CALCULUS GilbertStrang Massac huset t sInst it ut eofTechnology WELLES LEY- CAMBRIDGE PRESS Box 812060 WellesleyMA 02482 CALCULUS T h irdEdition GILBERTSTRANG Mass achusettsInstituteofTechnology WELLESLEY-CAMBRIDGEPRESS Box812060WellesleyMA02482 P re fa c e M y g oalistohelpyou learncalculus.Itisabeautifulsubjectan d itscentralideas arenotsohard.Everythingcomesfrom therelationbetweentwodifferentfunctions. Herearetwoimportantexamples: Function .1/ The distance acartravels Function .2/ Its speed Function .1/ The height ofagraphFunction .2/ Its slope Function(2)istellingus howquicklyFunction(1)is changing. Thedistancewill changequicklyorslowlybasedonthespeed. Theheightchanges quicklyorslowly basedontheslope.YouseethesameinclimbingFunction .1/ can betheheightof amountain andFunction .2/ isitssteepness.TheheightandthedistanceinFunction .1/ arerunningtotalsthataddupthechangesthatcomefromFunction .2/ . Theclearestexampleiswhenthespeedis CONSTANT .Thedistanceissteadily increasing.Ifyoutravelat 50 milesperhour,orat 80 kilometersperhour,then after 3 hoursyouknowthedistancetraveled. Icanwritetheanswerbymultiplying 3 times 50 or 3 times 80 .Icanwritetheformula usingalgebra,whichallows any constantspeed s and anytimeoftravel t : Thedistance f atconstantspeed s intraveltime t is f D s times t . Wedon'tneedcalculuswhen thespeed isconstantortheslopeisconstant: If s D slopeand x D distanceacross ; thedistanceupis y D s times x . Thoserules ndFunction .1/ from Function .2/: Wecanalso ndFunction .2/ from Function .1/: Toknowthespeedortheslope,divideinsteadofmultiplying: speed s D di s tance f traveltime t slope s D di s tanceup distanceacross Fromthedistance,we ndthespeed.Thisis Di f ferentialCalculus. Knowingthe speed s ,we ndthedistance f: Thisis IntegralCalculus. Algebraisenoughforthisexampleofconstantspeed.Butwhen s iscontinually changing,andwespeeduporslowdown,thenmultiplicationanddivisionarenot enough! A new ideaisneededandthatideaistheheartofcalculus. vi Pre fa ce DifferentialCalculus ndsFunction .2/ fr omFunction .1/ .Werecoverthe speedometerinformationfromknowingthetripdistanceatalltimes.

Integral Calculusgoestheotherway.Theintegraladdsup smallpieces,toget thetotaldistancetraveled.ThatintegrationbringsbackFunction .1/ . Function . 1 / i s f .t/ or y.x/ (2)Its d erivative s i s d f=dt or dy=dx ThederivativeinFunction .2 / i s the rateofchange ofFunction .1/ .Thebookwill explain themeaningofthesesymbols df=dt and dy=dx forthederivative. CHANGINGSPEEDANDCHANGINGSLOPE Letmetakea rststepintotherealproblem ofcalculus,when s isn o tconstant. NowFunction .1/ willnothaveastraightlinegraph.Thespeedandtheslopeofthe graphwillchange,butonlyeveryhour.Fromthenumbersyou canseethepattern: Distances 0 1 4 9 16SubtractthedistancestogetFunction (2) Speeds 1 3 57Addup thespeedstogetFunction (1) Goingfrom .1/ to .2/ wearesubtracting,asin 4 1 D 3 a n d 9 4 D 5 . T h o se d ifferences 3 a n d 5 a r e thespeedsinthesecondhourandthirdhour. G o in g fro m .2 / to .1 / w e a r e a dding,as in 1 C 3 C 5 D 9 : T h e tripmeteraddsupthe d istancesfromhours 1 a n d 2 a n d 3 . A d d ition istheoppositeofsubtraction. T h e e ssentialpointofcalculusistoseethissamepatternincontinuoustime. I t's n o t e noughtolookatthetotalorthechangeeveryhour oreveryminute. T h e d istanceandspeed canbe c ha ng ing a t eve r yinstant . I n thatcaseaddition a n dsubtractionarenot enough.Thecentral ideaofcalculusis c o ntinuo uschange . T h e rearesomanypairslike .1 / a n d .2 / n o tjustcarsandgraphsandmountains. T h isiswhat makescalculusimportant.Thefunctionsarechangin gcontinuouslynot

.2/ Conte nts

 TotheS tude nt I h o peyouwilllearncalculusfromthisbook.On thispageIwilla d mit thatIeven hopefor more.Ifyou ndthattheexplanationsareclear,andalsothe purpose is clearthatmeans youseenotonlyequationsbutideas.Thenthebookwasworth writing,andthecourseisasuccess. Iamtryingtosaythatthis subjectis alive.As longasthereareproblemstosolve, mathematicswillkeepgrowing.Itisnotwrappedupinsidesomegiantformula!

spirit,withaseriouspurposeIhopeyou enjoyit. ASMALLREWARD Somemistakesmay havecreptinto thesolutions.Istillhavethe t endimesthat GeorgeThomasofferedin 1952; forcorrectingtenerrors. Thisrewardishereby increasedto$ e .Itshouldbe e t butthatcouldgrow exponentially. Moreimportant : Allsuggestionsarewelcome .Pleasewriteaboutanypartofthebookorthevideos. Derivatives Su m : d dx .u C v / D d u dx C d v dx d dx s i n x D c os x d dx e c x D c e c x Product: d dx .u v / D u d v dx C v d u dx d dx c o s x D sin x d dx b x D b x l n b Q u otient: d dx u v D v d u= dx udv=dx v 2 d dx t a n x D s ec 2 x d dx l n x D 1 x P o w er: d dx .u n / D n u n 1 du dx d dx c o t x D csc 2 x d dx s i n 1 x D 1 ? 1 x 2 Ch ain: d dx z .y .x // D dz dy d y dx d dx s e c x D s ec x tan x d dx t a n 1 x D 1 1 C x 2 I n v erse: dx dy D 1 dy=dx d dx c s c x D csc x c o t x d dx s e c 1 x D 1 | x | ? x 2 1 Li mi tsandContinuity s in x x 1 1 cos x x 0 1 cos x x 2 1 2 a n 0 W | a n | " f o r a ll n N a n L W | a n L | " f o r a ll n N f .x / L W | f.x/ L | " f o r 0 | x a | f .x / f.a/ :Continuousat a if L D f .a / f. x/

f.a/ x a f 1 .a/ :Derivativeat a f. x/ f.a/ x a D f 1 .c / :MeanValueTheorem f. x C x/ f.x/ x f 1 .x/ :Derivativeat x f. x C x/ f.x x/ 2x f 1 .x/ :Centered lim f. x/ g.x/ D li m f 1 .x/ g 1 .x/ l' H p ital'sRulefor 0 0 Max i mumandMinimum Critical: f 1 .x/ D 0 o r n o f 1 orendpoint M inimum f 1 .x/ D 0 a n d f 2 .x/ 0 M a ximum f 1 .x/ D 0 a n d f 2 .x/ 0 I n e c tion point f 2 .x/ D 0 N ewton'sMethod x n C 1 D x n f.x n / f 1 .x n / I te ration x n C 1 D F .x n / a ttractedto xed point x D F .x / if | F 1 .x / | 1 Sta tionary in2D: B f= B x D 0 ; B f= B y D 0 M inimum f xx 0f xx f yy f 2 x y M a ximum f xx 0f xx f yy f 2 x y Sa d d lepoint f xx f yy f 2 x y N ewton in2D # g C g x x C g y y D 0 h C h x x C h y y D 0 A l gebra x=a y=b D bx ay x n D 1 x n n ? x D x 1= n .x 2 /.x 3 / D x 5 .x 2 / 3 D x 6 x 2 = x 3 D x 1 a x 2 C b x C c D 0 h a sroots x D b ? b 2 4ac 2a x 2 C 2B x C C D 0 h asroots x D B ? B 2 C Co mpleting square a x 2 C b x C c D a x C b 2a 2 C c b 2 4a Pa r tialfractions cx C d .x a/.x b/ D A x a C B x b M istakes a b C c a b C a c ? x 2 C a 2 x C a Fu n damentalTheorem of Calculus d dx r x a v.t /d t D v .x/ r b a df dx dx D f .b / f.a/ d dx r b. x / a.x/ v .t/dt D v .b.x// db dx v .a.x// da dx r b 0 y.x /d x D lim x 0 x y . x / C y .2 x / C C y .b / C i rcle,Line,andPlane x D r cos !t , y D r s i n !t ,speed !r y D mx C b or y y 0 D m .x x 0 / Pla n e a x C b y C c z D d o r a .x x 0 / C b .y y 0 / C c .z z 0 / D 0 N o r malvector a i C b j C c k D ista nceto .0 ; 0 ; 0/ : | d | = ? a 2 C b 2 C c 2 Li n e .x;y;z/ D .x 0 ;y 0 ;z 0 / C t.v 1 ;v 2 ;v 3 / Noparameter: x x 0 v 1 D y y 0 v 2 D z z 0 v 3 Pr o jection: p D b a a a a ; | p | D | b | cos Trigonometric Identities si n 2 x C c os 2 x D 1 ta n 2 x C 1 D s ec 2 x (dividebycos 2 x) 1 C cot 2 x D csc 2 x (dividebysin 2 x) sin 2x D 2 sin x cos x (doubleangle) cos 2x D cos 2 x sin 2 x D 2 c o s 2 x 1 D 1 2 sin 2 x sin .s t/ D sin s c o s t cos s sin t (A d d ition c o s .s t/ D c o s s c o s t sin s sin t f o r mulas) ta n .s C t / D . ta n s C ta n t /= .1 tan s ta n t / c 2 D a 2 C b 2 2ab c o s (L aw o fcosines) a = sin A D b = sin B D c = sin C (L aw o fsines) a c o s C b sin D ? a 2 C b 2 cos . tan 1 b a / co s . x/ D c o s x a n d sin . x/ D sin x sin . 2 x/ D c o s x a n dcos . 2 x/ D sin x sin . x/ D sin x a n dcos . x/ D cos x Tri g o no metri c Integrals r sin 2 x d x D x sin x c os x 2 D r 1 cos 2x 2 dx D x 2 sin 2x 4 r co s 2 x dx D x C sin x cos x 2 D r 1 C co s 2 x 2 dx D x 2 C si n 2 x 4 r ta n 2 x dx D tan x x r c o t 2 x d x D cot x x r sin n x d x D sin n 1 x c os x n C n 1 n r si n n 2 x d x r c o s n x d x D C c os n 1 x s in x n C n 1 n r co s n 2 xdx r tan n xdx D tan n 1 x n 1 r ta n n 2 x d x r se c n x d x D s e c n 2 x ta n x n 1 C n 2 n 1 r se c n 2 x d x r ta n x d x D ln | cos x | r c o t x d x D ln | sin x | r se c x d x D ln | sec x C ta n x | r c sc x d x D ln | csc x cot x | D ln | csc x C c o t x | r se c 3 x d x D 1 2 se c x t an x C 1 2 ln | sec x C ta n x | r sin p x sin q x d x D s in . p q/x 2.p q/ sin . p C q/ x 2.p C q/ r c o s px cos qxdx D sin .p q/x 2.p q/ C sin .p C q/x 2.p C q/ r s i n px cos qxdx D cos . p q/x 2.p q/ cos . p C q/ 2.p C q/ I n tegrationbyParts r l n x d x D x l n x x r x n ln x d x D x n C 1 ln x n C 1 x n C 1 .n C 1 / 2 r x n e x d x D x n e x n r x n 1 e x d x r e c x sin k x d x D e cx c 2 C k 2 .c s i n k x k c o s k x / r e cx c o s k x d x D e cx c 2 C k 2 .c c o s k x C k sin k x/ r x sin xdx D sin x x c o s x r x c o s x d x D c o s x C x sin x r x n sin x d x D x n c o s x C n r x n 1 c o s x d x r x n c o s x d x D C x n sin x n r x n 1 sin x d x r sin 1 x d x D x sin 1 x C ? 1 x 2 r ta n 1 x d x D x ta n 1 x 1 2 ln .1 C x 2 / I n tegralswith x 2 and a 2 an d D D b 2 4ac r dx x 2 C a 2 D 1 a ta n 1 x a r dx a 2 x 2 D 1 2a ln | x C a x a | D 1 a ta n h 1x a r dx ? x 2 C a 2 D ln | x C ? x 2 C a 2 | r dx ? a 2 x 2 D sin 1x a r ? x 2 C a 2 dx D x 2 ? x 2 C a 2 C a 2 2 ln | x C ? x 2 C a 2 | r ? a 2 x 2 dx D x 2 ? a 2 x 2 C a 2 2 si n 1x a r dx x ? x 2 a 2 D 1 a co s 1 a x r dx x ? x 2 C a 2 D 1 a ln | ? x 2 C a 2 a x | r dx ax 2 C bx C c D 1 ? D ln | 2ax C b ? D 2ax C b C ? D | ;D 0 D 2 ? D ta n 1 2 ax C b ? D ; D 0 D 2 2ax C b ;D D 0 r d x ? ax 2 C bx C c D 1 ? a ln | 2ax C b C 2 ? a ? ax 2 C bx C c | D 1 ? a sin 1 2ax b ? D ;a 0 D e niteIntegrals r 8 0 x n e x dx D n D .n C 1/ r 8 0 e a 2 x 2 d x D ? =2a r 1 0 x m .1 x/ n d x D mn .m C n C 1/ r 8 0 s in 2 x x 2 dx D r 8 0 s in x x dx D 2 r = 2 0 s in n xdx D r =2 0 cos n xdx D 1 2 3 4 n 1 n . 2 / n ev e n or 2 3 4 5 n 1 n n od d 1 CHAPTER0 Hi ghl i ghts ofCa l c ul us 0.1 Dist ance andSpeed == H e ightandSlope Calculusisaboutfunctions .I u sethatwordfunctionsinthe rstsentence, because wecan'tgoforwardwithoutit.Likeallotherwords,welearnthis onein twodifferentways:We de ne theword andwe use theword. Ibelievethatseeingexamplesoffunctions,andusingthewordtoexplainthose examples,isafastandpowerfulwaytolearn.Iwillstartwiththreeexamples: Linearfunction y.x/ D 2x Squaringfunction y.x/ D x 2 Exponentialfunction y.x/ D 2 x The rstpointisthatthosearenotthesame!Theirformulasinvolve 2 and x invery differentways.When

