Deterministic $(\frac{2}{3}-\varepsilon)$-approximation of Matroid Intersection Using Nearly-Linear Independence-Oracle Queries

Tatsuya Terao (Kyoto University)

WADS 2025 @ Toronto

What is Matroid?

Def: A finite set V and non-empty collection $\mathcal{I}$ of subset of V

s.t.

① $S' \subseteq S \in \mathcal{I} \Rightarrow S' \in \mathcal{I}$

② $S, T \in \mathcal{I}, |S| > |T| \Rightarrow \exists e \in S \setminus T$ s.t. $T \cup \{e\} \in \mathcal{I}$

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e.g. • Linear matroid

$$\begin{pmatrix} \boxed{1} & \boxed{1} & \boxed{2} \\ 2 & 1 & 2 \end{pmatrix}$$

$V = \{①, ②, ③\}$

$\mathcal{I} = \{\phi, \{①\}, \{②\}, \{③\}, \{①,②\}, \{①,③\}\}$

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$S \in \mathcal{I}$: S is independent

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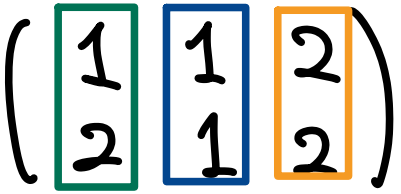

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- partition matroid
- laminar matroid
- graphic matroid

etc...

What is Matroid Intersection?

input: two matroids $M_1 = (V, I_1)$, $M_2 = (V, I_2)$

output: maximum common independent set $S \in I_1 \cap I_2$

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— Generalization of Bipartite Matching

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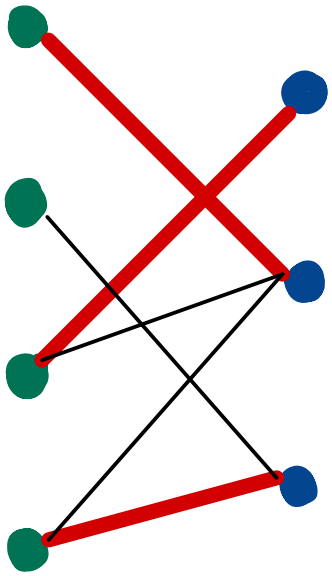
eg. Maximum Bipartite Matching

V = edges

I_1 = each left vertex has at most 1 edge

I_2 = each right vertex has at most 1 edge



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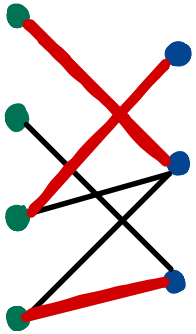
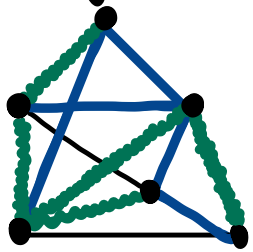
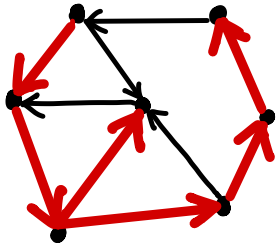
Bipartite Matching



Packing Spanning Tree



Arborescences



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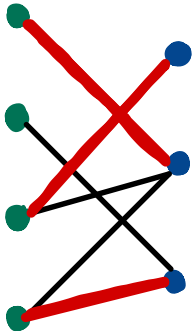

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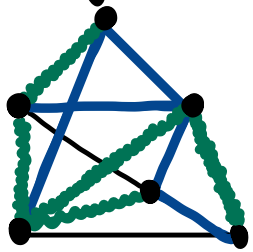

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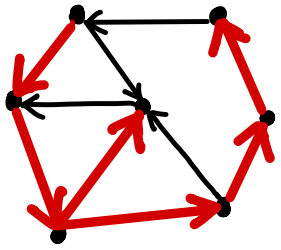

