Faster Matroid Partition Algorithms

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Tatsuya Terao :Faster Matroid Partition Algorithms,

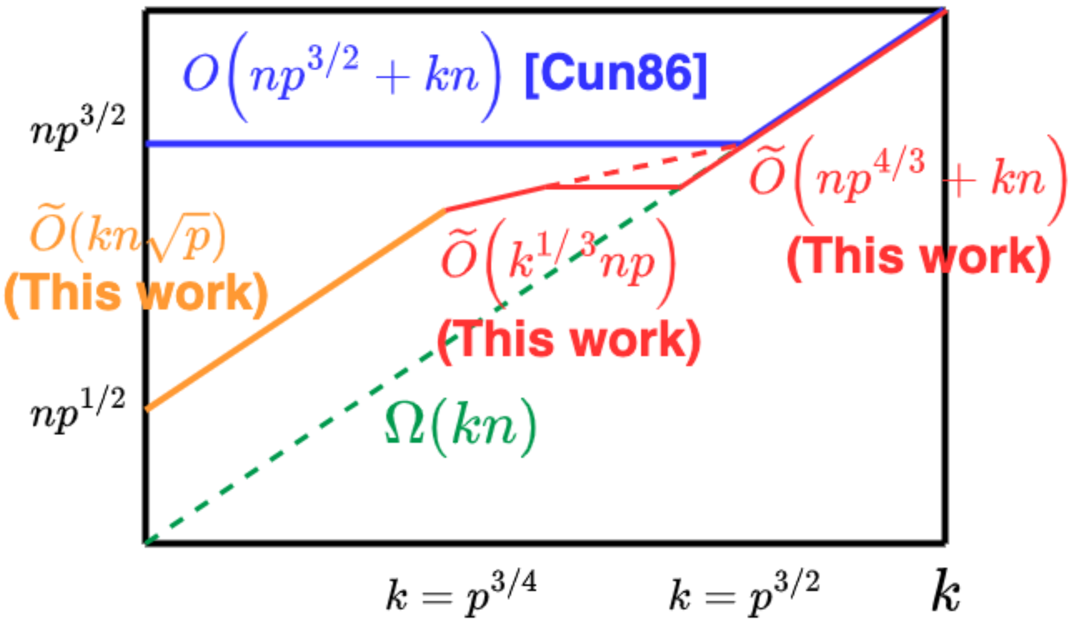
A preliminary version appears in **ICALP 2023, a full version is published at TALG.** (arXiv:2303.05920)

Summary

Result

Three fast algorithms for **matroid partition**

- Algorithm 1.
  $\tilde{O}(kn\sqrt{p})$ **independence** queries
- Algorithm 2.
  $\tilde{O}(k'^{1/3}np + kn)$ **independence** queries
- Algorithm 3.
  $\tilde{O}((n+k)\sqrt{p})$ **rank** queries



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p = solution size, $k' = \min\{k, p\}$

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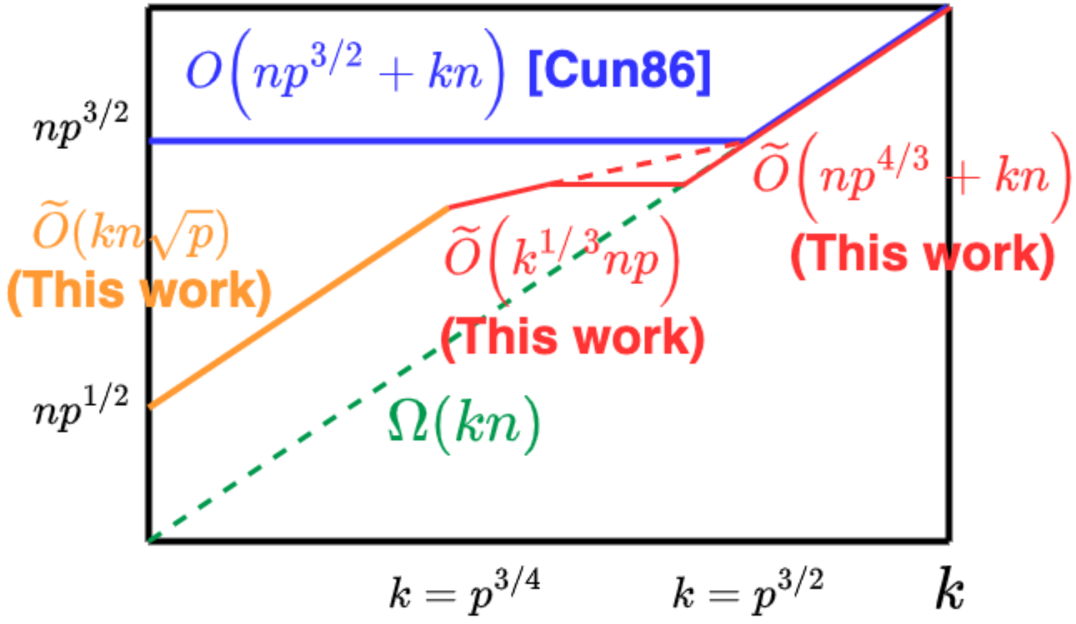

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Summary

Result

Three fast algorithm

the first improvement since [Cunningham'86]

- Algorithm 1.
  $\widetilde{O}(kn\sqrt{p})$ **independence** queries
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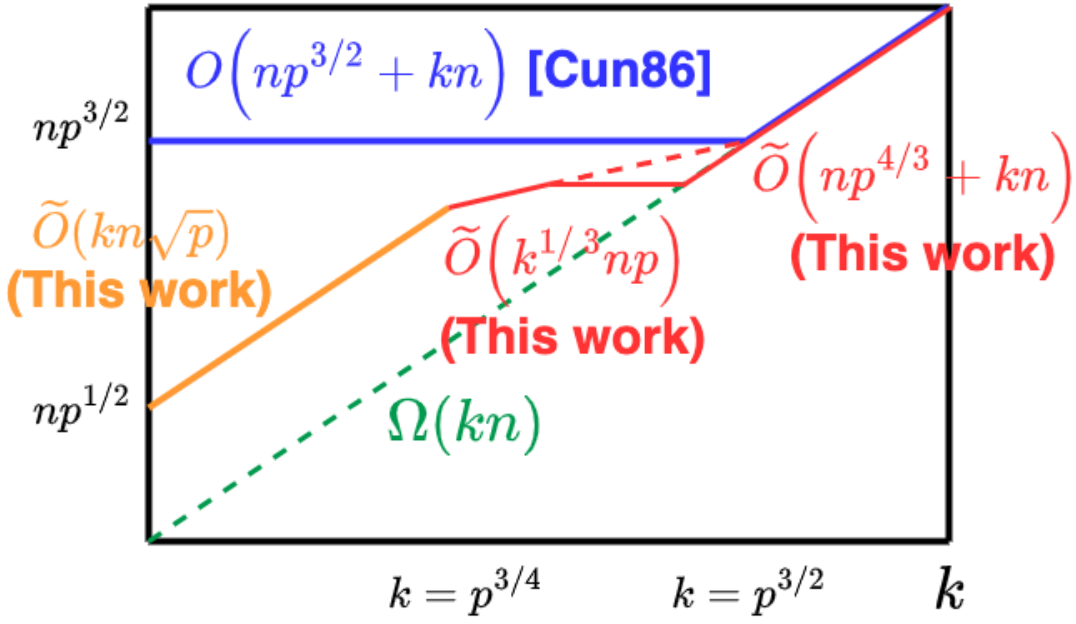


A new approach

Edge Recycling Augmentation

Outline

- **Summary**
- **Preliminaries**
  - -Matroid
  - -Matroid Intersection
  - -Matroid Partition
- **Result**
  - -Faster Matroid Partition Algorithms
- **Idea**
  - -Blocking Flow
  - -Edge Recycling Augmentation
- **Conclusion**

Matroid $\mathcal{M} = (V, \mathcal{I})$

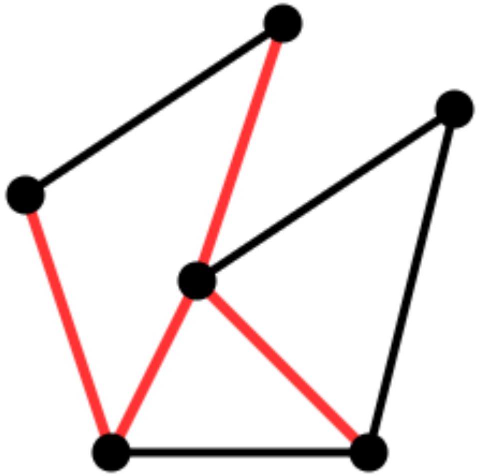
Def

A finite set V and non-empty family of **independent** sets $\mathcal{I} \subseteq 2^V$ such that

- $S' \subseteq S \in \mathcal{I} \implies S' \in \mathcal{I}$
- $S, T \in \mathcal{I}$, $|S| > |T| \implies \exists\, e \in S - T$ s.t. $T \cup \{e\} \in \mathcal{I}$

E.g.

- **Graphic Matroid**



V = edges
$\mathcal{I}$ = forests

- **Linear Matroid**

$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 3 & 1 & 2 & 3 \\ 2 & 0 & 1 & 3 \\ 1 & 2 & 3 & 0 \end{bmatrix}$$

V = row vectors
$\mathcal{I}$ = linearly independent

Matroid $\mathcal{M} = (V, \mathcal{I})$

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Algorithm accesses a matroid through an **oracle**

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Algorithm accesses a matroid through an **oracle**

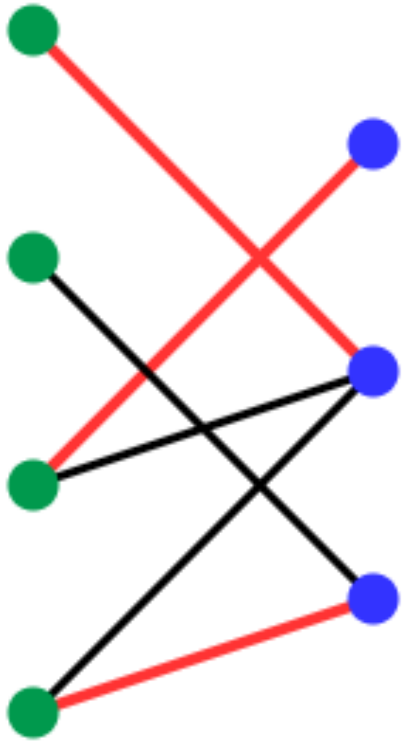
- **Independence** oracle query: Is $S \in \mathcal{I}$?