Idrawtheirgraphs(thisisagoodway to understandfunctions)

series,Iam asking youto believethateverythingworks. Wecanadd theseries toget e x ; and wecanaddallderivativestoseethattheslopeof e x is e x : Forme,theadvantageofusing onlythepowers x n isoverwhelming. 16 0H i ghlightsofCalculus CONSTRUCTING y D e x Iwillsolve dy=d x D y astepatatime.Atthestart, y D 1 meansthat dy=dx D 1 : Start y D 1 dy=dx D 1 Change y y D 1 C x dy=dx D 1 Change dy dx y D 1 C x d y = dx D 1 C x Afterthe rstchange, y D 1 C x hasthecorrectderivative dy=dx D 1: ButthenI had tochange dy=dx tokeepitequalto y: AndIcan'tstopthere: y D 1 C x dy=dx D 1 C x Update y to 1 C x C 1 2 x 2 T h e nupdate dy dx t o 1 C x C 1 2 x 2 T h e extra 1 2 x 2 ga v ethecorrect x intheslope.Then 1 2 x 2 al s ohadtogointo dy=dx , tokeepitequalto y: Nowweneed anewtermwiththisderivative 1 2 x 2 : Th e termthatgives 1 2 x 2 ha s x 3 d ividedby 6: Thederivativeof x n is nx n 1 , so I m u st d ivideby n (to c a ncelcorrectly).Th enthederivativeof x 3 = 6 is 3 x 2 = 6 D 1 2 x 2 as w ewanted.Afterthatcomes x 4 divided by24: x 3 6 D x 3 .3/.2/.1/ ha s slope x 2 .2/.1/ x 4 24 D x 4 .4/.3/.2/.1/ ha s slope 4x 3 .4/.3/.2/.1/ D x 3 6 : Th e patternbecomesmoreclear.The x n termisdividedby n factorial ,which is n D .n/.n 1/:::.1/: T h e rst vefactorialsare 1 ; 2 ; 6 ; 24;120 : Thederivative o f t hat term x n = n is t he previous term x n 1 =.n 1/ (b e c ausethe n 's c a ncel). A s lo n g aswedon'tstop,thissumofin nitely manytermsdoesachieve dy = d x D y : y.x/ D e x D 1 C x C 1 2 x 2 C 1 6 x 3 C C 1 n x n C (1) Ifwe substitute x D 10 i n tothisseries,dothe in nitelymanytermsaddtoa nite number e 10 ? Yes .Thenumbers n growmuchfasterthan10 n (oranyother x n). Sotheterms x n =n in thisexponentialseriesbecomeextremelysmallas n 8 : A n a lysisshowsthatthesumoftheseries(whichis y D e x) d o esachieve dy = d x D y : N o t e 1 L e tmejustre memberaseriesthatyou know, 1 C 1 2 C 1 4 C 1 8 C D 2 : I f I r eplace 1 2 by x , t hisbecomesthe geometricseries 1 C x C x 2 C x 3 C and it a ddsup to 1 = .1 x/: T h isisthemostimportantseriesin mathematics,butit r u n s in toapr o b lemat x D 1 :thein nitesum 1 C 1 C 1 C 1 C doesn'tconverge. I e mphasizethattheseriesfor e x is alwayssafe, because thepowers x n a r e dividedb y therapidlygrowingnumbers n D n fa ctorial . T h isisagreat ex a mpletomeet,longbeforeyoulearnmoreaboutconvergenceanddivergence. N o t e 2 H e r e is a n otherwaytolookatthatseriesfor e x : Sta r t w ith x n a n dta ke itsde rivative n times.Firstget nx n 1 a n d then n.n 1/x n 2 . Fin allythe n th d e rivativeis n.n 1/.n 2/:::.1/x 0 ; w h ic his n fa ctorial . Wh e n wed ividebythat n u mber, t he n t h derivativeof x n =n is equalto 1 : N ow lo okat e x : A ll itsderivativesarestill e x : T h ey a llequal1at x D 0 : Th e se riesismatching everyderivativeofthefunction e x a t thestartingpoint x D 0 : 0.3The Exponential y D e x 17 S e t x D 1 intheexponentialseries .Th istellsustheamazing number e 1 D e : Thenumber e e D 1 C 1 C 1 2 C 1 6 C 1 24 C 1 120 C (2) The rstthreetermsaddto 2.5.The rst vetermsalmostreach2.71. W e n everreach 2.72.Withquiteafewterms (how many?)youcanpass 2.71828.Itis certainthat e isnotafraction.Itneverappearsinalgebra,butitisthekeynumberforcalculus.

MULTIPLYINGBYADDINGEXPONENTS We write e 2 inth esame way thatwewrite 3 2 : Isittrue that e times e equals e 2 ? Uptonow, e and e 2 comefromsetting x D 1 and x D 2 inthein niteseries. Thewonderfulfactisthatforevery x ,theseriesproducesthe x thpowerofthe number e: When x D 1,weget e 1 w h ic his1 =e : Se t x D 1 e 1 D 1 e D 1 1 C 1 2 1 6 C 1 24 1 120 C Ifwemultiply thatseriesfor 1 =e b y th eseriesfor e , w e g e t 1 : T h e b e stway istogostraightforallmultiplicationsof e x timesanypower e X : T h e r u le o f addingexponentssaysthattheansweris e x C X : T h e se ries mustsaythis too! Wh e n x D 1 a n d X D 1 , th isruleproduces e 0 f r o m e 1 times e 1 : Addtheexponents .e x /.e X / D e x C X (3) Weonlyknow e x an d e X f romthein niteseries.Forthisall-importantrule,wecan multiplythoseseriesandrecognizetheanswerastheseriesfor e x C X : Makeastart: Multiplyeachterm e x times e X Hopingfor e x C X e x D 1 C x C 1 2 x 2 C 1 6 x 3 C e X D 1 C X C 1 2 X 2 C 1 6 X 3 C .e x /.e X / D 1 C x C X C 1 2 x 2 C xX C 1 2 X 2 C (4) Ce rtainly yousee x C X : D o y o u se e 1 2 .x C X/ 2 i n equation .4/ ?Noproblem: 1 2 .x C X/ 2 D 1 2 .x 2 C 2x X C X 2 / m atchesthesecond degreeterms. Thesteptothirddegreetakesalittlelonger,butitalsosucceeds: 1 6 .x C X/ 3 D 1 6 x 3 C 3 6 x 2 X C 3 6 xX 2 C 1 6 X 3 ma t chesthenexttermsin(4). Forhighpowersof x C X weneedthe binomialtheorem (orahealthytrustthat mathematicscomesoutright).When e x multiplies e

X ,thecoef cientof x n X m willbe1 =n times1 =m: Nowlook forthatsameterm intheseriesfor e x C X : .x C X/ n C m .n C m/ i n cludes x n X m .n C m/ t i mes .n C m/ n m wh i chgives x n X m n m : (5) 18 0H i ghlightsofCalculus Thatbinomialnumber .n C m/= nm isknown tosuccessfulgamblers.Itcountsthe numberofwaystochoose n acesoutof n C m aces.Outof4aces,youcouldchoose 2acesin 4=22 D 6 ways.Toamathematician,thereare6waystochoose 2x 's outof xxxx .Thisnumber6willbethecoef cientof x 2 X 2 in .x C X/ 4 : That 6 showsupinthefourthdegree term.Itis dividedby4!(toproduce 1=4). When e x multiplies e X , 1 2 x 2 m u l tiplies 1 2 X 2 (w h ichalso produces 1=4).Allterms aregood,butwearenot goingthereweaccept .e x /.e X / D e x C X asnowcon rmed. Note Adifferentwaytoseethisrulefor .e x /.e X / isb asedon dy=dx D y: Starting from y D 1 at x D 0 ,follow thisequation.Atthepoint x ,youreach y D e x : Nowgo anadditionaldistance X toarriveat e x C X : Noticethattheadditionalpartstartsfrom e x (insteadofstartingfrom1).That startingvalue e x willmultiply e X intheadditionalpart.So e x times e X mustbethe sameas e x C X : (Thisisadifferentialequationsproofthattheexponentsareadded. Personally,Iwashappy tomultiply theseriesandmatch theterms.) Theruleimmediatelygives e x times e x : Theansweris e x C x D e 2x : Ifwemultiply againby e x , we nd .e x / 3 : This isequalto e 2x C x D e 3x : Weare ndinganewrule forallpowers .e x / n D .e x /.e x / .e x / : Multiplyexponents .e x / n D e nx (6) Thisiseasy to seefor n D 1; 2 ;3;::: andthen n D 1, 2, 3,... I t remainstruefor a llnumbers x a n d n . T h a tlastsentenceaboutallnumbersis important!Calculus cannotdevelop p r o p e rlywithoutworkingwith allexponents(not justwholenumbersorfractions). T h e in niteseries(1)de nes e x f o r eve r y x a n dweareonourway.Hereisthe g r a p h: F unction . 1 / D F unction . 2 / D e x D e x p .x /: -2 -1 0 1 2 l n 2 x e 1 D :368 : :: e D 2 :718 : :: e ln 2 D 2 e 2 D 7:388 : :: y D e x dy dx D e x e 0 D 1 .e x /.e X / D e x C X . e x / n D e n x e ln y D y 0.3The Exponential y D e x 19 T H EE XPONENTIALS 2 x AND b x Weknowthat 2 3 D 8 and 2 4 D 16 . But whatisthemeaningof 2 ?Onewaytoget closetothatnumberistoreplace by 3:14 whichis 314=100: Aslongaswehavea fraction intheexponent,wecanlivewithoutcalculus: Fractionalpower 2 314=100 D 314 th powerofthe 100 th root 2 1=100 : Butthisisonlycloseto 2 : Andincalculus,wewillwanttheslopeofthecurve y D 2 x : Thegoodwayistoconnect 2 x with e x ; whoseslopeweknow(itis e x again). Soweneed toconnect 2 with e: Thekey numberisthe logarithmof2 .Thisiswritten ln 2 anditisthepower of e thatproduces 2: Itisspeciallymarkedonthegraphof e x : Naturallogarithm of2 e ln 2 D 2 Thisnumber ln 2 isabout 7=10 . Acalculatorknowsitwith muchhigheraccuracy. Inthegraph of y D e x ,thenumberln 2 onthe x axisproduces y D 2 on the y axis. Thisisanexamplewherewewanttheoutput y D 2 andweaskfortheinput x D ln 2: Thatistheoppositeofknowing x andaskingfor y .Thelogarithm x D ln y isthe inverse oftheexponential y D e x : Thisideawillbeexplainedin Section 4:3 andintwovideolecturesinversefunctionsarenotalwayssimple. Now 2 x hasameaningforevery x: Whenwehavethenumber ln 2; meetingthe requirement 2 D e ln 2 ; wecantakethe x thpowerofboth sides: Powersof2from powersof e 2 D e ln 2 and 2 x D e x ln 2 : (7) Allpowersof e arede nedbythein niteseries.Thenewfunction 2 x alsogrows exponentially,butnotas fastas e x (because 2 is smaller than e).Probably y D 2 x couldhavethesamegraphas e x ,ifI stretchedoutthe x axis. Thatstretching multipliestheslopebytheconstantfactorln 2: Hereisthealgebra: Slopeof y D 2 x d dx 2 x D d dx e x ln 2 D . l n 2/e x ln 2 D . ln 2/2 x : Foranypositivenumber b ,thesameapproachleadstothefunction y D b x . First, ndthenaturallogarithmln b .Thisisthenumber(positiveornegative) so that b D e ln b .Thentakethe x thpowerofboth sides: Connect b to e b D e ln b a nd b x D e x ln b a n d d dx b x D . l n b /b x (8) When b is e (t heperfectchoice),ln b D ln e D 1: When b is e n ,thenln b D ln e n D n: The logarithmis theexponent . Thankstotheseriesthatde nes e x forevery x , thatexponentcanbeany numberatall. Allowmetomention Euler'sGreatFormula e ix D cos x C i sin x .Theexponent ix hasbecomean imaginarynumber .(Youknowthat i 2 D 1:) I fwefaithfully u se c os x C i sin x a t 9 0 and 1 8 0 (where x D = 2 a n d x D) , w e a r r ive atthese a mazing facts: Imaginary exponents e i = 2 D i and e i D 1 : (9) Thoseequationsarenotimaginary,theycomefromthegreatseriesfor e x : 20 0H i ghlightsofCalculus CONTINUOUSCOMPOUNDINGOFINTEREST Thereisadifferentandimportantwayto reach e and e x (n ot byanin niteseries). Wesolvethekeyequation dy=dx D y insmallsteps.Asthesestepsapproachzero (alimitisalwaysinvolved!)thesmall-stepsolution becomestheexact y D e x : Icanexplain thisideaintwodifferentlanguages.Each stepmultipliesby 1 C x : 1. Compound interest .Aftereachstep x ,theinterestis addedto y: Thenthe nextstepbeginswithalargeramount,and y increasesexponentially. 2. Finitedifferences .Thecontinuous dy=dx isreplaced bysmallsteps Y= x : dy dx D y c h a ngesto Y.x C x/ Y.x/ x D Y .x / w ith Y.0/ D 1: (10) ThisisEuler'smethodofapproximation. Y.x/ approaches y.x/ as x 0. L e tmecomputecompoundinterestwhen1 yearisdividedinto 12 months,and th en 365 days.Theinterestrateis100%and you startwith Y .0 / D $ 1 : I f y o u only g e tinterestonce,attheendoftheyear,thenyouhave Y .1 / D $ 2 : I f in terestisadded every month,younowget 1 12 of 1 00%eachtime(12times). So Y ismultipliedeachmonthby1 C 1 12 : (T h ebankadds 1 12 fo r every1youhave.) Dothis12 timesandthe nalvalue$2isimprovedto$2.61: After12months Y.1/ D 1 C 1 12 12 D $ 2 .61 Nowadd interestevery day. Y.0/ D $ 1 ismultiplied 365 timesby1 C 1 365 : Af t er365 days Y.1/ D 1 C 1 365 36 5 D $ 2.71(closeto e) Veryfewbanks useminutes,andnobodydivides theyear into N D 31;536;000 seconds.Itwouldaddlessthanapennyto$2.71.Butmanybanksarewillingto use continuouscompounding , thelimitas N 8 : A f te r o n e y earyouhave$ e : Anotherlimitgives e 1 C 1 N N e D 2 : 7 1 8 : : : a s N 8 (11) Youcouldinvestatthe100%ratefor x ye a rs.Noweachofthe N stepsisfor x=N years. Againthebankmultipliesateverystepby1 C x N : Th e 1keepswhat you have,the x=N addstheinterestinthatstep.After N stepsyou arecloseto e x : Aformulafor e x 1 C x N N e x a s N 8 (12) Finally,Iwillchangethe interestrate to a: G o for x yearsatthe interestrate a: Thedifferentialequationchangesfrom dy=dx D y to dy=dx D ay : Theexponential functionstillsolvesit,butnowthatsolutionis y D e ax : Changetherateto a dy dx D ay i s solved by y.x/ D e ax (13) 0.3The Exponential y D e x 21 Y o u canwritedown theseries e ax D 1 C ax C 1 2 .ax / 2 C andtakeitsderivative: d dx .e a x / D a C a 2 x C D a .1 C a x C / D a e ax (1 4) T h e d e rivativeof e ax b r in gsdowntheextrafactor a : So y D e ax so lves dy = d x D ay : T h eExponential y D e x Lookingforafunction y .x / t hatequalsitsownderivative dy dx Adifferentialequation!Westartat x D 0 w i t h y D 1 In niteSeries y .x / D 1 C x C x 2 2 C x 3 3 C C x n n C Takederivative dy dx

