EGA I

A. GROTHENDIECK

WHAT THIS IS. This is my poor translation of Grothendieck's EGA I. This will probably consist of lots of online translations and incorrect grammar. You have been warned!

S'il te plaît pardonne-moi, Grothendieck.

Ryan Keleti :)

CONTENTS

Introduction

To Oscar Zariski and André Weil.

This memoir, and the many others that must follow, are intended to form a treatise on the foundations of algebraic geometry. They do not assume, in principle, any particular knowledge of this discipline, and it has even been that such knowledge, despite its obvious advantages, could sometimes (by the too-exclusive habit that the birational point of view it implies) to be harmful to the one who wants to become familiar with the point of view and techniques presented here. However, we assume that the reader has a good knowledge of the following topics:

(a) *Commutative algebra*, as it is exhibited for example in volumes under preparation of the *Elements* of N. Bourbaki (and, pending the publication of these volumes, in Samuel-Zariski [13] and Samuel [11], [12]).
(b) *Homological algebra*, for which we refer to Cartan-Eilenberg [2] (cited as (M)) and Godement [4] (cited as (G)), as well as the recent article by A. Grothendieck [6] (cited as (T)).
(c) *Sheaf Theory*, where our main references will be (G) and (T); this theory provides the essential language for interpreting in "geometric" terms the essential notions of commutative algebra, and to "globalize" them.
(d) Finally, it will be useful for the reader to have some familiarity with *functorial language*, which will be constantly used in this Treatise, and for which the reader may consult (M), (G) and especially (T); the principles of this language and the main results of the general theory of functors will be described in more detail in a book currently in preparation by the authors of this Treatise.

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It is not the place, in this Introduction, to give a more or less summarily description from the point of view of "schemes" in algebraic geometry, nor the long list of reasons which made its adoption necessary, and in particular the systematic acceptance of nilpotent elements in the local rings of "manifolds" that we consider (which necessarily shifts the idea of rational mappings into the background, in favor of those of regular mappings or "morphisms"). This Treatise aims precisely to systematically develop the language of schemes, and will demonstrate, we hope, its necessity. Although it would be easy to do so, we will not try to give here an "intuitive" introduction to the notions developed in Chapter 1. For the reader who would like to have a glimpse of the preliminary study of the subject matter of this Treatise, we refer them to the conference by A. Grothendieck at the International Congress of Mathematicians in Edinburgh in 1958 [7], and the expose [8] of the same author. The work [14] (cited as (FAC)) of J.-P. Serre can also be considered as an intermediary exposition between the classical point of view and the point of view of schemes in algebraic geometry, and, as such, its reading may be an excellent preparation to that of our *Elements*.

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We give below the general outline planned for this Treaty, subject to later modifications, especially concerning the later chapters.

In principal, all chapters are considered open to changes, and supplementary paragraphs can always be added later; such paragraphs would appear in separate fascicles in order to minimise the inconveniences accompanying whatever mode of publication adopted. When the writing of such a paragraph is foreseen or in progress during the publication of a chapter, it will be mentioned in the summary of the chapter in question, even if, owing to certain orders of urgency, its actual publication clearly ought to have been later. For the use of the reader, we give in "Chapter 0" the necessary tools in commutative algebra, homological algebra, and sheaf theory, that will be used throughout this Treatise, that are more or less well known but for which it was not possible to give convenient references. It is recommended for the reader to not read Chapter 0 except whilst reading the Treatise proper, when the results to which we refer seem unfamiliar. Besides, we think that in this way, the reading of this Treatise could be a good method for the beginner to familiarise themselves with commutative algebra and homological algebra, whose study, when not accompanied with tangible applications, is considered tedious, or even depressing, by many.

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To finish, we believe it useful to warn the reader that, as was the case with all the authors themselves, they will almost certainly have difficulty before becoming accustomed to the language of schemes, and to convince themselves that the usual constructions that suggest geometric intuition can be translated, in essentially only one sensible way, to this language. As in many parts of modern mathematics, the first intuition seems further and further away, in appearance, from the correct language needed to express the mathematics in question with complete precision and the desired level of generality. In practice, the psychological difficulty comes from the need to replicate some familiar set-theoretic constructions to a category that is already quite different from that of sets (the category of preschemes, or the category of preschemes over a given prescheme): cartesian products, group laws, ring laws, module laws, fibre bundles, principal homogeneous fibre bundles, etc. It will most likely be difficult for the mathematician, in the future, to shy away from this new effort of abstraction, maybe rather negligible, on the whole, in comparison with that endowed by our fathers, to familiarise themselves with the Theory of Sets.

