Some subtleties of spherical coordinates

Jonathan Delgado

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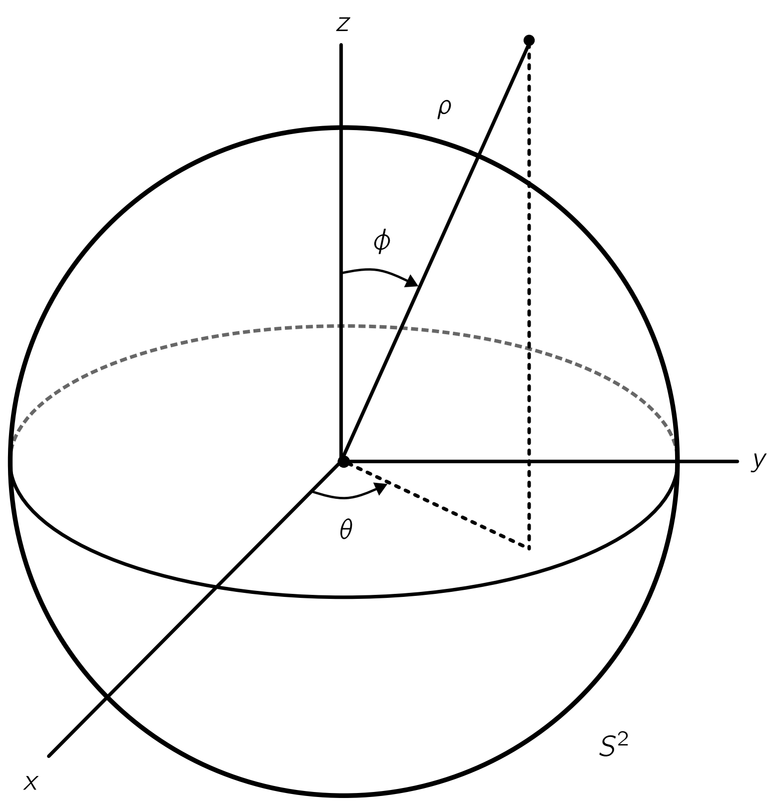
Abstract

The following are some reflections on why the polar angle coordinate ϕ does not globally define a 1-form on S^2 and hence on $\mathbb{R}^3 \setminus \{0\}$.

1 Review of Spherical Coordinates

Recall that any point $(x, y, z) \in \mathbb{R}^3$ can be described by its distance from the origin, $\rho = \sqrt{x^2 + y^2 + z^2}$, its **polar angle** ϕ[1], and its **azimuthal angle** θ. From this picture we have $\rho \in [0, \infty)$, $\phi \in [0, \pi]$,



and $\theta \in [0, 2\pi)$. Despite the care we take with these possible values, we have still introduced several

[1] We are following the mathematics convention: https://en.wikipedia.org/wiki/Spherical_coordinate_system, as opposed to the physics convention which swaps the roles of ϕ and θ.

degeneracies: various choices of "coordinates" that represent the same point. For example, at $\rho = 0$, we identify

$$(0, \phi, \theta) \sim (0, \phi', \theta').$$

Similarly, on the z-axis,

$$(\rho, 0, \theta) \sim (\rho, 0, \theta') \quad \text{and} \quad (\rho, \pi, \theta) \sim (\rho, \pi, \theta').$$

As a result, spherical coordinates only really describe a coordinate system in the precise sense (as charts on a topological manifold) provided

$$(\rho, \phi, \theta) \in (0, \infty) \times (0, \pi) \times [0, 2\pi).$$

This covers $\mathbb{R}^3 \setminus (z\text{-axis})$, and we can only guarantee the existence of $d\rho$, $d\phi$, and $d\theta$, on this domain.

2 The problem of the 1-form

From the equation for z we have

$$\phi = \arccos\left(\frac{z}{\rho}\right) = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

The domain for arccos is $[-1, 1]$ and the inequality $|z| \leq \rho$ ensures that ϕ is a globally defined smooth function on $\mathbb{R}^3 \setminus (z\text{-axis})$. We have more, since this expression for ϕ actually shows that it's well-defined on $\mathbb{R}^3 \setminus \{0\}$. This makes it tempting to conclude that $d\phi$ exists on $\mathbb{R}^3 \setminus \{0\}$. It turns out that this is not the case, $d\phi$ is not even a 1-form on S^2.

