EGA I

A. GROTHENDIECK

WHAT THIS IS. This is a community translation of Grothendieck's EGA I. As it is a work in progress by multiple people, it will probably have a few mistakes — if you spot any then please feel free to let us know!

Warning. EGA uses 'prescheme' for what is now usually called a scheme, and 'scheme' for what is now usually called a separated scheme.

On est désolés, Grothendieck.

— Ryan Keleti, Tim Hosgood

https://github.com/ryankeleti/en.ega.i.

Contents

Introduction

To Oscar Zariski and André Weil.

This memoir, and the many others that must follow, are intended to form a treatise on the foundations of algebraic geometry. They do not assume, in principle, any particular knowledge of this discipline, and it has even been that such knowledge, despite its obvious advantages, could sometimes (by the too-exclusive habit that the birational point of view it implies) to be harmful to the one who wants to become familiar with the point of view and techniques presented here. However, we assume that the reader has a good knowledge of the following topics: **5**

(a) *Commutative algebra*, as it is exhibited for example in volumes under preparation of the *Elements* of N. Bourbaki (and, pending the publication of these volumes, in Samuel-Zariski [13] and Samuel [11], [12]).
(b) *Homological algebra*, for which we refer to Cartan-Eilenberg [2] (cited as (M)) and Godement [4] (cited as (G)), as well as the recent article by A. Grothendieck [6] (cited as (T)).
(c) *Sheaf Theory*, where our main references will be (G) and (T); this theory provides the essential language for interpreting in "geometric" terms the essential notions of commutative algebra, and to "globalize" them.
(d) Finally, it will be useful for the reader to have some familiarity with *functorial language*, which will be constantly used in this Treatise, and for which the reader may consult (M), (G) and especially (T); the principles of this language and the main results of the general theory of functors will be described in more detail in a book currently in preparation by the authors of this Treatise.

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It is not the place, in this Introduction, to give a more or less summarily description from the point of view of "schemes" in algebraic geometry, nor the long list of reasons which made its adoption necessary, and in particular the systematic acceptance of nilpotent elements in the local rings of "manifolds" that we consider (which necessarily shifts the idea of rational mappings into the background, in favor of those of regular mappings or "morphisms"). This Treatise aims precisely to systematically develop the language of schemes, and will demonstrate, we hope, its necessity. Although it would be easy to do so, we will not try to **6**
give here an "intuitive" introduction to the notions developed in Chapter 1. For the reader who would like to have a glimpse of the preliminary study of the subject matter of this Treatise, we refer them to the conference by A. Grothendieck at the International Congress of Mathematicians in Edinburgh in 1958 [7], and the expose [8] of the same author. The work [14] (cited as (FAC)) of J.-P. Serre can also be considered as an intermediary exposition between the classical point of view and the point of view of schemes in algebraic geometry, and, as such, its reading may be an excellent preparation to that of our *Elements*.

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We give below the general outline planned for this Treatise, subject to later modifications, especially concerning the later chapters.

Chapter I. — The language of schemes.
— II. — Elementary global study of some classes of morphisms.
— III. — Cohomology of algebraic coherent sheaves. Applications.
— IV. — Local study of morphisms.
— V. — Elementary procedures of constructing schemes.
— VI. — Descent. General method of constructing schemes.
— VII. — Schemes of groups, principal fibre bundles.
— VIII. — Differential study of fibre bundles.
— IX. — The fundamental group.
— X. — Residues and duality.
— XI. — Theories of intersection, Chern classes, Riemann-Roch theorem.
— XII. — Abelian schemes and Picard schemes.
— XIII. — Weil cohomology.

In principal, all chapters are considered open to changes, and supplementary paragraphs can always be added later; such paragraphs would appear in separate fascicles in order to minimise the inconveniences accompanying whatever mode of publication adopted. When the writing of such a paragraph is foreseen or in progress during the publication of a chapter, it will be mentioned in the summary of the chapter in question, even if, owing to certain orders of urgency, its actual publication clearly ought to have been later. For the use of the reader, we give in "Chapter 0" the necessary tools in commutative algebra, homological algebra, and sheaf theory, that will be used throughout this Treatise, that are more or less well known but for which it was not possible to give convenient references. It is recommended for the reader to not read Chapter 0 except whilst reading the Treatise proper, when the results to which we refer seem unfamiliar. Besides, we think that in **7**
this way, the reading of this Treatise could be a good method for the beginner to familiarise themselves with commutative algebra and homological algebra, whose study, when not accompanied with tangible applications, is considered tedious, or even depressing, by many.

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It is outside of our capabilities to give a historic overview, or even a summary thereof, of the ideas and results described. The text will contain only those references considered particularly useful for comprehension, and we indicate the origin only of the most important results. Formally, at least, the subjects discussed in our work are reasonably new, which explains the scarcity of references made to the Fathers of algebraic geometry from the 19th to the beginning of the 20th century, whose works we know only by hear-say. It is suitable, however, to say some words here about the works which have most directly influenced the authors and contributed to the development of scheme-theoretic point of view. We absolutely must mention the fundamental work (FAC) of J.-P. Serre first, which has served as an introduction to algebraic geometry for more that one young student (one of the authors of this Treatise being one), deterred by the dryness of the classic *Foundations* of A. Weil [18]. It is there that it is shown, for the first time, that the "Zariski topology" of an "abstract" algebraic variety is perfectly suited to applying certain techniques from algebraic topology, and notably to be able to define a cohomology theory. Further, the definition of an algebraic variety given therein is that which translates most naturally to the idea that we develop here[1]. Serre himself had incidentally noted that the cohomology theory of affine algebraic varieties could translate without difficulty by replacing the affine algebras over a field by arbitrary commutative rings. Chapters I and II of this Treatise, and the first two paragraphs of chapter III, can thus be considered, for the most part, as easy translations, in this bigger framework, of the principal results of

[1] Just as J.-P. Serre informed us, it is right to note that the idea of defining the structure of a manifold by the data of a sheaf of rings is due to H. Cartan, who took this idea as the starting point of his theory of analytic spaces. Of course, just as in algebraic geometry, it would be important in "analytic geometry" to give the right to use nilpotent elements in local rings of analytic spaces. This extension of the definition of H. Cartan and J.-P. Serre has recently been broached by H. Grauert [5], and there is room to hope that a systematic report of analytic geometry in this setting will soon see the light of day. It is also evident that the ideas and techniques developed in this Treatise retain a sense of analytic geometry, even though one must expect more considerable technical difficulties in this latter theory. We can foresee that algebraic geometry, by the simplicity of its methods, will be able to serve as a sort of formal model for future developments in the theory of analytic spaces.

(FAC) and a later article of the same author [15]. We have also vastly profited from the *Séminaire de Géométrie algébrique* de C. Chevalley [1]; in particular, the systematic usage of "constructible sets" introduced by him has turned out to be extremely useful in the theory of schemes (cf. chap. IV). We have also borrowed from him the study of morphisms from the point of view of dimension (chap. IV), that translates with negligible change **8**
to the framework of schemes. It also merits noting that the idea of "schemes of local rings", introduced by Chevalley, naturally lends itself to being extended to algebraic geometry (not having, however, all the flexibility and generality that we intend to give it here); for the connections between this idea and our theory, see the chap. I, s. 8. One such extension has been developed by M. Nagata in a series of memoires [9], which contain many special results concerning algebraic geometry over Dedekind rings[1].