② Applications extend beyond Combinatorial Optimization

- electrical engineering
- network coding

input: $(V, I_1), (V, I_2)$

max $|S|$ s.t. $S \in I_1 \cap I_2$

Matroid Intersection can be solved in Polynomial time

[Edmonds'70, Aigner-Dowling'71, Lawler'75]

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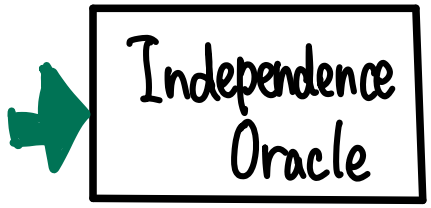
Matroid is given via an independence oracle

Query: Is $S \in I$?



Answer: Yes or No

Time Complexity = # of Queries

Time Complexity of Matroid Intersection

input : $(V, I_1), (V, I_2)$

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$n := |V|$, $r := \max_{S \in I_1 \cap I_2} |S|$

# of queries
to independence oracle

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1970s	Edmonds, Lawler, Aigner-Dowling	$O(nr^2)$
1986	Cunningham	$O(nr^{3/2})$

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1986	Cunningham	$O(nr^{3/2})$
2015	Lee-Sidford-Wong	$\tilde{O}(n^2)$
2019	Nguyễn, Chakrabarty-Lee-Sidford-Singla-Wong	$\tilde{O}(nr)$
2021	Blikstad-v.d.Brand-Mukhopadhyay-Nanongkai	$\tilde{O}(n^{9/5})$
2021	Blikstad ※	$\tilde{O}(nr^{3/4})$

※: [Blikstad21] : $\tilde{O}(nr^{3/4})$ if randomized, $\tilde{O}(nr^{5/6})$ if deterministic

Today's Focus: Nearly Linear-Time Algo.

$O(n \cdot \text{polylog}(n))$-time

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α-approx.: $S \in I_1 \cap I_2$
s.t. $|S| \geq \alpha \cdot OPT$

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e.g.

sparse cut : Khandekar-Rao-Vazirani '09, Madry '10
k-cut : Quanrud '19
k-ECSS : Chalermsook-Huang-Naraykai-Saranurak-Sukprasert-Yingchareonthawornchai '22
Weighted matching : Preis '99, Vinkemeier-Haugardy '03, Duan-Pettie '14, Zhang-Henzinger '23

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Nearly-Linear Time Algo. for Matroid Intersection

Folklore	$\frac{1}{2}$-approx.	$O(n)$ queries

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Quanrud '24	$(1-\varepsilon)$-approx.	$\tilde{O}_\varepsilon(n + r^{3/2})$ queries
Blikstad-Tu '25	$(1-\varepsilon)$-approx.	$\tilde{O}_\varepsilon(n)$ queries

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All recent algo. are randomized!

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Greedy algo.

$S \leftarrow \emptyset$
for $v \in V$ do
  if $S \cup \{v\} \in I_1 \cap I_2$ then
    $S \leftarrow S \cup \{v\}$
return S

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Main Thm.

$O(\frac{n}{\varepsilon} + r \log r)$-query $(\frac{2}{3} - \varepsilon)$-approx. algo.

