CS201A: Math for CS I/Discrete Mathematics

Endsem exam

Max marks:150
Time:180 mins.

23-Nov-2017

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1. *Answer all 8 questions. It has 4 pages + 1 page for the standard normal distribution table.*
2. **Please start each answer to a question on a fresh page. And keep answers of parts of a question together.**
3. *Just writing a number/final value/figure will not get you full credit. You must give justifications/ derivations in each case.*
4. *You can consult* **only your own handwritten notes**. *Nothing else is allowed. Keep any electronic gadgets in your bag and the bag on or near the stage.*
5. *Where needed use the standard normal distribution table at the end of the question paper.*

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1. (a) You are the instructor of CS201 some time in the future and one of the questions in the exam is:

   Prove that:
   Every integer $n > 1$ is a product of a unique non-decreasing sequence of prime numbers..
   For example, $18 = 2 \times 3 \times 3$, $20 = 2 \times 2 \times 5$, $23 = 23$ etc.
   A student produces the following inductive proof:
   Base case: $n = 2$. 2 is a prime and the unique sequence is trivially 2.
   Strong inductive hypothesis: Assume that for integers $\leq n$ the claim holds.
   Inductive step for $n+1$: If $n+1$ is a prime then the claim holds trivially since the non-decreasing sequence contains only $n+1$. On the other hand if $n+1$ is composite then there exist integers $y, z < n$ such that $n+1 = y \times z$. Since $y, z < n$ the claim holds for y and for z. We can merge the non-decreasing sequences for y and z into a single non-decreasing sequence whose product is $n+1$.
   Hence proved.

   i. When the answer scripts are returned the student finds s/he has got a zero and s/he approaches you with a regrading request. How will you defend your decision? In other words what is wrong with the proof?
   ii. Is there a simple way to repair the student's proof?

   (b) i. Let S be an infinite set and $A = \{a_1, \ldots, a_n\}$ be a finite set. Argue that there exists a bijection from $S \cup A$ to S.
   ii. Find a bijection from the semi-closed interval $(0, 1] \subset \mathbb{R}$ to the interval $[0, \infty) \subset \mathbb{R}$.

(c) Argue that a graph is a tree iff there is a unique path between every pair of nodes in the graph.

[(3,3),(5,3),6=20]

2. Assume all graphs in this question are connected.

Define the dual of a planar graph $G = (V, E)$ as the graph $G^* = (V^*, E^*)$ where every vertex $v^* \in V^*$ corresponds to a distinct face in G. So, $|V^*| = |F|$ (no. of faces in G). We connect vertices $v_1^*, v_2^* \in G^*$ by an edge $e^* \in E^*$ whenever an edge $e \in E$ is a separating or boundary edge for faces F_1, F_2 in G corresponding to vertices v_1^*, v_2^* respectively. Note that v_1^*, v_2^* may coincide for example when G has only one face so F_1, F_2 are same.

(a) Through an example show that a simple graph G can have a dual G^* that is not simple - that is it can have multi-edges and self loops.

(b) Again through an example show that two different planar embeddings of a planar graph can have non-isomorphic duals.

(c) Define the length of a face F_1 of G to be the total length of the closed walk in G that circumscribe or bound face F_1. If F is the set of faces in G argue that: $2|E| = \sum_{F_i \in F} \text{length}(F_i)$.

(d) Assume G is a bi-partite planar graph. What kind of graph is G^*? Justify your answer.

[4,6,5,5=20]

3. (a) Let set $S = \{a, b, c, d, e, f\}$. Find the number of ways in which we can get two not necessarily distinct pair of sets $A, B \subseteq S$ such that $A \cup B = S$. The order is not important - so $\{a, c\}, \{b, c, d, e, f\}$ is the same as $\{b, c, d, e, f\}, \{a, c\}$.

(b) Let $n > 1$ be an odd integer. Argue that the sequence ${}^nC_1, {}^nC_2, \ldots, {}^nC_{\frac{n-1}{2}}$ has an odd number of odd numbers.

(c) Consider the 3×3 numbered grid below. Each square in the grid will be painted either BLACK or WHITE. The colour for each square is decided by tossing a fair coin. Find the probability that the grid does not have a 2×2 BLACK square (that is all 4 squares are painted BLACK).

1	2	3
4	5	6
7	8	9

(Hint: Principle of inclusion-exclusion.)

[7,5,8=20]

4. (a) For each recurrence below apply the Master theorem, if applicable, and determine a solution bound for the recurrence. If not applicable say why it cannot be applied.

i. $T(n) = 2^n T(\frac{n}{2}) + n$.

ii. $T(n) = 16T(\frac{n}{4}) + n!$.