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It goes without saying that a book on algebraic geometry, and especially a book dealing with the fundamentals, is of course influenced, [...], by mathematicians such as O. Zariski and A. Weil. In particular, the *Théorie des fonctions holomorphes* de Zariski [20], properly flexible thanks to the cohomological methods and an existence theorem (chap. III, ss. 4 et 5), is (along with the method of descent described in chap. VI) one of the principal tools used in this Treatise, and it seems to us one of the most powerful at our disposal in algebraic geometry.

The general technique in which it is employed can be sketched as follows (a typical example of which will be given in chap. XI, in the study of the fundamental group). We have a proper morphism (chap. II) $f\colon X \to Y$ of algebraic varieties (more generally, of schemes) that we wish to study on the neighbourhood of a point $y \in Y$, with the aim of resolving a problem P relative to a neighbourhood of y. We follow successive steps:

(1) We can suppose that Y is affine, such that X becomes a scheme defined on the affine ring A of Y, and we can even replace A by the local ring of y. This reduction is always easy in practice (chap. V) and brings us to the case where A is a *local* ring.
(2) We study the problem in question when A is a local *artinien* ring. So that the problem P still makes sense when A is not assumed to be integral, sometimes we have to reformulate P, and it appears that we often thus obtain a better understanding of the problem during this stage, in an "infinitesimal" way.
(3) The theory of formal schemes (chap. III, ss. 3, 4, and 5) lets us pass from the case of an artinien ring to a *complete local ring*.
(4) Finally, if A is an arbitrary local ring, considering "multiform sections" of suitable schemes over X approximates the idea of a given "formal" section (chap. IV), and this will let us pass, by extension of **9**
scalars to the completion of A, from a known result of [...]to an analogous result for a finite simple (e.g. unramified) extension of A.

This sketch shows the importance of the systematic study of schemes defined over an artinien ring A. The point of view of Serre in his formulation of the theory of local class fields, and the recent works of Greenberg, seem to suggest that such a study could be undertaken by functorially attaching, to some such scheme X, a scheme X' over the residue field k of A (assumed perfect) of dimension equal (in nice cases) to $n \dim X$, where n is the height of A.

As for the influence of A. Weil, it suffices to say that it is the need to develop the tools necessary to formulate, with full generality, the definition of "Weil cohomology," and to tackle the proof[1] of all the formal properties necessary to establish the famous conjectures in diophantine geometry [19], that has been one of the principal motivations of the writing of this Treatise, as has the desire to find the natural setting of the usual ideas and methods of algebraic geometry, and to give the authors the chance to understand these ideas and methods.

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[1]Amongst the works that come close to our point of view of algebraic geometry, we pick out the work of E. Kàhler [22] and a recent note of Chow and Igusa [3], which go back over certain results of (FAC) in the context of Nagata-Chevalley theory, as well as giving a Künneth formula.

[1]To avoid any misunderstanding, we point out that this task has barely been undertaken at the moment of writing this Introduction, and still hasn't led to the proof of the Weil conjectures.

To finish, we believe it useful to warn the reader that, as was the case with all the authors themselves, they will almost certainly have difficulty before becoming accustomed to the language of schemes, and to convince themselves that the usual constructions that suggest geometric intuition can be translated, in essentially only one sensible way, to this language. As in many parts of modern mathematics, the first intuition seems further and further away, in appearance, from the correct language needed to express the mathematics in question with complete precision and the desired level of generality. In practice, the psychological difficulty comes from the need to replicate some familiar set-theoretic constructions to a category that is already quite different from that of sets (the category of preschemes, or the category of preschemes over a given prescheme): cartesian products, group laws, ring laws, module laws, fibre bundles, principal homogeneous fibre bundles, etc. It will most likely be difficult for the mathematician, in the future, to shy away from this new effort of abstraction, maybe rather negligible, on the whole, in comparison with that endowed by our fathers, to familiarise themselves with the Theory of Sets.

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The references are given following the numerical system; for example, in III, 4.9.3, the III indicates the chapter, the 4 the paragraph, the 9 the section of the paragraph. If we reference a chapter from within itself then we omit the chapter number.

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Chapter 0. — Preliminaries

1. Rings of Fractions

1.0. Rings and Algebras

(1.1.1) Let $\mathfrak{a}$ be an ideal of a ring A; the *root* (*radical*) of $\mathfrak{a}$, denoted by $\mathfrak{r}(\mathfrak{a})$, is the set of $x \in A$ such that $x^n \in \mathfrak{a}$ for an integer $n > 0$; it is an ideal containing $\mathfrak{a}$. We have $\mathfrak{r}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a})$; the relation $\mathfrak{a} \subset \mathfrak{b}$ leads to $\mathfrak{r}(\mathfrak{a}) \subset \mathfrak{r}(\mathfrak{b})$; the root of a finite intersection of ideals is the intersection of their roots. If φ is a homomorphism of a ring A' into A, then we have $\mathfrak{r}(\varphi^{-1}(\mathfrak{a})) = \varphi^{-1}(\mathfrak{r}(\mathfrak{a}))$ for any ideal $\mathfrak{a} \subset A$. For an ideal to be the root of an ideal, it is necessary and sufficient that it be an intersection of prime ideals. The root of an ideal $\mathfrak{a}$ is the intersection of the *minimal* prime ideals among those containing $\mathfrak{a}$; if A is Noetherian, these minimal prime ideals are finite in number.

The root of the ideal (0) is also called the *nilradical* of A; it is the set $\mathfrak{N}$ of the nilpotent elements of A. It is said that the ring A is *reduced* if $\mathfrak{N} = (0)$; for every ring A, the quotient $A/\mathfrak{N}$ of A by its nilradical is a reduced ring.

(1.1.2) Recall that the *radical* $\mathfrak{R}(A)$ of a ring A (not necessarily commutative) is the intersection of the maximal left ideals of A (and also the intersection of maximal right ideals). The radical of $A/\mathfrak{R}(A)$ is (0).

1.1. Root (radical) of an ideal. Nilradical and radical of a ring.

(1.1.1) Let $\mathfrak{a}$ be an ideal of a ring A; the *root* (*radical*) of $\mathfrak{a}$, denoted by $\mathfrak{r}(\mathfrak{a})$, is the set of $x \in A$ such that $x^n \in \mathfrak{a}$ for an integer $n > 0$; it is an ideal containing $\mathfrak{a}$. We have $\mathfrak{r}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a})$; the relation $\mathfrak{a} \subset \mathfrak{b}$ leads to $\mathfrak{r}(\mathfrak{a}) \subset \mathfrak{r}(\mathfrak{b})$; the root of a finite intersection of ideals is the intersection of their roots. If φ is a homomorphism of a ring A' into A, then we have $\mathfrak{r}(\varphi^{-1}(\mathfrak{a})) = \varphi^{-1}(\mathfrak{r}(\mathfrak{a}))$ for any ideal $\mathfrak{a} \subset A$. For an ideal to be the root of an ideal, it is necessary and sufficient that it be an intersection of prime ideals. The root of an ideal $\mathfrak{a}$ is the intersection of the *minimal* prime ideals among those containing $\mathfrak{a}$; if A is Noetherian, these minimal prime ideals are finite in number.