Idea: Analogy from (Bipartite) Matching

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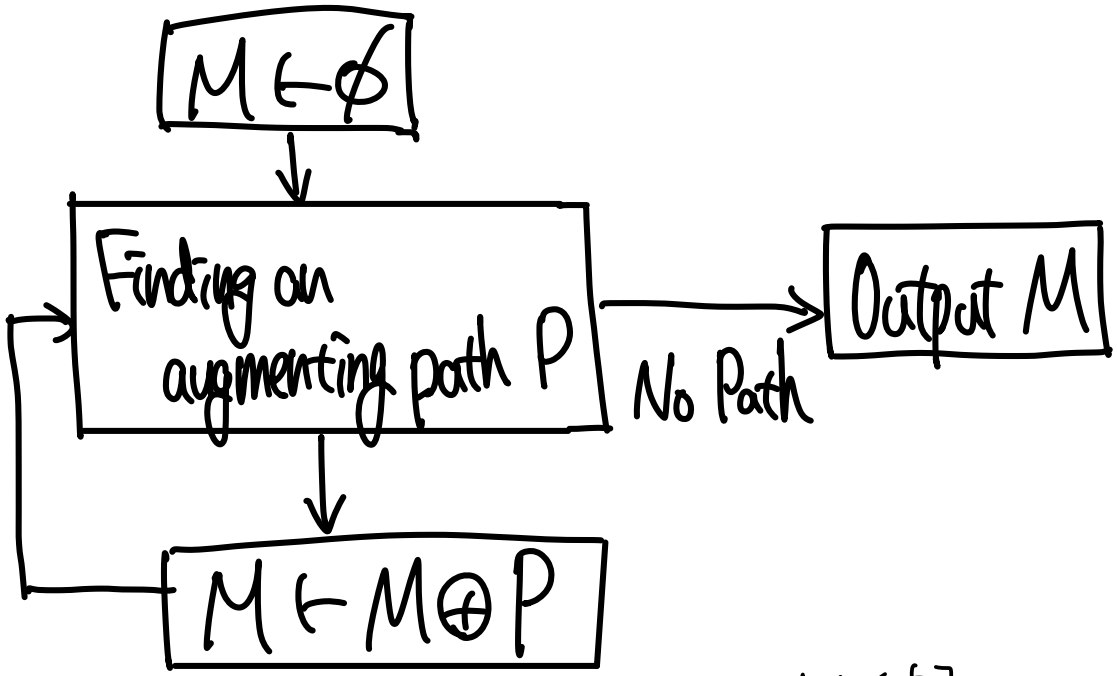
Algo.



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[Edmonds '65]

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Algo.

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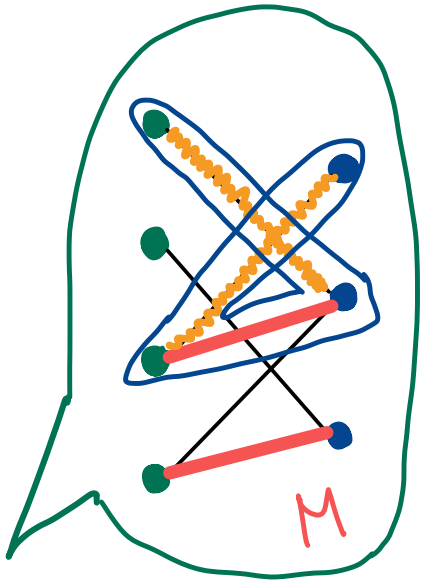
Finding an augmenting path P

No Path

Output M

$M \leftarrow M \oplus P$



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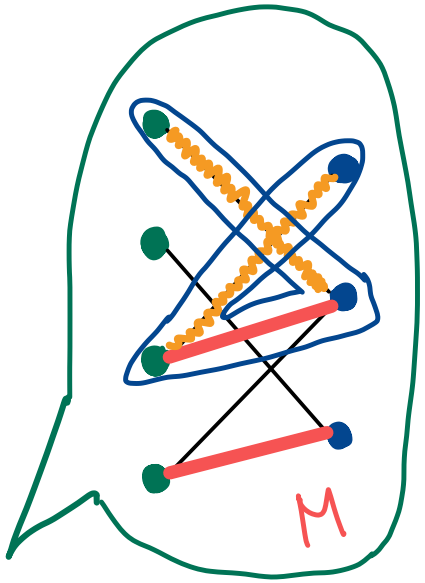

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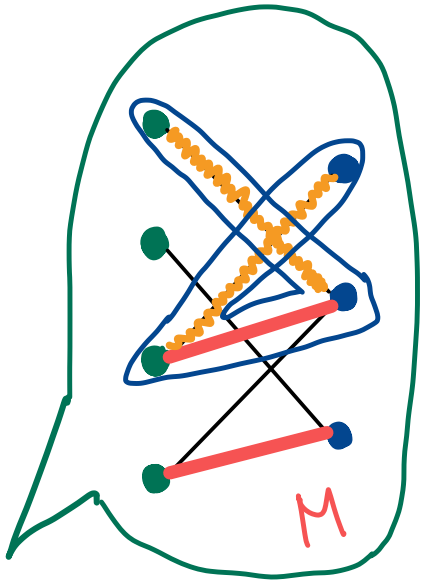

