of that function become concentrated around $x = 0$; therefore, the function F_Φ resembles a sharp step up from zero to 1 at $x = 0$ as n becomes large.

4.4 Delta Sequences

In the previous section, we developed a *sequence* of functions indexed by the integers, n that became increasingly peaked and narrow as n increased. These functions were defined by

$$f_1(x) = (1 - x^2)^n$$

It turns out that there are any number of sequences of functions that one can make that, in the appropriate limit, the sequence defines a delta function. For example, the function and

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$$f(x) = \begin{cases} 0 & \text{for } x < -\frac{1}{2} \\ \frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)} [\cos(n\pi x)]^n & \text{for } -\frac{1}{2} \le x \le \frac{1}{2} \\ 0 & \text{for } x > \frac{1}{2} \end{cases} \tag{4.19}$$

is also a delta sequence (this is plotted in Fig. 4.8).

The fact that there are multiple kinds of delta sequences that can lead to the same result indicates that the delta function is not a single entity; it is actually an equivalence class of sequences, any of which can be used equally well to define the (generalized) delta function. A delta sequence, then, can be defined as follows.

cap x3 **Definition 4.1 (delta convergent sequence.).** A delta (convergent) sequence, $\delta_n(x)$, is any sequence of functions such that

$$\lim_{n\to\infty} \left(\int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx \right) = f(0) \tag{4.20}$$

There are an infinite number of such functions. For example, if g(x) is any non-negative function such ital that

$$\int_{-\infty}^{\infty} g(x)\, dx = 1 \tag{4.21}$$

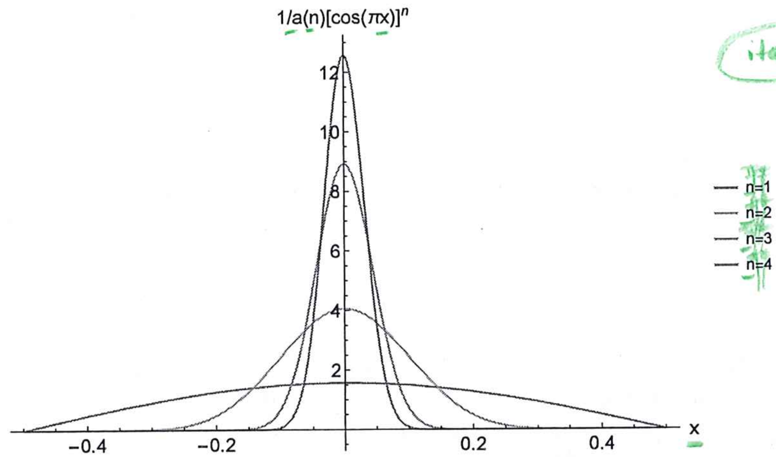

Fig. 4.8 Another delta sequence. The cosine function raised to the power n and normalized by its area (as a function of n) is a delta sequence.

then rescaling this function in the form

$$\delta_n(x) = ng(nx), \quad n = 1,2,3,\ldots \tag{4.22}$$

is a delta convergent sequence.

4.5 Properties of the delta function

Ultimately, the purpose for defining the delta function is because of its unique analytical properties, which make the representation of point sources in physical system very convenient. There are two primary analytical properties of the delta function that make it convenient in applications. Let I be either a compact ($\{I : x \in [a,b]\}$) or non-compact ($\{I : x \in [-\infty,\infty]\}$) interval containing the point x_0. Furthermore, we must assume that *the function is at least continuous* at x_0 (i.e., there cannot be a jump discontinuity there). Then the following two properties are true for the delta function.

Property 1: Unit integral.

$$\int_{x\in I} \delta(x)\, dx = 1 \tag{4.23}$$

Property 2: The sifting property.

$$\int_{x\in I} f(x)\delta(x-x_0)\, dx = f(x_0) \tag{4.24}$$

The first property arises by construction; the delta sequences are always structured such that they have unit area, so the limiting function must also have unit area. The second of these properties is the so-called *sifting* property of the delta function. It is just an application of the definition given by Eq. 4.20 but one where the delta function is shifted by the amount x_0. The simplicity of the integration of the delta function is one of its most attractive features.

Example 4.2 (The sifting property).

