

# Some subtleties of spherical coordinates

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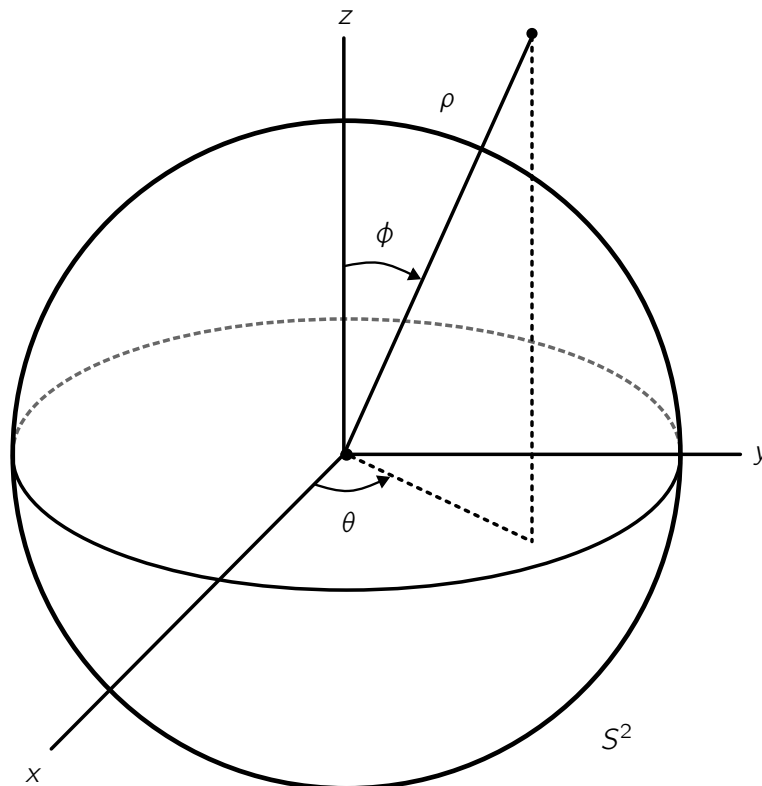
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## Abstract

The following are some reflections on why the polar angle coordinate  $\phi$  does not globally define a 1-form on  $S^2$  and hence on  $\mathbb{R}^3 \setminus \{0\}$ .

## 1 Review of Spherical Coordinates

Recall that any point  $(x, y, z) \in \mathbb{R}^3$  can be described by its distance from the origin,  $\rho = \sqrt{x^2 + y^2 + z^2}$ , its **polar angle**  $\phi^1$ , and its **azimuthal angle**  $\theta$ . From this picture we have  $\rho \in [0, \infty)$ ,  $\phi \in [0, \pi]$ ,



and  $\theta \in [0, 2\pi)$ . Despite the care we take with these possible values, we have still introduced several

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<sup>1</sup>We are following the mathematics convention: [https://en.wikipedia.org/wiki/Spherical\\_coordinate\\_system](https://en.wikipedia.org/wiki/Spherical_coordinate_system), as opposed to the physics convention which swaps the roles of  $\phi$  and  $\theta$ .

degeneracies: various choices of “coordinates” that represent the same point. For example, at  $\rho = 0$ , we identify

$$(0, \phi, \theta) \sim (0, \phi', \theta').$$

Similarly, on the z-axis,

$$(\rho, 0, \theta) \sim (\rho, 0, \theta') \quad \text{and} \quad (\rho, \pi, \theta) \sim (\rho, \pi, \theta').$$

As a result, spherical coordinates only really describe a coordinate system in the precise sense (as charts on a topological manifold) provided

$$(\rho, \phi, \theta) \in (0, \infty) \times (0, \pi) \times [0, 2\pi).$$

This covers  $\mathbb{R}^3 \setminus (z\text{-axis})$ , and we can only guarantee the existence of  $d\rho$ ,  $d\phi$ , and  $d\theta$ , on this domain.

## 2 The problem of the 1-form

From the equation for  $z$  we have

$$\phi = \arccos\left(\frac{z}{\rho}\right) = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right).$$

The domain for  $\arccos$  is  $[-1, 1]$  and the inequality  $|z| \leq \rho$  ensures that  $\phi$  is a globally defined smooth function on  $\mathbb{R}^3 \setminus (z\text{-axis})$ . We have more, since this expression for  $\phi$  actually shows that it's well-defined on  $\mathbb{R}^3 \setminus \{0\}$ . This makes it tempting to conclude that  $d\phi$  exists on  $\mathbb{R}^3 \setminus \{0\}$ . It turns out that this is not the case,  $d\phi$  is not even a 1-form on  $S^2$ .