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The references are given following the numerical system; for example, in III, 4.9.3, the III indicates the chapter, the 4 the paragraph, the 9 the section of the paragraph. If we reference a chapter from within itself then we omit the chapter number.

Chapter 0. — Preliminaries

1. Rings of Fractions

1.0. Rings and Algebras

(1.0.1) All the rings considered in this Treatise will have a *unit element*; all the modules on such a ring will be assumed to be *unitary*; the ring homomorphisms will always be assumed to *transform the unit element into a unit element*; unless otherwise stated, a sub-ring of a ring A will be assumed to *contain the unit element of* A. We will consider especially *commutative* rings, and when we speak of a ring without specification, it will be implied that it is commutative. If A is a ring not necessarily commutative, by A-module we will we mean a left module, unless stated otherwise.

(1.0.2) Let A, B be two rings, not necessarily commutative, $\varphi : A \to B$ a homomorphism. Any left (resp. right) B-module M can be provided with a left (resp. right) A-module structure by $a \cdot m = \varphi(a) \cdot m$ (resp. $m \cdot a = m \cdot \varphi(a)$); when it will be necessary to distinguish M as an A-module or a B-module, we will denote by $M_{[\varphi]}$ the left (resp. right) A-module as defined. If L is an A-module, then a homomorphism $u : L \to M_{[\varphi]}$ is a homomorphism of commutative groups such that $u(a \cdot x) = \varphi(a) \cdot u(x)$ for $a \in A$, $x \in L$; we will also say that it is a φ-*homomorphism* $L \to M$, and that the pair (φ, u) (or, by misuse of langauge, u) is a *di-homomorphism* of (A, L) in (B, M). The pairs (A, L) formed by a ring A and an A-module L thus form a *category* for which the morphisms are di-homomorphisms.

(1.0.3) Under the hypothesis of (1.0.2), if $\mathfrak{J}$ is a left (resp. right) ideal of A, we denote by $B\mathfrak{J}$ (resp. $\mathfrak{J}B$) the left (resp. right) ideal $B\varphi(\mathfrak{J})$ (resp. $\varphi(\mathfrak{J})B$) of B generated by $\varphi(\mathfrak{J})$; it is also the image of the canonical homomorphism $B \otimes_A \mathfrak{J} \to B$ (resp. $\mathfrak{J} \otimes_A B \to B$) of left (resp. right) B-modules.

(1.0.4) If A is a (commutative) ring, B a non necessarily commutative ring, the data of a structure of an A-*algebra* on B is equivalent to the data of a ring homomorphism $\varphi : A \to B$ such that $\varphi(A)$ is contained in the center of B. For all ideals $\mathfrak{J}$ of A, $\mathfrak{J}B = B\mathfrak{J}$ is then a two-sided ideal of B, and for every B-module M, $\mathfrak{J}M$ is then a B-module equal to $(B\mathfrak{J})M$.

(1.0.5) We will not return to the notions of *module finite type* and *algebra* (commutative) *of finite type*; to say that an A-module M is of finite type means that there exists an exact sequence $A^p \to M \to 0$. We say that an A-module M admits a *finite presentation* if it is isomorphic to the cokernel of a homomorphism $A^p \to A^q$, in other words, there exists an exact sequence $A^p \to A^q \to M \to 0$. We note that for a *Noetherian* ring A, every A-module of finite type admits a finite presentation.

Let us recall that an A-algebra B is called *integral* over A if every element in B is a root in B of a monic polynomial with coefficients in A; equivalently, every element of B is contained in a subalgebra of B which is an A-*module of finite type*. When this is so, and B is commutative, the subalgebra of B generated by a finite part of B is an A-module of finite type; for a commutative algebra B to be integral and of finite type over A, it is necessary and therefore sufficient that B be an A-module of finite type; we also say that B is an *integral* A-*algebra of finite type* (or simply *finite* if there is no confusion). It will be observed that in these definitions, it is not assumed that the homomorphism $A \to B$ defining the structure of an A-algebra is injective.