The sifting property of the delta function is called this because it offers a method to extract (i.e., to *sift out*) a single point of a function by integration. Consider the following example.

$$\int_3^5 y^3 \ln(y-2)\sin(y^2)e^{-y}\delta(y-4)\,dy$$

Ordinarily, this integral would be impossible to compute analytically (without the delta function, a symbolic integration software cannot resolve the integrand of this integral). However, because of the sifting property of the delta function, this integral is simple. Essentially, the integral is zero everywhere, except at the single point where the independent variable of integration is equal to 4. At that point, we have

$$\delta(y-4) = \delta(4-4) = \delta(0)$$

Thus, the single point that contributes to the integral is the one where the argument of the delta function is zero, that is, the point $y = 4$. Thus, the result is

$$\int_3^5 y^3 \ln(y-2)\sin(y^2)e^{-y}\delta(y-4)\,dy = 4^3\ln(4-2)\sin(4^2)e^{-4}$$

Which is just the integrand evaluated at $y = 4$.

As a second, more abstract example, consider the following integration. As above, assume that x_0 is any point defined within the domain of the integration. Then, even the Gaussian function can be easily evaluated when integrated against a delta function.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2a^2\pi}} \exp\left(-\frac{1}{2}\frac{x^2}{a^2}\right)\delta(x-x_0)\,dx = \frac{1}{\sqrt{2a^2\pi}} \exp\left(-\frac{1}{2}\frac{x_0^2}{a^2}\right)$$

We will see examples similar to this one in the course of studying the solution to partial differential equations. There, the Gaussian is known as the heat kernel, and the delta function would represent the distribution of heat specified as an initial condition.

4.6 An explanation of the sifting property for smooth functions

For piecewise smooth functions, understanding how and why the delta function behaves the way it does is reasonably straightforwards. We need only use the ideas of limits, and the expansion defined by the Taylor series.

In the following development, we assume that the function f is analytic near zero. For concreteness, we will examine the ~~following~~ function, f

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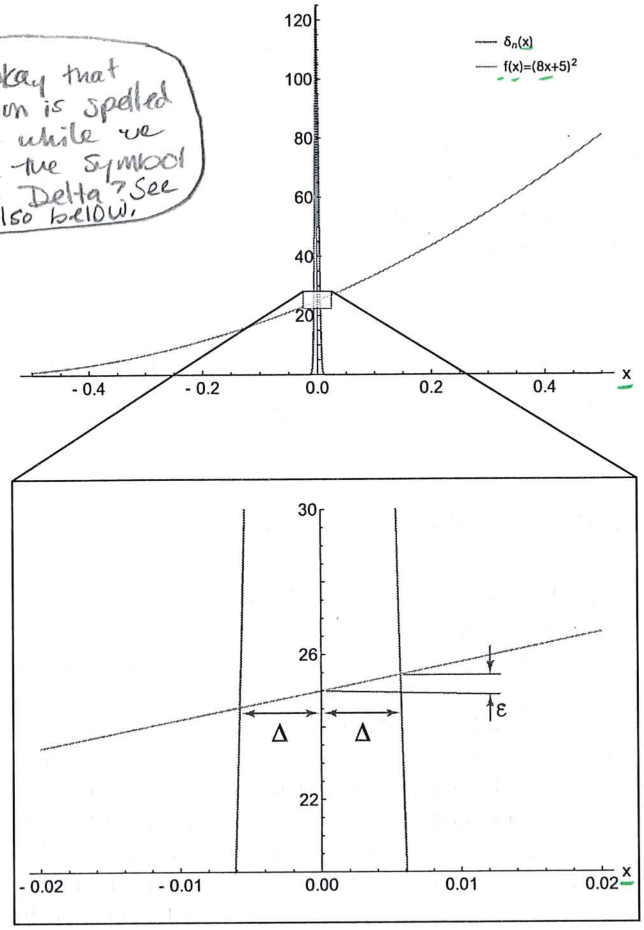
Fig. 4.9 A function, f in the vicinity of a delta sequence function. The function changes by less than or equal to the amount 2ε over the interval given by $\{I : -\Delta < x < \Delta\}$. The value of *epsilon* can be made as small as we like by making Δ smaller. By definition, there is always a delta sequence with width less than 2Δ around zero; we need only make the integer n large enough.

