Quantum Harmonic Oscillator

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The Harmonic Oscillator

- the classic example of a harmonic oscillator in classical mechanics is a mass m attached to a spring with spring constant k

$$F = -kx = m\,\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

- this is an ODE with solution

$$x(t) = A\sin(\omega t) + B\cos(\omega t),$$

- where $\omega = \sqrt{k/m}$

The Harmonic Oscillator Potential

- the potential energy is parabolic, and given by $V(x) = \frac{1}{2}kx^2$,
- practically any potential is approximately parabolic, when x is close to some local minimum, x_0
- this can be shown by the taylor series centered around x_0

The Harmonic Oscillator Potential

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + \ldots$$

- ignore the constant $V(x_0)$, as that has no effect on the force
- $V'(x_0) = 0$, because it is a local minimum
- ignore the higher order terms, leaving

$$V(x) \cong \frac{1}{2}V''(x_0)(x - x_0)^2$$

Ladder Operators

- the time independent Schrödinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V(x)\psi = E\psi$$

- using $\hat{p}^2 = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}$, and $V(x) = \frac{1}{2}m\omega^2 x^2$, we can rewrite the wave equation as

$$\frac{1}{2m}\left[\hat{p}^2 + (m\omega x)^2\right]\psi = E\psi$$

- so the Hamiltonian is

$$\hat{H} = \frac{1}{2m}\left[\hat{p}^2 + (m\omega x)^2\right]$$

Ladder Operators

- if $\hat{p}$ were a scalar, and not an operator, then we could rewrite the Hamiltonian as

$$\hat{H} = \frac{1}{2m}(\imath\hat{p} + m\omega x)(-\imath\hat{p} + m\omega x)$$

- this is obviously false, but it is still worth inspecting this quantity on the left
- to make the math simpler later, we multiply by $1/(\hbar\omega)$

$$\frac{1}{2\hbar m\omega}(\imath\hat{p} + m\omega x)(-\imath\hat{p} + m\omega x)$$

$$\left[\frac{1}{\sqrt{2\hbar m\omega}}(\imath\hat{p} + m\omega x)\right]\left[\frac{1}{\sqrt{2\hbar m\omega}}(-\imath\hat{p} + m\omega x)\right] := a_- a_+$$

Ladder Operators

- we have just defined the raising operator (a_+) and the lowering operator (a_-)

$$a_\pm := \frac{1}{\sqrt{2\hbar m\omega}}[\mp\imath\hat{p} + m\omega x]$$

- remember, if $\hat{p}$ behaved like a number, then $a_- a_+ = 1/(\hbar\omega)\hat{H}$
- let us now see what $a_- a_+$ *actually* is

Product of Ladder Operators

$$
\begin{aligned}
a_- a_+ &= \frac{1}{2\hbar m\omega}(\imath\hat{p} + m\omega x)(-\imath\hat{p} + m\omega x) \\
&= \frac{1}{2\hbar m\omega}[\hat{p}^2 + (m\omega x)^2 - \imath m\omega(x\hat{p} - \hat{p}x)] \\
&= \frac{1}{2\hbar m\omega}[\hat{p}^2 + (m\omega x)^2 - \imath m\omega[x, \hat{p}]] \\
&= \frac{1}{2\hbar m\omega}[\hat{p}^2 + (m\omega x)^2 - \imath m\omega(\imath\hbar)] \\
&= \frac{1}{2\hbar m\omega}[\hat{p}^2 + (m\omega x)^2] + \frac{1}{2} \\
&= \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}
\end{aligned}
$$

Hamiltonian in Terms of Ladder Operators

- we have just shown

$$a_- a_+ = \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}$$

- solving for $\hat{H}$ gives

$$\hat{H} = \hbar\omega \left(a_- a_+ - \frac{1}{2} \right)$$

- working out the derivation on the previous slide again, with the order of operators reversed gives

$$\hat{H} = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right)$$

Raising Operator

- ψ satisfies the Schrödinger equation with energy E

$$\hat{H}\psi = E\psi$$

- now we will show that $a_+\psi$ satisfies the Schrödinger equation with energy $E + \hbar\omega$

Raising Operator

$$\begin{aligned}
\hat{H}(a_+\psi) &= \hbar\omega\left(a_+a_- + \frac{1}{2}\right)(a_+\psi) = \hbar\omega\left(a_+a_-a_+ + \frac{1}{2}a_+\right)\psi \\
&= \hbar\omega a_+\left(a_-a_+ + \frac{1}{2}\right)\psi = a_+\left[\hbar\omega\left(a_+a_- + \frac{1}{2} + 1\right)\psi\right] \\
&= a_+(\hat{H} + \hbar\omega)\psi = a_+(E + \hbar\omega)\psi = (E + \hbar\omega)(a_+\psi).
\end{aligned}$$

Lowering Operator

$$
\begin{aligned}
\hat{H}(a_-\psi) &= \hbar\omega\left(a_-a_+ - \frac{1}{2}\right)(a_-\psi) = \hbar\omega\left(a_-a_+a_- - \frac{1}{2}a_-\right)\psi \\
&= \hbar\omega a_-\left(a_+a_- - \frac{1}{2}\right)\psi = a_-\left[\hbar\omega\left(a_-a_+ - \frac{1}{2} - 1\right)\psi\right] \\
&= a_-(\hat{H} - \hbar\omega)\psi = a_-(E - \hbar\omega)\psi = (E - \hbar\omega)(a_-\psi).
\end{aligned}
$$

The Lowest Rung

- if we apply the lowering operator repeatedly, eventually we will reach a negative energy
- however, there occurs a point where $a_- \psi_0 = 0$
  - this is non-normalizable, and therefore an invalid solution
- this helps us determine $\psi_0(x)$

$$\frac{1}{\sqrt{2\hbar m\omega}} \left(\hbar \frac{\mathrm{d}}{\mathrm{d}x} + m\omega x \right) \psi_0 = 0$$

$$\psi_0(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \exp\left(-\frac{m\omega}{2\hbar} x^2 \right)$$

Solution to All Stationary States

- recursive solution

$$\psi_0(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$
$$\psi_n(x) = A_n(a_+)^n\psi_0(x), \qquad \text{with } E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

- we can solve for the normalization constant A_n individually for each solution, but it turns out that

$$A_n = \frac{1}{\sqrt{n!}}$$

Thank you!