D 0 C 1 C x C x 2 2 C C x n 1 .n 1/ C Termbyterm dy dx ag r eeswith y Limitstep D add upthisseries n D .n/.n 1/ .1/ g r ow s muchfasterthan x n so thetermsgetvery small At x D 1 thenumber y .1 / iscalled e: Set x D 1 intheseriesto nd e e D 1 C 1 C 1 2 C 1 6 C 1 24 C D 2 : 7 1 8 28::: GOAL Showthat y .x / agreeswith e x forall x Seriesgivespowersof e Check thattheseriesfollowstheruletoaddexponentsasin e 2 e 3 D e 5 Directly multiplyseries e x ti m es e X toget e x C X 1 C x C 1 2 x 2 ti m es 1 C X C 1 2 X 2 pr o ducestherightstartfor e x C X 1 C .x C X/ C 1 2 .x C X/ 2 C HIGHERTERMSALSOWORK Theseriesgivesus e x fo r EVERY x ,notjustwholenumbers CHECK de x dx D li m e x C x e x x D e x li m e x 1 x D e x YE S ! SECONDKEY RULE .e x / n D e nx f o revery x and n Anotherapproachto e x us e smultiplication insteadofanin nitesum Startwith$ 1 .E a rninterestevery day atyearly rate x Multiply 365 t i mesby 1 C x 365 : En d theyearwith$ 1 C x 365 36 5 Nowpay n timesintheyear.Endtheyearwith 1 C x n n $ e x a s n 8 Wearesolving Y x D Y in n s h ortsteps x: Thelimitsolves dy dx D y: 22 0H i ghlightsofCalculus PracticeQuestions 1.Whatisthederivativeof x 1 0 10 ? W h atisthederivativeof x 9 9 ? 2 . H owtoseethat x n n g e t ssmallas n 8 ? Startwith x 1 a n d x 2 2 , p o ssiblybig. Butwemultiplyby x 3 ; x 4 ; whichgetssmall. 3.Whyis 1 e x t h e sameas e x ? U se e q u ation (3)andalsouse(6). 4 . Wh y is e 1 D 1 1 C 1 2 1 6 C between 1 3 a n d 1 2 ? T h en 2 e 3: 5.Canyou solve dy dx D y st a rtingfrom y D 3 at x D 0 ? Why is y D 3e x t h erightanswer?Noticehow 3; multiplies e x : 6.Canyou solve dy dx D 5y s t artingfrom y D 1 at x D 0 ? Why is y D e 5x t h erightanswer?Notice 5 intheexponent! 7.Whydoes e x 1 x ap p roach 1 as x getssmaller?Usethe e x series. 8.Drawthegraphof x D ln y , j ustby ippingthegraphof y D e x acrossthe 45 line y D x . Rememberthat y sta yspositivebut x D ln y c a nbenegative. 9 . Wh atistheexactsum of 1 C ln 2 C 1 2 . l n 2 / 2 C 1 3 . l n 2 / 3 C ? 1 0 . I f y o u r e p la celn 2 b y 0 :7; w h a t isthesum ofthosefourterms? 1 1 . Fr o m E u le r'sGreatFormula e ix D c o s x C i sin x ; w h a t n umberis e 2 i ? 1 2 . H owcloseis 1 C 1 10 10 t o e ? 1 3.Whatisthelimitof 1 C 1 N 2N a s N 8 ? 0.4VideoSummariesandPracticeProblems 23 0.4VideoSum m ar iesandPr act iceProblem s This sectionistohelpreaderswhoalsolookatthe H i g hlights of Calculus video lectures.

The rst vevideosarejustreleasedon ocw.mit.edu asIwritethese words.Sections0.10.20.3discussedthecontentofthreelecturesinfulldetail.The summariesandpracticeproblemsfortheothertwowillcome rstinthissection: 4 .MaximumandMinimumandSecond Derivative 5 .BigPictureofIntegrals ThatLecture 5is atasteof IntegralCalculus .Asecondsetofvideolecturesgoes deeperinto DifferentialCalculus therulesforcomputingandusingderivatives. Thissecond setisrightnowwiththevideoeditors,tozoom inwhenIwriteonthe blackboardandzoomoutforthebigpicture.Ijustborrowed avideocamera from MIT'sOpenCourseWareandsetitupinanemptyroom.Iamnotgoodatlookingat theaudienceanyway,soitwaseasierwithnobodywatching!

Ihopeitwillbehelpfultoprintherethesummariesandpracticeproblemsthatare plannedtoaccompanythosevideos.Herearethetopics: 6 .DerivativeoftheSineandCosine 7 .ProductandQuotientRules 8 .ChainRulefortheSlopeof f.g.x// 9 .InverseFunctionsandLogarithms 10 .Growth Ratesand LogGraphs 11 .LinearApproximationand Newton'sMethod 12 .DifferentialEquationsofGrowth 13 .DifferentialEquationsofMotion 14 .PowerSeriesand Euler'sFormula 15 .SixFunctions,SixRules,SixTheorems Thatlastlecturesummarizesthetheoryof differentialcalculus.Theotherlectures explain thesteps.Herearethe rstlineswrittenforthemax-minvideo.

MaximumandMinimumandSecondDerivative To ndthemaximumand minimumvaluesofafunction y.x / Solve dy dx D 0 to ndpoints x where slo pe D z e ro Testeach x forapossibleminimumormaximum Example y.x/ D x 3 12x dy dx D 3x 2 12 So lve 3 x 2 D 1 2 Theslopeis dy dx D 0 at x D 2 a n d x D 2 Atthosepoints y.2 / D 8 24 D 16 D min a n d y . 2/ D 8 C 2 4 D 1 6 D ma x 24 0H i ghlightsofCalculus x D 2 is aminimum Look at d dx dy dx D se c ondderivative d 2 y dx 2 D d e r ivativeof 3x 2 12: T h issecondderivativeis 6 x : d 2 y dx 2 0 dy dx i n c reasesSlopegoesfromdown toupat x D 2 Thebending isupwardsandthis x isa minimum d 2 y dx 2 0 dy dx d e c reasesSlopegoesfrom up todown at x D 2 Thebending isdownwardsand x isa ma x imum Findthemaximumof y.x / D s in x C cos x using dy dx D co s x sin x Theslopeiszerowhencos x D si n x a t x D 4 5 d eg r e es D 4 ra d ians Thatpoint x has y D sin 4

C co s 4 D ? 2 2 C ? 2 2 D ? 2 Thesecondderivativeis d 2 y dx 2 D sin x cos x At x D 4 th i sis 0y is b e ndingdown x isa ma x imum d 2 y dx 2 0 w h e n thecurvebendsup d 2 y dx 2 0 w h e n thecurvebendsdown Direction ofbendingchangesata po i ntofin ection where d 2 y dx 2 D 0 Which x givestheminimumof y D .x 1/ 2 C .x 2/ 2 C .x 6/ 2 ? You canwrite y D .x 2 2x C 1 / C .x 2 4x C 4 / C .x 2 12x C 3 6 / Theslopeis dy dx D 2x 2 C 2 x 4 C 2 x 12 D 0 a t theminimumpoint x Then 6x D 1 8 a n d x D 3 M inimumpoin tistheaverageof 1 , 2 , 6 Key for max = m i n word problemsistochooseasuitablemeaningfor x 0.4VideoSummariesandPracticeProblems 25 PracticeQuestions 1.Which x givestheminimumof y .x / D x 2 C 2 x ? So lve dy dx D 0 : 2 . Find d 2 y dx 2 f o r y .x/ D x 2 C 2x: Thisis 0 so th eparabolabendsup. 3 . Fin d themaximumheightof y .x / D 2 C 6 x x 2 : So lve dy dx D 0 : 4 . Find d 2 y dx 2 t o s howthatthisparabolabendsdown. 5.For y.x/ D x 4 2x 2 sh owthat dy dx D 0 at x D 1;0;1: Find y. 1/ , y .0 / , y .1 /: Ch eckmax versusminbythesignof d 2 y = d x 2 : 6 . A t a minimumpointexplain why dy dx D 0 an d d 2 y dx 2 0: 7 . Bend ingdown d 2 y dx 2 0 c h angestobendingup d 2 y dx 2 0 a t a p o int o f :Atthispoint d 2 y dx 2 D 0 Do e s y D sin x havesuchapoint? 8.Suppose x C X D 12: Whatisthemaximumof x times X ? Thisquestion asksforthemaximumof y D x.1 2 x/ D 1 2 x x 2 : Findwheretheslope dy dx D 12 2x is zero.Whatis x times X ? TheBigPictureofIntegrals Key problem Recovertheintegral y.x / f romitsderivative dy dx Findthetotaldistancetraveled fromarecordofthespeed FindFunction .1/ D t o talheightknowingFunction .2/ D slopesincethestart SimplestwayRecognize dy dx as d erivativeofaknown y.x/ If dy dx D x 3 th e n its integral y.x/ was 1 4 x 4 C C D Fu n ction(1) If dy dx D e 2x t h en y D 1 2 e 2x C C IntegralCalculusisthereverseofDifferentialCalculus y.x/ D dy dx dx a d dsupthewholehistory ofslopes dy dx to nd y.x/ IntegralislikesumDerivativeislikedifference 26 0H i ghlightsofCalculus Sums y 0 y 1 y 2 y 3 y 4 Di f ferences y 1 y 0 y 2 y 1 y 3 y 2 y 4 y 3 Noticecancellation .y 1 y 0 / C .y 2 y 1 / D y 2 y 0 D c h angeinheight Divideandmultiply thedifferencesbythestepsize x Sum of y x x D y 1 y 0 x x C y 2 y 1 x x i s still y 2 y 0 Nowlet x 0 Su m c h angestointegral dy dx d x D y e n d y st a r t FundamentalTheoremofCalculus dy dx dx D y .x / C C Theintegralreversesthederivativeand bringsback y.x /