Au: Okay that epsilon is spelled out while we use the symbol for Delta? See also below.



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$$f(x) = (8x+5)^2 \tag{4.25}$$

9/ In Fig. 4.9, the function is plotted, and one member of the of delta sequence functions is plotted for reference.

For the interval around zero, we can write the following Taylor series expansion for f because f is assumed to be analytic there.

$$f(\varepsilon) = f(0) + \Delta f'(0) + \frac{\Delta^2}{2!} f''(0) + \ldots \tag{4.26}$$

cap/ Note that we can take Δ to be as small as we like because we can always find a delta sequence that has width less than 2Δ simply by making n large enough. From chapter 1, we know that none of the derivatives of f must be finite in the region around zero (this is a fact arising from the fact that f is analytic). Therefore, we can always find a *Delta* sufficiently small such that 9/

$$f(0) \gg \Delta f'(0)$$
$$f(0) \gg \tfrac{1}{2!}\Delta f''(0)$$
$$\dots$$

This completes the construction. What we have shown is that we can always find a member of the delta sequence of functions such that the width of the function is small enough such that we can estimate f by its value at $x = 0$, with an error that can be made as small as we like. This validates the use of the delta function as having the property, in the limit, of

$$\int_{x\in I} f(x)\delta(x)\,dx = f(0) \tag{4.27}$$

The extension of the development to applications of the delta function at other points requires only that delta function be shifted (an affine transformation) appropriately.

4.7 A computable example of the limit of a delta sequence and the sifting property

One of the difficulties of working with delta sequences is that most of the results are understood (in the general cases, anyway) though abstract constructions. In other words, it is not frequent that one can find, for example, a computable example that shows how a delta sequence converges in the sense that we have been discussing.

There are some examples that are computable, with some effort. ~~In the example following,~~ one example of a delta sequence is provided. The goal for the example is to show that the sifting property of the delta function is valid when the sequence of functions ~~are~~ integrated against a continuous function $f(x)$. This is done in two steps. In the first step, the delta sequence is integrated against $f(x)$. In the second step, the limit as $n \to \infty$ is evaluated after integration to show that the result is indeed the sifting property of the delta function. It is important to note that the process of limits and integration do not necessarily commute! In fact, in this case they distinctly do not commute. If we first take the limit of the delta, and then attempt to ~~to~~ integrate, we find a situation where the delta function is supported at only a single point; regardless of what kind of integration one proposes, such an integral is necessarily zero!

Example 4.3 (A computable example of the sifting property).
There are many functions, both compact (i.e., defined on a closed interval) and non-compact (defined for the entire real number line). For those who might be interested, compilations of such functions are available in the literature (e.g., Dang and Ehrhardt (2012)). The following function is one that is interesting, and also one that can be handled analytically

$$\delta_n(x) = \begin{cases} \frac{1}{a_n}(1-4x^2)^n & \text{for } \frac{1}{2} \le x \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

where

$$a_n = \frac{\sqrt{\pi}\,\Gamma(n+1)}{2\Gamma\left(n+\frac{3}{2}\right)}$$

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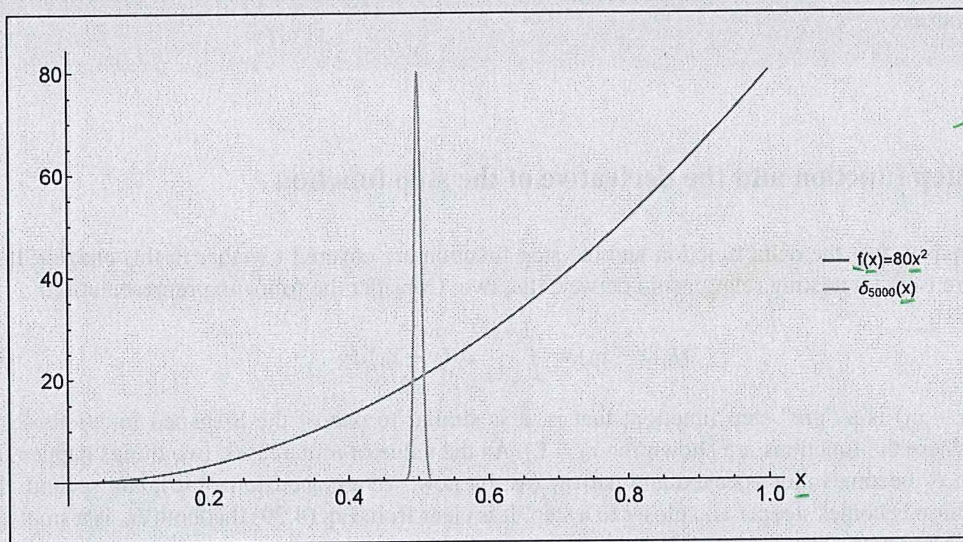
This function is, in fact, the one provided at the start of this chapter. Examples of the shape of this function with increasing n are given in Fig. 4.5.