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1.2. Modules and rings of fractions.

(1.2.1) We say that a subset S of a ring A is *multiplicative* if $1 \in S$ and if the product of two elements of We say **13**
that a part S of a ring A is *multiplicative* if $1 \in S$ and if the product of two elements of S is in S. The examples which will be the most important for the following are: 1st the set S_f of powers f^n $(n \geqslant 0)$ of an element $f \in A$; 2nd the complement $A - \mathfrak{p}$ of a *prime* ideal $\mathfrak{p}$ of A.

(1.2.2) Let S be a multiplicative subset of a ring A, M an A-module; in the set $M \times S$, the relation between couples (m_1, s_1), (m_2, s_2):

"There exists $s \in S$ such that $s(s_1 m_2 - s_2 m_1) = 0$"

is an equivalence relation. We denote by $S^{-1}M$ the quotient set of $M \times S$ by this relation, by m/s the canonical image in $S^{-1}M$ of the pair (m, s); we call the *canonical* mapping of M in $S^{-1}M$ the mapping $i_M^S : m \mapsto m/1$ (also denoted i^S). This mapping is generally neither injective nor surjective; its kernel is the set of $m \in M$ such that there exists an $s \in S$ for which $sm = 0$.

On $S^{-1}M$ we define an additive group law by taking

$$(m_1/s_1) + (m_2/s_2) = (s_2 m_1 + s_1 m_2)/(s_1 s_2)$$

(we check that it is independent of the expressions of the elements of $S^{-1}M$ considered). On $S^{-1}A$ we further define a multiplicative law by taking $(a_1/s_1)(a_2/s_2) = (a_1a_2)/(s_1s_2)$, and finally an external law on $S^{-1}M$, having $S^{-1}A$ as a set of operators, by setting $(a/s)(m/s') = (am)/(ss')$. It is thus verified that $S^{-1}A$ is provided with a ring structure (called *the ring of fractions of* A, *with denominators in* S) and $S^{-1}M$ the structure of an $S^{-1}A$-module (called *the module of fractions of* M, *with denominators in* S); for all $s \in S$, $s/1$ is invertible in $S^{-1}A$, its inverse being $1/s$. The canonical mapping i_A^S (resp. i_M^S) is a homomorphism of rings (resp. a homomorphism of A-modules, $S^{-1}M$ being considered A-module by means of the homomorphism $i_A^S : A \to S^{-1}A$).

(1.2.3) If $S_f = \{f^n\}_{n\geqslant 0}$ for a $f \in A$, we write A_f and M_f instead of $S_f^{-1}A$ and $S_f^{-1}M$; when A_f is considered as algebra over A, we can write $A_f = A[1/f]$. A_f is isomorphic to the quotient algebra $A[T]/(fT-1)A[T]$. When $f = 1$, A_f and M_f identify canonically with A and M; if f is niipotent, A_f and M_f are reduced to 0. When $S = A - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A, we write $A_\mathfrak{p}$ and $M_\mathfrak{p}$ instead of $S^{-1}A$ and $S^{-1}M$; $A_\mathfrak{p}$ is a *local ring* whose maximal ideal $\mathfrak{q}$ is generated by $i_A^S(\mathfrak{p})$, and we have $(i_A^S)^{-1}(\mathfrak{q}) = \mathfrak{p}$; by passing to the quotients, i_A^S gives a monomorphism of the integral domains $A/\mathfrak{p}$ into the field $A_\mathfrak{p}/\mathfrak{q}$, which identifies with the field of fractions of $A/\mathfrak{p}$.

(1.2.4) The ring of fractions $S^{-1}A$ and the canonical homomorphism i_A^S are a solution of a *universal mapping problem*: any homomorphism u of A into a ring B such that $u(S)$ is composed of *invertible* elements in B factorizes in one way

$$u : A \xrightarrow{i_A^S} S^{-1}A \xrightarrow{u^*} B$$

where u^* is a ring homomorphism. Under the same hypotheses, let M be an A-module, N a B-module, **14**
$v : M \to N$ a homomorphism of A-modules (for the B-module structure on N defined by $u : A \to B$); then v is factorizes in a single way

$$v : M \xrightarrow{i_M^S} S^{-1}M \xrightarrow{v^*} N$$

where v^* is a homomorphism of $S^{-1}A$-modules (for the $S^{-1}A$-module structure on N defined by u^*).

(1.2.5) We define a canonical isomorphism $S^{-1}A \otimes_A M \xrightarrow{\sim} S^{-1}M$ of $S^{-1}A$- modules, sending the element $(a/s) \otimes m$ to the element $(am)/s$, the isomorphism reciprocally applying m/s to $(1/s) \otimes m$.

(1.2.7) When A is an *integral domain*, for which K denotes its field of fractions, the canonical mapping $i_A^S : A \to S^{-1}A$ is injective for any multiplicative subset S not containing 0, and $S^{-1}A$ then identifies canonically with a subring of K containing A. In particular, for every prime ideal $\mathfrak{p}$ of A , $A_\mathfrak{p}$ is a local ring containing A, with maximal ideal $\mathfrak{p}A_\mathfrak{p}$, and we have $\mathfrak{p}A_\mathfrak{p} \cap A = \mathfrak{p}$.

(1.2.8) If A is a *reduced* ring (1.1.1), so is $S^{-1}A$: indeed, if $(x/s)^n = 0$ for $x \in A$, $s \in S$, it means that there exists $s' \in S$ such that $s'x^n = 0$, hence $(s'x)^n = 0$, which, by hypothesis, entails $s'x = 0$, so $x/s = 0$.

1.3. Functorial properties.

(1.3.1) Let M, N be two A-modules, u an A-homomorphism $M \to N$. If S is a multiplicative subset of A, we define a $S^{-1}A$-homomorphism $S^{-1}M \to S^{-1}N$, denoted by $S^{-1}u$, by putting $S^{-1}u(m/s) = u(m)/s$; if $S^{-1}M$ and $S^{-1}N$ are canonically identified with $S^{-1}A \otimes_A M$ and $S^{-1}A \otimes_A N$ (1.2.5), $S^{-1}u$ is considered as $1 \otimes u$. If P is a third A-module, v an A-homomorphism $N \to P$, we have $S^{-1}(v \circ u) = (S^{-1}v) \circ (S^{-1}u)$; in other words, $S^{-1}M$ is a *covariant functor in* M, of the category of A-modules into that of $S^{-1}A$-modules (A and S being fixed).