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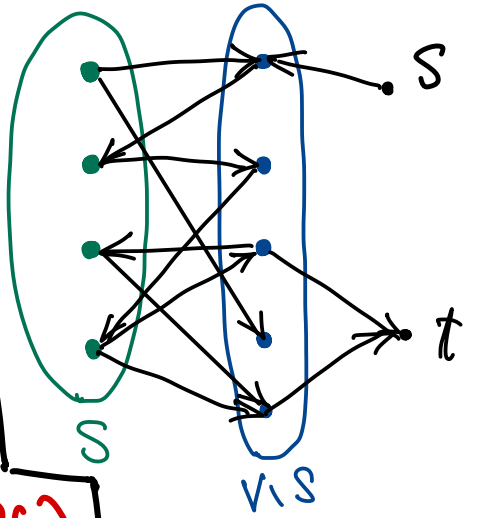

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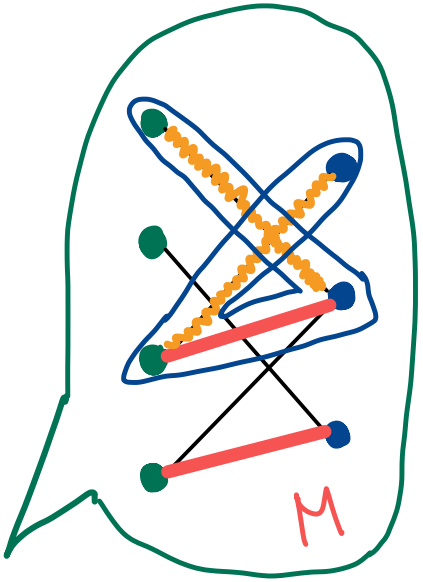

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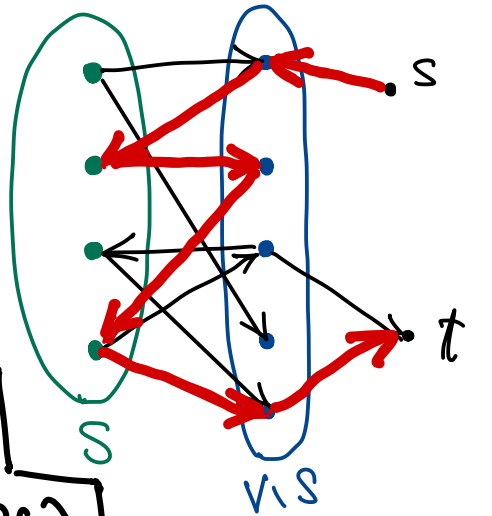

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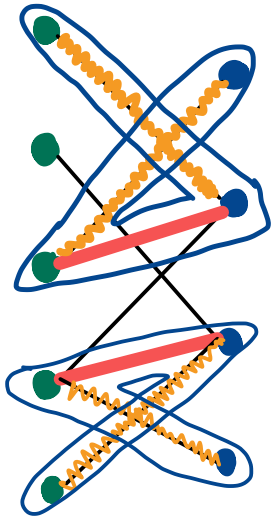
Fast Algo.