One of the nice properties of this function is that it can be integrated analytically. Thus, we can consider the following.



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bf Fig. 4.10: The function $f(x) = 80x^2$ plotted against the delta function $\delta_n(x - \frac{1}{2})$ defined above, with $n = 5000$.

If we integrate our pre-delta function against an explicit function, say $f(x) = 80x^2$, then it may be possible to evaluate this integral. In this example, we will shift the pre-delta function by one-half; an example of the pre-delta function and the function $f(x)$ are plotted in Fig. 4.10. Mathematically, the problme is represented as follows tr

$$\int_{x=0}^{x=1} f(x)\delta_n(x - \tfrac{1}{2})\,dx = \int_{x=0}^{x=1} \underbrace{\underbrace{80x^2}_{f(x)} \frac{1}{a_n}(1 - 4(x - \tfrac{1}{2})^2)^n}_{\delta_n(x-\frac{1}{2})}\,dx$$

It turns out that this definite integral can be computed for any value of n. The result is

$$\int_{x=0}^{x=1} f(x)\delta_n(x - \tfrac{1}{2})\,dx = 80\frac{2+n}{2(3+2n)}$$

This is the result of the integration of any of the infinity of pre-delta functions, $\delta_n(x - \frac{1}{2})$, against the function $f(x) = 80x^2$ for any value of n. To find the final result, we need only take the limit of $n \to \infty$

$$\lim_{n\to\infty} 80\frac{2+n}{2(3+2n)} = 80 \cdot \frac{1}{4} = 20$$

This means that we have for our particular choice of pre-delta function, we have the result

$$\lim_{n\to\infty}\left(\int_{x=0}^{x=1} 80x^2\left[\frac{1}{a_n}(1-4(x-\tfrac{1}{2})^2)^n\right]dx\right)=20$$

and this result is exactly the result for $f(\frac{1}{2})=80(\frac{1}{2})^2$. Thus, this provides at least one explicit example showing how the limit of delta functions yields exactly the result specified by the sifting property of the delta function.

4.8 The step function and the derivative of the step function

There is a reason that the delta function and the step function are covered together in this chapter. It turns out that there is an interesting relationship between the two. Consider the following representation.

$$H_n(x-x_0)=\int_{y=-\infty}^{y=x}\delta_n(y-x_0)\,dy \tag{4.28}$$

where $H_n(x-x_0)$ is a "pre" step function; that is, it is similar to one of the S-shaped functions show in Fig. 4.11 (where the functions are shown for $x_0=0$). As the value of n increases, two things happen. First, the function δ_n becomes more peaked and narrow (as, by now, we are accustomed to), and second, the S-shaped function becomes steeper and closer to a step. It is clear from Eq. (4.29) that both H_n is a smooth and differentiable function. The derivative is given by

$$\frac{d}{dx}H_n(x-x_0)=\delta_n(x-x_0) \tag{4.29}$$

So we can consider the delta function to be the derivative of the step function. Ordinarily, we should only attempt to interpret the delta function when it is under an integral; in other words, it is not a regular function but a generalized one, and technically it is not defined outside of an integral. However, it is a common and even customary abuse of the notation to use the delta function as a symbol outside of an integral, particularly in physics, engineering, and the mathematics of partial differential equations. However, it must always be kept in mind that the *meaning* of this is that the delta function represents a sequence of functions whose limit must be interpreted only during the process of integration.