IntegrationandDifferentiationareinverseoperations FundamentalTheoremintheoppositeorder d dx x 0 s .t /dt D s .x / KEYWhatisthemeaning ofanintegral x 0 s .t /dt ? A d d u p sh o r t : Example s.t/ D 6t s howsincreasing speedandslope.Find y .t/: Method1 y D 3t 2 h a stherequired derivative 6t (thisisthesimplestway!) Method2 Th e triangleunderthegraph of s.t/ D 6t hasarea 3t 2 From 0 to t ; b ase D t and height D 6t and area D 1 2 t.6 t /: [Mostshapesaremoredif cult!Areacomesfrom integrating s.t / o r s.x/] Method3 (fundamental) A d dupshorttimestepseachatconstantspeed Inastep t; t h edistanceiscloseto s.t / t t isthestarting timeforthatstepand s .t / is thestartingspeed Thisisnotexactbecausethespeedchangesalittlewithintime t Thetotaldistancebecomesexactas t 0 a n d sum integral Pictureofeachstepshowsatallthinrectangle s.t / t D h e ighttimesbase D a r e aofrectangle t D sta rtpointofthetimestep Sum of s.t / t D to talareaofallrectangles Now t 0 T h e r e ctangles llupthetriangle Integralof s.t/dt D e x actarea y.t/ underthegraph 0.4VideoSummariesandPracticeProblems 27 FundamentalTheorem Area y.t/ h asthedesired derivative s .t/ Reason: y i s thethinareaunder s.t/ between t and t C t t i sthebaseofthatthinrectangle y t i s t heheightofthatthinrectangle Thisheight y = t approaches s.t/ asthebase t 0 PracticeQuestions 1.Whatfunctions y.t / h avetheconstantderivative s.t/ D 7 ? 2.Whatistheareafrom 0 to t underthegraph of s.t/ D 7 ? 3.From t D 0 to 2; ndtheintegral 2 0 7 d t D : 4. W hatfunction y.t/ hasthederivative s.t/ D 7 C 6t ? 5.From t D 0 to 2; ndarea D integral 2 0 .7 C 6 t / dt: 6 . A t th isinstant t D 2 , w h a tis d . a rea / dt ? 7.From 0 to t ; t heareaunderthecurve s D e t ISNOT y D e t : If t issmall,theareamustbesmall.Thewrong answer e t i s notsmall! 8.From 0 to t ; t hecorrectareaunder s D e t is y D e t 1: Theslope dy dt is and nowthestartingarea y.0 / i s 9.Sameforsums.Notice y 0 in .y 1 y 0 / C .y 2 y 1 / C .y 3 y 2 / D : Thesum of y D y t t b e comestheintegralof dy dt dt Thearea under s.t / f rom0to t becomes y.t/ y.0/: 28 0H i ghlightsofCalculus DerivativeoftheSineandCosine Thislectureshowsthat d dx . s i n x/ D cos x and d dx . c o s x/ D sin x Wehavetomeasuretheangle x i n r a dians 2 radians D full 360 degrees Allthewayaroundthecircle(2 r a dians) Length D 2 whentheradiusis1 Partwayaround thecircle(x radians) Length D x when theradiusis1 =2 3 = 2 2 y D si n x sl ope 1 at x D 0 0 1 C 1 x Slopecos x at x D 0 s l o pe 1 D cos 0 at x D =2 s l ope 0 D cos =2 at x D sl o pe 1 D c o s =2 2 y D c o s x x C 1 0 1 Slope sin x at x D 0 sl o pe D 0 D sin 0 at x D =2 s l ope 1 D sin = 2 at x D sl o pe D 0 D sin Problem: y x D si n .x C x/ sin x x is n otassimpleas .x C x/ 2 x 2 x Good ideatostartat x D 0 Sh o w y x D si n x x ap p roaches1 Drawarighttrianglewithangle x t o see sin x x x r D 1 st r aightpiece curvedarc straightpieceisshortest straightlength D si n x curvedlength D x IDEA si n x x 1 a n d sin x x cos x w ill squeeze sin x x 1 a s x 0 Toprove si n x x cos x w h ic h is tan x x Go toabiggertriangle Angle x x F ullangle 2 tan x Trianglearea D 1 2 .1/. t an x/ greaterthan Circulararea D x 2 (w h olecircle) D 1 2 .x / 0.4VideoSummariesandPracticeProblems 29 Thesqueezecos x sin x x 1 te llsusthat sin x x ap p roaches1 . sin x/ 2 . x/ 2 1 me ans .1 cos x / x .1 C c o s x/ x So 1 cos x x 0 C o sine c urvehasslope D 0 Fortheslopeat other x r e m ember aformulafromtrigonometry: sin .x C x/ D sin x cos x C cos x sin x Wewant y D s i n .x C x/ sin x D iv idethatby x y x D . s i n x/ cos x 1 x C . c o s x/ sin x x N o w let x 0 Inthelimit dy dx D . s i n x/.0/ C . cos x/.1/ D cos x D Derivativeofsin x For y D cos x t h eformulaforcos .x C x/ leadssimilarlyto dy dx D sin x PracticeQuestions 1.Whatistheslopeof y D si n x a t x D and at x D 2 ? 2.Whatistheslopeof y D cos x at x D =2 and x D 3=2 ? 3.Theslopeof . sin x/ 2 is 2 sin x cos x: Theslopeof . cos x/ 2 is 2 c o s x sin x : Combined,theslopeof . s i n x/ 2 C . cos x/ 2 is zero .Whyisthistrue ? 4.Whatisthe se c ondderivative of y D sin x (derivativeofthederivative)? 5.Atwhatangle x does y D sin x C cos x havezeroslope? 6.Hereareamazing in niteseriesforsin x an d cos x: e ix D cos x C i sin x sin x D x 1 x 3 3 2 1 C x 5 5 4 3 2 1 (o d d p o werso f x) c o s x D 1 x 2 2 1 C x 4 4 3 2 1 (ev e n p owersof x) 7 . Takethederivativeofthesineseriestoseethecosineseries. 8 . Takethederivativeofthecosineseriestosee minus th esineseries. 9 . T h oseseriestell usthat for small angles sin x x a n d c o s x 1 1 2 x 2 : Wi t htheseapproximationscheckthat . sin x/ 2 C . cos x/ 2 iscloseto 1: 30 0H i ghlightsofCalculus ProductandQuotientRules Goal T o ndthederivativeof y D f .x /g.x / from df dx a n d d g dx I d e a Write y D f.x C x /g.x C x/ f.x/g.x/ b y se p arating f a n d g T h a tsame y is f .x C x / g .x C x / g.x/ C g .x / f .x C x / f.x/ y x D f .x C x / g x C g .x / f x P r o ductRule dy dx D f .x / d g dx C g .x

/ d f dx E x a mple y D x 2 sin x Pro ductRule dy dx D x 2 c o s x C 2 x sin x Apictureshowsthetwounshadedpiecesof y D f.x C x/g C g.x/f g g.x / f .x/ f toparea D . f . x / C f / g sidearea D g . x / f Example f.x / D x n g .x/ D x y D f.x /g.x/ D x n C 1 ProductRule dy dx D x n dx dx C x dx n dx D x n C xn x n 1 D .n C 1 /x n T h e c orrectderivativeof x n le adstothecorrectderivativeof x n C 1 QuotientRule If y D f .x / g.x/ th e n dy dx D g.x / d f dx f.x/ dg dx g 2 EX A MPLE d dx si n x cos x D . c o s x. cos x/ sin x . sin x // c o s 2 x T h issaysthat d dx ta n x D 1 cos 2 x D se c 2 x (Notice . cos x/ 2 C . sin x/ 2 D 1) EXAMPLE d dx 1 x 4 D x 4 ti m es 0 1 times 4 x 3 x 8 D 4 x 5 Th i sis nx n 1 Pr ovetheQuotientRule y D f .x C x / g.x C x/ f.x/ g.x/ D f C f g C g f g Wr i tethis y as g.f C f/ f.g C g / g.g C g/ D g f fg g.g C g/ N o wdividethat y by x As x 0 w e h avetheQu o tientRule 0.4VideoSummariesandPracticeProblems 31 PracticeQuestions 1.ProductRule:Findthederivativeof y D .x 3 /.x 4 /: S i m plify andexplain. 2.ProductRule:Findthederivativeof y D .x 2 /.x 2 /: Simplify andexplain. 3 . Q u o tientRule:Findthederivativeof y D c o s x sin x : 4 . Q uotientRule:Showthat y D sin x x h a s amaximum(zero slope)at x D 0: 5.ProductandQuotient!Findthederivativeof y D x sin x cos x : 6 . g .x / hasaminimumwhen d g dx D 0 an d d 2 g dx 2 0 T h e g raph isbendingup y D 1 g.x/ ha s a maximum atthatpoint:Showthat dy dx D 0 an d d 2 y dx 2 0 ChainRulefortheSlopeof f.g .x // y D g.x/z D f .y / thechainis z D f .g .x // y D x 5 z D y 4 thechainis z D .x 5 / 4 D x 20 Averageslope z x D z y y x Ju s tcancel y Instantslope dz dx D dz dy dy dx D CH A INRULE (likecancelling dy) You MUSTchange y to g .x / inthe nalanswer Exampleofchain z D y 4 D .x 5 / 4 dz dy D 4y 3 d y dx D 5x 4 Chainrule dz dx D dz dy d y dx D .4y 3 /.5 x 4 / D 2 0y 3 x 4 Replace y by x 5 t o getonly x dz dx D 20 .x 5 / 3 x 4 D 2 0x 19 CHECK z D .x 5 / 4 D x 20 d o eshave dz dx D 20 x 1 9 1. Find d z dx fo r z D c os .4x/ Write y D 4x and z D cos y so dz dx D 2. F i nd dz dx fo r z D .1 C 4 x/ 2 Write y D 1 C 4x and z D y 2 so dz dx D CHECK .1 C 4x / 2 D 1 C 8 x C 16x 2 so dz dx D 32 0H i ghlightsofCalculus PracticeQuestions 3. F ind d h dx f o r h .x/ D . sin 3x/. cos 3x/ Productrule rstThen theChainruleforeachfactor dh dx D . s i n 3x/ d dx . c o s 3x/ C . cos 3x/ d dx . s i n 3x/ D . sin 3x/. CHAIN / C . cos 3x/. CHAIN / D 4. Toughchallenge:Findthe secondderivative of z.x/ D f.g.x// FIRST DERIV dz dx D dz dy d y dx Fu n ction of y.x/ timesfunction of x PRODUCT RULE d 2 z dx 2 D dz dy d dx dy dx C dy dx d dx dz dy SE C OND DERIV dz dy d 2 y dx 2 C dy dx d 2 z dy 2 d y dx dy dx tw i ce Check y D x 5 z D y 4 D x 20 dz dx D 20 x 1 9 d 2 z dx 2 D 38 0 x 18 SECOND

DE R IV .4y 3 /.20x 3 / C .5x 4 /.12y 2 /.5x 4 / 80 C 300 D 380 OK InverseFunctionsandLogarithms Afunction assignsan ou t put y D f.x/ toeach input x Aone-to-onefunctionhasdifferentoutputs y fo r differentinputs x Forthe inversefunction t h einputis y and theoutputis x D f 1 .y / ExampleIf y D f.x / D x 5 t hen x D f 1 .y / D y 1 5 KE Y If y D ax C b thensolvefor x D y b a D in v ersefunction Noticethat x D f 1 .f .x // a n d y D f .f 1 .y // The chainrule w i llconnectthederivativesof f 1 a n d f Thegreatfunction ofcalculusis y D e x Itsinversefunction isthe n a turallogarithm x D ln y Rememberthat x is t heexponentin y D e x Therule e x e X D e x C X te l lsusthat ln .yY/ D ln y C ln Y Addlogarithmsbecauseyouadd exponents: ln .e 2 e 3 / D 5 .e x / n D e nx (m ultiply exponent)tellsusthat ln .y n / D n ln y 0.4VideoSummariesandPracticeProblems 33 Wecanchangefrombase e to b ase 10 :New function y D 10 x Theinversefunction isthelogarithmtobase 10 C a llitlog: x D log y Then log 1 00 D 2 a n dlog 1 100 D 2 a n dlog 1 D 0 Wewillsoon ndthebeautifulderivativeofln y d dy . l n y / D 1 y You canchangeletterstowritethatas d dx . l n x / D 1 x PracticeQuestions 1.Whatis x D f 1 .y/ if y D 50x ? 2 . Wh a tis x D f 1 .y/ if y D x 4 ?Why dowekeep x 0 ? 3 . D r awagraphofanincreasingfunction y D f .x /: T h ishasdifferentoutputs y f o r d iff e r e n t x : F lipthegraph (sw itchtheaxes) t o se e x D f 1 .y / 4 . T h is graph hasthesame y f ro m t wo x 's . Thereisno f 1 .y / f . x / i sNOTone-to-one x y f .x/ f 1 . y / isNOTafunction x y 5.Thenaturallogarithmof y D 1= e i sln .e 1 / D ? Wh a tisln . ? e/ ? 6. T henaturallogarithmof y D 1 isln 1 D ?andalsobase10haslog 1 D ? 7.Thenaturallogarithmof .e 2 / 50 is?Thebase 10 logarithmof .10 2 / 50 is? 8.Ibelievethat ln y D . ln 10/. log y/ becausewecanwrite y intwo ways y D e ln y and also y D 10 log y D e . ln 10/. log y/ : Explain those laststeps. 9.Changefrombase e and base 10 to base 2 : Now y D 2 x means x D log 2 y : Whatarelog 2 32 and log 2 2 ?Whyislog 2 .e/ 1 ? 34 0H i ghlightsofCalculus GrowthRatesandLogGraphs Inorderoffastgrowth as x g e t slarge log x x ; x 2 ; x 3 2 x ; e x ; 1 0 x x ; x x logarithmicpolynomial exponential factorial Choose x D 1 0 0 0 D 10 3 sothatlog x D 3 OK touse x x x e x log 1 000 D 3 1 0 3 ;10 6 ;10 9 10 300 ;10 434 ;10 1000 10 2566 ;10 3000 Why is 1 000 1 0 00 D 10 3000 ?Logarithmsarebestforbignumbers Logarithmsareexponents! l o g 1 0 9 D 9 loglog x isVERYslow Logarithms 3 ; 6 ; 93 0 0 ; 434 ; 1000 2566 ; 3000 Polynomialgrowth ! Exponentialgrowth ! Factorialgrowth Decaytozero forNEGATIVEpowersand exponents 1 x 2 D x 2 d e caysmuch moreslowlythantheexponential 1 e x D e x Logarithmicscaleshows x D 1; 1 0;100 equally spaced.NO ZERO! 3 2 10123 lo g x x lo g ? 10 D 1 2 1= 1 000 1=100 1=101101001000 Question If x D 1 ; 2;4;8 are plotted,whatwouldyousee? Answer THEYAREEQUALLYSPACED TOO! log - loggraphs (l ogscaleupandalsoacross) If y D Ax n ,h o wtosee A and n onthegraph? Plotlog y versuslog x t o getastraightline log y D log A C n l o g x Slopeonalog-loggraph istheexponent n y D x 1: 5 A D 1 n D 1:5 l og y D 1:5 l og x l og A D 0 l og x 1 1 sl ope n D 1:5 For y D Ab x us e s emilog (x versuslog y isnowaline)log y D log A C x log b 0.4VideoSummariesandPracticeProblems 35 Newtypeofquestion Ho w quicklydoes f x ap p roach df dx as x 0 ? Theerror E D f x df dx w i l lbe E A. x/ n Wh a tis n ? Usualone-sided f x D f .x C x / f.x/ x o n l y has n D 1 Centered difference f .x C x / f.x x/ 2 x h a s n D 2 Centered ismuchbetterthanone-sided E . x/ 2 v s E x I D E AFOR f.x/ D e x PROJECTat x D 0 One-sided E vscentered E Graphlog E vs log x Shouldseeslope 1 or 2 PracticeQuestions 1.Does x 10 0 g rowfasterorslowerthan e x as x getslarge ? 2.Does 100 ln x growfasterorslowerthan x as x getslarge ? 3.Puttheseinincreasing orderforlarge n : 1 n ;n l o g n;n 1:1 ; 10 n n 4. P uttheseinincreasing orderforlarge x : 2 x ; e x ; 1 x 2 ; 1 x 10 5.Describethelog-loggraph of y D 10 x 5 (graphlog y vslog x) Why don'twesee y D 0 at x D 0 o n thisgraph? Whatistheslopeofthestraightlineonthelog-loggraph? Thelinecrossestheverticalaxiswhen x D and y D Then log x D 0 an d log y D Thelinecrossesthehorizontalaxiswhen x D and y D 1 Then log x D andlog y D 0 6.Drawthesemilog graph (aline)of y D 10 e x (graphlog y versus x) 7.Thatlinecrossthe x D 0 axisatwhichlog y ?Whatistheslope? 36 0H i ghlightsofCalculus