(1.0.6) An *integral domain* is a ring in which the product of a finite family of elements $\neq 0$ is $\neq 0$; equivalently, in such a ring we have $0 \neq 1$ and the product of two elements $\neq 0$ is non zero. A *prime* ideal of a ring A is an ideal $\mathfrak{p}$ such that $A/\mathfrak{p}$ is integral; this therefore entails $\mathfrak{p} \neq A$. For a ring A to have at least one prime ideal, it is necessary and sufficent that $A \neq \{0\}$.

(1.0.7) A *local* ring is a ring A in which there exists a single maximal ideal, which is then the complement of the invertible elements and contains all the ideals $\neq A$. If A and B are two local rings, $\mathfrak{m}$ and $\mathfrak{n}$ their respective maximal ideals, we say that a homomorphism $\varphi : A \to B$ is *local* if $\varphi(\mathfrak{m}) \subset \mathfrak{n}$ (or, equivalently, $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$). By passing to quotients, such a homomorphism then defines a momomorphism of the residue field $A/\mathfrak{m}$ into the residue field $B/\mathfrak{n}$. The composition of two local homomorphisms is a local homomorphism.

1.1. Root (radical) of an ideal. Nilradical and radical of a ring.

(1.1.1) Let $\mathfrak{a}$ be an ideal of a ring A; the *root* (*radical*) of $\mathfrak{a}$, denoted by $\mathfrak{r}(\mathfrak{a})$, is the set of $x \in A$ such that $x^n \in \mathfrak{a}$ for an integer $n > 0$; it is an ideal containing $\mathfrak{a}$. We have $\mathfrak{r}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a})$; the relation $\mathfrak{a} \subset \mathfrak{b}$ leads to $\mathfrak{r}(\mathfrak{a}) \subset \mathfrak{r}(\mathfrak{b})$; the root of a finite intersection of ideals is the intersection of their roots. If φ is a homomorphism of a ring A' into A, then we have $\mathfrak{r}(\varphi^{-1}(\mathfrak{a})) = \varphi^{-1}(\mathfrak{r}(\mathfrak{a}))$ for any ideal $\mathfrak{a} \subset A$. For an ideal to be the root of an ideal, it is necessary and sufficient that it be an intersection of prime ideals. The root of an ideal $\mathfrak{a}$ is the intersection of the *minimal* prime ideals among those containing $\mathfrak{a}$; if A is Noetherian, these minimal prime ideals are finite in number.

The root of the ideal (0) is also called the *nilradical* of A; it is the set $\mathfrak{N}$ of the nilpotent elements of A. It is said that the ring A is *reduced* if $\mathfrak{N} = (0)$; for every ring A, the quotient $A/\mathfrak{N}$ of A by its nilradical is a reduced ring.

(1.1.2) Recall that the *radical* $\mathfrak{R}(A)$ of a ring A (not necessarily commutative) is the intersection of the maximal left ideals of A (and also the intersection of maximal right ideals). The radical of $A/\mathfrak{R}(A)$ is (0).

1.2. Modules and rings of fractions

(1.2.1) We say that a subset S of a ring A is *multiplicative* if $1 \in S$ and if the product of two elements of S is in S. The examples which will be the most important for the following are: 1st the set S_f of powers f^n $(n \geqslant 0)$ of an element $f \in A$; 2nd the complement $A - \mathfrak{p}$ of a *prime* ideal $\mathfrak{p}$ of A.

(1.2.2) Let S be a multiplicative subset of a ring A, M an A-module; in the set $M \times S$, the relation between couples (m_1, s_1), (m_2, s_2):

$$\text{"There exists } s \in S \text{ such that } s(s_1 m_2 - s_2 m_1) = 0\text{"}$$

is an equivalence relation. We denote by $S^{-1}M$ the quotient set of $M \times S$ by this relation, by m/s the canonical image in $S^{-1}M$ of the pair (m, s); we call the *canonical* mapping of M in $S^{-1}M$ the mapping $i_M^S : m \mapsto m/1$ (also denoted i^S). This mapping is generally neither injective nor surjective; its kernel is the set of $m \in M$ such that there exists an $s \in S$ for which $sm = 0$.