(1.3.2) The functor $S^{-1}M$ is *exact*; in other words, if the following

$$M \xrightarrow{u} N \xrightarrow{v} P$$

is exact, so is the following

$$S^{-1}M \xrightarrow{S^{-1}u} S^{-1}N \xrightarrow{S^{-1}v} S^{-1}P.$$

In particular, if $u : M \to N$ is injective (resp. surjective), the same is true for $S^{-1}u$; if N and P are two **15**
submodules of M, $S^{-1}N$ and $S^{-1}P$ identify canonically with submodules of $S^{-1}M$, and we have

$$S^{-1}(N+P) = S^{-1}N + S^{-1}P \quad \text{and} \quad S^{-1}(N \cap P) = (S^{-1}N) \cap (S^{-1}P).$$

(1.3.3) Let $(M_\alpha, \varphi_{\beta\alpha})$ be an inductive system of A-modules; then $(S^{-1}M_\alpha, S^{-1}\varphi_{\beta\alpha})$ is an inductive system of $S^{-1}A$-modules. Expressing the $S^{-1}M_\alpha$ and $S^{-1}\varphi_{\beta\alpha}$ as tensor products (1.2.5 and 1.3.1), it follows from the permutability of tensor product and inductive limit operations that we have a canonical isomorphism

$$S^{-1} \varinjlim M_\alpha \xrightarrow{\sim} \varinjlim S^{-1}M_\alpha$$

which is further expressed by saying that the functor $S^{-1}M$ (in M) *commutes with inductive limits*.

(1.3.4) Let M, N be two A-modules; there is a canonical *functorial* isomorphism (in M and N)

$$(S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) \xrightarrow{\sim} S^{-1}(M \otimes_A N)$$

which transforms $(m/s) \otimes (n/t)$ into $(m \otimes n)/st$.

(1.3.5) We also have a *functorial* homomorphism (in M and N)

$$S^{-1} \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

which, at u/s, corresponds to the homomorphism $m/t \mapsto u(m)/st$. When M has a finite presentation, the preceding homomorphism is an *isomorphism*: it is immediate when M is of the form A^r, and goes on to the general case starting from the following exact sequence $A^p \to A^q \to M \to 0$, and using the exactness of the functor $S^{-1}M$ and the left-exactness of the functor $\mathrm{Hom}_A(M, N)$ in M. Note that this case always occurs when A is *Noetherian* and the A-module M is *of finite type*.

1.4. Change of multiplicative subset.

(1.4.1) Let S, T be two multiplicative subsets of a ring A such that $S \subset T$; there exists a canonical homomorphism $\rho_A^{T,S}$ (or simply $\rho^{T,S}$) of $S^{-1}A$ into $T^{-1}A$, sending the element denoted a/s of $S^{-1}A$ to the element denoted a/s in $T^{-1}A$; we have $i_A^T = \rho_A^{T,S} \circ i_A^S$. For every A-module M, there exists in the same way an $S^{-1}A$-linear map of $S^{-1}M$ into $T^{-1}M$ (the latter considered as an $S^{-1}A$-module thanks to the homomorphism $\rho_A^{T,S}$), which matches the element m/s of $S^{-1}M$ to the element m/s of $T^{-1}M$; we note that the map $\rho_M^{T,S}$, or simply $\rho^{T,S}$, and we still have $i_M^T = \rho_M^{T,S} \circ i_M^S$; in canonical identification (1.2.5), $\rho_M^{T,S}$ identifies with $\rho_A^{T,S} \otimes 1$. The homomorphism $\rho_M^{T,S}$ is a *functorial morphism* (or natural transformation) of the functor $S^{-1}M$ into the functor $T^{-1}M$, in other words, the diagram

$$\begin{tikzcd} S^{-1}M \arrow[r, "S^{-1}u"] \arrow[d, "\rho_M^{T,S}"'] & S^{-1}N \arrow[d, "\rho_N^{T,S}"] \\ T^{-1}M \arrow[r, "T^{-1}u"] & T^{-1}N \end{tikzcd}$$

is commutative, for every homomorphism $u : M \to N$; $T^{-1}u$ is entirely determined by $S^{-1}u$, because for $m \in M$ **16**
and $t \in T$, we have

$$(T^{-1}u)(m/t) = (t/1)^{-1}\rho^{T,S}((S^{-1}u)(m/1)).$$

(1.4.2) With the same notation, for two A-modules M, N, the diagrams (cf. (1.3.4) and (1.3.5))

$$\begin{tikzcd} (S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) \arrow[r, "\sim"] \arrow[d] & S^{-1}(M \otimes_A N) \arrow[d] \\ (T^{-1}M) \otimes_{T^{-1}A} (T^{-1}N) \arrow[r, "\sim"] & T^{-1}(M \otimes_A N) \end{tikzcd} \qquad \begin{tikzcd} S^{-1} \mathrm{Hom}_A(M, N) \arrow[r] \arrow[d] & \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \arrow[d] \\ T^{-1} \mathrm{Hom}_A(M, N) \arrow[r] & \mathrm{Hom}_{T^{-1}A}(T^{-1}M, T^{-1}N) \end{tikzcd}$$

are commutative.

(1.4.3) There is an important case in which the homomorphism $\rho^{T,S}$ is *bijective*, we know that then every element of T is divisor of an element of S; we then identify by $\rho^{T,S}$ the modules $S^{-1}M$ and $T^{-1}M$. We say that S is *saturated* if every divisor in A of an element of S is in S; by replacing S with the set T of all the divisors of the elements of S (a set which is multiplicative and saturated), we see that we can always, if we wish, be limited to the consideration of modules of fractions $S^{-1}M$, where S is saturated.

(1.4.4) If S, T, U are three multiplicative subsets of A such that $S \subset T \subset U$, we have

$$\rho^{U,S} = \rho^{U,T} \circ \rho^{T,S}.$$

(1.4.5) Consider an *increasing filtered family* (S_α) of multiplicative subsets of A (we write $\alpha \leqslant \beta$ for $S_\alpha \subset S_\beta$), and let S be the multiplicative subset $\bigcup_\alpha S_\alpha$; let us put $\rho_{\beta\alpha} = \rho_A^{S_\beta, S_\alpha}$ for $\alpha \leqslant \beta$; according to (1.4.4), the homomorphisms $\rho_{\beta\alpha}$ define a ring A' as the *inductive limit* of the inductive system of rings $(S_\alpha^{-1}A, \rho_{\beta\alpha})$. Let ρ_α be the canonical map $S_\alpha^{-1}A \to A'$, and let $\varphi_\alpha = \rho_A^{S,S_\alpha}$; as $\varphi_\alpha = \varphi_\beta \circ \rho_{\beta\alpha}$ for $\alpha \leqslant \beta$ according to (1.4.4), we can uniquely define a homomorphism $\varphi : A' \to S^{-1}A$ such that the diagram

$$\begin{tikzcd} & S_\alpha^{-1}A \arrow[d, "\rho_{\beta\alpha}"] \arrow[ddl, "\rho_\alpha"'] \arrow[ddr, "\varphi_\alpha"] & \\ & S_\beta^{-1}A \arrow[dl, "\rho_\beta"] \arrow[dr, "\varphi_\beta"'] & \\ A' \arrow[rr, "\varphi"'] & & S^{-1}A \end{tikzcd} \qquad (\alpha \leqslant \beta)$$