* Blocking Flow
  - Find multiple shortest augmenting paths in one phase [Hopcroft-Karp '73]



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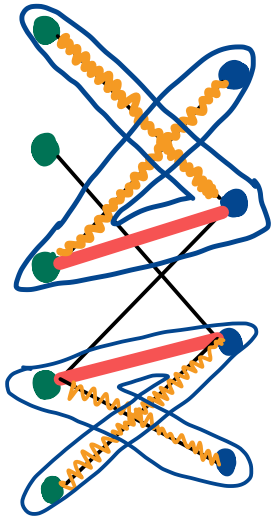

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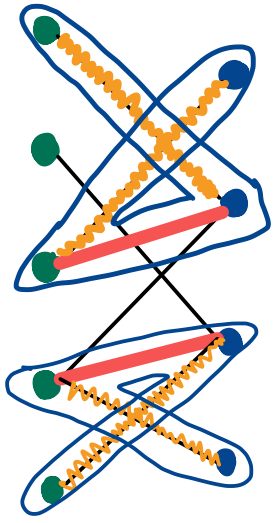

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Fast Algo.

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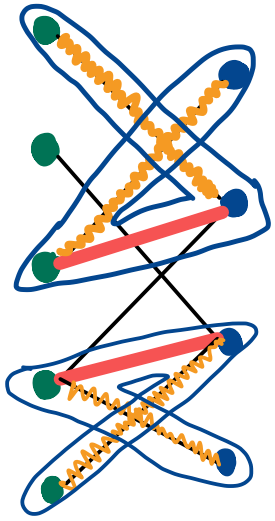
  - Find multiple shortest augmenting paths in one phase [Hopcroft-Karp '73]



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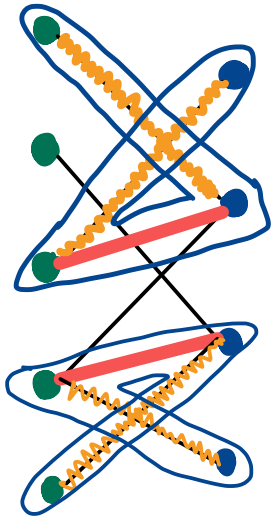

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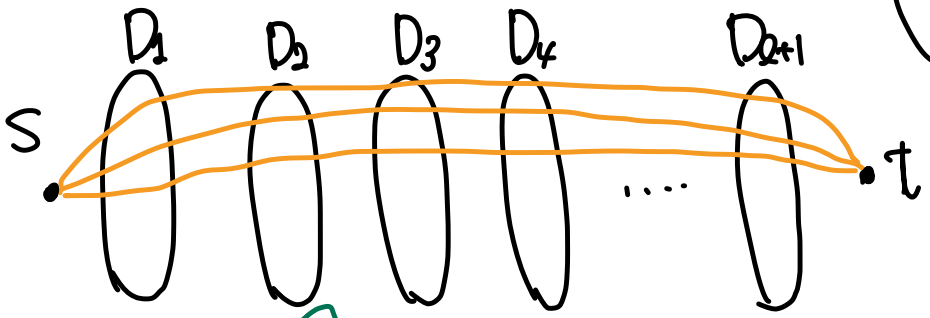

distance layers in the exchange graph

Not vertex-disjoint paths

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$\Pi_\ell := (B_1, A_1, B_2, \cdots, B_{\ell+1})$

① $A_R \subseteq D_{2R}$, $B_R \subseteq D_{2R-1}$

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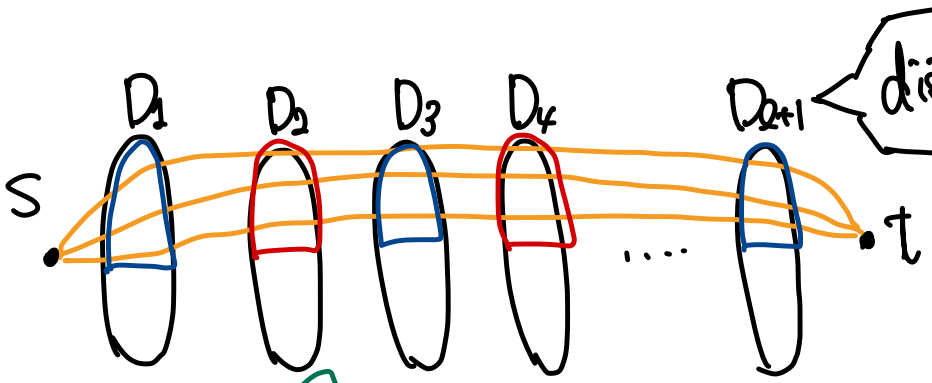