### 2.1 Direct computation

We can argue that  $d\phi$  fails to exist by a direct computation. For  $\rho > 0$  we have

$$\begin{aligned} d\phi &= d \arccos\left(\frac{z}{\rho}\right) = -\frac{1}{\sqrt{1 - \frac{z^2}{\rho^2}}} d\left(\frac{z}{\rho}\right) = -\frac{1}{\sqrt{\frac{x^2 + y^2}{\rho^2}}} \frac{\rho dz - z d\rho}{\rho^2} \\ &= -\frac{1}{\sqrt{\frac{(\rho \sin \phi)^2}{\rho^2}}} \frac{\rho dz - z d\rho}{\rho^2} = \frac{1}{\sin \phi} \left( \frac{z}{\rho^2} d\rho - \frac{1}{\rho} dz \right). \end{aligned}$$

Computing  $d\rho$ ,

$$2\rho d\rho = d\rho^2 = 2(xdx + ydy + zdz).$$

Combining these we have

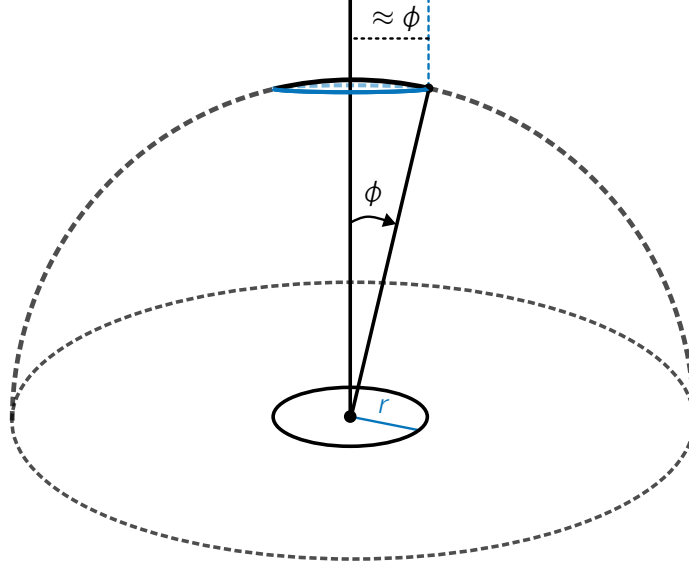
$$\begin{aligned} d\phi &= \frac{1}{\sin \phi} \left( \frac{z}{\rho^2} d\rho - \frac{1}{\rho} dz \right) \\ &= \frac{1}{\sin \phi} \left( \frac{z}{\rho^3} (xdx + ydy + zdz) - \frac{1}{\rho} dz \right) \\ &= \frac{1}{\sin \phi} \left( \frac{xzdx + yzdy - (x^2 + y^2)dz}{\rho^3} \right). \end{aligned}$$

Since we're away from the origin, the factor on the right is not an issue. So the only problem for  $d\phi$  happens at  $\phi \in \{0, \pi\}$ . In this case, the right vanishes as  $\phi \rightarrow 0$  and  $\phi \rightarrow \pi$  but computing limits shows that the derivative still diverges.

While this calculation shows why the derivative does not exist on the z-axis, our goal now is to go over a shorter and more geometric argument for why it does not exist.

## 2.2 A geometric argument

We work on  $S^2$ , for arbitrary  $\rho$  changing  $\phi$  does not change the distance of the point so the argument will transfer over identically. Focusing on the positive  $z$ -axis, let's see what happens as  $\phi$  gets close to 0. The distance any given point on  $S^2$  ( $z > 0$ ) has from the  $z$ -axis is  $\sin \phi$ . For small values of  $\phi$  the



small angle approximation  $\sin \phi \approx \phi$  (justified by Taylor expansion) says that the point is approximately  $\phi$ -distance away from the axis. On the other hand, spherical coordinates also tell us that

$$\sin \phi = r = \sqrt{x^2 + y^2}.$$

Our small angle approximation tells us  $\phi \approx r$  and the problem of differentiating  $\phi$  near the  $z$ -axis is exactly the problem of differentiating  $r$  in polar coordinates. In particular

$$\phi \approx \sqrt{x^2 + y^2},$$

and the derivative of this explodes at 0, as seen by

$$dr = \frac{1}{\sqrt{x^2 + y^2}} d(x^2 + y^2).$$

## 3 Consequences

The observation  $\phi \approx r = \sqrt{x^2 + y^2}$  for small values of  $\phi$  offers a suggested way to smooth it over by instead considering  $d\phi^2$ . In this case

$$d\phi^2 = 2\phi d\phi$$

and a direct computation shows that  $\phi d\phi \rightarrow 0$  as  $\phi \rightarrow 0$  for the same reason  $r dr \rightarrow 0$  as  $r \rightarrow 0$ .

Notice that this still fails to fix the issue on the negative  $z$ -axis since here  $\phi \rightarrow \pi$  which does not remove the singular behavior of its derivative given by the same picture. To justify the small-angle approximation, we need to instead consider the complementary angle  $\psi = \pi - \phi$ . Now  $\phi \rightarrow \pi$  is the same as considering  $\psi \rightarrow 0^+$  and our previous argument holds. This means that, away from the positive  $z$ -axis,

$$d\psi^2 = (\pi - \phi)d(\pi - \phi) = -(\pi - \phi)d\phi$$

is well-defined.