(b) The recurrence equation $T(n) = 2T(\frac{n}{2}) + n \log n$ does not fit the Master theorem we discussed in class ($T(n) = aT(\frac{n}{b}) + f(n)$) where $f(n)$ was n^c. Use iterative unfolding to get the $\Theta(.)$ solution for the given recurrence equation.

(c) Consider integer $n > 0$ and define an *ordered partition* of n as a breakup of n into one or more positive integers that add up to n but where order is important. That is for $n = 4$, $2+1+1$, $1+2+1$, $1+1+2$ are different partitions. Derive the expression for the number of ordered partitions for n without using generating functions.

[(3,3),9,5=20]

5. (a) Let $\mathcal{F}$ be a σ-Field and let $A, B \in \mathcal{F}$. Are $A \backslash B$ (A difference B) and $A \triangle B$ (symmetric difference between A and B) in $\mathcal{F}$? Justify.

   (b) Let $E_1, \ldots, E_n$ be events. You are given that at least one of E_i, $1 \leq i \leq n$ is certain to occur and definitely no more than two of the events can occur. Let $P(E_i) = p$ and $P(E_i \cap E_j) = q$, $i \neq j$. Argue that the lower bound on p is $\frac{1}{n}$ and the upper bound on q is $\frac{2}{n}$.

[(2,2),(3,3)=10]

6. You have 5 coins. Two of them have Head on both sides, one has Tail on both sides and two are normal (that is Head on one side and Tail on the other). Answer the following (carefully specify each event for clarity since there are a sequence of conditional probabilities):

   (a) You close your eyes pick a coin and toss it. What is the probability the bottom face is a Head?

   (b) You open your eyes and see that the top face is a Head. What is the probability the bottom face is Head?

   (c) You shut your eyes and again toss the same coin. What is the probability that the bottom face is a Head?

   (d) You open your eyes and see that the top face is a Head. What is the probability that the bottom face is a Head?

   (e) You discard the above coin, pick a random coin from the remaining and toss it. What is the probability the top face is a Head?

[4×5=20]

7. (a) You have a fair die with 6 faces marked 1 to 6. You continue to roll the die repeatedly and only stop when either you roll a 1 or you voluntarily decide to stop at some point. When you stop you get a score that is equal to the value of the last roll. So your last score is either 1 or the value of the last roll before you decided to stop.

      i. What stopping strategy will you choose to maximize your expected score.

      ii. If the score was the square of the last rolled value what stopping strategy will maximize your expected score.

   (b) What exactly is the meaning of an α-significance value?

   (c) The target thickness for some sheet metal for use in manufacturing some part is 245mm. A sample of 50 sheets is obtained and the thickness of each one is determined, resulting in a sample mean thickness of 246.18 mm and a sample standard deviation of 3.60mm. Does this data suggest that the true average sheet thickness is something other than the

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Define the dual of a planar graph $G = (V, E)$ as the graph $G^\star = (V^\star, E^\star)$ where every vertex $v^\star \in V^\star$ corresponds to a distinct face in G. So, $|V^\star| = |F|$ (no. of faces in G). We connect vertices $v_1^\star, v_2^\star \in G^\star$ by an edge $e^\star \in E^\star$ whenever an edge $e \in E$ is a separating or boundary edge for faces F_1, F_2 in G corresponding to vertices $v_1^\star, v_2^\star$ respectively. Note that $v_1^\star, v_2^\star$ may coincide for example when G has only one face so F_1, F_2 are same.

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(b) What exactly is the meaning of an α-significance value?

(c) The target thickness for some sheet metal for use in manufacturing some part is 245mm. A sample of 50 sheets is obtained and the thickness of each one is determined, resulting in a sample mean thickness of 246.18 mm and a sample standard deviation of 3.60mm. Does this data suggest that the true average sheet thickness is something other than the target value at $\alpha = 0.02$? (Use the table for the standard normal distribution at the end of the question paper where necessary.)

[(5,5),5,5=20]

8. (a) If events $E_1, \ldots, E_n$ are independent events then show that

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = 1 - \prod_{i=1}^{n}(1 - P(E_i))$$

(b) Let X be a Poisson random variable. Then $P(X = i) = e^{-\lambda}\frac{\lambda^i}{i!}$, $i = 0, 1, 2, \ldots$ where parameter $\lambda > 0$. Derive the mean, and variance of X.

(c) Assume that the height in inches of all college basketball players in India is a normally distributed random variable with $\mu = 71$ and $\sigma^2 = 6.25$.

i. What percentage of college basketball players are over 74 inches tall?

ii. What percentage of college basketball players in the six foot club are over 77 inches tall?

[5,(5,5),(2,3)=20]

Table III
Normal Distribution

The following table presents the standard normal distribution. The probabilities tabled are

$$P(X \leq x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw.$$

Note that only the probabilities for $x \geq 0$ are tabled. To obtain the probabilities for $x < 0$, use the identity $\Phi(-x) = 1 - \Phi(x)$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998