In $S^{-1}M$ we define an additive group law by taking

$$(m_1/s_1) + (m_2/s_2) = (s_2 m_1 + s_1 m_2)/(s_1 s_2)$$

(we check that it is independent of the expressions of the elements of $S^{-1}M$ considered). On $S^{-1}A$ we further define a multiplicative law by taking $(a_1/s_1)(a_2/s_2) = (a_1 a_2)/(s_1 s_2)$, and finally an external law on $S^{-1}M$, having $S^{-1}A$ as a set of operators, by setting $(a/s)(m/s') = (am)/(ss')$. It is thus verified that $S^{-1}A$ is provided with a ring structure (called *the ring of fractions of* A, *with denominators in* S) and $S^{-1}M$ the structure of an $S^{-1}A$-module (called *the module of fractions of* M, *with denominators in* S); for all $s \in S$, $s/1$ is invertible in $S^{-1}A$, its inverse being $1/s$. The canonical mapping i_A^S (resp. i_M^S) is a homomorphism of rings (resp. a homomorphism of A-modules, $S^{-1}M$ being considered A-module by means of the homomorphism $i_A^S : A \to S^{-1}A$).

(1.2.3) If $S_f = \{f^n\}_{n \geqslant 0}$ for a $f \in A$, we write A_f and M_f instead of $S_f^{-1}A$ and $S_f^{-1}M$; when A_f is considered as algebra over A, we can write $A_f = A[1/f]$. A_f is isomorphic to the quotient algebra $A[T]/(fT - 1)A[T]$. When $f = 1$, A_f and M_f identify canonically with A and M; if f is niipotent, A_f and M_f are reduced to 0. When $S = A - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A, we write $A_\mathfrak{p}$ and $M_\mathfrak{p}$ instead of $S^{-1}A$ and $S^{-1}M$; $A_\mathfrak{p}$ is a *local ring* whose maximal ideal $\mathfrak{q}$ is generated by $i_A^S(\mathfrak{p})$, and we have $(i_A^S)^{-1}(\mathfrak{q}) = \mathfrak{p}$; by passing to the quotients, i_A^S gives a monomorphism of the integral domains $A/\mathfrak{p}$ into the field $A_\mathfrak{p}/\mathfrak{q}$, which identifies with the field of fractions of $A/\mathfrak{p}$.

(1.2.4) The ring of fractions $S^{-1}A$ and the canonical homomorphism i_A^S are a solution of a *universal mapping problem*: any homomorphism u of A into a ring B such that $u(S)$ is composed of *invertible* elements in B factorizes in one way

$$u : A \xrightarrow{i_A^S} S^{-1}A \xrightarrow{u^*} B$$

where u^* is a ring homomorphism. Under the same hypotheses, let M be an A-module, N a B-module, $v : M \to N$ a homomorphism of A-modules (for the B-module structure on N defined by $u : A \to B$); then v is factorizes in a single way

$$v : M \xrightarrow{i_M^S} S^{-1}M \xrightarrow{v^*} N$$

where v^* is a homomorphism of $S^{-1}A$-modules (for the $S^{-1}A$-module structure on N defined by u^*).

(1.2.5) We define a canonical isomorphism $S^{-1}A \otimes_A M \xrightarrow{\sim} S^{-1}M$ of $S^{-1}A$- modules, sending the element $(a/s) \otimes m$ to the element $(am)/s$, the isomorphism reciprocally applying m/s to $(1/s) \otimes m$.

(1.2.7) When A is an *integral domain*, for which K denotes its field of fractions, the canonical mapping $i_A^S : A \to S^{-1}A$ is injective for any multiplicative subset S not containing 0, and $S^{-1}A$ then identifies canonically with a subring of K containing A. In particular, for every prime ideal $\mathfrak{p}$ of A , $A_\mathfrak{p}$ is a local ring containing A, with maximal ideal $\mathfrak{p}A_\mathfrak{p}$, and we have $\mathfrak{p}A_\mathfrak{p} \cap A = \mathfrak{p}$.

(1.2.8) If A is a *reduced* ring (1.1.1), so is $S^{-1}A$: indeed, if $(x/s)^n = 0$ for $x \in A$, $s \in S$, it means that there exists $s' \in S$ such that $s'x^n = 0$, hence $(s'x)^n = 0$, which, by hypothesis, entails $s'x = 0$, so $x/s = 0$.