is commutative. In fact, φ is an *isomorphism*; it is indeed immediate by construction that φ is surjective. On the other hand, if $\rho_\alpha(a/s_\alpha) \in A'$ is such that $\varphi(\rho_\alpha(a/s_\alpha)) = 0$, this means that $a/s_\alpha = 0$ in $S^{-1}A$, that is, to say that there exists $s \in S$ such that $sa = 0$; but there is a $\beta \geqslant \alpha$ such that $s \in S_\beta$, and consequently, as $\rho_\alpha(a/s_\alpha) = \rho_\beta(sa/ss_\alpha) = 0$, we find that φ is injective. The case for an A-module M is treated likewise, and thus we have defined canonical isomorphisms

$$\varinjlim S_\alpha^{-1}A \xrightarrow{\sim} (\varinjlim S_\alpha)^{-1}A, \qquad \varinjlim S_\alpha^{-1}M \xrightarrow{\sim} (\varinjlim S_\alpha)^{-1}M,$$

the second being *functorial* in M.

(1.4.6) Let S_1, S_2 be two multiplicative subsets of A; then S_1S_2 is also a multiplicative subset of A. Let us 17
denote by S_2' the canonical image of S_2 in the ring $S_1^{-1}A$, which is a multiplicative subset of this ring. For every A-module M there is then a functorial isomorphism

$$S_2'^{-1}(S_1^{-1}M) \xrightarrow{\sim} (S_1S_2)^{-1}M$$

which maps $(m/s_1)/(s_2/1)$ to the element $m/(s_1s_2)$.

1.5. Change of ring.

(1.5.1) Let A, A' be two rings, φ a homomorphism $A' \to A$, S (resp. S') a multiplicative subset of A (resp. A'), such that $\varphi(S') \subset S$; the composition homomorphism $A' \xrightarrow{\varphi} A \to S^{-1}A$ factors as $A' \to S'^{-1} \xrightarrow{\varphi^{S'}} S^{-1}A$ by virtue of (1.2.4); where $\varphi^{S'}(a'/s') = \varphi(a')/\varphi(s')$. If $A = \varphi(A')$ and $S = \varphi(S')$, $\varphi^{S'}$ is *surjective*. If $A' = A$ and if φ is the identity, $\varphi^{S'}$ is none other than the homomorphism $\rho_A^{S,S'}$ defined in (1.4.1).

(1.5.2) Under the hypothesis of (1.5.1), let M be an A-module. There exists a canonical functorial morphism

$$\sigma : S'^{-1}(M_{[\varphi]}) \longrightarrow (S^{-1}M)_{[\varphi^{S'}]}$$

of $S'^{-1}A'$-modules, sending each element m/s' of $S'^{-1}(M_{[\varphi]})$ to the element $m/\varphi(s')$ of $(S^{-1}M)_{[\varphi^{S'}]}$; in fact, we verify immediately that this definition does not depend on the expression m/s' of the element considered. *When* $S = \varphi(S')$, the homomorphism σ is *bijective*. When $A' = A$ and φ is the identity, σ is none other than the homomorphism $\rho_M^{S,S'}$ defined in (1.4.1).

When $M = A$ is taken in particular, the homomorphism φ defines on A an A'-algebra structure; $S'^{-1}(A_{[\varphi]})$ is then provided with a ring structure, for which it identifies with $(\varphi(S'))^{-1}A$, and the homomorphism $\sigma : S'^{-1}(A_{[\varphi]}) \to S^{-1}A$ is a homomorphism of $S'^{-1}A'$-algebras.

(1.5.3) Let M and N be two A-modules; by composing the homomorphisms defined in (1.3.4) and (1.5.2), we obtain a homomorphism

$$(S^{-1}M \otimes_{S^{-1}A} S^{-1}N)_{[\varphi^{S'}]} \longleftarrow S'^{-1}((M \otimes A)_{[\varphi]})$$

which is an isomorphism when $\varphi(S') = S$. Similarly, by composing the homorphisms in (1.3.5) and (1.5.2), we obtain a homomorphism

$$S'^{-1}((\mathrm{Hom}_A(M,N))_{[\varphi]}) \longrightarrow (\mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N))_{[\varphi^{S'}]}$$

which is an isomorphism when $\varphi(S') = S$ and M admits a finite presentation.

(1.5.4) Let us now consider an A'-module N', and form the tensor product $N' \otimes_{A'} A_{[\varphi]}$, which can be considered as an A-module by setting $a \cdot (n' \otimes b) = n' \otimes (ab)$. There is a functorial isomorphism of $S^{-1}A$-modules

$$\tau : (S'^{-1}N') \otimes_{S'^{-1}A'} (S^{-1}A)_{[\varphi^{S'}]} \xrightarrow{\sim} S^{-1}(N' \otimes_{A'} A_{[\varphi]})$$

which maps the element $(n'/s') \otimes (a/s)$ to the element $(n' \otimes a)/(\varphi(s')s)$; indeed, we verify separately that when we replace n'/s' (resp. a/s) by another expression of the same element, $(n' \otimes a)/(\varphi(s')s)$ does not change; on the other hand, we can define a reciprocal homomorphism of τ by sending $(n' \otimes a)/s$ to the element $(n'/1) \otimes (a/s)$: we use the fact that $S^{-1}(N' \otimes_{A'} A_{[\varphi]})$ is canonically isomorphic to $(N' \otimes_{A'} A_{[\varphi]}) \otimes_A S^{-1}A$ (1.2.5), so also to $N' \otimes_{A'} (S^{-1}A)_{[\varphi]}$, by designating by ψ the composite homomorphism $a' \mapsto \varphi(a')/1$ of A' into $S^{-1}A$. **18**

(1.5.5) If M' and N' are two A'-modules, by composing the isomorphisms (1.3.4) and (1.5.4), we obtain an isomorphism

$$S'^{-1}M \otimes_{S'^{-1}A'} S'^{-1}N' \otimes_{S'^{-1}A'} S^{-1}A \xrightarrow{\sim} S^{-1}(M' \otimes_{A'} N' \otimes_{A'} A).$$

Likewise, if M' admits a finite presentation, we have by (1.3.5) and (1.5.4) an isomorphism

$$\mathrm{Hom}_{S'^{-1}A'}(S'^{-1}M', S'^{-1}N') \otimes_{S'^{-1}A'} S^{-1}A \xrightarrow{\sim} S^{-1}(\mathrm{Hom}_{A'}(M', N') \otimes_{A'} A).$$

