Two-Particle Systems

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- To use the Schrödinger equation we must use know the Hamiltonian:

$$H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\mathbf{r}_1, \mathbf{r}_2, t)$$

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- The probability of finding each particle in a given volume $d^3\mathbf{r}_i$ is

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- Here, ψ satisfies the time-independent Schrödinger wave equation:

$$-\frac{\hbar^2}{2m_1} \boldsymbol{\nabla}_1^2 \psi - \frac{\hbar^2}{2m_2} \boldsymbol{\nabla}_2^2 \psi + V\psi = E\psi$$

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- With particles, we can only know that one of the two is in state ψ_a, and the other is in state ψ_b, but we cannot tell them apart
- We represent this by a wave function which doesn't assume either particle is in a particular state:

$$\psi \pm (\mathbf{r}_1, \mathbf{r}_2) = A\left[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) \pm \psi_b(\mathbf{r}_1)\psi_a(\mathbf{r}_2)\right]$$

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- It works out that Bosons are particles with integer spin, and Fermions are particles with half-integer spin.
- As a consequence, we see we cannot have two Fermions cannot occupy the same state else $\psi_a = \psi_b$ and

$$\psi_-(\mathbf{r}_1, \mathbf{r}_2) = A\left[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) - \psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2)\right] = 0$$

- This result is the famous *Pauli Exclusion Principle*

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$$\psi_-(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2) - \psi_b(x_1)\psi_a(x_2)] \tag{3}$$

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$$\langle (x_1 - x_2)^2 \rangle_d = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b$$

Exchange Forces: Identical Particles

- Case 2: Identical Particles (using wavefunctions from 2 and 3)

$$
\begin{aligned}
\langle x_1^2 \rangle =& \frac{1}{2}[\int x_1^2 |\psi_a(x_1)|^2 dx_1 \int |\psi_b(x_2)|^2 dx_2 \\
&+ \int x_1^2 |\psi_b(x_1)|^2 dx_1 \int |\psi_a(x_2)|^2 dx_2 \\
&\pm \int x_1^2 \psi_a(x_1)^* \psi_b(x_1) dx_1 \int \psi_b(x_2)^* \psi_a(x_2) dx_2 \\
&\pm \int x_1^2 \psi_b(x_1)^* \psi_a(x_1) dx_1 \int \psi_a(x_2)^* \psi_b(x_2) dx_2] \\
=& \frac{1}{2}\left[\langle x^2 \rangle_a + \langle x^2 \rangle_b \pm 0 \pm 0 \right] = \frac{1}{2}\left(\langle x^2 \rangle_a + \langle x^2 \rangle_b \right)
\end{aligned}
$$

- Just the same,

$$\langle x_2^2 \rangle = \frac{1}{2}\left(\langle x^2 \rangle_b + \langle x^2 \rangle_a \right)$$

$$\begin{aligned}
\langle x_1 x_2 \rangle = \frac{1}{2}\Big[&\int x_1 |\psi_a(x_1)|^2 \, dx_1 \int x_2 |\psi_b(x_2)|^2 \, dx_2 \\
&+ \int x_1 |\psi_b(x_1)|^2 \, dx_1 \int x_2 |\psi_a(x_2)|^2 \, dx_2 \\
&\pm \int x_1 \psi_a(x_1)^* \psi_b(x_1) \, dx_1 \int x_2 \psi_b(x_2)^* \psi_a(x_2) \, dx_2 \\
&\pm \int x_1 \psi_b(x_1)^* \psi_a(x_1) \, dx_1 \int x_2 \psi_a(x_2)^* \psi_b(x_2) \, dx_2 \Big] \\
= \frac{1}{2}&\left(\langle x \rangle_a \langle x \rangle_b + \langle x \rangle_b \langle x \rangle_a \pm \langle x \rangle_{ab} \langle x \rangle_{ba} \pm \langle x \rangle_{ba} \langle x \rangle_{ab} \right) \\
= \langle x &\rangle_a \langle x \rangle_b \pm |\langle x \rangle_{ab}|^2,
\end{aligned}$$

where

$$\langle x \rangle_{ab} \equiv \int x \psi_a(x)^* \psi_b(x) \, dx.$$

Evidently

$$\langle (x_1 - x_2)^2 \rangle_\pm = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b \mp 2|\langle x \rangle_{ab}|^2.$$

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- If we compare the results of the distinguishable particles and the identical, we see the only difference occurring in the last term:

$$\langle(\Delta x)^2\rangle_\pm = \langle(\Delta x)^2\rangle_d \mp 2|\langle x\rangle_{ab}|^2$$

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- The term $\langle x\rangle_{ab}$ vanishes if the two wave functions do not overlap
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- Therefore, it's safe to say that if two electrons have nonoverlapping wave functions, they are in fact distinguishable

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- The complete state is described by the electrons position and also a spinor: $\psi(r)\chi(s)$

Thank you!