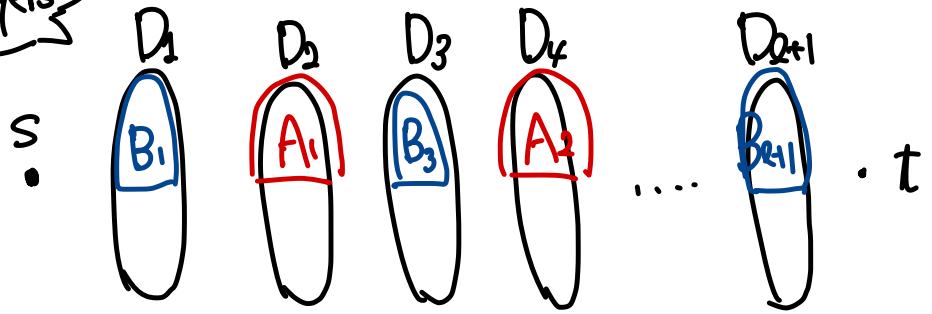

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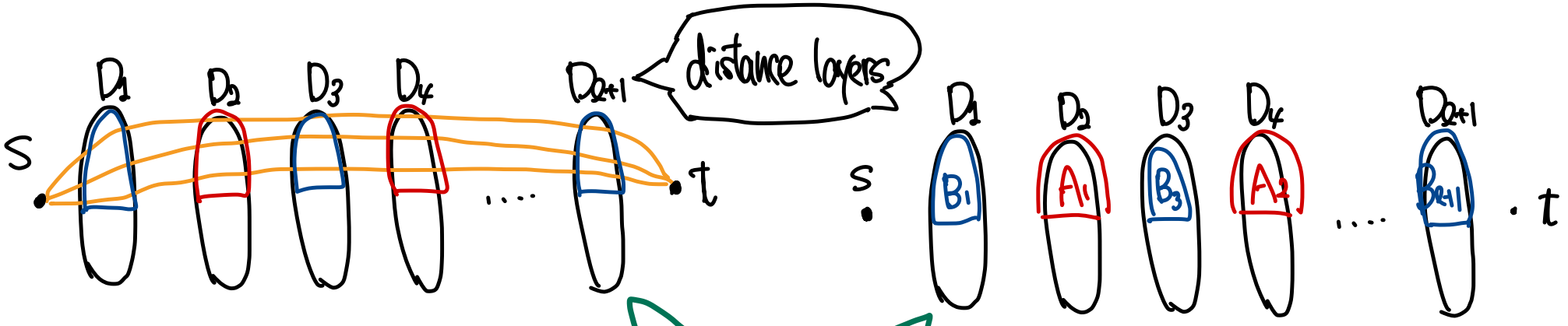

Each set corresponds to another set!

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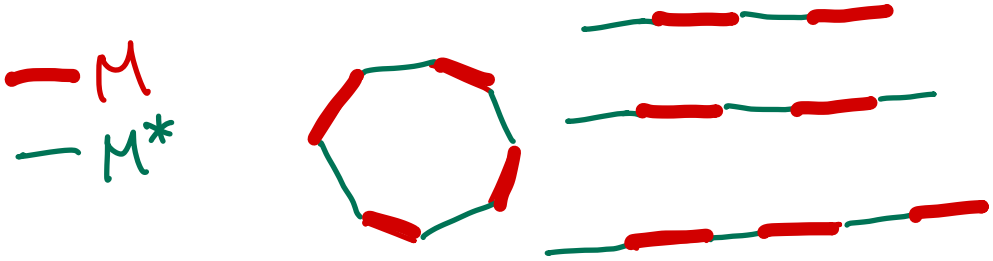