(1.5.6) Under the hypothesis of (1.5.1), let T (resp. T') be a second multiplicative subset of A (resp. A') such that $S \subset T$ (resp. $S' \subset T'$) and $\varphi(T') \subset T$. Then the diagram

$$\begin{tikzcd}
S'^{-1}A' \arrow[r, "\varphi^{S'}"] \arrow[d, "\rho^{T',S'}"'] & S^{-1}A \arrow[d, "\rho^{T,S}"] \\
T'^{-1}A' \arrow[r, "\varphi^{T'}"'] & T^{-1}A
\end{tikzcd}$$

is commutative. If M is an A-module, the diagram

$$\begin{tikzcd}
S'^{-1}(M_{[\varphi]}) \arrow[r, "\sigma"] \arrow[d, "\rho^{T',S'}"'] & (S^{-1}M)_{[\varphi^{S'}]} \arrow[d, "\rho^{T,S}"] \\
T'^{-1}(M_{[\varphi]}) \arrow[r, "\sigma"'] & (T^{-1}M)_{[\varphi^{T'}]}
\end{tikzcd}$$

is commutative. Finally, if N' is an A'-module, the diagram

$$\begin{tikzcd}
(S'^{-1}N') \otimes_{S'^{-1}A'} (S^{-1}A)_{[\varphi^{S'}]} \arrow[r, "\sim", "\tau"'] \arrow[d] & S^{-1}(N' \otimes_{A'} A_{[\varphi]}) \arrow[d, "\rho^{T,S}"] \\
(T'^{-1}N') \otimes_{T'^{-1}A'} (T^{-1}A)_{[\varphi^{T'}]} \arrow[r, "\sim", "\tau"'] & T^{-1}(N' \otimes_{A'} A_{[\varphi]})
\end{tikzcd}$$

is commutative, the left vertical arrow obtained by applying $\rho_{N'}^{T',S'}$ to $S'^{-1}N'$ and $\rho_A^{T,S}$ to $S^{-1}A$.

(1.5.7) Let A'' be a third ring, $\varphi' : A'' \to A'$ a ring homomorphism, S'' a multiplicative subset of A'' such **19**

that $\varphi'(S'') \subset S'$. Set $\varphi'' = \varphi \circ \varphi'$; then we have

$$\varphi''^{S''} = \varphi^{S'} \circ \varphi'^{S''}.$$

Let M be an A-module; evidently we have $M_{[\varphi'']} = (M_{[\varphi]})_{[\varphi']}$; if σ' and σ'' are the homomorphisms defined by φ' and φ'' as σ is defined in (1.5.2) by φ, we have the transitivity formula

$$\sigma'' = \sigma \circ \sigma'.$$

Finally, let N'' be an A''-module; the A-module $N'' \otimes_{A''} A_{[\varphi'']}$ identifies canonically with $(N'' \otimes_{A''} A'_{[\varphi']}) \otimes_{A'} A_{[\varphi]}$, and likewise the $S^{-1}A$-module $(S''^{-1}N'') \otimes_{S''^{-1}A''} (S^{-1}A)_{[\varphi''^{S''}]}$ identifies canonically with $((S''^{-1}N'') \otimes_{S''^{-1}A''} (S'^{-1}A')_{[\varphi'^{S''}]}) \otimes_{S'^{-1}A'} (S^{-1}A)_{[\varphi^{S'}]}$. With these identifications, if τ' and τ'' are the isomorphisms defined by φ' and φ'' as τ is defined in (1.5.4) by φ, we have the transitivity formula

$$\tau'' = \tau \circ (\tau' \otimes 1).$$

(1.5.8) Let A be a subring of a ring B; for every *minimal* prime ideal $\mathfrak{p}$ of A, there exists a minimal prime ideal $\mathfrak{q}$ of B such that $\mathfrak{p} = A \cap \mathfrak{q}$. Indeed, $A_\mathfrak{p}$ is a subring of $B_\mathfrak{p}$ (1.3.2) and has *a single* prime ideal $\mathfrak{p}'$ (1.2.6); since $B_\mathfrak{p}$ is not reduced to 0, it has at least one prime ideal $\mathfrak{q}'$ and we have necessarily $\mathfrak{q}' \cap A_\mathfrak{p} = \mathfrak{p}'$; the prime ideal $\mathfrak{q}_1$ of B, a reciprocal image of $\mathfrak{q}'$ is thus such that $\mathfrak{q}_1 \cap A = \mathfrak{p}$, and *a fortiori* we have $\mathfrak{q} \cap A = \mathfrak{p}$ for every minimal prime ideal $\mathfrak{q}$ of B contained in $\mathfrak{q}_1$.

1.6. Indentification of the module M_f as an inductive limit.

(1.6.1) Let M be an A-module, f an element of A. Consider a sequence (M_n) of A-modules, all identical to M, and for each pair of integers $m \leqslant n$, let φ_{nm} be the homomorphism $z \mapsto f^{n-m}z$ of M_m into M_n; it is immediate that $((M_n), (\varphi_{nm}))$ is an *inductive system* of A-modules; let $N = \varinjlim M_n$ be the inductive limit of this system. We define a canonical A-isomorphism, *functorial* of N on M_f. For this reason, let us note that, for all n, $\theta_n : z \mapsto z/f^n$ is an A-homomorphism of $M = M_n$ into M_f, and it follows from the definitions that we have $\theta_n \circ \varphi nm = \theta_m$ for $m \leqslant n$. There exists therefore an A-homomorphism $\theta : N \to M_f$ such that, if φ_n denotes the canonical homomorphism $M_n \to N$, we have $\theta_n = \theta \circ \varphi_n$ for all n. Since, by hypothesis, every element of M_f is of the form z/f^n for at least n, it is clear that θ is surjective. On the other hand, if $\theta(\varphi_n(z)) = 0$, in other words $z/f^n = 0$, there exists an integer $k > 0$ such that $f^k z = 0$, so $\varphi_{n+k,n}(z) = 0$, which results in $\varphi_n(z) = 0$. We can therefore identify M_f and $\varinjlim M_n$ by means of θ.

(1.6.2) Now write $M_{f,n}$, φ^f_{nm} and φ^f_n instead of M_n, φ_{nm} and φ_n. Let g be a second element of A. As f^n divides $f^n g^n$, we have a functorial homomorphism

$$\rho_{fg,f} : M_f \longrightarrow M_{fg} \quad \text{(1.4.1 and 1.4.3)};$$

if we indentify M_f and M_{fg} with $\varinjlim M_{f,n}$ and $\varinjlim M_{fg,n}$ respectively, $\rho_{fg,f}$ identifies with the *inductive limit* **20**
of the maps $\rho^n_{fg,f} : M_{f,n} \to M_{fg,n}$, defined by $\rho^n_{fg,f}(z) = g^n z$. Indeed, this follows immediately from the commutivity of the diagram

$$\begin{array}{ccc} M_{f,n} & \xrightarrow{\rho^n_{fg,f}} & M_{fg,n} \\ {\scriptstyle \varphi^f_n}\downarrow & & \downarrow{\scriptstyle \varphi^{fg}_n} \\ M_f & \xrightarrow{\rho_{fg,f}} & M_{fg}. \end{array}$$

1.7. Support of a module.

(1.7.1) Given an A-module M, we call the *support* of M and denote by $\mathrm{Supp}(M)$ the set of prime ideals $\mathfrak{p}$ of A such that $M_\mathfrak{p} \neq 0$. For $M = 0$, it is necessary and sufficient that $\mathrm{Supp}(M) = \varnothing$, because if $M_\mathfrak{p} = 0$ for all $\mathfrak{p}$, the annihilator of an element $x \in M$ cannot be contained in any prime ideal of A, so A is total.

