On the sum of reciprocals of three integers

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1 Introduction

This is a generalization of a simple problem I found on the internet, it states: Find all positive integers a, b, c such that:

$$\begin{cases} a+b+c=100 \\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{9} \end{cases} \tag{1}$$

Outline This article is organized as follows. Section 2 gives account of the general problem. The proof is described in Section 3. Finally, Section 4 gives some examples.

2 The general problem

Let q be a power of a prime number, and n a positive integer such that: $\gcd(q, n) = 1$ find all positive integers a, b, c such that:

$$\begin{cases} a+b+c=n \\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{q} \end{cases} \tag{2}$$

3 Solution

Let p the only prime divisor of q. WLOG we can assume that $\gcd(p,c)=1$ so $\gcd(c-q,qc)=1$. From $\frac{a+b}{ab}=\frac{c-q}{qc}$ we deduce

$$\boxed{\begin{cases} a+b = k(c-q) \\ ab \quad = qkc \end{cases}} \tag{3}$$

for some positive integer k.
In addition to that $a+b+c=n$ yields $n-q=k(c-q)+c-q$ Therefore:

$$\boxed{(k+1)(c-q)=n-q} \tag{4}$$

Particularly we know that we should find $c-q$ among the divisors of the number $n-q$ as for the integers a and b we can easily deduce their values from system 3

4 Special cases

In this section we will conserve the same notations, we will refer to the previous results for the study of a couple of special cases of system 2

4.1 $n=100$ and $q=9$

This is the special case mentioned in the introduction, equation 4 gives: $(k+1)(c-9)=100-9=91=13\times 7$ thus we have three cases:

- $c-9=1$ and $k+1=91 \Rightarrow c=10$ and $k=90$:
  In that case system 3 becomes $a+b=90$ and $ab=9\times 90\times 10=8100$ no need to continue any further since the corresponding second degree equation has no solutions.

- $c-9=7$ and $k+1=13 \Rightarrow c=16$ and $k=12$:
  System 3 becomes $a+b=12\times 7=84$ and $ab=16\times 12\times 9=1728$. The corresponding second degree equation have two solutions 36 and 48. Thus, the solutions are the permutations of $(36,48,16)$ we verify that those are indeed solutions.

- $c-9=13$ and $k+1=7 \Rightarrow c=22$ and $k=6$:
  System 3 becomes $a+b=13\times 6=78$ and $ab=22\times 6\times 9=1188$. This system has solutions but they are irrational.

4.2 $n-q$ is a prime number

From equation 4 we can infer that $k+1=n-q$ is a prime number and $c-q=1$ (remember $k>0$), but from system 3 we deduce $a+b=k(c-q)=k$ and $ab=kq(q+1)$ so a,b verify this relation: $ab=m(a+b) \Rightarrow$ $\boxed{(a-m)(b-m)=m^2}$ where $m=q(q+1)$ and $\boxed{c=q+1}$