LinearApproximationandNewton'sMethod Startat x D a w i t hknown f.a/ D heightand f 1 .a/ D slo pe KEYIDEA f 1 .a/ f.x/ f.a/ x a w h e n x is near a Tangentlinehasslope f 1 .a/ So lve f o r f .x / f .x / f.a/ C .x a/f 1 .a/ meansapproximately c u rve linenear x D a Examplesoflinearapproximationto f.x / 1. f.x/ D e x f .0 / D e 0 D 1 and f 1 .0/ D e 0 D 1 a r e k nown at a D 0 Followthetangentline e x 1 C .x 0/1 D 1 C x 1 C x isthelinearpartoftheseriesfor e x 2. f.x/ D x 1 0 a nd f 1 .x/ D 1 0 x 9 f .1 / D 1 a n d f 1 .1/ D 1 0 k n own at a D 1 Followthetangentline x 10 1 C .x 1/10 n e ar x D 1 Take x D 1:1.1:1/ 1 0 i sapproximately 1 C 1 D 2 Newton'sMethod (l o okingfor x tonearly solve f.x/ D 0) Goback to f 1 .a/ f.x/ f.a/ x a f.a/ and f 1 .a/ a r e a gain known Solvefor x wh e n f.x/ D 0 x a f.a/ f 1 .a/ N e w t o n x L in ecrossing nearcurvecrossing a ex a ct x Newt on x f.x/ 0.4VideoSummariesandPracticeProblems 37 ExamplesofNewton'sMethodSolve f.x / D x 2 1:2 D 0 1. a D 1 g ives f .a / D 1 1:2 D :2 a n d f 1 .a/ D 2 a D 2 Tangentlinehits 0 a t x 1 D . :2/ 2 N e w ton's x willbe 1:1 2.Forabetter x ; N e wtonstartsagain fromthatpoint a D 1:1 Now f .a/ D 1 : 1 2 1:2 D : 0 1 a n d f 1 .a/ D 2 a D 2 : 2 Thenewtangentlinehas x 1:1 D :01 2:2 F o r this x;x 2 isvery closeto 1:2 PracticeQuestions 1.Thegraphof y D f.a / C .x a/f 1 .a/ is astraight At x D a theheightis y D At x D a theslopeis d y =dx D Thisgraph ist ttothegraph of f.x / a t x D a For f.x/ D x 2 a t a D 3 t hislinearapproximationis y D 2. y D f.a/ C .x a/f 1 .a/ h a s y D 0 w h e n x a D Instead ofthecurve f.x / c rossing 0 ,Newtonhastangentline y crossing 0 f.x/ D x 3 8:12 a t a D 2 h a s f .a / D and f 1 .a/ D 3 a 2 D Newton'smethodgives x 2 D f.a/ f 1 .a/ D ThisNewton x D 2: 0 1 nearlyhas x 3 D 8:12: Itactuallyhas .2:01/ 3 D : DifferentialEquationsofGrowth dy dt D cy C o mpletesolution y.t/ D Ae ct forany A Startingfrom y.0/y.t/ D y.0/e ct A D y.0/ Nowincludeaconstantsourceterm s Th i sgivesanewequation dy dt D cy C s s 0 is saving, s 0 is spending, c y is interest Completesolution y.t / D s c C Ae c t (any A givesasolution) y D s c is a constantsolutionwith cy C s D 0 and dy dt D 0 an d A D 0 Forthatsolution,thespending s ex a ctly balancestheincome cy Choose A tostartfrom y .0 / at t D 0y.t/ D s c C y.0 / C s c e ct 38 0H i ghlightsofCalculus Nowadd anonlinearterm sP 2 c o ming fromcompetition P.t/ D worldpopulationattime t (f orexample)followsanewequation dP dt D c P sP 2 c D b irthra teminusdeathrate LOGISTICEQN P 2 s i n ceeachperson competeswitheach person Tobring back alinearequation set y D 1 P Then d y dt D dP=dt P 2 D . cP C sP 2 / P 2 D c P C s D cy C s y D 1=P p roducedourlinearequation(no y 2) with c n o t C c y.t/ D s c C Ae ct D s c C y.0 / s c e ct D old solutionwithchangeto c At t D 0 wecorrectly get y .0 / CORRECTSTART As t 8 and e ct 0 w e get y . 8 / D s c an d P . 8 / D c s Thepopulation P.t / i ncreasesalong an S -curve approaching c s P D c 2s ha s P 2 D 0 I n e c tion pointBending changesfromuptodown CHECK d 2 P dt 2 D d dt cP sP 2 D .c 2sP/ dP dt D 0 at P D c 2s Worldpopulationapproachesthelimit c s 12 b illion(FORTHISMODEL!) Populationnow 7 b illionTryGoogleforWorldpopulation 0.4VideoSummariesandPracticeProblems 39 PracticeQuestions dy dt D c y s h a s s D sp e nding ratenotsavingsrate(withminussign) 1.Theconstantsolution is y D when d y dt D 0 I n t hatcaseinterestincomebalancesspending: cy D s 2.Thecompletesolutionis y.t/ D s c C Ae c t : W h y is A D y.0/ s c ? 3 . I fyou startwith y.0/ s c w h y doeswealthapproach 8 ? I f you startwith y .0 / s c wh y doeswealthapproach 8 ? 4 . T h ecompletesolutionto dy dt D s is y .t / D st C A Whatsolution y.t/ startsfrom y.0/ at t D 0 ? 5.If dP dt D sP 2 a n d y D 1 P ex p lain why dy dt D s Pu r ecompetition.Showthat P.t/ 0 as t 8 6.If dP dt D cP sP 4 n d alinearequation for y D 1 P 3 DifferentialEquationsofMotion Adifferentialequation for y.t / c aninvolve dy=dt and also d 2 y=dt 2 Hereareexampleswithsolutions C an d D c anbeanynumbers d 2 y dt 2 D y a n d d 2 y dt 2 D ! 2 y So lu tions y D C c o s t C D sin t y D C c o s ! t C D sin ! t Nowinclude dy = dt andlook forasolution method m d 2 y dt 2 C 2r d y dt C ky D 0 h a sadamping term 2r dy dt : Tr y y D e t Substituting e t g i ves m 2 e t C 2r e t C k e t D 0 Cancel e t t oleavethekeyequationfor m 2 C 2r C k D 0 Thequadraticformulagives D r ? r 2 km m Tw o solutions 1 and 2 Thedifferentialequation issolvedby y D Ce 1 t C D e 2 t Specialcase r 2 D km h a s 1 D 2 Then t enters y D Ce 1 t C D t e 1 t 40 0H i ghlightsofCalculus EXAMPLE1 d 2 y dt 2 C 6 dy dt C 8y D 0 m D 1 a nd 2r D 6 and k D 8 1 ; 2 D r ? r 2 km m i s 3 ? 9 8 T h e n 1 D 2 2 D 4 S ol uti on y D Ce 2t C D e 4t O verdampingwithnooscillation E X AMPLE2 Ch ang eto k D 10 D 3 ? 9 10 h a s 1 D 3 C i 2 D 3 i Oscillations f r o m th e imaginarypartof D e c ay f r o m th e r e a l p art 3 S ol uti on y D Ce 1 t C De 2 t D Ce . 3 C i / t C D e . 3 i/t e i t D c o s t C i sin t le adsto y D .C C D /e 3t c o s t C .C D/e 3t sin t E X AMPLE3 Ch ang eto k D 9 Now D 3; 3 (r e p e ated root) S ol uti on y D Ce 3t C D t e 3t in cludesthefactor t PracticeQuestions 1.For d 2 y dt 2 D 4 y n d twosolutions y D Ce at C De bt : Whatare a and b ? 2.For d 2 y dt 2 D 4 y n d two solutions y D C c o s ! t C D sin ! t : Wh a tis ! ? 3 . Fo r d 2 y dt 2 D 0 y n d twosolutions y D Ce 0t and (???) 4.Put y D e t into 2 d 2 y dt 2 C 3 dy dt C y D 0 to nd 1 and 2 (real numbers) 5.Put y D e t into 2 d 2 y dt 2 C 5 dy dt C 3y D 0 t o nd 1 and 2 (complex numbers) 6.Put y D e t into d 2 y dt 2 C 2 dy dt C y D 0 to nd 1 and 2 (equal numbers) Now y D Ce 1 t C D t e 1 t : Thefactor t appearswhen 1 D 2 0.4VideoSummariesandPracticeProblems 41 P o we rSeriesandEuler'sFormula At x D 0 ,the n t h derivativeof x n isthenumber n !Otherderivativesare 0: Multiply the n t h d erivativesof f.x/ by x n =n tomatchfunctionwithseries TAYLOR SERIES f.x/ D f .0/ C f 1 .0/ x 1 C f 2 .0/ x 2 2 C C f . n / .0 / x n n C EXAMPLE1 f .x / D e x A lld erivatives D 1 at x D 0 Matchwith x n =n ! TaylorSeries ExponentialSeries D e x D 1 C 1 x 1 C 1 x 2 2 C C 1 x n n C EXAMPLE2 f D sin xf 1 D c o s x f 2 D sin x f 3 D cos x A t x D 0 th isis 0 1 0 1010 1 RE PEAT sin x D 1 x 1 1 x 3 3 C 1 x 5 5 ODD POWERSsin . x/ D sin x EXAMPLE3 f D cos x produces 1 0 1010 10 RE PEAT cos x D 1 1 x 2 2 C 1 x 4 4 EVENPOWERS d dx . c o s x/ D sin x Imaginary i 2 D 1 and then i 3 D i F ind t he exponential e i x e ix D 1 C i x C 1 2 .ix / 2 C 1 3 .ix / 3 C D 1 x 2 2 C C i x x 3 3 C T h o seare c o s x C i sin x E U L E R'SG REATFORMULA e ix D c o s x C i sin x e i c o s i sin e i Real part e i D cos C i sin e i C e i D 2 cos e i D 1 c o mbines 4 g r e a tnumbers TwomoreexamplesofPowerSeries(TaylorSeries for f .x /) f.x/ D 1 1 x D 1 C x C x 2 C x 3 C Geometricseries f.x/ D ln .1 x/ D x 1 C x 2 2 C x 3 3 C x 4 4 C Integralofgeometricseries 42 0H i ghlightsofCalculus Summary:SixFunctions,SixRules,SixTheorems Integrals S i x Functions Derivatives x n C 1 =.n C 1/;n 1 x n nx n 1 cos x sin x c o s x sin x c o s x sin x e c x =c e c x c e c x x ln x x ln x 1 = x R a mp f unctionStepfunctionDeltafunction 0 x 0 1 0 In nitespike hasarea D 1 SixRulesofDifferentialCalculus 1. T h ederivativeof af.x/ C bg.x/ is a df dx C b dg dx Su m 2 . Thederivativeof f.x/g.x/ is f.x/ d g dx C g.x / d f dx Pr o duct 3. Thederivativeof f.x/ g.x/ is g d f dx f d g dx g 2 Q u otient 4. Thederivativeof f.g.x// is df dy dy dx wh e re y D g.x/ Chain 5.