Matroid Intersection

Input : two matroids $\mathcal{M}_1 = (V, \mathcal{I}_1), \mathcal{M}_2 = (V, \mathcal{I}_2)$
Find : maximum **common independent set** $S \in \mathcal{I}_1 \cap \mathcal{I}_2$

E.g. Bipartite Matching

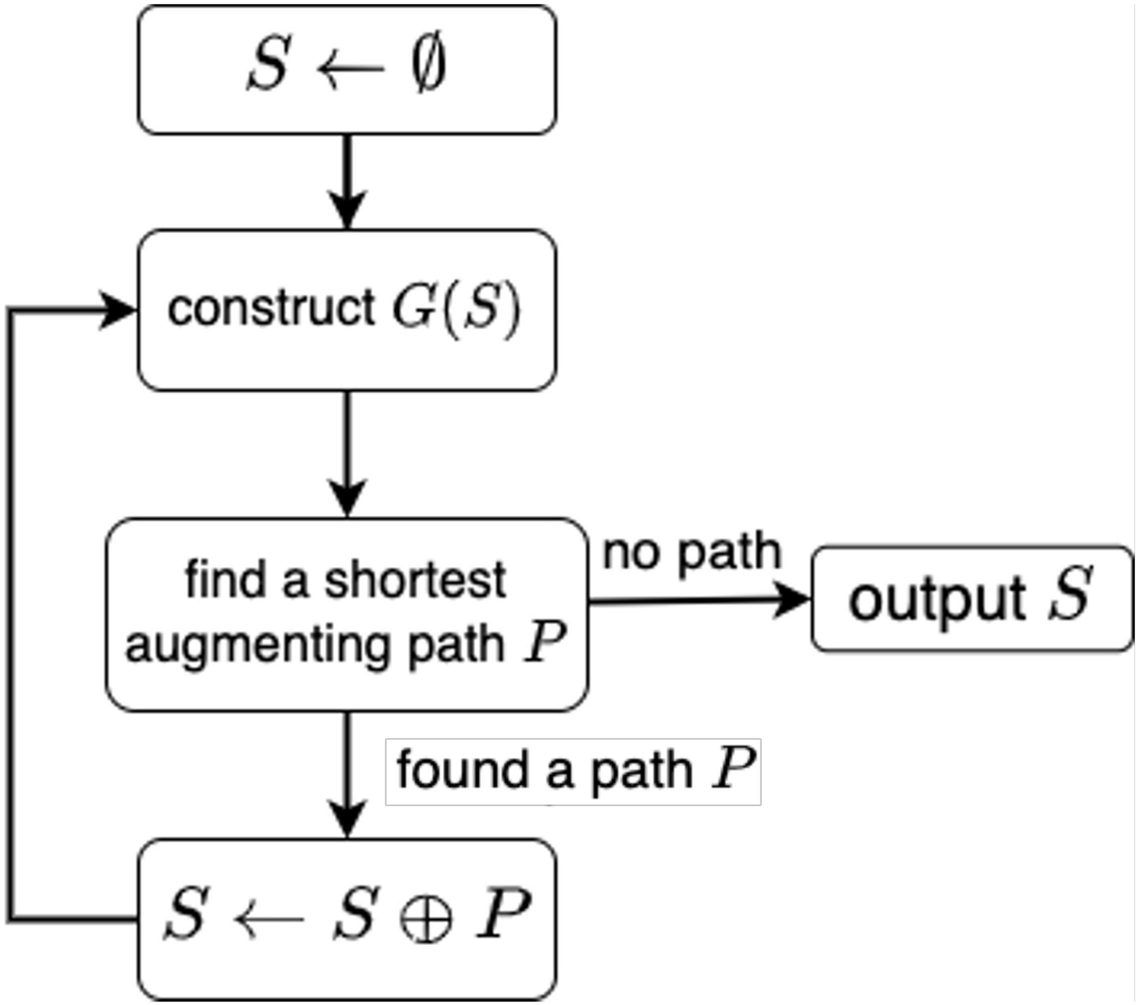
$\boldsymbol{V} =$ **edges**
$\boldsymbol{\mathcal{I}_1} =$ **each left vertex has at most 1 edge**
$\boldsymbol{\mathcal{I}_2} =$ **each right vertex has at most 1 edge**

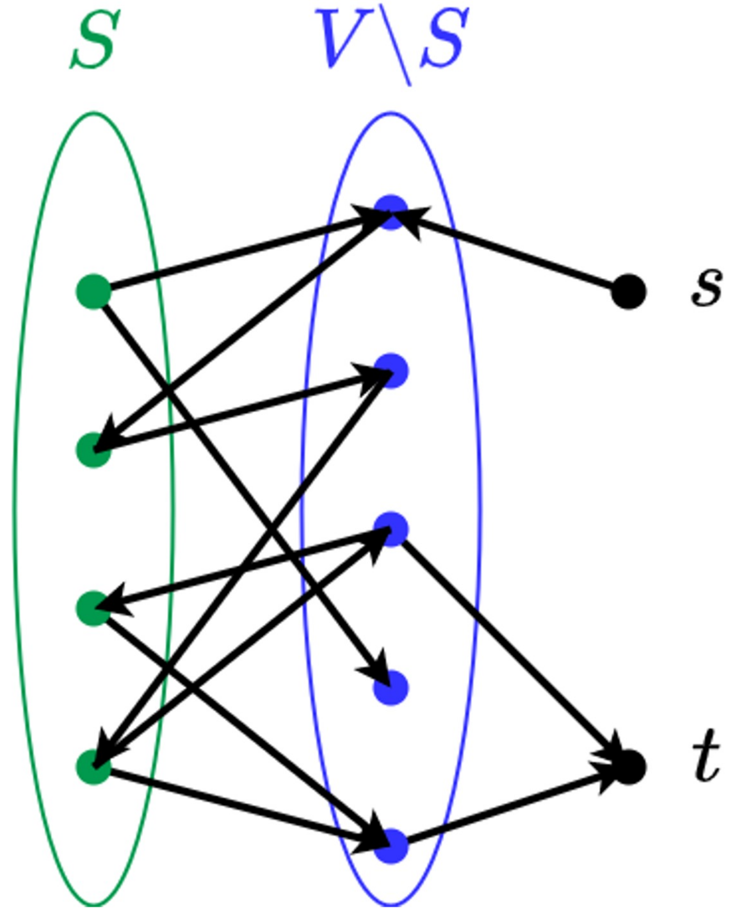


Algorithm for Matroid Intersection

given: $(V,\mathcal{I}_1),(V,\mathcal{I}_2)$
max $|S|$ s.t. $S \in \mathcal{I}_1 \cap \mathcal{I}_2$

[**Edmonds 1970, Aigner-Dowling 1971, Lawler 1975**]





Exchange graph $G(S)$

given: $(V, \mathcal{I}_1), (V, \mathcal{I}_2)$
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Algorithm for Matroid Intersection

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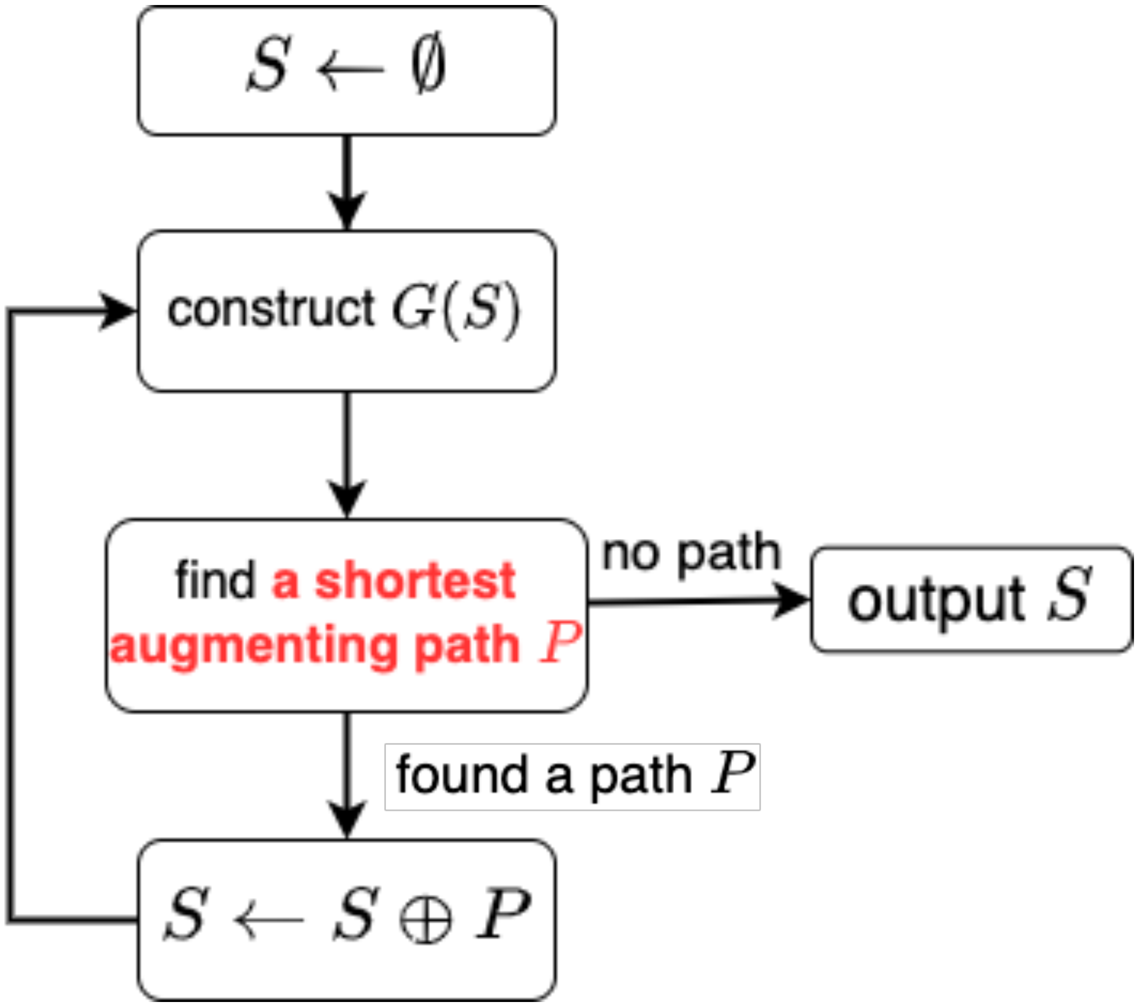

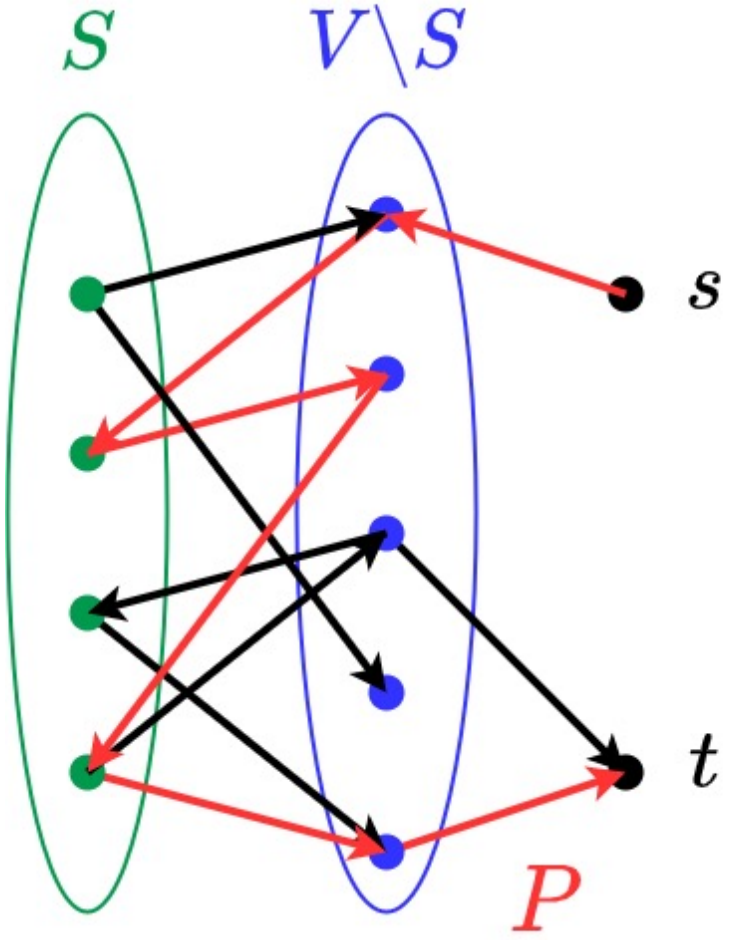

Exchange graph $G(S)$

Prior Work on Matroid Intersection

given: $(V, \mathcal{I}_1), (V, \mathcal{I}_2)$
max $|S|$ s.t. $S \in \mathcal{I}_1 \cap \mathcal{I}_2$
$n = |V|$, $r =$ sol. size