(1.7.2) If $0 \to N \to M \to P \to 0$ is an exact sequence of A-modules, we have

$$\mathrm{Supp}(M) = \mathrm{Supp}(N) \cup \mathrm{Supp}(P)$$

because for every prime ideal $\mathfrak{p}$ of A, the sequence $0 \to N_{\mathfrak{p}} \to M_{\mathfrak{p}} \to P_{\mathfrak{p}} \to 0$ is exact (1.3.2) and for $M_{\mathfrak{p}} = 0$, it is necessary and sufficient that $N_{\mathfrak{p}} = P_{\mathfrak{p}} = 0$.

(1.7.3) If M is the sum of a family (M_λ) of submodules, $M_{\mathfrak{p}}$ is the sum of $(M_\lambda)_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of A (1.3.3 and 1.3.2), so $\mathrm{Supp}(M) = \bigcup_\lambda \mathrm{Supp}(M_\lambda)$.

(1.7.4) If M is an A-module *of finite type*, $\mathrm{Supp}(M)$ is the set of prime ideals *containing the annihilator of* M. Indeed, if M is cyclic and generated by x, say that $M_{\mathfrak{p}} = 0$ means that there exists $s \notin \mathfrak{p}$ such that $s \cdot x = 0$, so that $\mathfrak{p}$ does not contain the annihilator of x. If now M admits a finite system $(x_i)_{1 \leqslant i \leqslant n}$ of generators and if $\mathfrak{a}_i$ is the annihilator of x_i, it follows from (1.7.3) that $\mathrm{Supp}(M)$ is th set of $\mathfrak{p}$ containing one of $\mathfrak{a}_i$, or, equivalently, the set of $\mathfrak{p}$ containing $\mathfrak{a} = \bigcap_i \mathfrak{a}_i$, which is the annihilator of M.

(1.7.5) If M and N are two A-modules *of finite type*, we have

$$\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N).$$

It can be seen that if $\mathfrak{p}$ is a prime ideal of A, the condition $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \neq 0$ is equivalent to "$M_{\mathfrak{p}} \neq 0$ and $N_{\mathfrak{p}} \neq 0$" (taking into account (1.3.4)). In other words, it is about seeing that if P, Q are two modules of finite type over a *local* ring B, not reduced to 0, then $P \otimes_B Q \neq 0$. Let $\mathfrak{m}$ be the maximal ideal of B. By virtue of Nakayama's lemma, the vector spaces $P/\mathfrak{m}P$ and $Q/\mathfrak{m}Q$ are not reduced to 0, so it is the same with the tensor product $(P/\mathfrak{m}P) \otimes_{B/\mathfrak{m}} (Q/\mathfrak{m}Q) = (P \otimes_B Q) \otimes_B (B/\mathfrak{m})$, hence the conclusion.

In particular, if M is an A-module of finite type, $\mathfrak{a}$ an ideal of A, $\mathrm{Supp}(M/\mathfrak{a}M)$ is the set of prime ideals containing both $\mathfrak{a}$ and the annihilator $\mathfrak{n}$ of M (1.7.4), that is, the set of prime ideals containing $\mathfrak{a} + \mathfrak{n}$.

2. Irreducible spaces. Noetherian spaces.

2.1. Irreducible spaces.

(2.1.1) We say that a topological space X is *irreducible* if it is nonempty and if it is not a union of two distinct 21
closed subspaces of X. It is the same to say that $X \neq \varnothing$ and that the intersection of two nonempty open sets (and consequently of a finite number of open sets) of X is nonempty, or that every nonempty open set is everywhere dense, or that any closed subset is *rare*, or finally that all open sets of X are *connected*.

(2.1.2) For a subspace Y of a topological spave X to be irreducible, it is necessary and sufficient that its closure $\overline{Y}$ be irreducible. In particular, any subspace which is the closure $\overline{\{x\}}$ of a singleton is irreducible; we will express the relation $y \in \overline{\{x\}}$ (equivalent to $\overline{\{y\}} \subset \overline{\{x\}}$) by saying that there is a *specialization of* x or that there is a *generalization of* y. When there exists in an irreducible space X a point x such that $X = \overline{\{x\}}$, we will say that x is a *generic point* of X. Any nonempty open subset of X then contains x, and any subspace containing x admits x for a generic point.

(2.1.3) Recall that a *Kolmogoroff space* is a topological space X satisfying the axiom of separation:

(T_0) If $x \neq y$ are any two points of X, there is an open set containing one of the points x, y and not the other.

If an irreducible Kolmogoroff space admits a generic point, it admits *only* one since a nonempty open set contains any generic point.

Recall that a topological space X is said to be *quasi-compact* if, from any collection of open sets of X, one can extract a finite cover of X (or, equivalently, if any decreasing filter family of nonempty closed sets has a nonempty intersection). If X is a quasi-compact space, then any nonempty closed subset A of X contains a *minimal* nonempty closed set M, because the set of nonempty closed subsets of A is inductive for the relation $\supset$; if in addition X is a Kolmogoroff space, M is necessarily reduced to a single point (or, as we say by abuse of language, is a *closed point*).

(2.1.4) In an irreducible space X, every nonempty open subspace U is irreducible, and if X admits a generic point x, x is also a generic point of U.

Let (U_α) be a cover (whose set of indices is nonempty) of a topological space X, consisting of nonempty open sets; if X is irreducible, it is necessary and sufficient that U_α is irreducible for all α, and that $U_\alpha \cap U_\beta \neq \varnothing$ for any α, β. The condition is clearly necessary; to the that it is sufficient, it suffices to prove that if V is a nonempty open subset of X, then $V \cap U_\alpha$ is nonempty for all α, since then $V \cap U_\alpha$ is dense in U_α for all α, and consequently V is dense in X. Now there is at least one index γ such that $V \cap U_\gamma \neq \varnothing$, so $V \cap U_\gamma$ is dense in U_γ, and as for all α, $U_\alpha \cap V_\alpha \neq \varnothing$, we also have $V \cap U_\alpha \cap U_\gamma \neq \varnothing$.

(2.1.5) Let X be an irreducible space, f a continuous map of X into a topological space Y. Then $f(X)$ is irreducible, and if x is a generic point of X, $f(x)$ is a generic point of $f(X)$ and hence also of $\overline{f(X)}$. In particular, if in addition Y is irreducible and with a single generic point y, for $f(X)$ to be everywhere dense, it is necessary and sufficient that $f(x) = y$. **22**

(2.1.6) Any irreducible subspace of a topological space X is contained in a maximal irreducible subspace, which is necessarily closed. Maximal irreducible subspaces of X are called the *irreducible components* of X. If Z_1, Z_2 are two irreducible components distinct from the space X, $Z_1 \cap Z_2$ is a closed *rare* set in each of the subspaces Z_1, Z_2; in particular, if an irreducible component of X admits a generic point (2.1.2) such a point can not belong to any other irreducible component. If X has only a *finite* number of irreducible components Z_i $(1 \leqslant i \leqslant n)$, and if, for each i,we put $U_i = Z_i \cap C(\bigcup_{j \neq i} Z_j)$, the U_i are open, irreducible, disjoint, and their union is dense in X. Let U be an open subset of a topological space X. If Z is an irreducible subset of X that intersects with U, $Z \cap U$ is open and dense in Z, thus irreducible; conversely, for any irreducible closed subset Y of U, the closure $\overline{Y}$ of Y in X is irreducible and $\overline{Y} \cap U = Y$. We conclude that there is a *bijective correspondence* between the irreducible components of U and the irreducible components of X which intersect U.