4.9 Does the delta function have a Fourier Series Representation?

This simultaneously deep and yet simple question. The delta function is not a function at all, so asking whether or not it has a Fourier series representation may seem a bit odd. However, the "pre" delta functions δ_n are in fact regular functions. Depending upon which delta sequence is chosen, these pre-delta functions may even be C^∞ smooth.

In the problems from the previous chapter, the Fourier series for a Gaussian function was examined. It turns out that we can adopt this function to be a delta sequence as follows. First, we define the sequences as a function of m (here, we have switched from n to m to prevent confusion when examining the Fourier series) as follows.

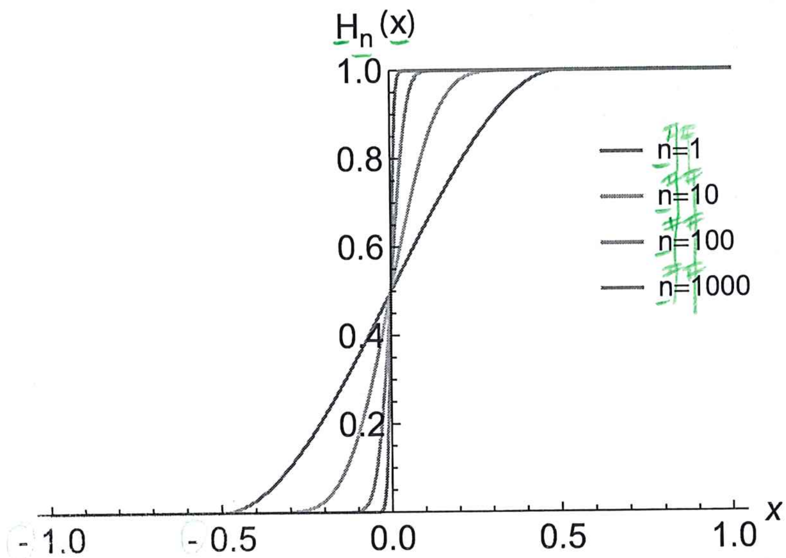

Fig. 4.11 Step function sequences. A *sequence* of functions that approaches the *integral* of the delta function in the limit. This function converges to the step function as $n \to \infty$.

$$\delta_m(x) = \frac{\alpha m}{\sqrt{\pi}\operatorname{erf}(\alpha m)} \exp\left(-\alpha^2 m^2 x^2\right) \tag{4.30}$$

$$A_0' = \frac{1}{2}\int_{-1}^{1} \delta_m(x)\,dx = \frac{1}{2}$$

$$A_n = \frac{1}{1}\int_{-1}^{1} \delta_m(x)\cos(n\pi x)\,dx = \exp\left(-\frac{\pi^2 n^2}{4\alpha^2 m^2}\right)$$

Now note that in the limit of $m \to \infty$, we obtain the delta function rather than the delta sequences. The Fourier coefficients take the values

$$A_0' = \frac{1}{2}\int_{-1}^{1} \delta_m(x)\,dx = \frac{1}{2}$$

$$A_n = 1$$

This leads to a series where all of the amplitudes are the same value - they are all unity (except A_0'). What this says is that the delta function contains all possible frequencies, and the value of the amplitude does not, in general, decrease. This is a very unusual result! For more regular functions, it is possible to show that the amplitudes of the Fourier coefficients must decrease in value (this is related to Parseval's theorem), and, in fact, even yield a finite value when summed. However, for the delta function, we do not have this relation; the sum of the amplitudes tends toward infinity. While it is true that the Fourier series for the delta function violates Parseval's theorem, we should not be too surprised. The delta function is not a regular function, and therefore the conditions required for proving Pareseval's theorem are not met by the delta function.

It is interesting to look at the plot of the function predicted by summing up the Fourier series for the delta function; this plot is given in Fig. 4.12. The Fourier transform of the delta function is used in applications, such as in signal processing or modeling of point sources. We will see more about Fourier analysis of the delta function when we study the Fourier transform later on.