2.1 Direct computation

We can argue that $d\phi$ fails to exist by a direct computation. For $\rho > 0$ we have

$$\begin{aligned} d\phi = d\arccos\left(\frac{z}{\rho}\right) &= -\frac{1}{\sqrt{1-\frac{z^2}{\rho^2}}} d\left(\frac{z}{\rho}\right) = -\frac{1}{\sqrt{\frac{x^2+y^2}{\rho^2}}} \frac{\rho dz - z d\rho}{\rho^2} \\ &= -\frac{1}{\sqrt{\frac{(\rho\sin\phi)^2}{\rho^2}}} \frac{\rho dz - z d\rho}{\rho^2} = \frac{1}{\sin\phi}\left(\frac{z}{\rho^2} d\rho - \frac{1}{\rho} dz\right). \end{aligned}$$

Computing $d\rho$,

$$2\rho d\rho = d\rho^2 = 2(xdx + ydy + zdz).$$

Combining these we have

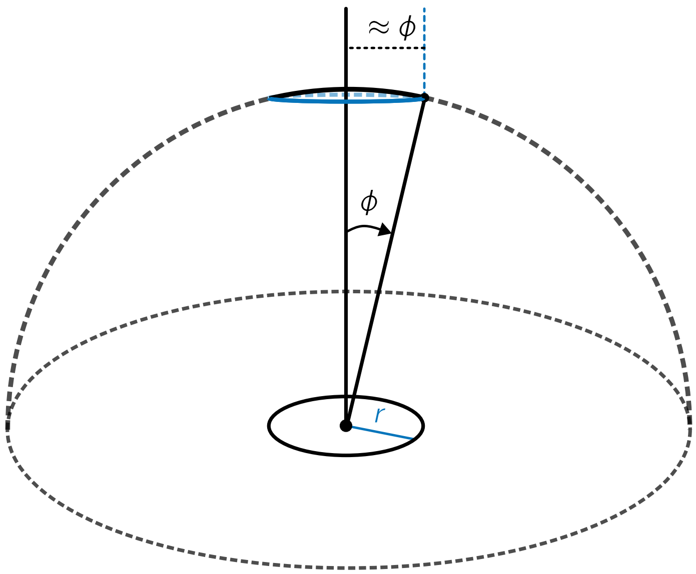
$$\begin{aligned} d\phi &= \frac{1}{\sin\phi}\left(\frac{z}{\rho^2} d\rho - \frac{1}{\rho} dz\right) \\ &= \frac{1}{\sin\phi}\left(\frac{z}{\rho^3}(xdx + ydy + zdz) - \frac{1}{\rho} dz\right) \\ &= \frac{1}{\sin\phi}\left(\frac{xzdx + yzdy - (x^2+y^2)dz}{\rho^3}\right). \end{aligned}$$

Since we're away from the origin, the factor on the right is not an issue. So the only problem for $d\phi$ happens at $\phi \in \{0, \pi\}$. In this case, the right vanishes as $\phi \to 0$ and $\phi \to \pi$ but computing limits shows that the derivative still diverges.

While this calculation shows why the derivative does not exist on the z-axis, our goal now is to go over a shorter and more geometric argument for why it does not exist.

2.2 A geometric argument

We work on S^2, for arbitrary ρ changing ϕ does not change the distance of the point so the argument will transfer over identically. Focusing on the positive z-axis, let's see what happens as ϕ gets close to 0. The distance any given point on S^2 ($z > 0$) has from the z-axis is $\sin\phi$. For small values of ϕ the



small angle approximation $\sin\phi \approx \phi$ (justified by Taylor expansion) says that the point is approximately ϕ-distance away from the axis. On the other hand, spherical coordinates also tell us that

$$\sin\phi = r = \sqrt{x^2 + y^2}.$$

Our small angle approximation tells us $\phi \approx r$ and the problem of differentiating ϕ near the z-axis is exactly the problem of differentiating r in polar coordinates. In particular

$$\phi \approx \sqrt{x^2 + y^2},$$

and the derivative of this explodes at 0, as seen by

$$dr = \frac{1}{\sqrt{x^2 + y^2}} d(x^2 + y^2).$$

3 Consequences

The observation $\phi \approx r = \sqrt{x^2 + y^2}$ for small values of ϕ offers a suggested way to smooth it over by instead considering $d\phi^2$. In this case

$$d\phi^2 = 2\phi d\phi$$

and a direct computation shows that $\phi d\phi \to 0$ as $\phi \to 0$ for the same reason $r dr \to 0$ as $r \to 0$.

Notice that this still fails to fix the issue on the negative z-axis since here $\phi \to \pi$ which does not remove the singular behavior of its derivative given by the same picture. To justify the small-angle approximation, we need to instead consider the complementary angle $\psi = \pi - \psi$. Now $\phi \to \pi$ is the same as considering $\psi \to 0^+$ and our previous argument holds. This means that, away from the positive z-axis,

$$d\psi^2 = (\pi - \phi) d(\pi - \phi) = -(\pi - \phi) d\phi$$

is well-defined.