Independence query complexity

1970s	Edmonds, Lawler, Aigner-Dowling	$O(nr^2)$
1986	Cunningham	$O(nr^{3/2})$
2015	Lee-Sidford-Wong	$\tilde{O}(n^2)$
2019	Nguyễn, Chakrabarty-Lee-Sidford-Singla-Wong	$\tilde{O}(nr)$
2021	Blikstad-v.d.Brand-Mukhopadhyay-Nanongkai	$\tilde{O}(n^{9/5})$
2021	Blikstad	$\tilde{O}(nr^{3/4})$

Algorithm for Matroid Intersection

given: $(V, \mathcal{I}_1), (V, \mathcal{I}_2)$
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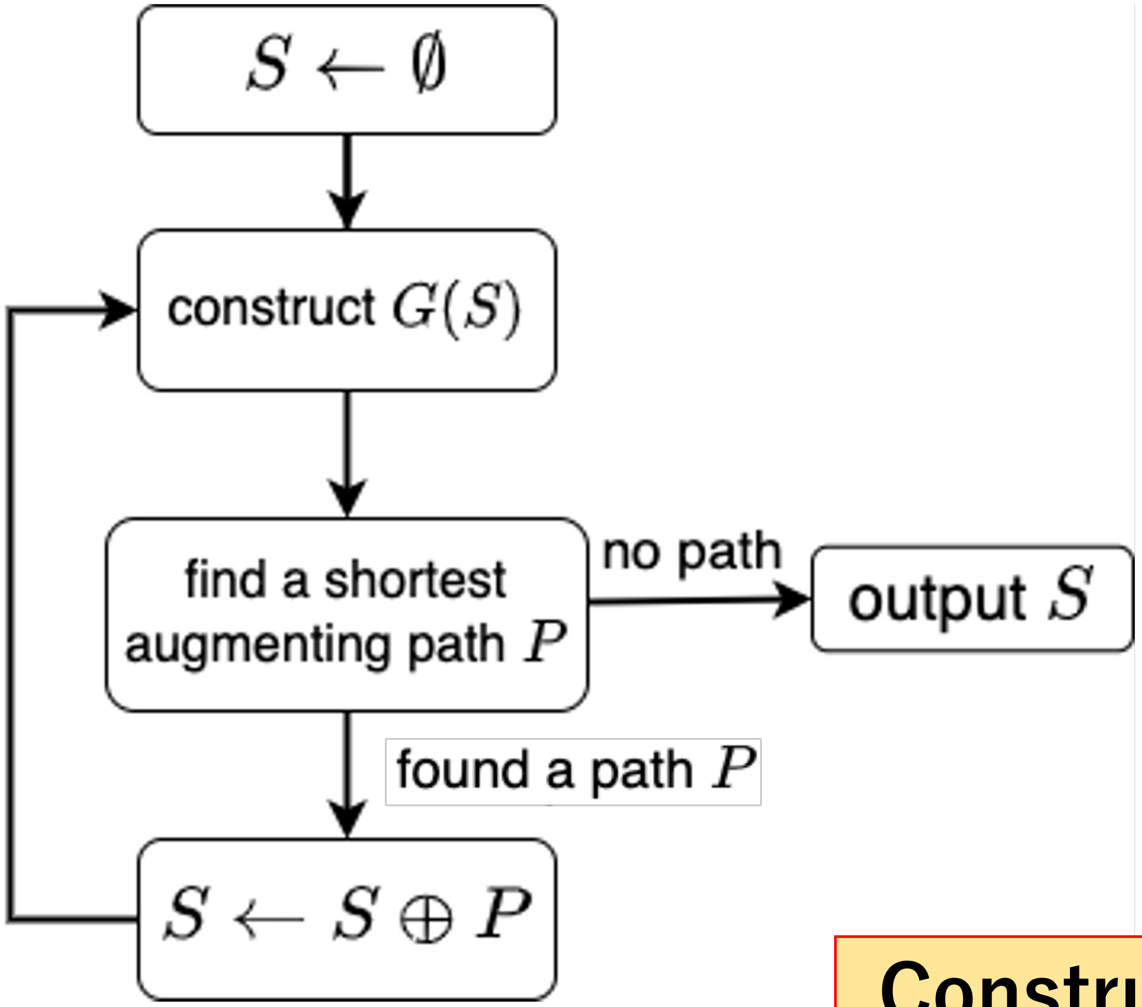
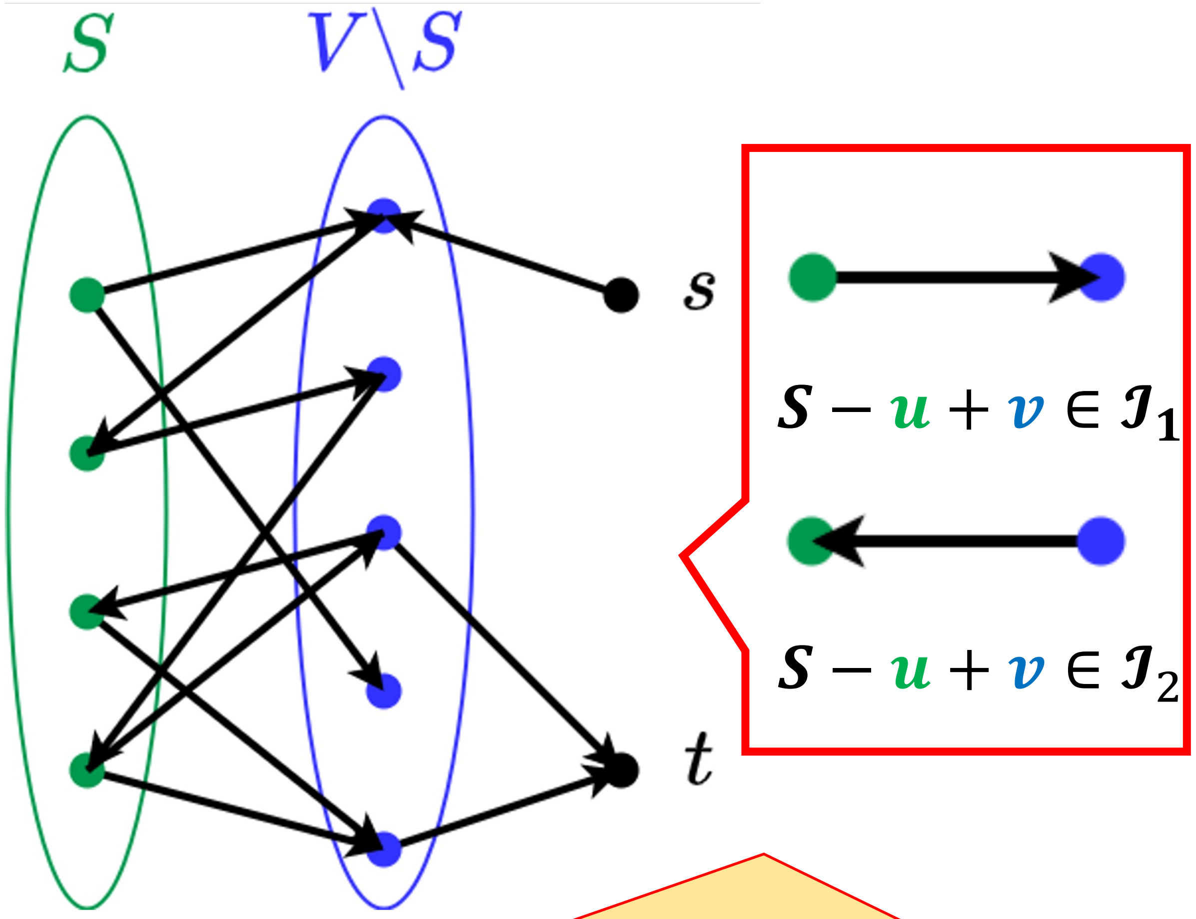
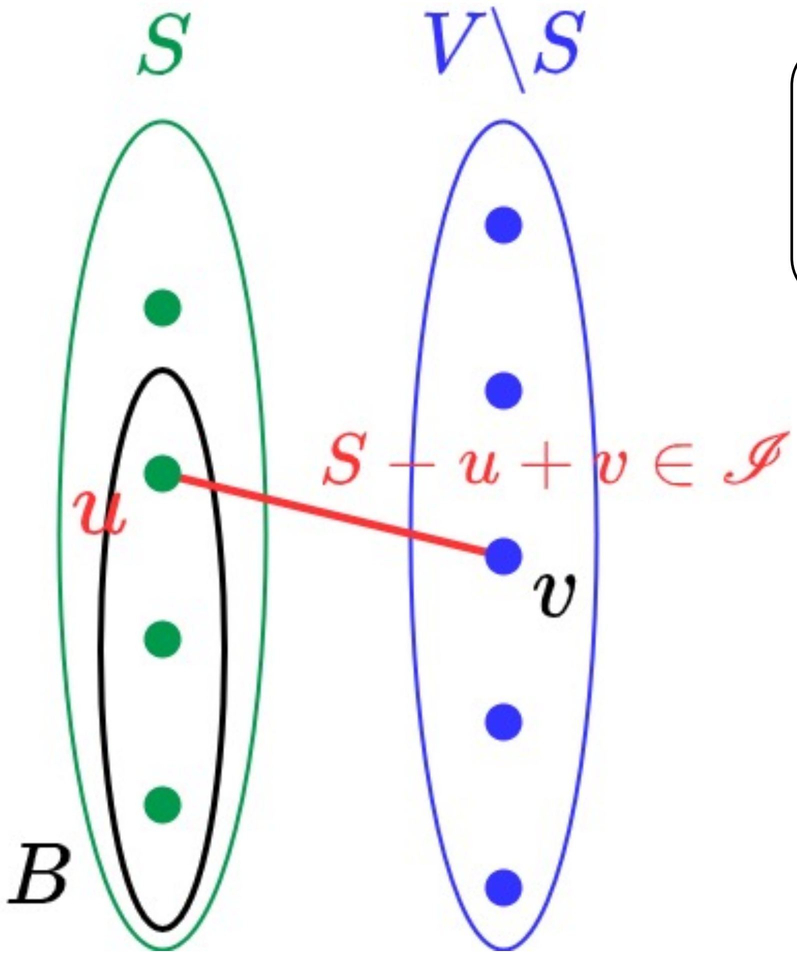




Construct exchange graph $G(S)$ explicitly

Tool for Faster Matroid Intersection

[Nguyễn 2019, Chakrabarty et al. 2019]



Input : $\mathcal{M} = (V, \mathcal{I})$, $S \in \mathcal{I}$, $v \in V \setminus S$, $B \subseteq S$
Find : $u \in B$ s.t. $S - u + v \in \mathcal{I}$