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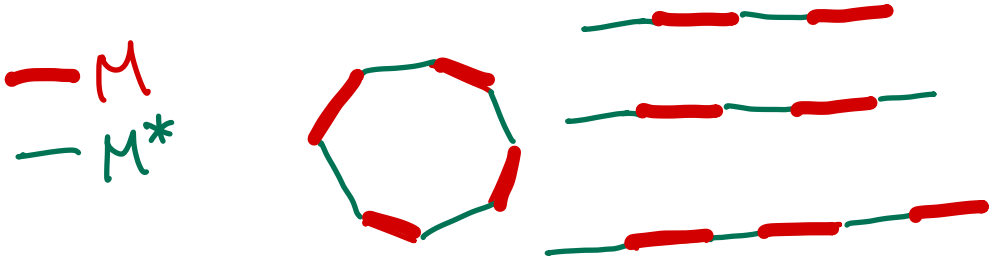

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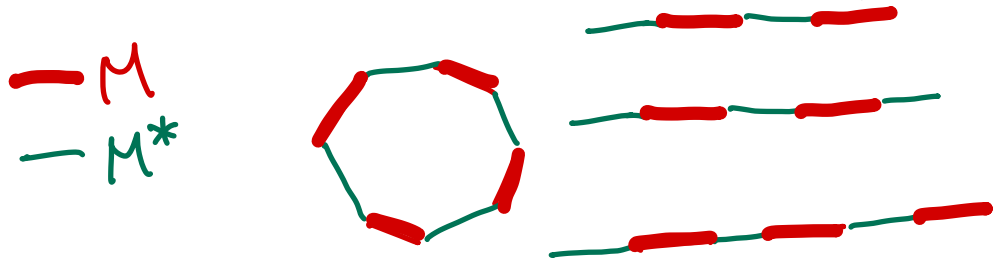

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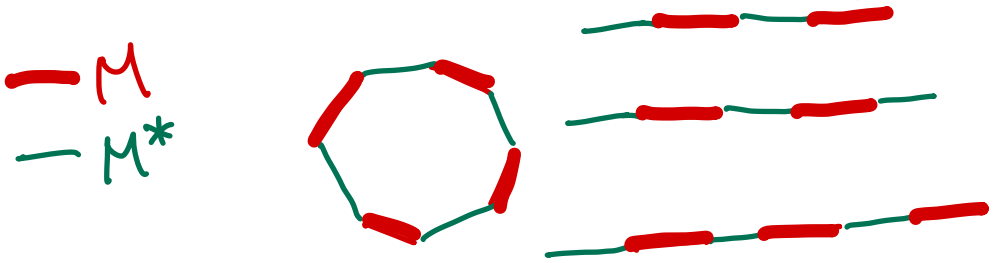

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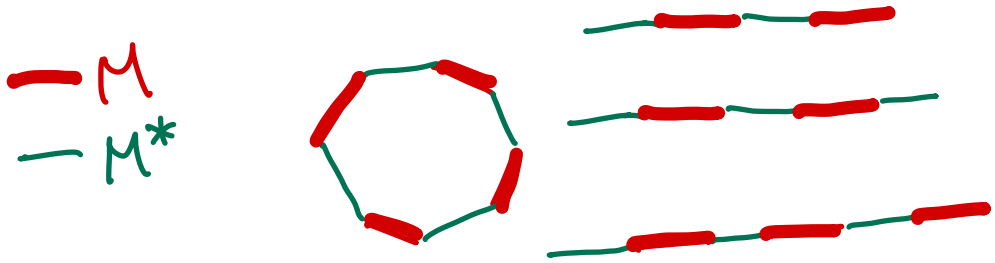

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Unexpectedly Interesting Aspect of Our Result

