# Gibbs sampling by completion

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Consider the example of Robert (1996), where the Gibbs sampler is illustrated with a completion technique. Completion of a density $f$ by a density $g$ means that the latter density is constructed such that $f$ appears as a marginal density of $g$. Assume we want to sample from the following density:

$$p(\theta|\theta_0) \propto \frac{\exp(-\theta^2/2)}{(1+(\theta-\theta_0)^2)^{\nu}},\ \theta \in \mathbb{R}, \nu > 0, \tag{1}$$

where $\theta_0$ is a known real number. To find the density $g$ that completes $p(\theta|\theta_0)$, note that $p(\theta|\theta_0)$ can be written by means of the following integral:

$$\begin{aligned}
p(\theta|\theta_0) &\propto \int_0^{+\infty} \exp(-\theta^2/2) \exp\left(-(1+(\theta-\theta_0)^2)\eta/2\right) \eta^{\nu-1}\, d\eta \\
&\propto \exp(-\theta^2/2) \int_0^{+\infty} \eta^{\nu-1} \exp\left(-\eta\left(\frac{1+(\theta-\theta_0)^2}{2}\right)\right)\, d\eta \\
&\propto \exp(-\theta^2/2)\Gamma(\nu)\left(\frac{1+(\theta-\theta_0)^2}{2}\right)^{-\nu} \\
&\propto \frac{\exp(-\theta^2/2)}{(1+(\theta-\theta_0)^2)^{\nu}}.
\end{aligned}$$

In other words, the integrand corresponding to the joint density $g(\theta,\eta) \propto \exp(-\theta^2/2) \exp\left(-(1+(\theta-\theta_0)^2)\eta/2\right) \eta^{\nu-1}$ with $\eta > 0$ completes $p(\theta|\theta_0)$. From here, we can easily obtain the conditional of $\eta$ given $\theta$:

$$\begin{aligned}
g_1(\eta|\theta) &= \frac{(0.5(1+(\theta-\theta_0)^2))^{\nu}}{\Gamma(\nu)} \eta^{\nu-1} \exp\left(-(1+(\theta-\theta_0)^2)\eta/2\right) \\
\Rightarrow (\eta|\theta) &\sim \mathcal{G}(\nu, 0.5(1+(\theta-\theta_0)^2)),
\end{aligned} \tag{2}$$

where $\mathcal{G}(a,b)$ is a Gamma distribution with shape parameter $a > 0$ and rate parameter $b > 0$, with mean $a/b$ and variance $a/b^2$. We can also show that the conditional distribution of $\theta$ given $\eta$ is given by:

$$
\begin{aligned}
g_2(\theta|\eta) &= \frac{\sqrt{1+\eta}}{\sqrt{2\pi}} \exp\left(-\frac{(1+\eta)}{2}\left(\theta - \frac{\theta_0\eta}{1+\eta}\right)^2\right) \\
(\theta|\eta) &\sim \mathcal{N}\left(\frac{\theta_0\eta}{1+\eta}, \frac{1}{1+\eta}\right). \qquad (3)
\end{aligned}
$$

From (2) and (3), one can easily build a Gibbs sampler to sample from $p(\theta|\theta_0)$.

```
#--------------------------------------------------------------------------------
#
# Gibbs sampling by completion
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#-------------------------------------------------------------------------------

theta0 <- 2
nu <- 3

#--- Target distribution to sample from
ptarget <- function(theta){
val <- exp(-0.5 * theta^2)/((1+(theta-theta0)^2)^nu)
return(val)
}

#--- Plot target
x <- seq(-1,4, length = 1000)
dx <- x[2]-x[1]
target_val <- sapply(x, ptarget)
cnorm <- 1/(sum(target_val * dx))
target_val <- cnorm * target_val
#sum(target_val * dx)
plot(x, target_val, type="l", xlab=expression(theta),
ylab="Posterior", lwd = 2)

#-- Gibbs sampler
theta_sample <- c()
eta_sample <- c()
theta_init <- 1
M <- 10000

for(m in 1:M){
eta_sample[m] <- rgamma(n = 1, shape = nu,
rate = 0.5 * (1+(theta_init-theta0)^2))
theta_sample[m] <- rnorm(n = 1,
mean = theta0 * (eta_sample[m]/(1+eta_sample[m])),
sd = sqrt(1/(1+eta_sample[m])))
```
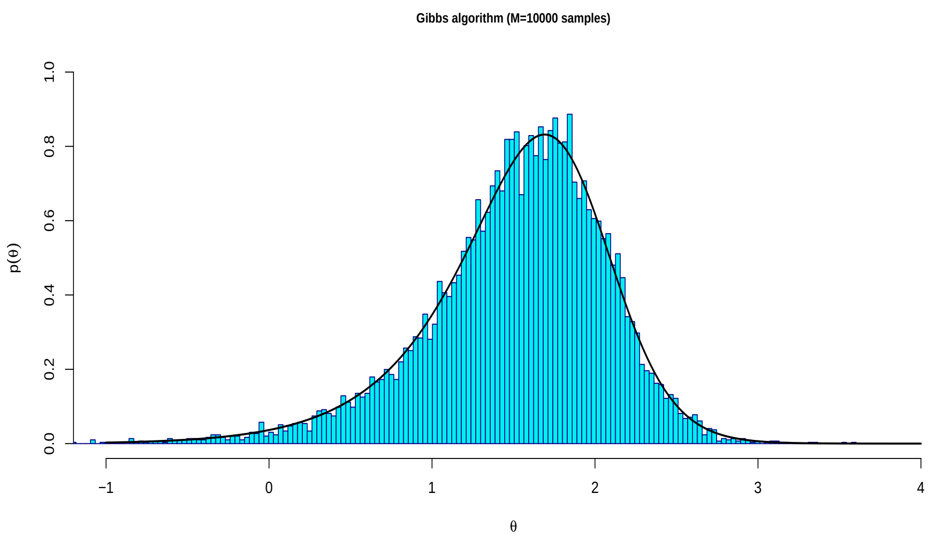

```
theta_init <- theta_sample[m]
}

hist(theta_sample,
breaks=seq(min(theta_sample),max(theta_sample),length.out = 200),
freq = FALSE,
xlim = c(-1,4),
ylim=c(0,1),
ylab=expression(p(theta)),
xlab=expression(theta),
col="cyan2",
border = "darkblue",
main="Gibbs algorithm (M=10000 samples)",
cex.main=0.8)
lines(x, target_val, type="l", lwd = 2)


summary(theta_sample)
```




# References


Robert, C. Méthodes de Monte Carlo par chaînes de Markov. Economica, 1996.