Tool for Faster Matroid Intersection

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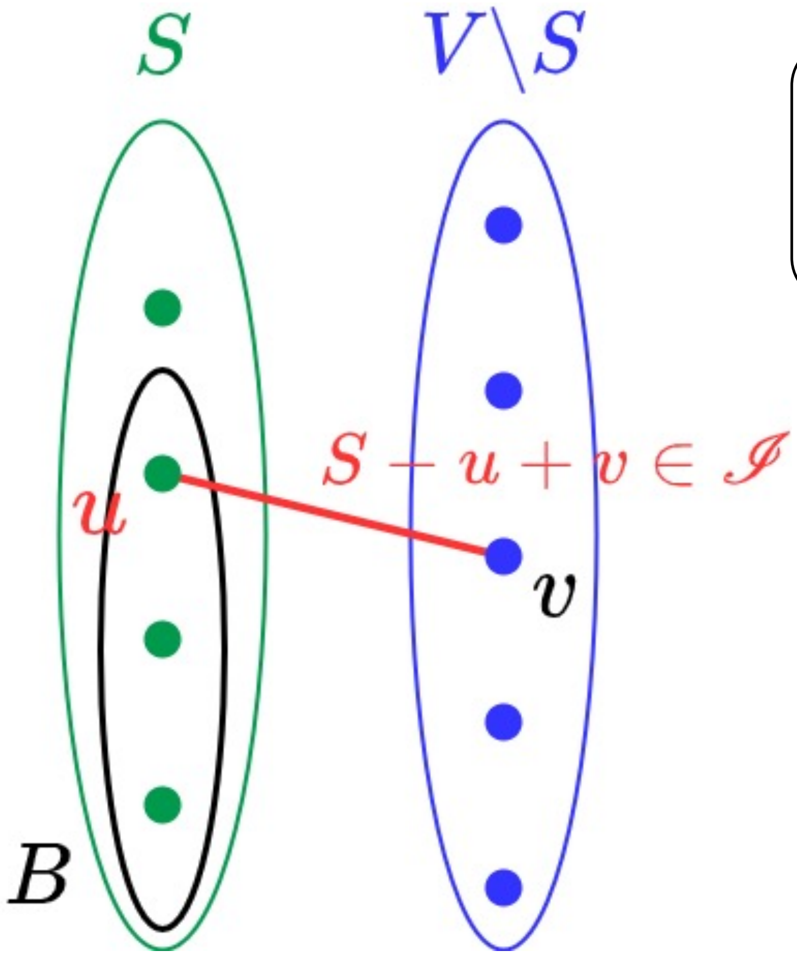

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$\boldsymbol{O(\log|B|)}$ independence query using **binary search**

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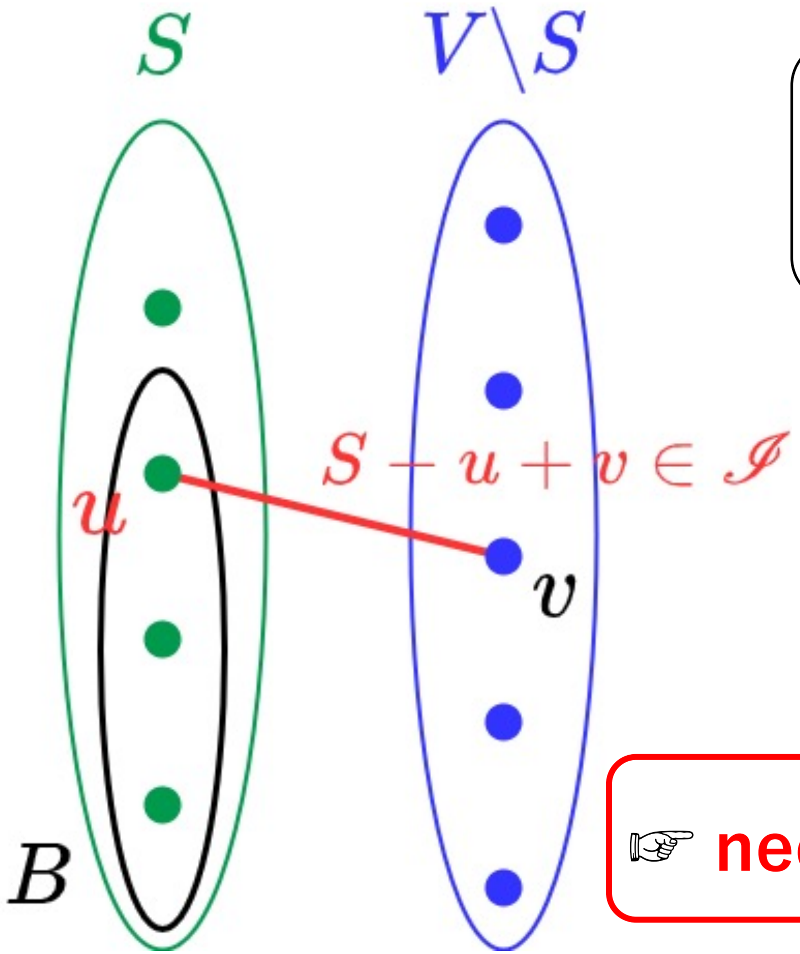

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$O(\log|B|)$ independence query using **binary search**

☞ **need not** construct **exchange graph $G(S)$ explicitly**

Matroid Partition

Input : $\boldsymbol{k}$ **matroids** $\mathcal{M}_1 = (V, \mathcal{I}_1), \ldots, \mathcal{M}_k = (V, \mathcal{I}_k)$
Find : maximum **partitionable** set $S \subseteq V$

There exists a **partition** $\boldsymbol{S} = \boldsymbol{S_1} \cup \cdots \cup \boldsymbol{S_k}$ s.t. $\boldsymbol{S_i} \in \boldsymbol{\mathcal{I}_i}$

Matroid Partition

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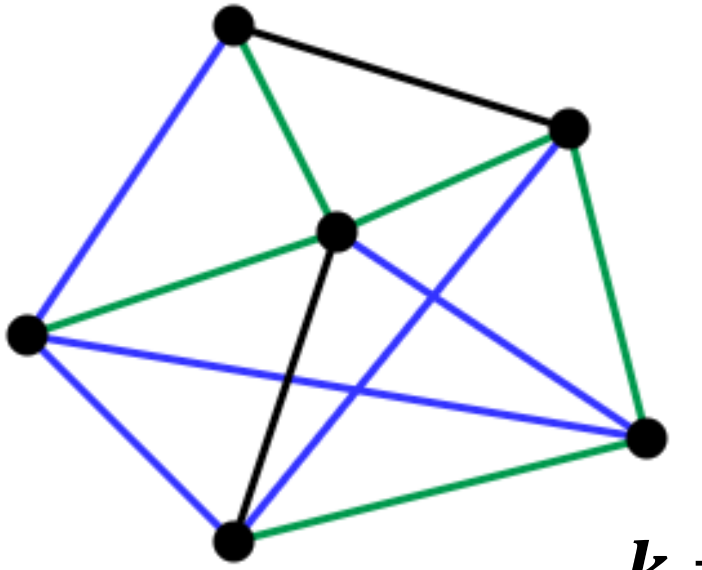
Find : maximum **partitionable** set $S \subseteq V$

There exists a partition $S = S_1 \cup \cdots \cup S_k$ s.t. $S_i \in \mathcal{I}_i$

E.g. k-forest

Find a maximum-size union of k forests



$\boldsymbol{k = 2}$

Matroid Partition and Matroid Intersection

Matroid partition can be solved by **the reduction to matroid intersection**

☞ Intersection of two matroids on $\boldsymbol{V} \times \{\mathbf{1}, \ldots, \boldsymbol{k}\}$

Matroid Partition and Matroid Intersection

Matroid partition can be solved by **the reduction to matroid intersection**

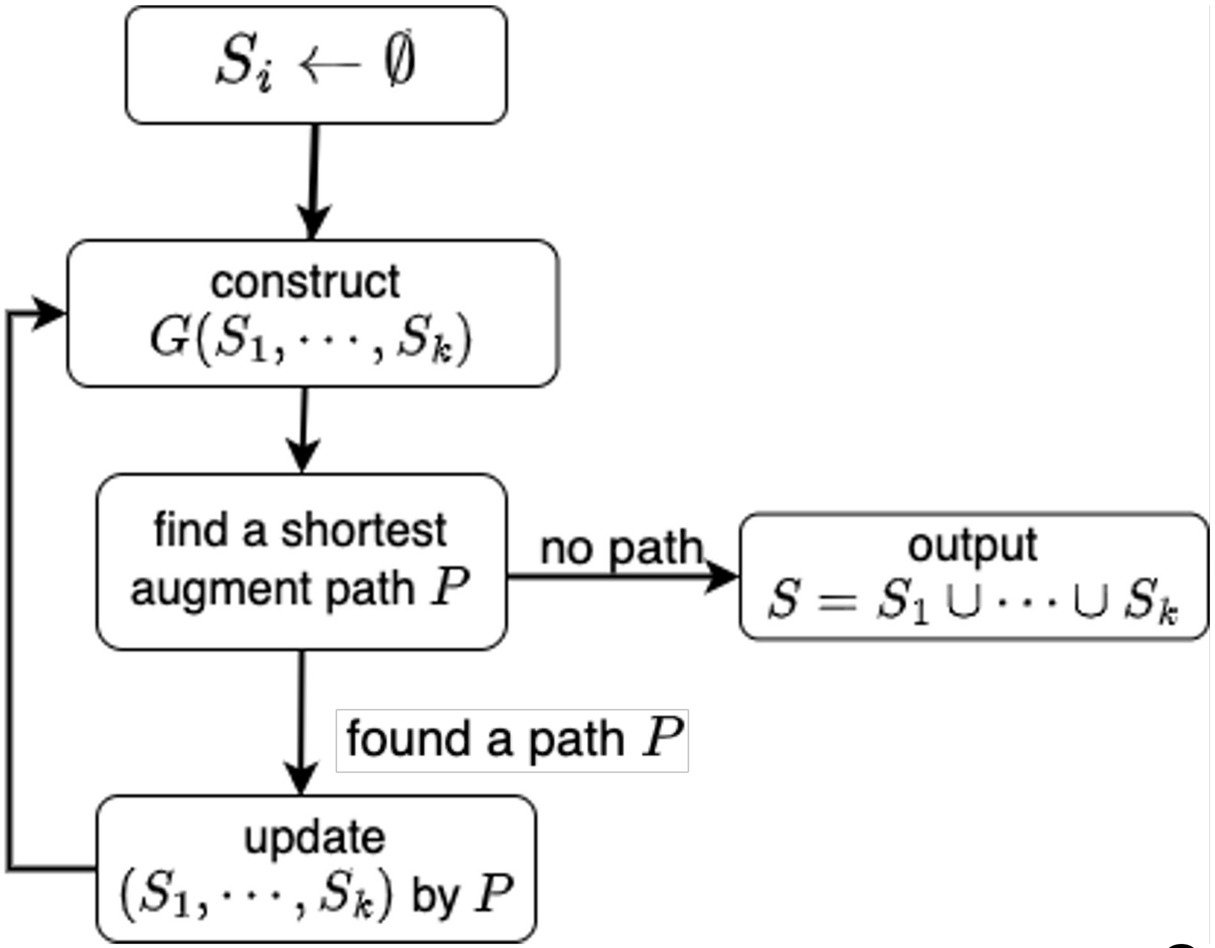
☞ Intersection of two matroids on $\boldsymbol{V} \times \{\mathbf{1}, \dots, \boldsymbol{k}\}$