Thederivativeof x D f 1 .y / is d x dy D 1 dy=dx In v erse 6. When f.x/ 0 and g .x / 0 a s x a , w h a t a bout f .x /= g .x / ? l' Hpital lim f .x / g.x/ D li m d f=dx d g=dx if t heselimitsexist.Normallythisis f 1 .a/ g 1 .a/ FundamentalTheoremofCalculus If f.x/ D x a s .t /dt th en de r iva t ive o f int egral D df dx D s.x / If df dx D s.x / t hen integralofderivative D b a s .x /d x D f .b/ f.a/ Bothpartsassumethat s.x / i sacontinuousfunction. AllValuesTheorem Su p pose f.x/ isacontinuous functionfor a x b : T h e nonthat interval, f .x / r e a chesitsmaximumvalue M a n ditsminimum m : A n d f .x / ta kesallvaluesbetween m a n d M (th e rearenojumps). 0.4VideoSummariesandPracticeProblems 43 MeanValueTheorem If f .x / hasaderivativefor a x b th en f.b/ f.a/ b a D df dx .c / a t some c between a and b Atsomemoment c , i n stantspeed D averagespeed TaylorSeries M a t challthederivatives f .n/ D d n f=dx n atthebasepoint x D a f.x/ D f .a/ C f 1 .a/.x a/ C 1 2 f 2 .a/.x a/ 2 C D 8 X n D 0 1 n f . n / .a /.x a/ n Stoppingat .x a/ n le avestheerror f n C 1 .c / .x a/ n C 1 = .n C 1 / [c issomewherebetween a a n d x][n D 0 istheMeanValueTheorem] TheTaylorserieslooksbestaround a D 0f .x / D 8 X n D 0 1 n f .n / .0 /x n BinomialTheorem sh o wsPascal'striangle .1 C x/ 1 C 1 x .1 C x / 2 1 C 2 x C 1 x 2 .1 C x / 3 1 C 3 x C 3 x 2 C 1 x 3 .1 C x/ 4 1 C 4 x C 6 x 2 C 4 x 3 C 1 x 4 ThosearejusttheTaylorseriesfor f.x / D .1 C x / p when p D 1;2;3;4 f .n / .x / D .1 C x/ p p.1 C x/ p 1 p .p 1/.1 C x / p 2 f .n/ .0 / D 1 p p.p 1/ Divideby n t o ndtheTaylorcoef cients D Binomialcoef cients 1 n f .n / .0 / D p.p 1/ .p n C 1 / n.n 1/ .1/ D p .p n/n D p n Theseriesstopsat x n wh e n p D n In niteseriesforother p Every .1 C x/ p D 1 C p x C p .p 1/ .2/.1/ x 2 C p.p 1/.p 2/ .3/.2/.1/ x 3 C 44 0H i ghlightsofCalculus PracticeQuestions 1.Checkthatthederivativeof y D x l n x x is dy = d x D ln x : 2 . T h e signfunctionis S .x / D # 1 f o r x 0 1 f o r x 0 Whatrampfunction F .x / h as d F dx D S .x / ? F i stheintegralof S: Why isthederivative d S dx D 2 d e l ta .x/ ?(In nitespikeat x D 0 witharea 2) 3.(l'H O opital)Whatisthelimitof 2 x C 3 x 2 5x C 7x 2 a s x 0 ? Wh atabout x 8 ? 4 . l' H O o p ital'sRulesaysthatlim x 0 f .x / x D ?? w hen f.0/ D 0: Here g.x/ D x: 5. DerivativeislikeDifferenceIntegralislikeSum D i fferenceofsumsIf f n D s 1 C s 2 C C s n , w h a tis f n f n 1 ? Su msofdifferencesWhatis .f 1 f 0 / C .f 2 f 1 / C C .f n f n 1 / ? T h o sea rethe

F undamentalTheorems o f D if f e re nceCalculus 6 . D r aw a non-con tinuousgraphfor 0 x 1 w h e r e y o urfunctiondoes NOT r e a ch itsmaximumvalue. 7 . Fo r f .x / D x 2 , w h ic h in-between point c gives f .5 / f.1/ 5 1 D df dx .c/ ? 8 . Ifyouraverage speedontheMass Pikeis 75 ,thenatsomeinstantyour speedometerwillread : 9. F indthreeTaylorcoef cients A;B;C for ? 1 C x (around x D 0) . .1 C x / 1 2 D A C Bx C C x 2 C 10.FindtheTaylor(D Binomial)seriesfor f D 1 1 C x ar o und x D 0.p D 1/: 0.5Graphs andGraphingCalculators 45 0.5 Gr aphsandGr aphing Calculat ors Thisbookstartedwiththesentence C a l culusisaboutfunctions .Whenthese functions aregivenbyformulaslike y D x C x 2 ,wenowknowaformulaforthe slope(andeventheslopeoftheslope).Whenweonlyhavearoughgraphof the function,wecan'texpectmorethanaroughgraphof theslope.Butgraphsarevery valuableinapplicationsofcalculus! Fromagraphof y.x/ ,thissectionextractsthebasicinformation aboutthegrowth rate(theslope)andtheminimum = maximumandthebending (andareatoo).A bigpartofthatinformationiscontainedin aplusorminussign .Is y.x/ increasing? Isits slopeincreasing? Is thearea underits graph increasing?Ineachcasesome

recordofthedistance.(Thatiscalled differentiation ,anditisthecentralideaof differentialcalculus .) Wealsowanttocompute thedistancefromahistoryofthe velocity.(Thatis integration ,and itisthegoalof integralcalculus .) Differentiation goes from f to v ;integrationgoesfrom v to f: Welook rstat examplesinwhichthesepairscanbecomputed and understood. CONSTANTVELOCITY Supposethevelocityis xedat v D 60 (m i lesper hour).Then f increasesatthis constantrate.Aftertwohoursthedistanceis f D 120 (miles).Afterfourhours f D 240 andafter t hours f D 60t: Wesaythat f increases linearly withtime itsgraph isastraightline. Fig.1.2 Const antvel ocit y v D 6 0 a nd l i nearl yi ncreasi ngdi st ance f D 60t: Noticethatthis examplestarts thecar atfullvelocity.Notimeisspentpickingup speed.(Thevelocityisastepfunction.)Noticealsothatthedistancestartsatzero; thecarisnew. Thosedecisionsmakethegraphsof v and f asneataspossible. Oneis thehorizontalline v D 60:

Theotheristhesloping line f D 60t: This v , f;t relation needsalgebrabutnotcalculus: If v isco nsta nta nd f sta rtsa tzerothen f D vt: Theoppositeisalsotrue.When f inc r easeslinearly, v isconstant. Thedivisionby timegivestheslope .Thedistanceis f 1 D 120 mileswhen thetimeis t 1 D 2 hours. Later f 2 D 240 miles at t 2 D 4 hours.Atbothpoints,theratio f=t is 60 miles = hour. Geometrically, thevelocityistheslopeofthedistancegraph : slope D chan geindistance changeintime D vt t D v: 1.1VelocityandDistance 53 Fig.1.3 S t rai ghtl i nes f D 2 0 C 6 0t (sl ope 60) and f D 30t (sl ope 30) . Th e slo peofthe f - g ra p h g ivesthe v - g ra p h . Fig u r e 1 . 3 sh owstwomorepossibilities: 1 . T h e distancestartsat 2 0 in steadof 0 : T h e distanceformulachangesfrom 6 0 t to 2 0 C 6 0 t: T h e n umber 2 0 c a ncelswhenwecompute ch a n ge in distanceso th eslopeisstill 6 0 : 2 . Wh e n v is n eg ative ,the graph of f g o e s d o wnward . T h e cargoesbackward a n dtheslopeof f D 30t is v D 30: I d o n 't thinkspeedometersgo b elowzero.Butdrivingbackwards,it'snotthatsafe to wa tch.If yougofastenough,Toyotasays theymeasureabsolutevaluesthe sp e edometerreads C 3 0 w h e n th e ve lo cityis 30: Fo r th e odometer,asfarasIknow it juststops.Itshould gobackward. V E LOCITY vs. D I STANCE :SLOPE vs. AREA Howdoyoucompute f from v ? Thepointofthequestionistosee f D vt onthe graphs .Wewanttostartwiththegraph of v anddiscoverthegraph of f: Amazingly, theoppositeofslopeis area . Thedistance f istheareaunderthe v - graph .When v isconstant,theregion underthegraphisarectangle.Its heightis v ,itswidthis t ,anditsareais v times t: This is integration ,togofrom v to f bycomputingthearea.Weareglimpsingtwo ofthecentralfactsofcalculus. 1A Theslopeofthe f - g ra phgivesthevelocity v .Theareaunderthe v -graph givesthedistance f . Thatiscertainlynotobvious,and Ihesitated a long time beforeIwroteitdown in

f is p iecewiselinearand v is piecewiseconstant. Fig.1.8 Thevel oci t yanddi st ancegrowexponent i all y(powersof 2) . W he rewillcalculuscomein?Itworkswiththesmoothcurve f.t/ D 2 t : This exponential growthiscriticallyimportantforpopulationandmoneyinabankandthe

nationaldebt.Youcanspotitbythefollowing test: v .t/ isproportionalto f.t/: 62 1I n troductiontoCalculus Remark Thef unction 2 t istrickierthan t 2 : For f D t 2 theslopeis v D 2t: Itis proportionalto t andnot t 2

: For f D 2 t theslopeis v D c 2 t , andwewon't ndthe constant c D :693::: untilChapter 6: (Thenumber c isthe naturallogarithmof 2:) Problem 37 estimates c withacalculatortheimportantthingisthatit'sconstant. OSCILLATINGVELOCITYANDDISTANCE Wehaveseenaforward-backmotion,velocity V fol l owedby V: T h a tisoscillation o f th e simplestkind.Thegraphof f g o e slinearlyupandlinearlydown.Figure1.9 sh owsanotheroscillation thatreturnstozero,butthepathismoreinteresting. T h e numbersin f a r e n ow 0 ; 1 ; 1;0; 1; 1;0: Sin c e f 6 D 0 th emotionbrings us b a ck tothestart.Thewholeoscillationcanberepeated. T h e d ifferencesin v a r e 1 ; 0 ; 1; 1;0;1: T h ey a d duptozero,whichagrees with f la s t f rs t : I t isthesameoscillationasin f (a n d a lso repeatable),butshiftedintime. T h e f - g r a p h r e se mbles(roughly)a sine c urve . T h e v - g r a p h r e se mbles(even more r o u g h ly)a c o sinecurve . T h e waveformsinnaturearesmooth curves,whiletheseare d igitizedthewayadigitalwatchgoesforwardinjumps. Yourecognizethat the c h angefromanalogtodigitalbroughtthecomputerrevolution.Thesamerevolution is coming inCD players.Digitalsignals(offoron, 0 o r 1) se e m towinevery time. T h e p ie cewise v a n d f sta rtagainat t D 6 : T h e o r d inarysineandcosinerepeatat t D 2 : A r e peatingmotionis p e riodic h e retheperiodis 6 o r 2 : (With t in d eg r e estheperiodis 3 6 0 a f u ll c ircle.Theperiodbe comes 2 w h e n anglesaremea- su r e din ra d ians . Wevirtuallyalwaysuseradianswhicharedeg r e estimes 2 = 3 60:) A wa tch hasaperiodof 1 2 h o u rs.Ifthedialshows A M a n d P M , th e p e riodis : Fig.1.9 Pi ecewi seconst antcosi neandpi ecewi sel i near si ne.They b ot hrepeat . ASHORTBURSTOFSPEED Thenextexampleisacarthatisdrivenfastforashorttime.Thesp eedis V untilthe distancereaches f D 1 ,whenthecarsuddenlystops.Thegraphof f goesuplinearly withslope V ,andthenacrosswithslopezero: v .t/ D # V u p to t D T 0 a f ter t D T f.t/ D " Vt u p to t D T 1 a f ter t D T T h isisanotherexampleoffunctionnotation.Noticethegeneraltime t a n dthe p a rticularstopping time T : T h e d istanceis f .t /: T h e d omainof f (th e inputs) in cludesalltimes t 0: T h e r a n geof f (th e o u tputs)includesalldistances 0 f 1: Fig u r e 1 .10allowsustocomparethreecarsaJeep andaCorvetteandaMaserati. T h ey h ave differentspeedsbuttheyallreach f D 1 : So theareasunderthe v - g r a p h s a r e a ll 1 : T h e r e ctangleshaveheight V a n dbase T D 1 = V : 1.2CalculusWithoutLimits 63 Fig.1.10 B u r st sofspeedwi t h V M T M D V C T C D V J T J D 1: Ste p funct i onhasi n ni t e sl ope. Optionalremark Itisnaturaltothinkaboutfasterandfasterspeeds,which means steeperslopes.The f -graph reaches 1 inshortertimes.Theextreme caseis a step function , whenthegraphof f goes straightup. Thisistheunitstep U.t/ ,whichis zero upto t D 0 andjumpsimmediatelyto U D 1 for t 0: W ha t istheslopeofthestepfunction? I t iszeroexcept atthejump.Atthatmoment, w h ic h is t D 0 , th e slopeis in nite . We d o n 'thaveanordinaryvelocity v .t / instead w e h ave animpulsethatmakesthecarjump.Thegraphisaspikeoverthesinglepoint t D 0 , a n d it isoftendenotedby so th eslopeofthestepfunctioniscalleda d e lta f unction . T h e a reaunderthein nitespikeis 1 : Yo uareabsolutelynotresponsiblefor thetheoryof deltafunctions!Calculus is a b outcurves,notjumps. O u r lastexa mpleisareal-worldapplication ofslopesandsratestoexplainhow ta xeswork.Noteespeciallythedifference betweentaxrates andtaxbracketsand

28 Ifyouknowt heaveragevel oci t y v ave .t / ,howcan you ndt he di st ance f.t / ?S t artfrom f.0/ D 0: 1.4Circular Motion 73 1.4CircularM ot ion Thissectionintroducescompletelynewdistancesandvelocities thesinesand c o s inesfromtrigonometry .AsIwritethat lastword,Iask myselfhowmuch trigonometryitisessential to know.Therewillbethebasicpictureofaright triangle,withsidescos t andsin t and 1: Therewillalsobethecrucialequation . cos t/ 2 C . sin t/ 2 D 1 ,whichisPythagoras'law a 2 C b 2 D c 2 : Thesquaresof two sidesaddtothesquareofthehypotenuse (andthe 1 isreally 1 2).Nothingelseis neededimmediately.Ifyoudon'tknowtrigonometry,don'tstopanimportantpart canbelearnednow. Youwillrecognizethewavy graphsofthe sineandcosine. We intendto ndthe slopesofthosegraphs . That can bedonewithoutusingtheformulasforsin .x C y/ andcos .x C y/ whichlatergivethesameslopesinamorealgebraicway.Hereitis only basicthingsthatareneeded. Andanyway,howcomplicatedcanatrianglebe? Remark

