Find the limit

3.25

$$\sum_{n=1}^{\infty}\left(\sqrt[2n+1]{x}-\sqrt[2n-1]{x}\right)=\left(\sqrt[3]{x}-\sqrt[1]{x}\right)+\left(\sqrt[5]{x}-\sqrt[3]{x}\right)+\dots$$

$$\sum_{k=1}^{n}\sqrt[2k+1]{x}-\sum_{k=1}^{n}\sqrt[2k-1]{x}=\sqrt[2n+1]{x}-\sqrt[1]{x}+\sum_{k=1}^{n-1}\sqrt[2k+1]{x}-\sum_{k=2}^{n}\sqrt[2k-1]{x}=\sqrt[2n+1]{x}-\sqrt[1]{x}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt[2k+1]{x}-\sqrt[2k-1]{x}\right)=\lim_{n\to\infty}\sqrt[2n+1]{x}-\sqrt[1]{x}=\begin{cases}1-x,\text{ if } x>0\\ -1-x,\text{ if } x<0\\ 0,\text{ if } x=0\end{cases}$$

Given the series partial sum find its n-th term and sum

3.26

$$\begin{aligned}S_n&=\tfrac{n+1}{n}=1+\tfrac{1}{n}\\ S_1&=1+\tfrac{1}{1}\\ S_2&=1+\tfrac{1}{2}=S_1-\tfrac{1}{2}\\ S_3&=1+\tfrac{1}{3}=S_2-\tfrac{1}{6}\\ \dots\end{aligned}$$

$$\begin{aligned}a_1&=1+1\\ a_2&=S_2-S_1=-\tfrac{1}{2}\\ a_3&=S_3-S_2=-\tfrac{1}{6}\\ \dots\end{aligned}$$

$$a_1=2$$
$$a_n=-\tfrac{1}{n(n-1)},\text{ if } n\neq 1$$

$$\lim_{n\to\infty}S_n=2-\sum_{n=2}^{\infty}\frac{1}{n(n-1)}$$

$$2-\sum_{k=2}^{n}\frac{1}{k(k-1)}=2-\sum_{k=1}^{n-1}\frac{1}{k(k+1)}$$

$$\lim_{n\to\infty} 2 - \sum_{k=1}^{n} \frac{1}{k(k+1)} = 2 - 1 = 1$$

3.27

$$\begin{aligned}
S_n &= \tfrac{-1+2^n}{2^n} = 1 - \tfrac{1}{2^n} \\
S_1 &= 1 - \tfrac{1}{2} \\
S_2 &= 1 - \tfrac{1}{4} \\
S_3 &= 1 - \tfrac{1}{8} \\
\ldots
\end{aligned}$$

$$\begin{aligned}
&a_1 = \tfrac{1}{2} \\
&a_2 = S_2 - S_1 = \tfrac{1}{4} - \tfrac{1}{2} = -\tfrac{1}{4} \\
&a_3 = S_3 - S_2 = \tfrac{1}{8} - \tfrac{1}{4} = -\tfrac{1}{8} \\
&\ldots
\end{aligned}$$

$$\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$S_n = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n} - \frac{1}{2^n}\right) = \frac{1}{2}\left(1 + S_n - \frac{1}{2^n}\right)$$

$$\begin{aligned}
S_n &= \tfrac{1}{2} + \tfrac{1}{2} S_n - \tfrac{1}{2^{n+1}} \\
S_n &= 1 - \tfrac{1}{2^{n+1}}
\end{aligned}$$

$$S = \lim_{n\to\infty} S_n = 1$$

3.28

$$S_n = \frac{(-1)^n}{n}$$

$$
\begin{aligned}
&a_1 = S_1 && = -1 \\
&a_2 = S_2 - S_1 && = \tfrac{1}{2} + 1 && = \tfrac{3}{2} \\
&a_3 = S_3 - S_2 && = -\tfrac{1}{3} - \tfrac{1}{2} && = -\tfrac{5}{6} \\
&a_4 = S_4 - S_3 && = \tfrac{1}{4} + \tfrac{1}{3} && = \tfrac{7}{12} \\
&\dots \\
&a_n = (-1)^n \tfrac{2n-1}{n(n-1)}
\end{aligned}
$$

$$S_n = -1 + \textstyle\sum_{k=2}^{n} (-1)^n \tfrac{2k-1}{k(k-1)} = -1 + \sum_{k=2}^{n} (-1)^n \tfrac{k+k-1}{k(k-1)} =$$

$$= -1 + \textstyle\sum_{k=2}^{n} (-1)^k \left(\tfrac{1}{k-1} + \tfrac{1}{k}\right) = -1 + \sum_{k=1}^{n} \left(\tfrac{(-1)^{k+1}}{k(k+1)}\right) =$$

$$= -1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{12} - \tfrac{1}{20} + \tfrac{1}{30} \cdots + \tfrac{(-1)^{n+1}}{n(n+1)} =$$

$$= -1 - \textstyle\sum_{k=1}^{n} \tfrac{1}{2k(2k+1)} + \sum_{k=1}^{n} \tfrac{1}{2k(2k-1)} =$$

$$= -1 + \textstyle\sum_{k=1}^{n} \tfrac{2k-1+2k+1}{2k(2k+1)(2k-1)} = -1 + \sum_{k=1}^{n} \tfrac{4k}{2k(4k^2-1)} = -1 + \sum_{k=1}^{n} \tfrac{2}{4k^2-1}$$

$$
\begin{aligned}
&S_{k_1} = \tfrac{1}{3} \\
&S_{k_2} = \tfrac{1}{15} + \tfrac{1}{3} = \tfrac{2}{5} \\
&S_{k_3} = \tfrac{1}{35} + \tfrac{2}{5} = \tfrac{3}{7} \\
&\dots \\
&S_{k_n} = \tfrac{k_n}{2k_n+1}
\end{aligned}
$$

$$\lim_{k_n \to \infty} S_{k_n} = \frac{1}{2}$$

$$S_n = -1 + \frac{2k_n}{2k_n + 1}$$

$$\lim_{n \to \infty} S_n = -1 + 1 = 0$$