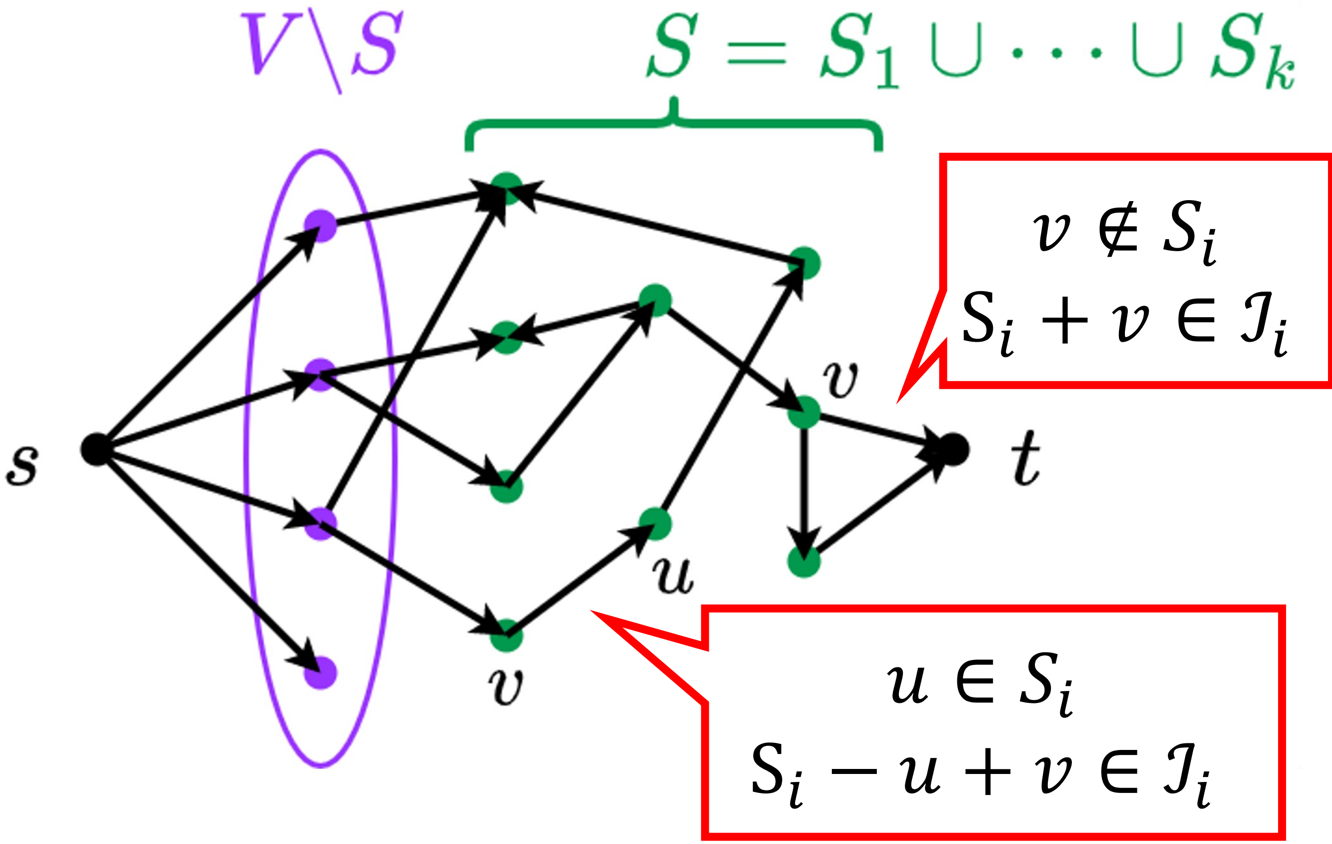
The size of ground set is $\boldsymbol{kn}$: large ⇨ **too many queries !**

given: $(V,\mathcal{I}_1),\ldots,(V,\mathcal{I}_k)$
max $|S_1 \cup \cdots \cup S_k|$ s.t. $S_i \in \mathcal{I}_i (\forall i)$

Algorithm for Matroid Partition

[Edmonds 1968]





Compressed exchange graph $G(S_1,\ldots,S_k)$

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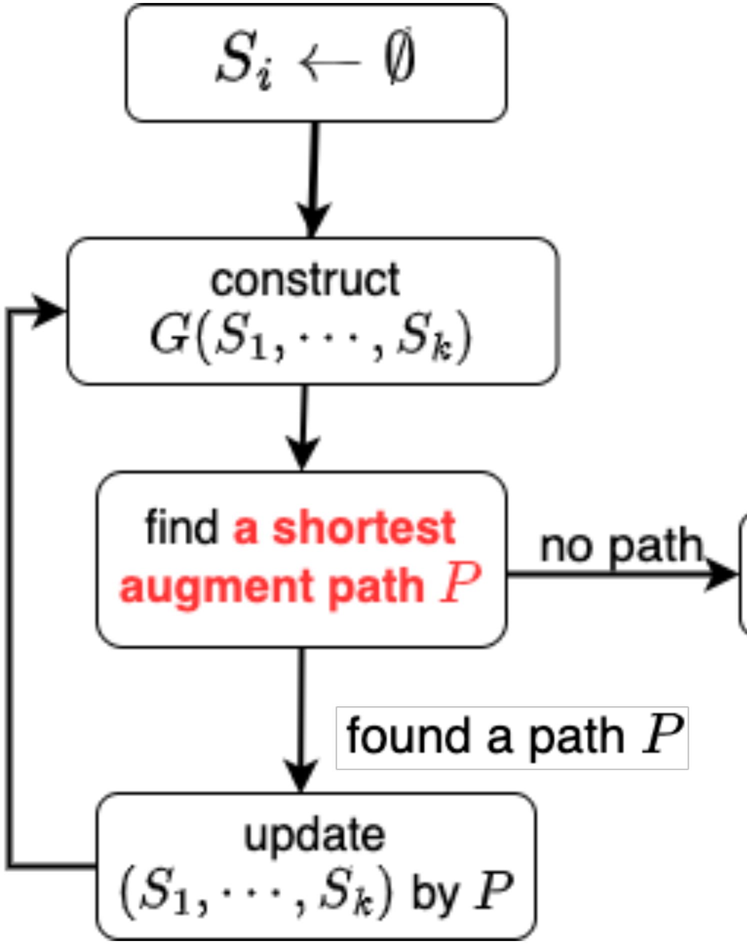

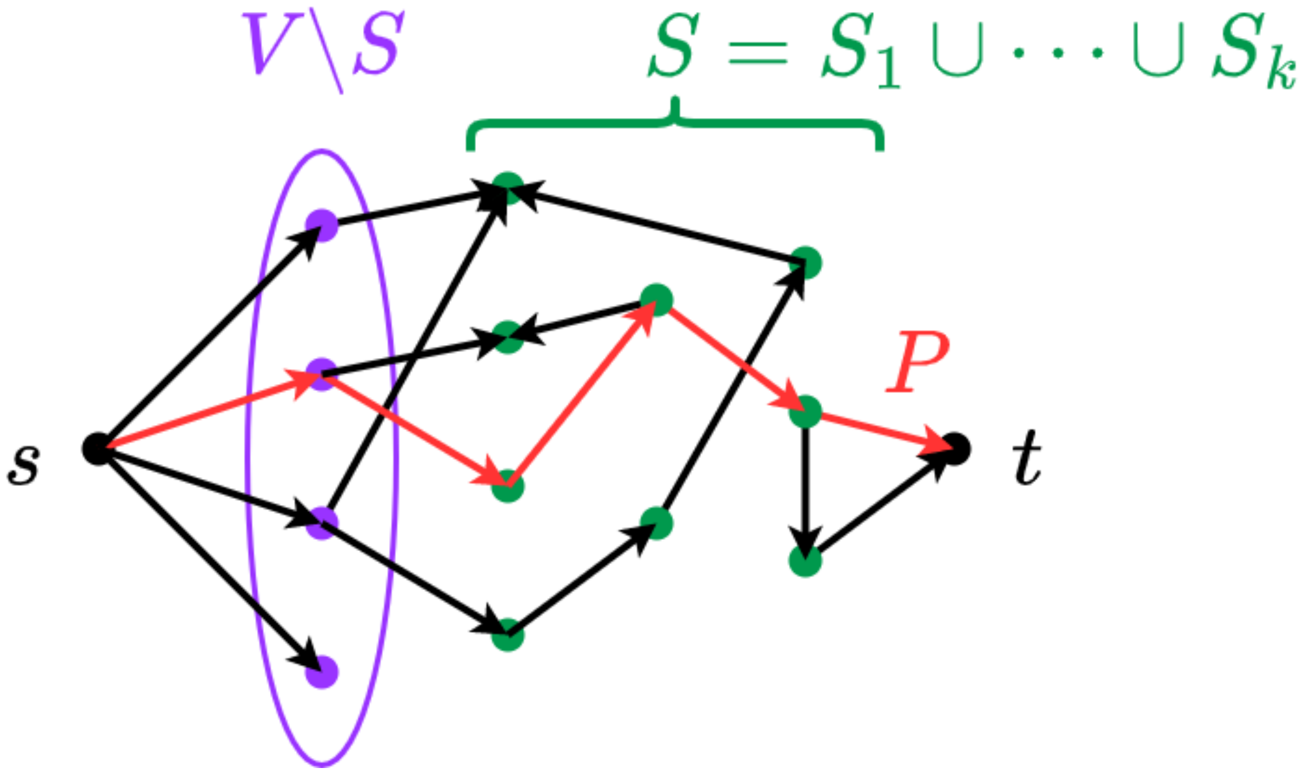

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$n = |V|$, $k =$ #matroids, $p =$ sol. size

Independence query Complexity

1968	Edmonds	$O(np^2 + kn)$
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2023	**This work**	$\tilde{O}(k^{1/3}np + kn)$

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Prior Work on Matroid Partition

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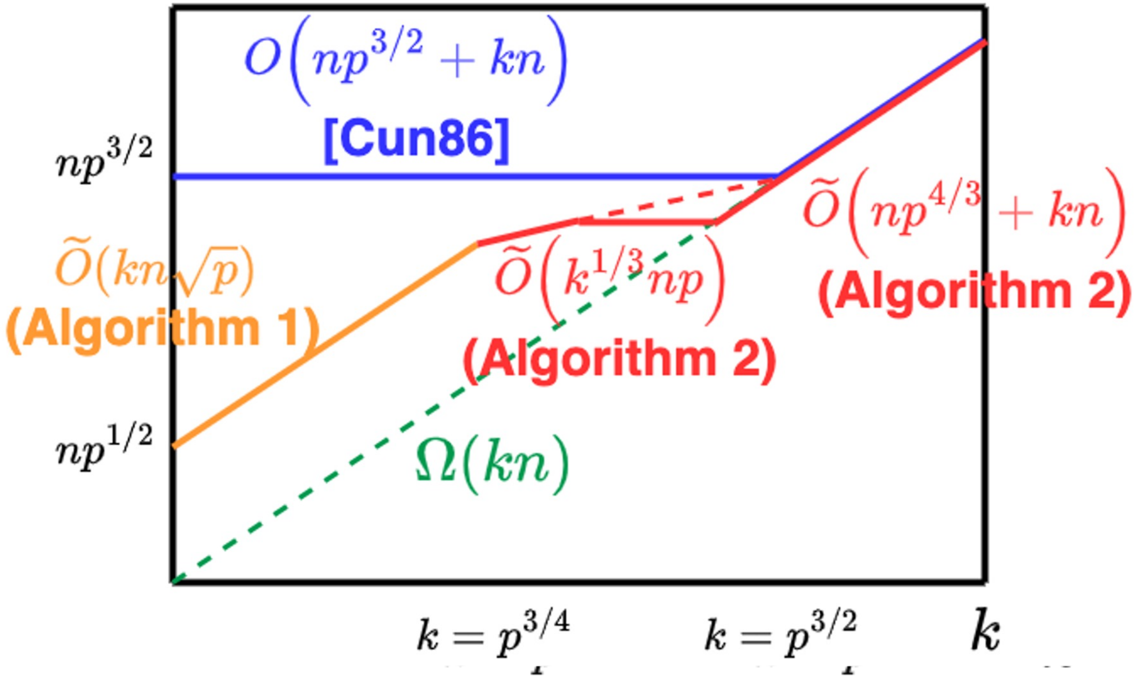

Algorithm 1: Blocking Flow + Binary Search

Thm1

Matroid partition can be solved using $\tilde{O}(kn\sqrt{p})$ **independence** queries

$n = |V|$, $k = \#$matroids
$p =$ solution size

given: $(V, \mathcal{I}_1), \dots, (V, \mathcal{I}_k)$
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Algorithm 1: Blocking Flow

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Matroid partition can be solved using $\tilde{O}(kn\sqrt{p})$ **independence** queries

Idea

Blocking Flow [Cunningham 1986]
☞ akin to Hopcroft-Karp / Dinic

+

Binary Seach
[Nguyễn 2019, Chakrabarty et al. 2019]

Finding **multiple** augmenting paths **of the same length** in one phase

given: $(V, \mathcal{J}_1), \dots, (V, \mathcal{J}_k)$
max $|S_1 \cup \cdots \cup S_k|$ s.t. $S_i \in \mathcal{J}_i (\forall i)$
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Algorithm 1: Blocking Flow