1.3. Functorial properties

(1.3.1) Let M, N be two A-modules, u an A-homomorphism $M \to N$. If S is a multiplicative subset of A, we define a $S^{-1}A$-homomorphism $S^{-1}M \to S^{-1}N$, denoted by $S^{-1}u$, by putting $S^{-1}u(m/s) = u(m)/s$; if $S^{-1}M$ and $S^{-1}N$ are canonically identified with $S^{-1}A \otimes_A M$ and $S^{-1}A \otimes_A N$ (1.2.5), $S^{-1}u$ is considered as $1 \otimes u$. If P is a third A-module, v an A-homomorphism $N \to P$, we have $S^{-1}(v \circ u) = (S^{-1}v) \circ (S^{-1}u)$; in other words, $S^{-1}M$ is a *covariant functor in* M, of the category of A-modules into that of $S^{-1}A$-modules (A and S being fixed).

(1.3.2) The functor $S^{-1}M$ is *exact*; in other words, if the following

$$M \xrightarrow{u} N \xrightarrow{v} P$$

is exact, so is the following

$$S^{-1}M \xrightarrow{S^{-1}u} S^{-1}N \xrightarrow{S^{-1}v} S^{-1}P.$$

In particular, if $u : M \to N$ is injective (resp. surjective), the same is true for $S^{-1}u$; if N and P are two submodules of M, $S^{-1}N$ and $S^{-1}P$ identify canonically with submodules of $S^{-1}M$, and we have

$$S^{-1}(N + P) = S^{-1}N + S^{-1}P \quad \text{and} \quad S^{-1}(N \cap P) = (S^{-1}N) \cap (S^{-1}P).$$

(1.3.3) Let $(M_\alpha, \varphi_{\beta\alpha})$ be an inductive system of A-modules; then $(S^{-1}M_\alpha, S^{-1}\varphi_{\beta\alpha})$ is an inductive system of $S^{-1}A$-modules. Expressing the $S^{-1}M_\alpha$ and $S^{-1}\varphi_{\beta\alpha}$ as tensor products (1.2.5 and 1.3.1), it follows from the permutability of tensor product and inductive limit operations that we have a canonical isomorphism

$$S^{-1} \varinjlim M_\alpha \xrightarrow{\sim} \varinjlim S^{-1}M_\alpha$$

which is further expressed by saying that the functor $S^{-1}M$ (in M) *commutes with inductive limits*.

(1.3.4) Let M, N be two A-modules; there is a canonical *functorial* isomorphism (in M and N)

$$(S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) \xrightarrow{\sim} S^{-1}(M \otimes_A N)$$

which transforms $(m/s) \otimes (n/t)$ into $(m \otimes n)/st$.

(1.3.5) We also have a *functorial* homomorphism (in M and N)

$$S^{-1}\operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

which, at u/s, corresponds to the homomorphism $m/t \mapsto u(m)/st$. When M has a finite presentation, the preceding homomorphism is an *isomorphism*: it is immediate when M is of the form A^r, and goes on to the general case starting from the following exact sequence $A^p \to A^q \to M \to 0$, and using the exactness of the functor $S^{-1}M$ and the left-exactness of the functor $\operatorname{Hom}_A(M, N)$ in M. Note that this case always occurs when A is *Noetherian* and the A-module M is *of finite type*.

Chapter 1. — The language of schemes

1. Preschemes and morphisms of preschemes

1.0. Definition of preschemes

(2.1.1) Given a ringed space $(X, \mathcal{O}_X)$, we say that an open subset V of X is an *affine open* if the ringed space $(V, \mathcal{O}_X|V)$ is an affine scheme (1.7.1).

Definition **(2.1.2)**. — *We define a prescheme to be a ringed space* $(X, \mathcal{O}_X)$ *such that every point of* X *admits an affine open neighbourhood.*

Proposition **(2.1.3)**. — *If* $(X, \mathcal{O}_X)$ *is a prescheme then the open affines give a base for the topology of* X.

In effect, if V is an arbitrary open neighbourhood of $x \in X$, then there exists by hypothesis an open neighbourhood W of x such that $(W, \mathcal{O}_X|W)$ is an affine scheme; we write A to mean its ring. In the space W, $V \cap W$ is an open neighbourhood of x; thus there exists $f \in A$ such that $D(f)$ is an open neighbourhood of x contained inside $V \cap W$ (1.1.10 (i)). The ringed space $(D(f), \mathcal{O}_X|D(f))$ is thus an affine scheme, isomorphic to A_f (1.3.6), whence the proposition.