Our algorithm can be extended to

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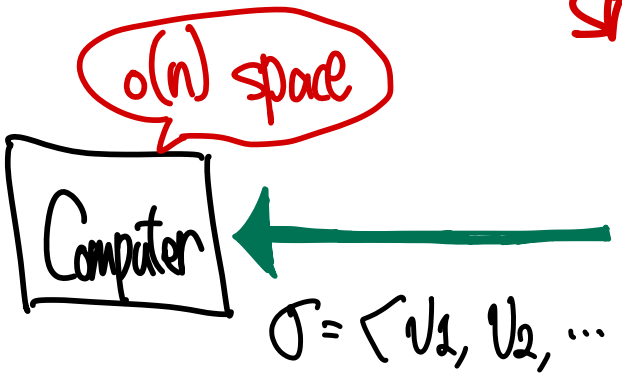

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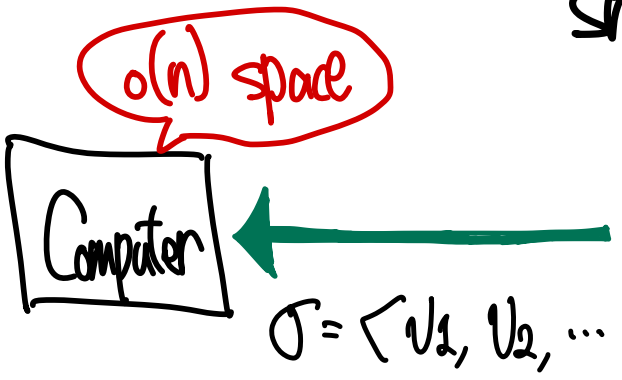

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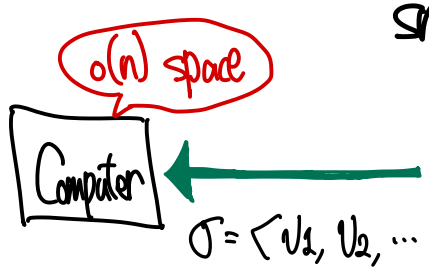

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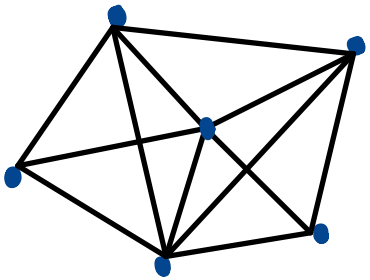

★ In particular, for graph problems, Semi-Streaming Algo. using O(# of vertices) space are well studied!

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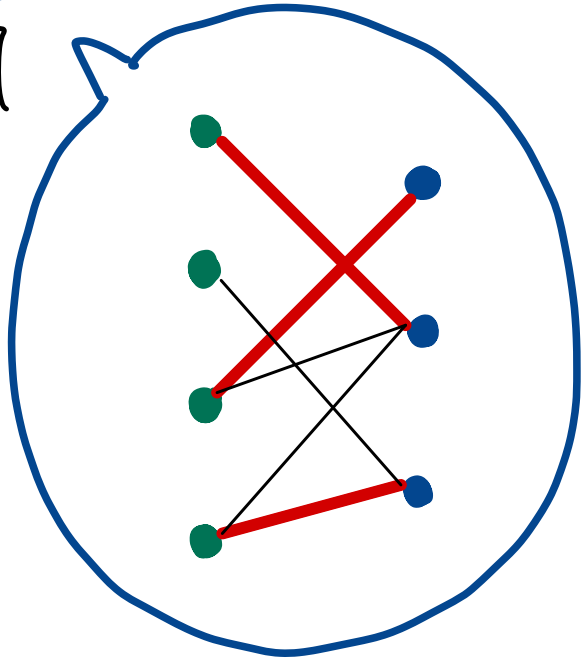
Semi-streaming algo. for matching

Paz-Schwartzman '17

Bernstein '20



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Feigenbaum-Kannan-McGregor-Suri-Zhang '04

First streaming graph algo. paper

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Conclusion

- Design
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open questions

- Deterministic $(1-\varepsilon)$-approx. $\tilde{O}_\varepsilon(n)$ queries Algo.
- $\text{poly}(1/\varepsilon)$ passes $(1-\varepsilon)$-approx. Semi-Streaming Algo.