Thm1

Matroid partition can be solved using $\tilde{O}(kn\sqrt{p})$ **independence** queries

Algorithm

Repeat:

Step 1: Breadth First Search

Step 2: Find multiple augmenting paths

Algorithm 1: Blocking Flow

given: $(V, \mathcal{I}_1), \ldots, (V, \mathcal{I}_k)$
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Thm1
Matroid partition can be solved using $\tilde{O}(kn\sqrt{p})$ **independence** queries

Algorithm

Repeat:
- Step 1: Breadth First Search ⇐ $\tilde{O}(kn)$ **queries**
- Step 2: Find multiple augmenting paths ⇐ $\tilde{O}(kn)$ **queries**

Fact: $\Theta(\sqrt{p})$ phases are required

given: $(V, \mathcal{I}_1), \dots, (V, \mathcal{I}_k)$
max $|S_1 \cup \cdots \cup S_k|$ s.t. $S_i \in \mathcal{I}_i (\forall i)$
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Algorithm 1: Blocking Flow

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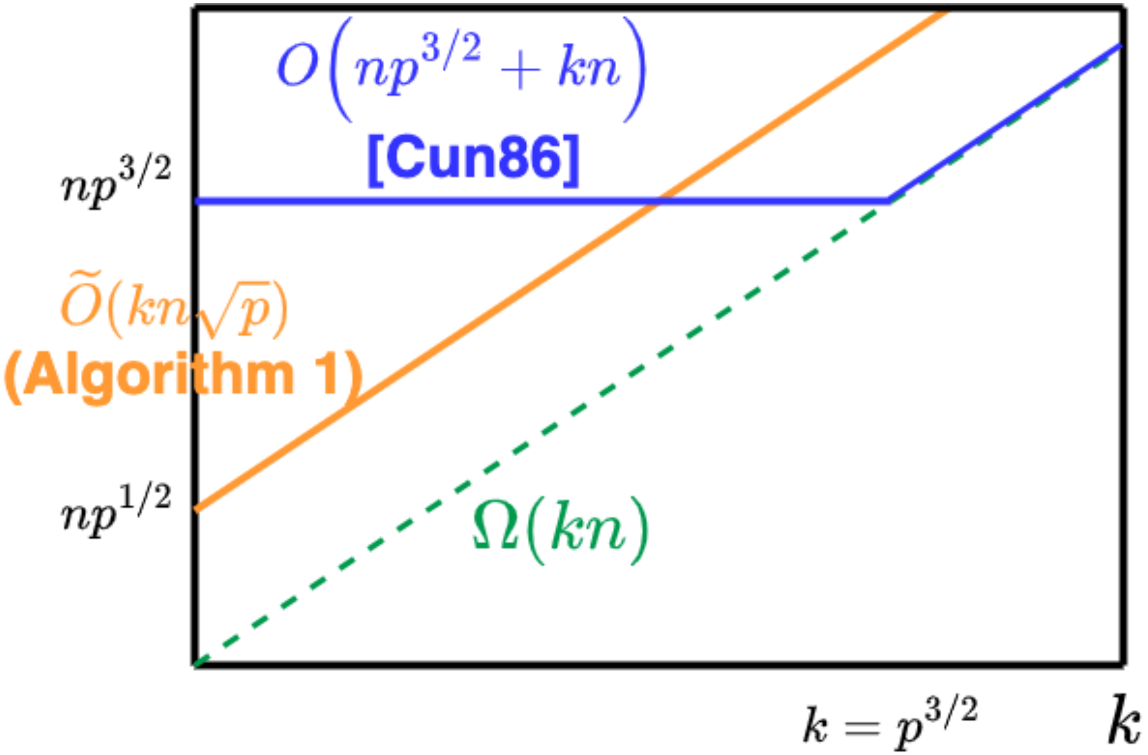
Matroid partition can be solved using $\tilde{O}(kn\sqrt{p})$ **independence** queries



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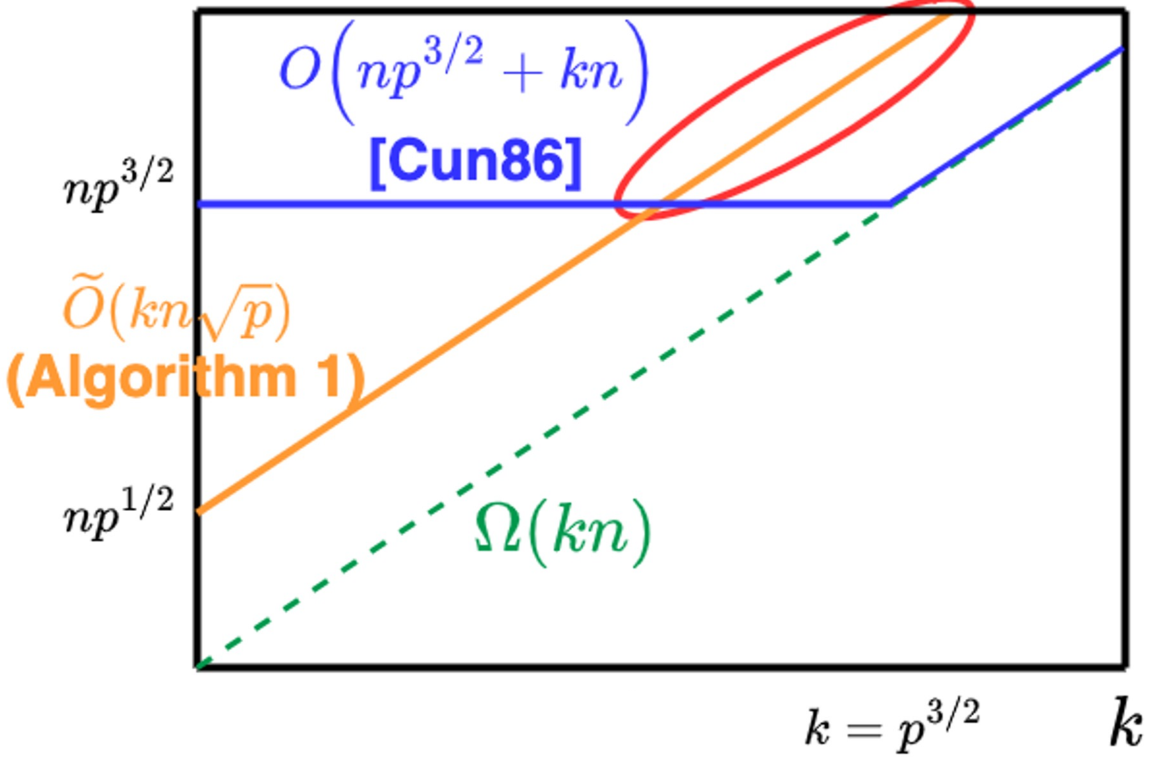

Despite of **binary search** technique, Alg. 1 is worse than [Cun 86].

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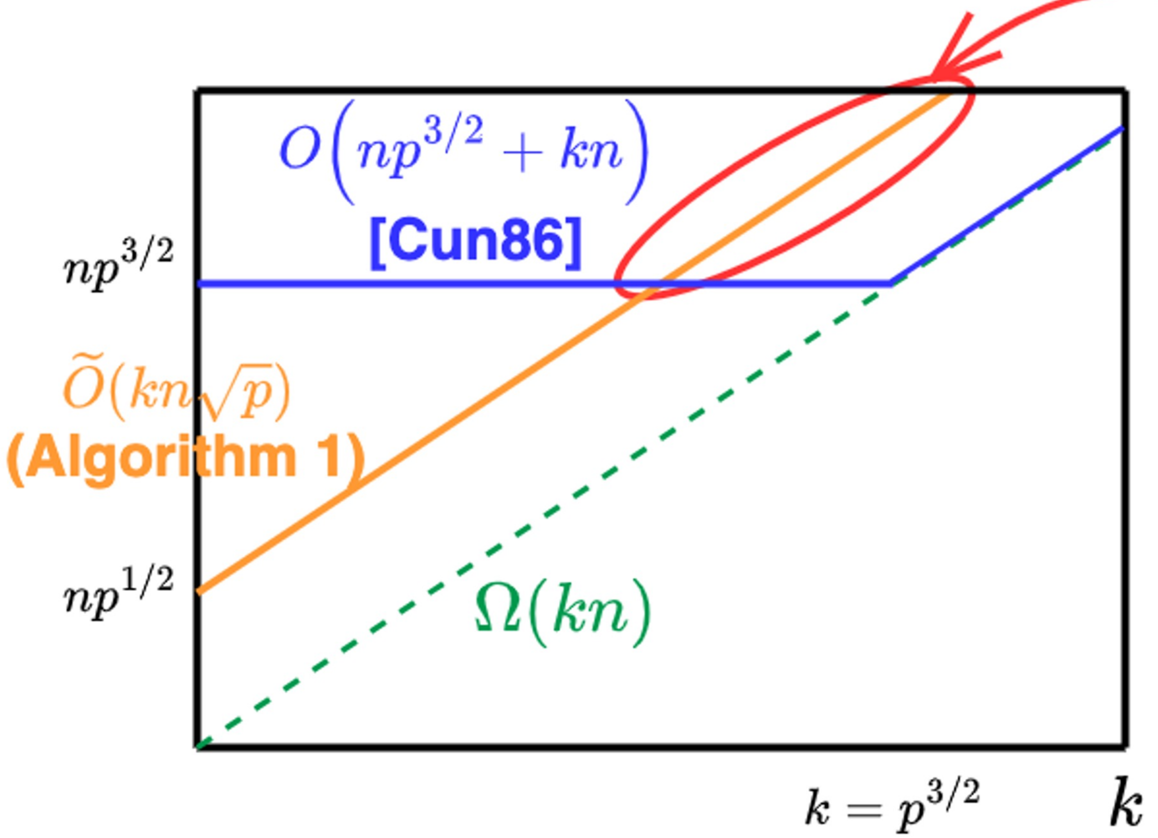

Despite of binary search technique, Alg. 1 is worse than [Cun 86].

