Dirac Notation

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Overview

Vectors

- A vector A can be described by specifying its components with respect to a set of axes

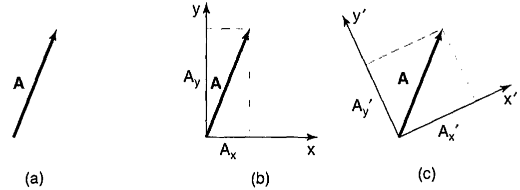

Figure 1: (a) vector, (b) x,y axis, (c) x',y' axis [Griffiths, 2005]

- $A : A_x = \hat{\imath} \cdot A,\ A_y = \hat{\jmath} \cdot A$ or $A : A'_x = \hat{\imath}' \cdot A,\ A'_y = \hat{\jmath}' \cdot A$
- Same vector A, different choice of basis
- The vector lives in a space independent of coordinates

The Wave Function

- The state of a system is also represented by a vector, $|S\rangle$
- This vector lives in Hilbert space
- Represented with respect to different bases

$$\Psi(x,t) = \langle x|S(t)\rangle \tag{1}$$

- $|x\rangle$ is the position eigenfunction with eigenvalues x

$$\Phi(p,t) = \langle p|S(t)\rangle \tag{2}$$

- $|p\rangle$ is the momentum eigenfunction with eigenvalues p
- Same vector S expanded over different bases of eigenfunctions

The Wave Function

- S may also be expanded in the energy eigenfunction basis (assuming discrete spectrum for simplicity)

$$c_n(t) = \langle n|S(t)\rangle) \tag{3}$$

- In this case, $|n\rangle$ is the *n*th eigenfunction of $\hat{H}$
- In all cases, it is the same state S
- Ψ, Φ and c_n contain the same exact information and are simply different ways of describing the same vector:

$$\begin{aligned}\Psi(x,t) &= \int \Psi(y,t)\delta(x-y)dy = \int \Phi(p,t)\frac{1}{\sqrt{2\pi\hbar}}e^{\imath px/\hbar}dp \\ &= \Sigma c_n e^{-\imath E_n t/\hbar}\psi_n(x)\end{aligned} \tag{4}$$

Linear Transformations

- Operators transform one vector into another:

$$|\beta\rangle = \hat{Q}|\alpha\rangle \tag{5}$$

- A vector is represented by it's components with respect to a basis $|e_n\rangle$ by:

$$|\alpha\rangle = \sum_n a_n |e_n\rangle \text{ with } a_n = \langle e_n|\alpha\rangle \tag{6}$$

- Operators are expressed by their matrix elements:

$$\langle e_m|\hat{Q}|e_n\rangle \equiv Q_{mn} \tag{7}$$

- With this notation, 5 becomes:

$$\sum_n b_n |e_n\rangle = \sum_n a_n \hat{Q}|e_n\rangle \tag{8}$$

Linear Transformations

- 8 can be rewritten by using the inner product with $|e_m\rangle$

$$\sum_n b_n \langle e_m | e_n \rangle = \sum_n a_n \langle e_m | \hat{Q} | e_n \rangle \tag{9}$$

- Which gives

$$b_m = \sum_n Q_{mn} a_n \tag{10}$$

- The elements of the matrix describe how the components transform
- We might ecounter systems with a finite number of linearly independent states
- Then, we have an N-dimensional vector space with operators in the form of $(N \times N)$ matricies

Dirac - bra and ket

- Dirac split the inner product bracket notation into two pieces: the bra, $\langle\alpha|$ and ket, $|\beta\rangle$
- The ket is a vector, but the bra is a linear function of vectors
- When the bra acts on the ket, it results in a complex number (the inner product)
- Similar to an operator, except an operator acting on a vector produces another vector, whereas the bra acting on a vector produces a number
- We can think of the bra as an instruction telling us to integrate:

$$\langle f| = \int f^*[\cdots]dx$$

- Where the $[\cdots]$ represents the function encountered in the ket

Dirac - bra and ket

- A vector in a finite-dimensional vector space can be expressed in columns as

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \tag{11}$$

- The bra corresponding to this vector is:

$$\langle\alpha| = (a_1^* a_2^* \cdots a_n^*) \tag{12}$$

- The collection of all bras describes a vector space known as *dual space*

Projection Operator

- If $|\alpha\rangle$ is normalized, we can define the *projection operator* to be

$$\hat{P} \equiv |\alpha\rangle\langle\alpha| \tag{13}$$

- which picks out the portion of any vector along $|\alpha\rangle$:

$$\hat{P}|\beta\rangle = \langle\alpha|\beta|\alpha\rangle \tag{14}$$

- Lets consider a discrete orthonormal basis $|e_n|\rangle$ such that

$$\langle e_m|e_n\rangle = \delta_{mn} \tag{15}$$

- Then the identity operator is

$$\sum_n |e_n\rangle\langle e_n| = 1 \tag{16}$$

Dirac Orthonormalized Continuous Basis

- If this operator acts on a vector $|\alpha\rangle$ we obtain

$$\sum_n |e_n\rangle\langle e_n|\alpha\rangle = |\alpha\rangle \tag{17}$$

- The same can be said for a Dirac orthonormalized continuous basis $|e_z\rangle$:

$$\langle e_z|e_{z'}\rangle = \delta(z - z') \tag{18}$$

- Then

$$\int |e_z\rangle\langle e_z|dz = 1 \tag{19}$$

References

David J. Griffiths (2005)
Introduction to Quantum Mechanics, 2nd Edition
Pearson Prentice Hall