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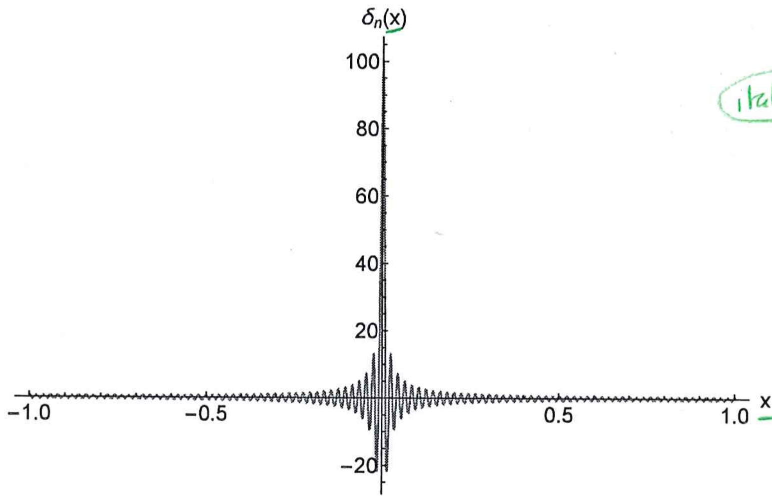

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Fig. 4.12 The Fourier series sum for the delta function. The delta function has an unusual Fourier series expansion. In this figure, the series is approximated by the first 101 terms in the sum. Note that the resulting sum does actually become peaked and narrow at $x = 0$, as expected for approximations to the delta function.

Problems

Practice Problems

For the following problems, evaluate the integral.

1.
$$\int_{-1}^{1} \delta(x)\,dx$$

2.
$$\int_{-\infty}^{\infty} f(y)\delta(y)\,dy$$

3.
$$\int_{-\infty}^{\infty} y^3\delta(y)\,dy$$

4.
$$\int_{-\infty}^{\infty} y^3\delta(x-y)\,dy$$

5.
$$\int_{z=-1}^{z=1} \delta(z)f(x-z)\,dz,$$
where $0 < x < 1$

Au: Okay? Sentence did not seem to make sense grammatically as written.

6.
$$\int_{z=0}^{z=1} \delta(x-z)f(z)\,dz,$$
where $0 < x < 1$

7.
$$\int_{-\infty}^{\infty} \frac{\sin(y)}{\sin(x)}\delta(x-y)\,dy$$

8.
$$\int_{-\infty}^{\infty} \frac{y^3-x^3}{x^2}\delta(x-y)\,dy$$

9.
$$\int_{a}^{b} \sin(e^x)\delta(x-x_0)\,dx,$$
where $a < x_0 < b$

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10. There are sequences that are the complement of delta sequences. These are the zero sequences. A delta sequence an extreme transformations of a function; when integrating a function against a delta sequence,

the only information that passes through comes from a single point. The zero sequences act in a complementary way; although its integral is always unity, when integrating a function against a zero sequence, in the limit the result is zero. As an example, the following is an zero sequence on the real line.

$$\iota_n(x) = \begin{cases} \frac{1}{2n} & \text{for } -n \leq x \leq n, \ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

11. Show that the function

$$(an)Z_n\left(\frac{x}{an}\right)$$

Is an zero function as $n \to \infty$

12. Show that every delta sequence can be transformed into an zero sequence by the substitution $n \to 1/n$.

Applied and More Challenging Problems

1. Heating of a surface at point can be represented by using the delta function. For this problem, suppose you are interested in finding the steady state temperature distribution for an insulated wire (so it does not loose heat to the environment) of length L and constant cross-sectional area, A. The wire is heated at its center point, $x = L/2$, by a powerful laser (where a small amount of insulation has been removed so that the wire can be heated). Suppose also that the left end of the wire is held in liquid nitrogen at $T_0 = 77$ Kelvin (K) and the right side is held at $T_1 = 194.5$ K in liquid CO_2. Assume $L = 1$ m.

$$K_T \frac{d^2T}{dx^2} = -q_0 \delta(x - L/2)$$
$$T(0) = T_0$$
$$T(L) = T_1$$

Note! The delta function (curiously) must have units of $1/Length$ if it is to have an integral of unity (without units). Thus, the equation above is dimensionally correct. For this problem, assume the heat flux, q_0 is $q_0 = 200{,}000$ W m^{-2}, and the thermal conductivity is K= 385 W m^{-1}K^{-1}. Please plot the final solution over the whole domain.

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