Q. Better Algorithm **when k is large** ?

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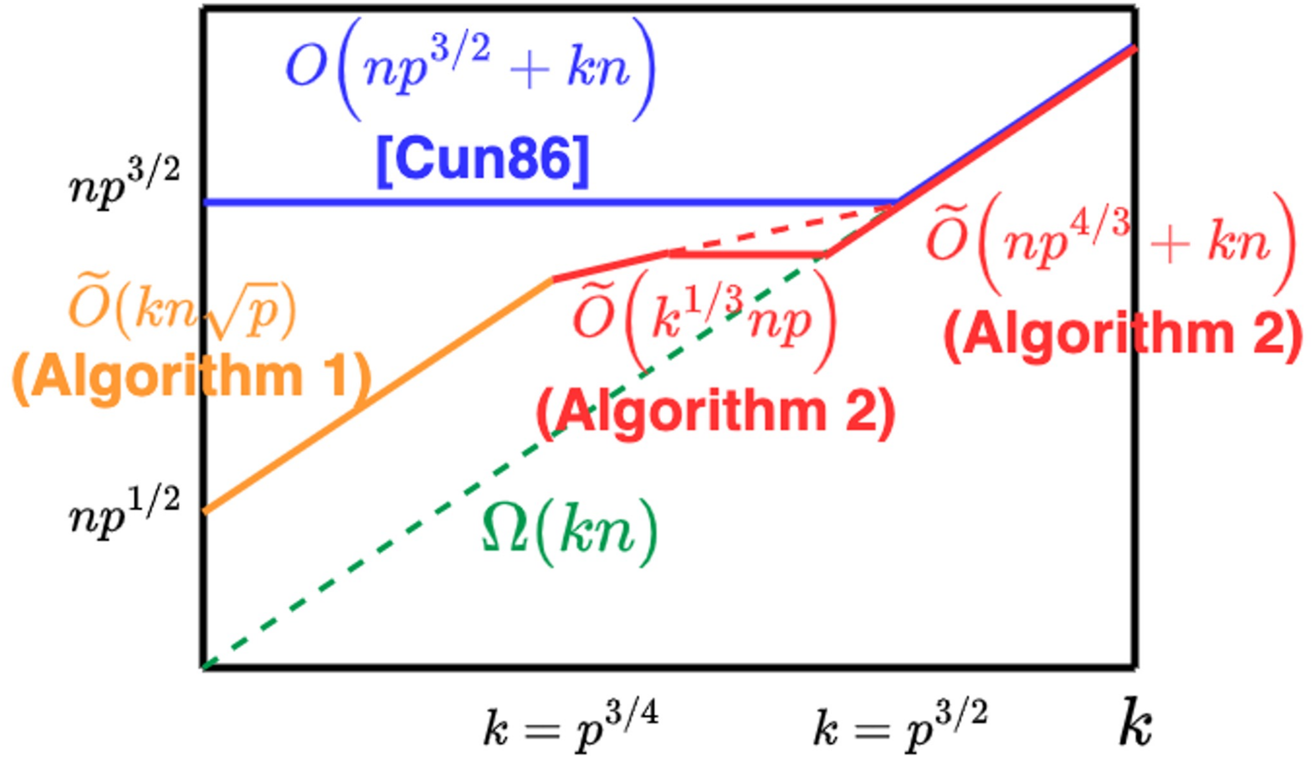
max $|S_1 \cup \cdots \cup S_k|$ s.t. $S_i \in \mathcal{I}_i (\forall i)$
$n = |V|$, $p =$ sol. size, k'=min$\{k, p\}$

Algorithm 2

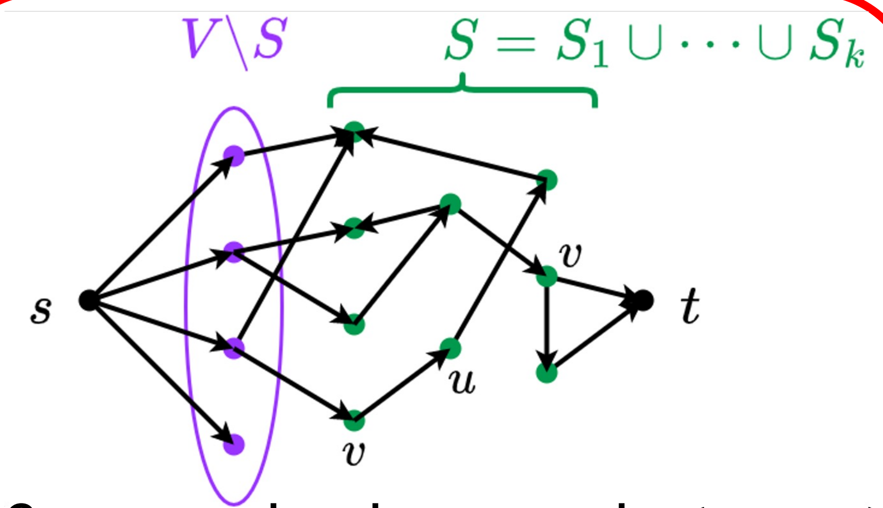
Thm2
Matroid partition can be solved using $\tilde{O}(k'^{1/3}np + kn)$ **independence** queries



One Phase of Edge Recycling Augmentation

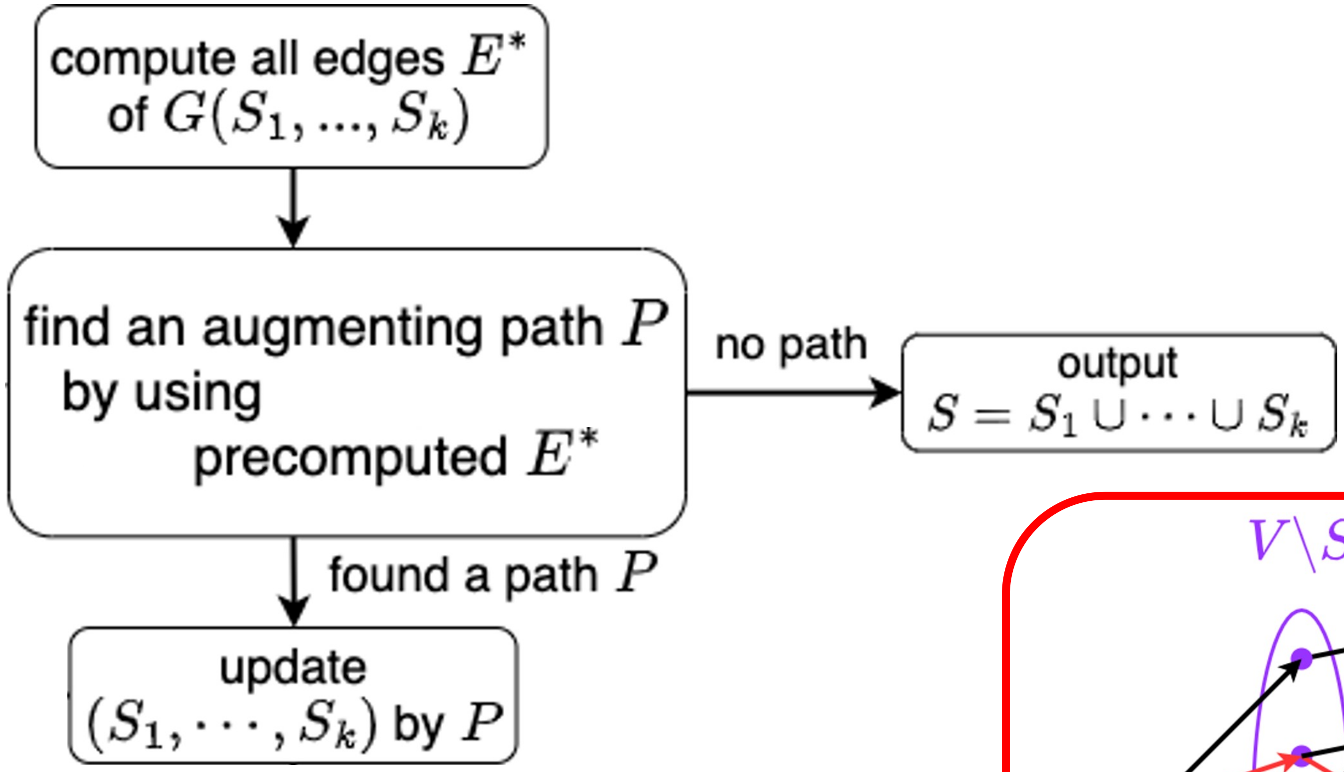
compute all edges E^* of $G(S_1, \ldots, S_k)$

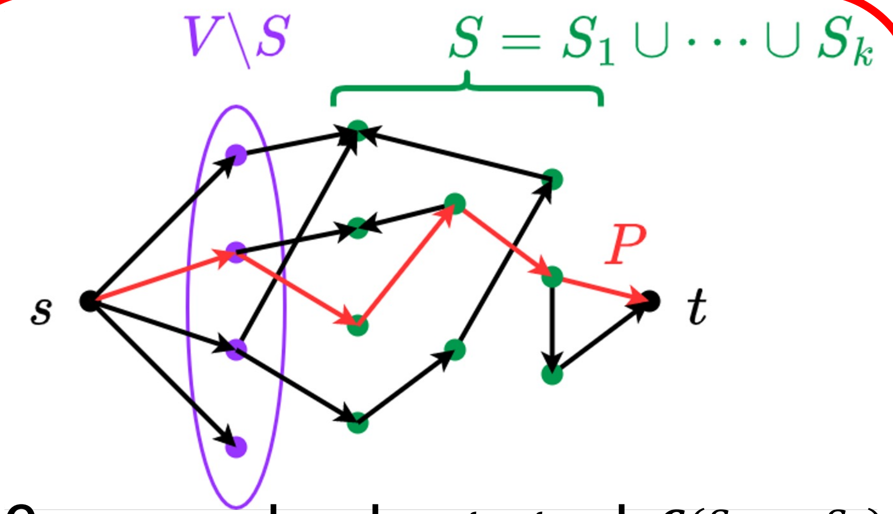
$\boldsymbol{O(np)}$ queries



Compressed exchange graph $\boldsymbol{G}(S_1, \ldots, S_k)$

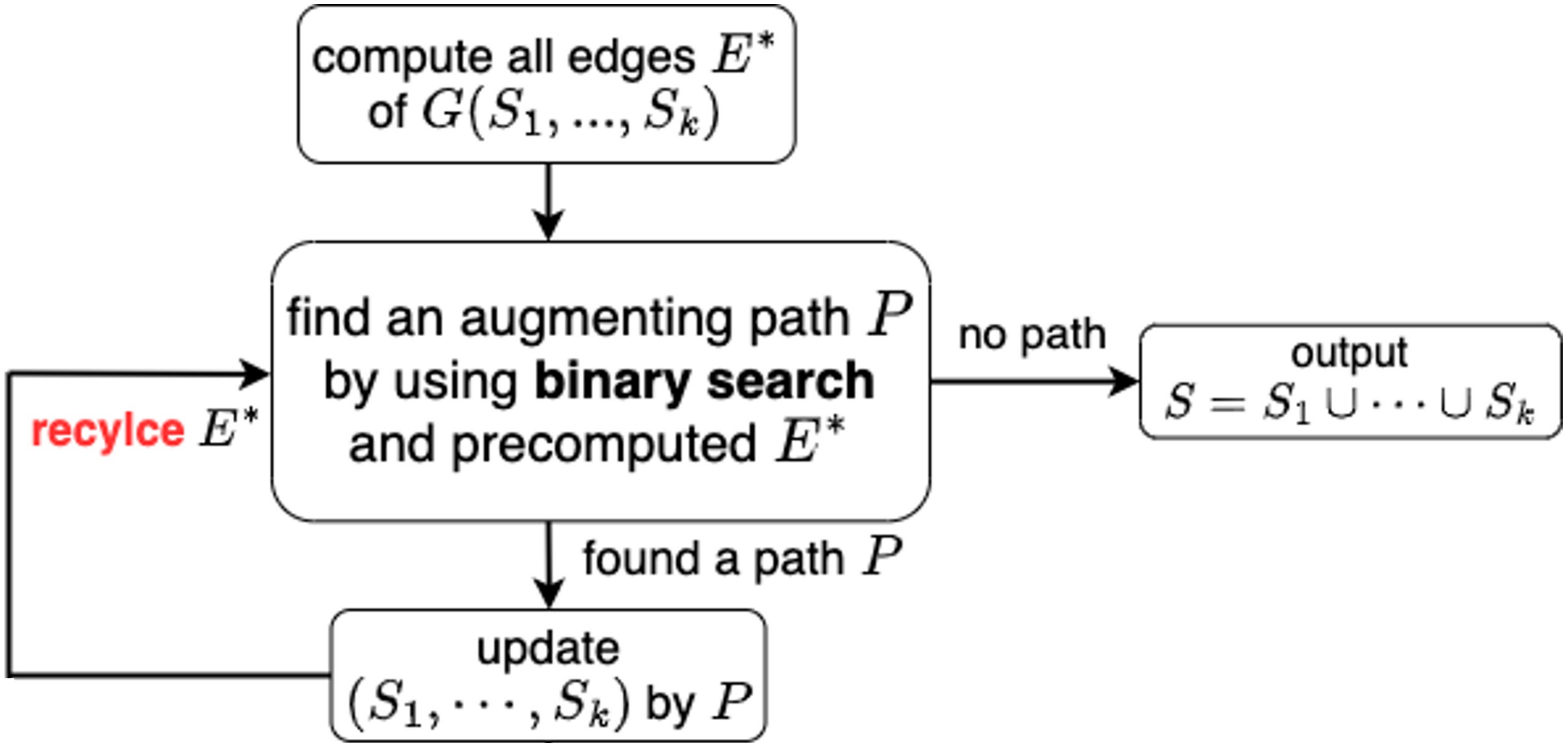
One Phase of Edge Recycling Augmentation