Youmightthinktrigonometryisonlyforsurveyorsandnavigators

(peoplewithtriangles).Notatall!Byfarthebiggestapplicationsareto rotation and vibration and oscillation. Itisfantastic thatsinesand cosinesare soperfectfor repeatingmotionaroundacircleorupanddown. Fig.1.15 As t heangl e t chan ges,t hegraphsshowt hesides of theri ghtt ri angl e. Ourunderlyinggoalistooffer onemoreexampleinwhichthevelocitycanbe computedbycommonsense.Calculus ismainlyanextensionof commonsense,but

hammeratMIT.Hesurvived, butthethrowerquittrack.)Calculuswill ndthatsame tangentdirection,whenthepointsat t and t C h comeclose. The velocitytriangle isin Figure1.16b . Itisthesame asthepositiontriangle, butrotatedthrough 90 : T h e hypotenuseistangenttothecircle,inthedirectionthe b a llismoving.Itslengthequals 1 (th e speed).Theangle t stillappears,butnowit is theanglewiththevertical. T he upwardcomponentofvelocityis c o s t ; whenthe up wa rd componentofpositionis sin t : T h a tisourcommon sensecalculation,based 1.4Circular Motion 75 o n a gureratherthanaformula.Therestofthissectiondependso nitandwecheck v D cos t atspecialpoints. Atthestartingtime t D 0 ,themovementisallupward.Theheightis sin 0 D 0 and the upwardvelocity iscos 0 D 1: Attime =2 ,the ballreachesthe top.The height issin =2 D 1 andtheupwardvelocityiscos =2 D 0: Atthatinstanttheballisnot movingupordown. Thehorizontalvelocitycontainsaminussign. At rsttheballtravelstothe left . The valueof x iscos t ,but thespeed inthe x direction is sin t : H a lf oftrigonometry is in th a t gure(thegoodhalf),andyouseehowsin 2 t C cos 2 t D 1 issobasic. That e q uationappliestoposition andvelocity,ateverytime. Applicationofplane geometry: T h e r ig httrianglesinFigure1.16 arethe same siz eandshape.Theylookcongruentandtheyaretheangle t a b oveth e b a llequals th eangle t a t thecenter.Thatisbecausethethreeanglesattheballadd to 1 8 0 . OS CIL L AT ION:UPAND DOWN M OT ION We nowusecircularmotiontostudy stra ight-linemotion .Thatlinewillbethe y axis. Insteadofaball goingaroundacircle,amasswillmoveupanddown.Itoscillates between y D 1 and y D 1: T he massisthe sha d owoftheball , asweexplaininа mo ment. T h e reisajumpyoscillationthatwedonotwant,with v D 1 a n d v D 1: T h a t b ang-bangvelocityislikea billiardball,bouncingbetweentwo wallswithout slowing down.Ifthe distance between the wallsis 2 ; th en at t D 4 th e b allisback to thestart.Thedistancegraph isazigzag (or saw to o th)fromSection 1 :2 : We p r e f e r a smoothermotion.Insteadofvelocitiesthatjumpbetween C 1 a n d 1 , a r e a l o sc illation slo wsdowntozero a n dgraduallybuilds upspeedagain.Themass is onaspring,which pullsitback.Thevelocitydropsto zeroasthespringisfully

massstopsat thetopandstartsdown.Astheballgoes aroundthebottom,themass stopsandturnsbackupthe y axis.Halfwayup(ordown),thespeedis 1: Figure1.17ashows themassatatypicaltime t: Theheightis y D f.t/ D sin t , levelwiththeball. Thisheightoscillatesbetween f D 1 and f D 1: Bu t themass 76 1I n troductiontoCalculus doesnotmovewithconstantspeed. Thes peed ofthemassischanging althoughthe speedoftheballisalways 1 .Thetimeforafullcycleisstill 2 ,butwithinthatcycle themass speedsupandslows down.Theproblem isto ndthechangingvelocity v: Sincethedistanceis f D sin t ,thevelocity willbethe slopeofthesinecurve . THESLOPEOFTHESINECURVE Atthetopandbottom(t D =2 a n d t D 3=2)theballchangesdirectionand v D 0: Theslopeatthetopand bottomofthesinecurveiszero . Attimezero,when theball isgoingstraightup,theslopeofthesinecurveis v D 1: At t D ,whentheballand massand f -grapharegoingdown,thevelocityis v D 1: T h e massgoesfastestatthe c e nter.Themassgoes slowest(infactitstops)whentheheightreaches amaximum o r minimum.Thevelocity triangleyields v a t every time t : To n d th e u p ward velocityofthe mass,look atthe upward velocityofthe ball. T h o sevelocitiesarethesame! Themassandballstaylevel,andweknow v f r o m c ircularmotion: T he upwardvelocityis v D c o s t : Fig u r e 1 . 1 8 showstheresultwewant.Ontheright, f D sin t g ivestheheight.On th eleftisthevelocity v D c o s t : T h a tvelocity istheslopeofth e f - c u r ve . T heheight a n d ve locity(red lines)areoscillatingtogether,buttheyareou t o f p h a sejustasthe p o sitiontriangleandvelocitytrianglewereatright angles.Thisisabsolutelyfantastic, th atincalculusthetwomostfamousfunctionsoftrigonometryformapair: T he slope o f the sine c urve isgivenbythecosinecurve . Whe n thed ista nceis f.t / D s in t , thevelo cityis v.t/ D co s t: Admissionofguilt :Th e slopeofsin t wasnotcomputedinthestandardway. Previouslywecompared .t C h/ 2 with t 2 , anddividedthatdistanceby h: Thisaverage velocityapproachedtheslope 2t as h becamesmall. For sin t wecould havedone thesame : averagevelocity D changeinsin t changein t D si n .t C h / sin t h : (1) T hisiswhereweneedtheformulaforsin .t C h/ ,comingsoon.Somehowtheratio in(1)shouldapproachcos t as h 0: (I t d o e s.)Thesineandcosine tthesame p a tternas t 2 a n d 2 t o u r shortcutwastowatchtheshadowofmotionaroundacircle. Fig.1.18 v D cos t when f D s i n t (red); v D si n t w hen f D cos t (bl ack) . Thatlookseasybutyouwillseelaterthatitisextremelyimportant. A t amaximum or minimumtheslope is zero .The curvelevels off. 1.4Circular Motion 77 Q u es tion1 Whatiftheballgoestwiceasfast,toreachangle 2t att ime t ? Answer Thespeedis now 2: The timeforafullcircleis only : Theball's positionis x D cos 2t and y D sin 2t: Thevelocityisstilltangenttothecirclebut the tangentisatangle 2t wheretheballis.Thereforecos 2t entersthe upward velocityand sin 2 t e n tersthehorizontalvelocity.Thedifferenceisthat the ve locitytriangleistwiceasbig . T h e u p wardvelocityisnotcos 2 t bu t 2 c o s 2 t : T h e h o r izontalvelocity is 2 sin 2 t : N o ticethese 2 's! Qu estion2 W ha t istheareaunderthecosinecurvefrom t D 0 to t D =2 ? Y oucananswerthat,ifyouaccepttheFundamentalTheoremofCalc ulus computingareasistheoppositeofcomputingslopes .Theslopeofsin t iscos t , sotheareaunder cos t istheincreaseinsin t: Noreasontobelievethat yet, but we useitanyway. Fromsin 0 D 0 tosin =2 D 1 , theincreaseis 1: Pleaserealizethepower of calculus.Noothermethod could computetheareaunderacosinecurvesofast. THESLOPEOFTHECOSINECURVE Icannotresistuncoveringanotherdistanceand velocity(anot h er f - v pair)withno extrawork. Thistime f isthecosine.Thetimeclockstarts at thetopofthecircle . Theoldtime t D =2 isnow t D 0: ThedottedlinesinFigure1.18showthenewstart. Buttheshadowhasexactly thesamemotiontheballkeepsgoingaroundthecircle, andthemassfollowsitupanddown.The f -graph and v -grapharestillcorrect,both withatimeshiftof =2: Thenew f - graphisthecosine . Thenew v -graphis minusthesine .Theslopeof thecosinecurvefollows the negative ofthesinecurve.Thatisanotherfamouspair, twinsofthe rst: Whenthed ista nceis f.t / D c o s t; thevelo cityis v.t/ D sin t : Yo ucouldseethatcoming,bywatchingtheballgoleftandright(i nsteadof upand down).Itsdistanceacrossis f D cos t: Itsvelocity acrossis v D sin t : T h a ttwin p a ircompletesthecalculusinChapter 1 (tr ig o nometrytocome).Wereviewtheideas: v is the ve locity th e slo pe o f th e d istancecurve th e limit o f averagevelocity overashorttime th e d e rivative o f f: f is the d istance th e a re a u n d ertheve lo city curve th e limit o f to taldistanceovermany shorttimes th e inte gral o f v : D ifferentialcalculus : Com p ute v from f . Integralcalculus : Comp ute f from v: Withconstantvelocity, f equ a ls vt: Withconstantacceleration, v D at and f D 1 2 at 2 : I n harmonicmotion, v D cos t and f D sin t: Onepartofourgoal isto extendthat listforwhichweneedthetools ofcalculus.Anotherandmoreimportantpartis to

(b)whatarei t s x and y coordi nates? (c)whatarei t s x and y vel oci t i es?Thi s parti sharder. 6 If anot herbal lst ays =2 r a di ansaheadof t hebal lwi t hspeed 3; ndi t sangl e,i t s x and y coordi nates,andi t svert icalvel oci t yat t i me t: 7 Amassmovesont he x axis underorovert heorigi nal bal l (on t heuni tci rcl ewi thspeed 1).W hati st heposi t i on x D f.t / ?F i nd x and v at t D =4: P l ot x and v upt o t D : Ontelevisionyouknowimmediatelywhenthe wordsarelive.The same withwriting. 1.4Circular Motion 79 8 D o est henewmass(underorovert hebal l)meett heol dmass (lev elwi t ht hebal l)?W hati sthedi stancebet weent hemassesat t i me t ? 9 Drawgraphsof f.t / D c os 3t andcos 2 t and 2 cos t , marki ngt het i meaxes.Howl ongunt i leach f repeat s? 10 Drawgraphsof f D sin .t C / and v D cos .t C /: Thi s osci l l at i onst ays l evelwi t hwhatbal l? 11 Drawgraphsof f D sin . =2 t / and v D cos . = 2 t /: T hi s osci l l at i onst ays l evelwi t habal lgoi ngwhi chwayst art i ngwhere? 12 D r aw a graph of f.t/ D s i n t C cos t: Est i mat ei t sgreat est hei ght(maxi mum f)andt het i mei treachest hatheight .By computi ng f 2 checkyourest i mat e. 13 Howfastshoul dyourunacrosst heci rcl et omeett hebal lagai n? It t ravel satspeed 1: 14 Amass fal l s fromt het opoft heuni tci rcl ewhent hebal lofspeed 1 pas sesby.W hataccel erat ion a i snecessaryt omeett hebal latt he bot t om? Fi n dth eareau n d er v D cos t fromth ech an gei n f D si n t : 15 from t D 0 to t D 17 fr om t D 0 to t D 2

1 6 f rom t D 0 to t D = 6 1 8 from t D =2 to t D 3 =2: 19 Thedi st ancecurve f D sin 4t y i el dst hevel ocit ycurve v D 4 cos 4t: Expl ai nbot h 4 's. 20 Thedi st ancecurve f D 2 cos 3t y i el dst hevel oci t ycurve v D 6 si n 3t : E xpl ai nt he 6: 21 T hevel oci t ycurve v D cos 4t y i el dst hedi st ancecurve f D 1 4 si n 4 t: Expl ai nt he 1 4 : 22 T h evel oci ty v D 5 sin 5t y i el ds whatdi st ance? 23 F i nd t hesl opeoft hesi necurveat t D =3 f r om v D cos t: Then ndanaveragesl opebydivi di ngsi n =2 si n =3 by t he t i medi fference = 2 =3: 24 T he sl opeof f D sin t at t D 0 i scos 0 D 1: Comput eaverage sl opes . si n t /=t for t D 1;:1;:01;:001: Th eb al lat x D cos t;y D si n t ci rcl es(1)cou n tercl ock wi se(2) wi thrad i us 1 (3)starti n gfrom x D 1;y D 0 (4)atsp eed 1: Fi n d (1)(2)(3)(4) forth emoti on s 2530. 25 x D cos 3t;y D si n 3t 26 x D 3 cos 4t; y D 3 si n 4t 27 x D 5 si n 2t; y D 5 co s 2t 28 x D 1 C cos t; y D s i n t 29 x D cos .t C 1/ ; y D si n .t C 1/ 30 x D cos . t /;y D si n . t / T h e osci l l ati on x D 0; y D si n t goes (1) u p an d d ow n(2) b etween 1 an d 1 (3) st arti n gfrom x D 0; y D 0 (4) at vel oci ty v D cos t : F i n d(1)(2) (3)(4)forth eosci l l ati on s 3136. 31 x D cos t;y D 0 33 x D 0 ;y D 2 sin . t C / 35 x D 0;y D 2 cos 1 2 t 32 x D 0; y D si n 5 t 34 x D cos t;y D co s t 36 x D cos 2 t;y D s i n 2 t 37 Ift hebal lon t heuni t ci rcl ereaches t degr ees att i me t , ndi t s posi t i onandspeedandupwardvel oci t y. 38 Chooset henumber k soth at x D cos k t;y D si n k t compl et esa rot at i onat t D 1: F i ndt hespeedandupwardvel oci t y. 39 Ifapi t cherdoesn'tpause beforest art i ngt ot hrow,abal ki s call ed.TheAmeri canLeaguedeci dedmat hemat i cal l yt hatt herei s alwaysast opbetweenbackwardandforwardmot i on,eveni ft he t i mei st ooshortt oseei t .(Thereforenobal k.)Ist hatt rue? 80 1I n troductiontoCalculus 1.5AReviewofTr igonom et ry Trigonometrybeginswitharighttriangle.Thesizeofthetriangleisnotasimportant a s t heangles.Wefocuson oneparticularanglecallit andon the ratios between thethreesides x;y;r: Theratiosdon'tchangeifthetriangleisscaledtoanothersize. Threesidesgivesixratios,which arethebasicfunctionsoftrigonometry: Fig.1.19 co s D x r D n e a rside hypotenuse s e c D r x D 1 co s s i n D y r D o p p ositeside hypotenuse c s c D r y D 1 sin t a n D y x D op p ositeside nearside co t D x y D 1 tan Of c oursethosesix ratiosarenot independent.Thethreeontheri ghtcomedirectly fromthethreeon theleft.Andthetangentisthesinedivided by thecosine: tan D sin cos