Compressed exchange graph $G(S_1, \ldots, S_k)$

One Phase of Edge Recycling Augmentation



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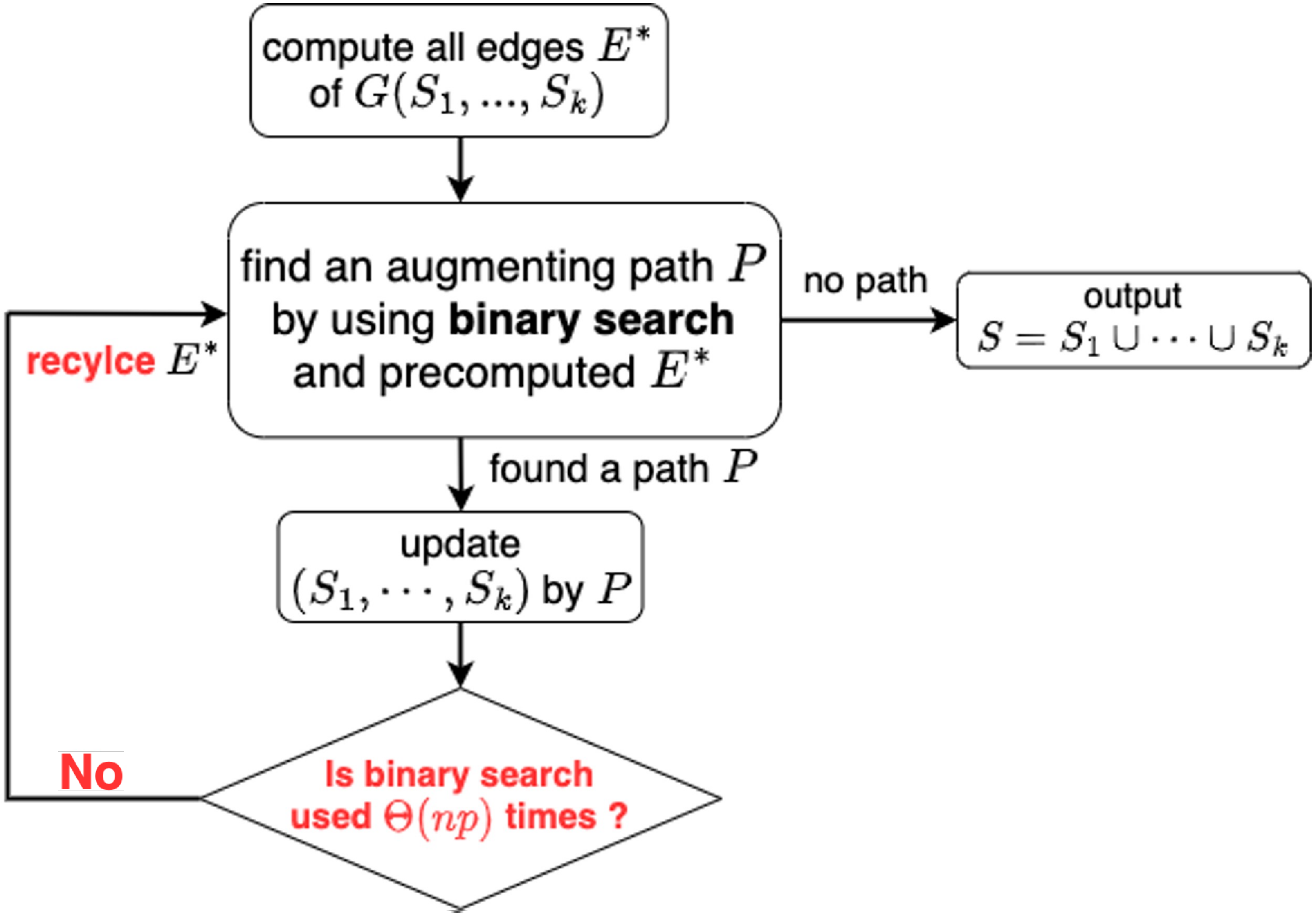

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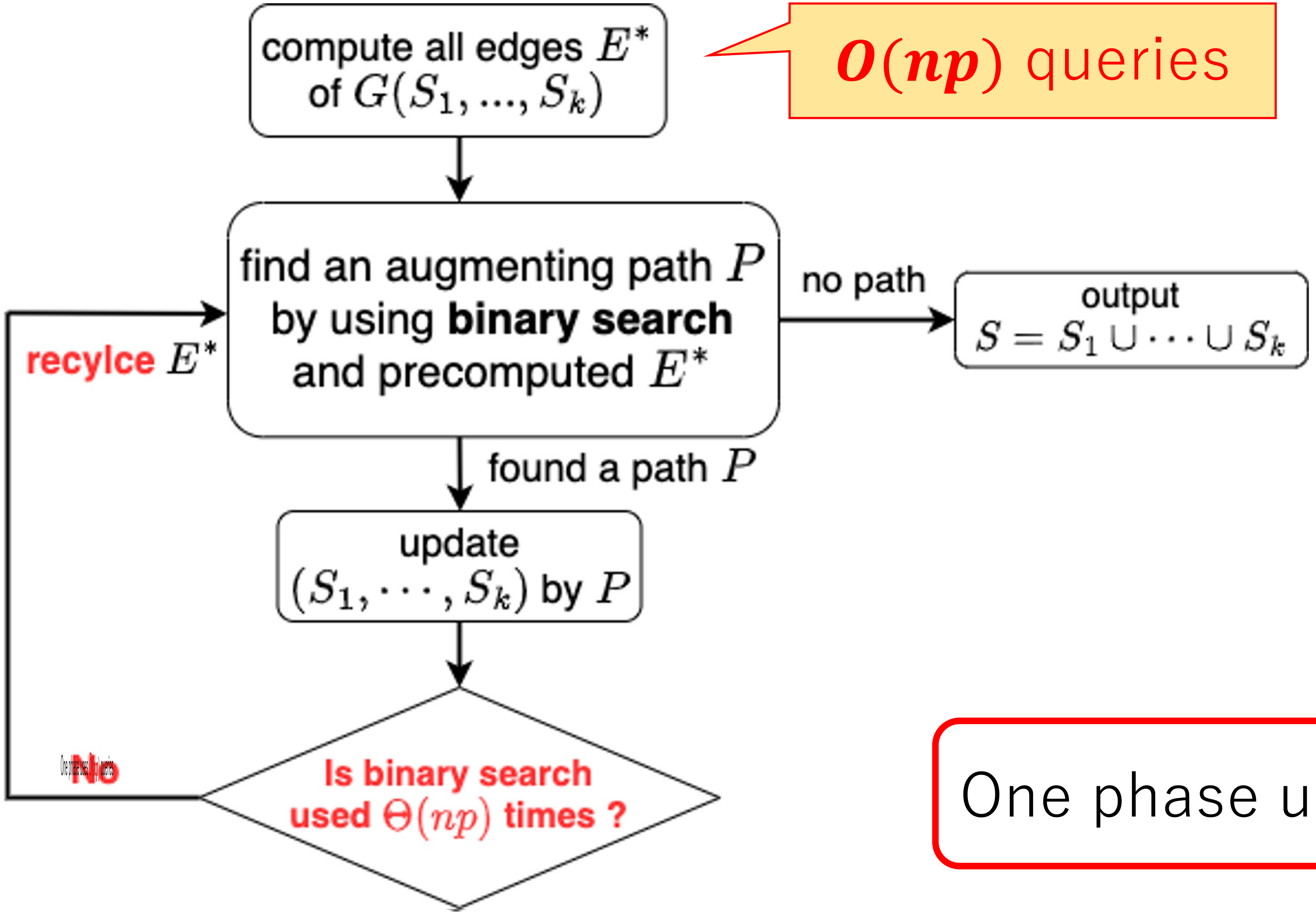

$\boldsymbol{O(np)}$ queries

One phase uses $\boldsymbol{\tilde{O}(np)}$ queries

given: $(V, \mathcal{I}_1), \dots, (V, \mathcal{I}_k)$
max $|S_1 \cup \cdots \cup S_k|$ s.t. $S_i \in \mathcal{I}_i (\forall i)$
$n = |V|$, $p =$ sol. size, $k' = \min\{k, p\}$

Algorithm 2: Hybrid Approach

Thm2
Matroid partition can be solved using $\boldsymbol{\tilde{O}(k'^{1/3} np + kn)}$ **independence** queries

given: $(V, \mathcal{I}_1), \dots, (V, \mathcal{I}_k)$
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Algorithm 2: Hybrid Approach

<u>Thm2</u>
Matroid partition can be solved using $\tilde{O}(k'^{1/3}np + kn)$ **independence** queries

Step 1. Apply **Blocking Flow** (Algorithm 1)

given: $(V, \mathcal{I}_1), \ldots, (V, \mathcal{I}_k)$
max $|S_1 \cup \cdots \cup S_k|$ s.t. $S_i \in \mathcal{I}_i (\forall i)$
$n = |\mathrm{V}|$, $p =$ sol. size, k'=min$\{k, p\}$

Algorithm 2: Hybrid Approach

Thm2
Matroid partition can be solved using $\tilde{O}(k'^{1/3}np + kn)$ **independence** queries

Step 1. Apply Blocking Flow (Algorithm 1)

Step 2. Apply **Edge Recycling Augmentation**

Algorithm 2: Hybrid Approach

given: $(V, \mathcal{I}_1), \dots, (V, \mathcal{I}_k)$
max $|S_1 \cup \cdots \cup S_k|$ s.t. $S_i \in \mathcal{I}_i (\forall i)$
$n = |V|$, $p =$ sol. size, $k' = \min\{k, p\}$

Thm2
Matroid partition can be solved using $\tilde{O}(k'^{1/3}np + kn)$ **independence** queries

Step 1. Apply **Blocking Flow** (Algorithm 1) in $\Theta(\frac{p}{k'^{2/3}})$ **phases**

Step 2. Apply Edge Recycling Augmentation

given: $(V, \mathcal{I}_1), \ldots, (V, \mathcal{I}_k)$
max $|S_1 \cup \cdots \cup S_k|$ s.t. $S_i \in \mathcal{I}_i (\forall i)$
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Algorithm 2: Hybrid Approach

Thm2

Matroid partition can be solved using $\tilde{O}(k'^{1/3} np + kn)$ **independence** queries

Step 1. Apply Blocking Flow (Algorithm 1) in $\Theta(\frac{p}{k'^{2/3}})$ **phases**

Step 2. Apply **Edge Recycling Augmentation**

Lemma: $\Theta(k'^{1/3})$ phases are required in Step 2

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Algorithm 2: Hybrid Approach

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Matroid partition can be solved using $\tilde{O}(k'^{1/3}np + kn)$ **independence** queries

Step 1. Apply Blocking Flow (Algorithm 1) in $\Theta(\frac{p}{k'^{2/3}})$ **phases**

One phase uses $\tilde{O}(k'n)$ queries

Step 2. Apply **Edge Recycling Augmentation**

One phase uses $\tilde{O}(np)$ queries

Lemma: $\Theta(k'^{1/3})$ phases are required in Step 2

Conclusion

Improve the **independence** query complexity of **Matroid Partition**

- Use **Binary Search Technique** [Nguyễn 2019, Chakrabarty et al. 2019]
- A new approach: **Edge Recycling Augmentation**

Q. Further improvement?

Q. Apply an idea of Edge Recycling Augmentation to other problems?