D y= r x=r D y x : No t ethattangentofan angleandtangentto acircleand tangentlineto agraph aredifferentusesofthesameword. Asthecosineof goesto zero,thetangentof goestoin nity.Theside x becomeszero, approaches 90 ; a n dthetriangleis in nitely steep.Thesineof 9 0 is y = r D 1 : Tr ia n gleshave aseriouslimitation.Theyareexcellentforanglesupto 9 0 ; a n d th ey areOKupto 1 8 0 ; bu tafterthattheyfail.Wecannotputa 2 4 0 angleintoa tr ia ngle.Thereforewechangenowtoacircle. Fig.1.20 Tr i gonometryonaci rcl e.Compare 2 sin wi t hsi n 2 andt an (peri ods 2;;). Anglesaremeasuredfromthepositive x axis(counterclockwise).Thus 90 is str a ightup, 1 8 0 istotheleft,and 3 6 0 isinthesamedirectionas 0 : (T h e n 4 5 0 is th esameas 9 0 :) E a c hangleyieldsapointonthecircleofradius r: T h e coordinates x a n d y o f th a tpointcanbenegative(bu tnever r) . A sthepointgoesaroundthecircle, th esixratioscos ; sin ; ta n ; ::: tr a ceoutsixgraphs.Thecosinewaveformis the sa measthesinewaveformjustshiftedby 9 0 : O n e mor e c h a ngecomeswiththe movetoa circle.Degreesare out.Radiansare in .Thedistancearoundthewholecircleis 2 r: T h e distancearou ndtoother points is r: We measuretheanglebythatmultiple . Fo r ahalf-circlethedistanceis r , 1.5A ReviewofTrigonometry 81 s o theangleis radi answhichis 180 : A q uarter-circleis = 2 r a d iansor 9 0 : T he d istancearoundtoangle is r times . Wh e n r D 1 th isistheultimateinsimplicity: Th e d istanceis : A 4 5 angleis 1 8 o f a c ircleand 2=8 radiansandthelength ofthecirculararcis 2=8: Similarlyfor 1 : 3 6 0 D 2 r a d ians 1 D 2 = 3 6 0 r a d ians 1 r a d ian D 3 6 0 =2 d eg r e es. A n anglegoing clockwiseis nega tive .Theangle =3 is 60 and takesus 1 6 o f t he wrong wayaroundthecircle.Whatistheeffectonthesixfunctions? Certainlytheradius r isnotchangedwhenwegoto : A lso x is n o t changed(see Fig u r e 1 . 2 0 a).Bu t y r eversessign,because is b elowtheaxiswhen C is a bove. T h ischangein y a ff e cts y = r a n d y = x bu tnot x = r : c o s . / D c o s s in . / D sin ta n . / D tan : T h e c o sineis even (nochange).Thesineandtangentare odd (changesign). Thesamepointis 5 6 of t he right wayaround.Therefore 5 6 of 2 r adians(or 300) g ives thesamedirectionas =3 r a d ians or 60 : A d ifferenceof 2 ma kes no d ifferenceto x ; y ; r . T h u ssin a n d cos a n dtheotherfourfunctionshaveperiod 2 : We c a ngo vetimes or ahundredtimes aroundthecircle,ad d ing 1 0 o r 2 0 0 to theangle,and thesixfunctionsrepeatthemselves. E X AMPLE E va luatethesixtrigonometricfunctionsat D 2=3 (or D 4=3) . T h isangleisshown inFigure1 . 2 0a(where r D 1) . T h e r a tiosare c o s D x = r D 1=2 sin D y = r D ? 3=2 tan D y = x D ? 3 se c D 2 c sc D 2 = ? 3 co t D 1= ? 3 Th o senumbersillustratebasicfactsaboutthesizesoffourfunctions: | cos | 1 | sin | 1 | sec | 1 | csc | 1: T h e ta ngentand cotangentcanfallanywhere,aslongascot D 1 = ta n : T h e n u mbersrevealmore.Thetangent ? 3 istheratioofsinetocosine.The s e cant 2 is 1 = c o s : T h e irsquaresare 3 a n d 4 (d iff e r ingby 1) . That maynotseem r e markable,butitis. Th e rearethreerelationshipsinthe sq u ares o f thosesixnumbers, a n d th eyarethekeyidentitiesoftrigonometry: c o s 2 C sin 2 D 11 C ta n 2 D sec 2 co t 2 C 1 D csc 2 Everything owsfromthePythagoras formula x 2 C y 2 D r 2 : Divi dingby r 2 gives .x=r/ 2 C .y=r/ 2 D 1: Thatiscos 2 C sin 2 D 1: Dividingby x 2 gives thesecond identity,which is 1 C .y=x/ 2 D .r=x/ 2 : Dividing by y 2 givesthethird.Allthreewill beneeded throughoutthebookandthe rstonehastobeunforgettable. DISTANCESANDADDITIONFORMULAS Tocomputethedistancebetweenpointswestay withPythagoras. T hepointsarein Figure1.21a.Theyareknownbytheir x and y coordinates,and d isthedistance between them.Thethirdpointcompletesarighttriangle. 82 1I n troductiontoCalculus Forthe x distancealongthebottomwedon'tneedhelp.Itis x 2 x 1 (o r | x 2 x 1 | sincedistancescan'tbenegative).Thedistanceupthesideis | y 2 y 1 | : Py th a goras immediatelygivesthedistance d : d is ta nc e b etween p o ints D d D a .x 2 x 1 / 2 C . y 2 y 1 / 2 : (1) Fig.1.21 Di s t ancebet weenpoi nt s andequaldi st ancesi nt wocircl es. Byap plyingthisdistanceformulaintwoidenticalcircles,wediscover thecosine of s t: (Subtr a c tinganglesisimportant.)InFigure1.21b,thedistancesquared is d 2 D . c h angein x / 2 C . c h angein y / 2 D . c o s s cos t / 2 C . sin s sin t / 2 : (2) Fig u r e 1 . 2 1cshowsthesamecircleandtriangle(butrotated).Thesamedistance sq u ared is d 2 D . c o s .s t/ 1/ 2

C . sin .s t// 2 : (3) N ow multiplyoutthesquaresin equations(2)and(3).Whenever . c o sine / 2 C . sin e / 2 a p pears,replac eitby 1 : T h e d istancesarethesame,so .2 / D .3 / : .2 / D 1 C 1 2 c o s s c o s t 2 sin s sin t .3 / D 1 C 1 2 c o s .s t/: A f te r c a nceling 1 C 1 a n dthen 2 , w e havethe a d d ition formula f o r c o s .s t/ : Theco sineo f s t e qua ls c o s s c o s t C s in s s in t: (4) Theco sineo f s C t equa lsco s s co s t sin s s in t: (5) Thee asiestis t D 0: Thencos t D 1 andsin t D 0: Theequationsreduceto cos s D cos s: Togofrom(4)to(5)inallcases,replace t by t: N o changeincos t , bu taminus a p pearswiththesine. I n th e sp ecialcase s D t , w e h avecos .t C t / D . c o s t /. c o s t / . sin t /. sin t /: T h isisamuch-usedformulaforcos 2 t : D o ub lea ng le : cos 2t D c o s 2 t sin 2 t D 2 c o s 2 t 1 D 1 2 s in 2 t: (6) Iamc onstantly usingcos 2 t C sin 2 t D 1 ,toswitchbetweensinesand cosines. Wealso needadditionformulasand double-angleformulasforthe sine of s t a n d s C t a n d 2 t : Fo r thatweconnectsinetocosine,ratherthan . sin e / 2 to . c o sine / 2 : T h e 1.5A ReviewofTrigonometry 83 c o nnectiongoes backtotheratio y=r in ouroriginaltriangle.Thisisthesineofthe angle andalsothecosineofthe complementaryangle =2 : sin D c o s . = 2 / a n dcos D sin . = 2 /: (7) T h e c o mplementaryangleis = 2 b e causethetwoanglesaddto = 2 (a r ig h tangle).Bymaking thisconnectioninProblem 1 9 ; f o r mulas(456)movefrom c o sinestosines: sin .s t/ D sin s c o s t cos s sin t (8) sin .s C t / D sin s c o s t C c o s s sin t (9) sin 2 t D sin .t C t / D 2 sin t c o s t (1 0) I wa n ttostopwiththesetenformulas,evenif morearepossible.Trigonometryis f u ll o f id entitiesthatconnectits sixfunctionsbasicallybecauseallthosefunctions c o me fromasinglerighttriangle.The x ; y ; r r a tios andtheequation x 2 C y 2 D r 2 c a nberewritteninmanyways.Butyouhavenowseentheformulasthatareneeded b y ca1culus. T h ey g ivederivativesinChapter 2 a n dintegralsinChapter 5 : A n d it is typicalofoursubjecttoaddsomething ofits owna limitinwhichanangle

Moir patternsmove.There aregood applicationsinengineeringand opticsbut wehavetogetbacktocalculus. CHAPTER2 De ri va ti ve s 2.1TheDer ivat iveofaFunct ion

Thischapterbeginswiththede nitionofthederivative.Twoexampleswere in Ch a pter 1 : Whenthedistanceis t 2 ,thevelocityis 2t: When f.t/ D sin t wefound v.t/ D co s t: Th evelo cityisn o wca lledth e d eriva tive o f f.t/: Aswemovetoa moreformalde nitionandnewexamples,weusenewsymbols f 1 and df = dt f o r the d e rivative. 2A Attime t , t he der iva tive f 1 .t/ o r df = dt o r v . t / is f 1 .t/ D lim t 0 f . t C t / f.t/ t : (1) Theratioontherightistheaveragevelocityoverashorttime t : T hederivative,on theleftside,isitslimitasthestep t (d elta t)approacheszero. Goslowlyandlookateachpiece.Thedistanceattime t C t is f.t C t/: The distanceattime t is f.t/: Subtractiongives the cha ng eind ista nce ,betweenthose times.Weoftenwrite f forthisdifference: f D f.t C t/ f.t/: Thea ver a g e ve lo cityisthera tio f = t c h angeindistancedivided bychangeintime. T h e limitoftheaveragevelocity isthederivative,ifthislimitexists: df dt D li m t 0 f t : (2) T h is istheneatnotationthatLeibnizinvented: f= t a p p ro a ch es df= dt: Behind theinnocentword limit isaprocessthatthiscoursewillhelpyouunderstand. Notethat f isnot times f ! Itisthecha ng ein f: Similarly t isnot times t: Itis thetimestep,positive ornegativeandeventually small.Tohavea one-lettersymbolwereplace t by h: Therightsidesof (1) and (2) contain averagespeeds.Onthegraph of f.t/ ,the distance u p isdivided bythedistance a cross .Thatgivestheaverageslope f= t: Theleftsidesof (1) and (2) are insta nta neo us speeds df= dt: Theygivetheslope attheinstant t: Thisisthederivative df= dt (when t and f shrinkto zero).Look again atthecalculation for f.t/ D t 2 : 87 88 2D e rivatives f t D f . t C t/ f.t/ t D t 2 C 2 t t C .t/ 2 t 2 t D 2 t C t : (3) Imp o rtantpoint:Thosestepsaretakenbefore t goestozero. If weset t D 0 to o so o n , welea rnno thing .Theratio f= t becomes 0 = 0 (whichismeaningless). Thenumbers f and t mustapproachzerotogether,notseparately.Heretheir ratiois 2t C t ,theaveragespeed. Torepeat:Successcamebywritingout .t C t/ 2 andsubtracting t 2 anddividing by t: Then andonly thencanweapproach t D 0 : Thelimitisthederivative 2t: Therearesever alnewthingsinformulas (1) and (2) .Someareeasybutimportant, othersaremoreprofound.Theideaofafunctionwewillcomebackto, andthe de nitionof alimit.Butthenotationscanbediscussedrightaway.Theyareused constantlyand youalsoneedtoknowhowtoreadthem aloud: f.t/ D f of t D thevalueofthefunction f at t ime t t D delta t

D thetimestepforward orbackwardfrom t f.t C t/ D f of t pl usdelta t D thevalueof f att ime t C t f =delta f D thechange f.t C t/ f.t/ f = t D delta f ove rdelta t D theaveragevelocity f 1 .t/ D f p r imeof t D thevalueofthederivativeattime t df= dt D dfd t D t hesameas f 1 (theinstantaneousvelocity) lim 0 D limitasdelta t g o e stozero D theprocessthatstartswith num bers f= t and producesthenumber df= dt: Fromthoselastwordsyouseewhatliesbehindthenotation df=d t: Thesymbol t indicatesanonzero(usuallyshort)lengthoftime.Thesymbol dt indicates anin nitesimal(evenshorter)lengthoftime.Somemathematiciansworkseparately with df and dt ,and df= dt istheirratio.Forus df= dt isasinglenotation(don't cancel d anddon'tcancel).Thederivative df= dt isthelimitof f= t: When th a tno ta tio n df= dt isa wkwa rd ,use f 1 o r v : Re m a rk T h e notationhides onethingweshouldmention.Thetimestepcanbe n eg a tive ju stas easilyas positive.Wecancomputetheaverage f = t ove r a time in terval b e fo re th etime t , in steadofafter.Thisratioalsoapproaches df = dt: T h e notationalso hidesanotherthing: Thed eriva tivemig htno tex ist . T h e averages f = t mightnotapproachalimit(ithastobethesamelimitgoingforward a n dbackwardfromtime t) . I n th a tcase f 1 .t/ is n o t de ned.Atthatinstantthereis n o c le arreading onthespeedometer.Thiswillhappen inExample 2 . EXAMPLE1 (Constantvelocity V D 2)The distance f is V times t: Thedistance attime t C t is V times t C t: Thed ifference f is V times t : f t D V t t D V so t helimitis df dt D V: T h ederivativeof Vt is V: Thederivativeof 2t is 2: Theaverages f= t arealways V D 2 ,inthisexceptionalcaseofaconstantvelocity.