(2.1.7) If a topological space X is a union of a *finite* number of irreducible closed subspaces Y_i, the irreducible components of X are the maximal elements of the set of Y_i, because if Z is an irreducible closed subset of X, Z is the union of $Z \cap Y_i$, from which one sees that Z must be contained in one of the Y_i. Let Y be a subspace of a topological space X, and suppose that Y has only a finite number of irreducible components Y_i, $(1 \leqslant i \leqslant n)$; then the closures $\overline{Y_i}$ in X are the irreducible components of Y.

(2.1.8) Let Y be an irreducible space admitting a single generic point y. Let X be a topological space, f a continuous mapping from X to Y. Then, for any irreducible component Z of X intersecting $f^{-1}(y)$, $f(Z)$ is dense in Y. The converse is not necessarily true; however, if Z has a generic point z, and if $f(Z)$ is dense in Y, we must have $f(z) = y$ (2.1.5); in addition, $Z \cap f^{-1}(y)$ is then the closure of $\{z\}$ in $f^{-1}(y)$ and is therefore irreducible, and like any irreducible subset of $f^{-1}(y)$ containing z is necessarily contained in Z (2.1.6), z is a generic point of $Z \cap f^{-1}(y)$. As any irreducible component of $f^{-1}(y)$ is contained in an irreducible component of X, we see that if any irreducible component Z of X intersecting $f^{-1}(y)$ admits a generic point, then there is a *bijective correspondence* between all these components and all the irreducible components $Z \cap f^{-1}(y)$ of $f^{-1}(y)$, the generic points of Z being identical to those of $Z \cap f^{-1}(y)$.

2.2. Noetherian spaces.

(2.2.1) We say that a topological space X is *Noetherian* if the set of open subsets of X satisfies the *maximal* condition, or, equivalently, if the set of closed subsets of X satisfies the *minimal* condition. We say that X is *locally Noetherian* if all $x \in X$ admit a neighborhood which is a Noetherian subspace. **23**

(2.2.2) Let E be an ordered set satisfying the *minimal* condition, and let $\mathbf{P}$ be a property of the elements of E subject to the following condition: if $a \in E$ is such that for any $x < a$, $\mathbf{P}(x)$ is true, then $\mathbf{P}(a)$ is true. Under these conditions, $\mathbf{P}(x)$ *is true for all* $x \in E$ ("principle of Noetherian recurrence"). Indeed, let F be the set of $x \in E$ for which $\mathbf{P}(x)$ is false; if F were not empty, it would have a minimal element a, and as then $\mathbf{P}(x)$ is true for all $x < a$, $\mathbf{P}(a)$ would be true, which is a contradiction.

We will apply this principle in particular when E is a *set of closed subsets of a Noetherian space*.

(2.2.3) Any subspace of a Noetherian space is Noetherian. Conversely, any topological space that is a finite union of Noetherian subspaces is Noetherian.

(2.2.4) Any Noetherian space is quasi-compact; conversely, any topological space in which all open sets are quasi-compact is Noetherian.

(2.2.5) A Noetherian space has only a *finite* number of irreducible components, as we see by Noetherian recurrence.

3. Supplement on Sheaves.

3.1. Sheaves with values in a category.

(3.1.1) Let **K** be a category, $(A_\alpha)_{\alpha\in I}$, $(A_{\alpha\beta})_{(\alpha,\beta)\in I\times I}$ two families of objects of **K** such that $A_{\beta\alpha} = A_{\alpha\beta}$, $(\rho_{\alpha\beta})_{(\alpha,\beta)\in I\times I}$ a family of morphisms $\rho_{\alpha\beta} : A_\alpha \to A_{\alpha\beta}$. We say that a pair formed by an object A of **K** and a family of morphisms $\rho_\alpha : A \to A_\alpha$ is a *solution to the universal problem* defined by the data of the families (A_α), $(A_{\alpha\beta})$, and $(\rho_{\alpha\beta})$ if, for every object B of **K**, the mapping which, at all $f \in \mathrm{Hom}(B, A)$ matches the family $(\rho_\alpha \circ f) \in \prod_\alpha \mathrm{Hom}(B, A_\alpha)$ is a *bijection* of $\mathrm{Hom}(B, A)$ to the set of all (f_α) such that $\rho_{\alpha\beta} \circ f_\alpha = \rho_{\beta\alpha} \circ f_\beta$ for any pair of indices (α, β). If such a solution exists, it is unique up to an isomorphism.

(3.1.2) We will not recall the defintion of a *presheaf* $U \mapsto \mathscr{F}(U)$ on a topological space X with values in a category **K** (G, I, 1.9); we say that such a presheaf is a *sheaf with values in* **K** if it satifies the following axiom:

(F) *For any covering* (U_α) *of an open set* U *of* X *by open sets* U_α *contained in* U*, if we denote by* ρ_α *(resp.* $\rho_{\alpha\beta}$*) the restriction morphism*

$$\mathscr{F}(U) \to \mathscr{F}(U_\alpha) \quad (\textit{resp. } \mathscr{F}(U_\alpha) \to \mathscr{F}(U_\alpha \cap U_\beta)),$$

the pair formed by $\mathscr{F}(U)$ *and the family* (ρ_α) *are a solution to the universal problem for* $(\mathscr{F}(U_\alpha))$, $(\mathscr{F}(U_\alpha \cap U_\beta))$, *and* 24
$(\rho_{\alpha\beta})$ *in* (3.1.1)[1].

Equivalently, we can say that, for each object T of **K**, the family $U \mapsto \mathrm{Hom}(T, \mathscr{F}(U))$ is a *sheaf of sets*.

(3.1.3) Assume that **K** is the category defined by a "type of structure with morphisms" Σ, the objects of **K** being the sets with structures of type Σ and morphisms those of Σ. Suppose that the category **K** also satisfies the following condition:

(E) If $(A, (\rho_\alpha))$ is a solution of a universal mapping problem *in the category* **K** for families (A_α), $(A_{\alpha\beta})$, $(\rho_{\alpha\beta})$, then it is also a solution of the universal mapping problem for the same families *in the category of sets* (that is, when we consider A, A_α, and $A_{\alpha\beta}$ as sets, ρ_α and $\rho_{\alpha\beta}$ as functions) [2].

Under these conditions, the condition (F) gives that, when considered as a presheaf *of sets*, $U \mapsto \mathscr{F}(U)$ is a *sheaf*. In addition, for a map $u : T \to \mathscr{F}(U)$ to be a morphism of **K**, it is necessary and sufficient, under (F), that each map $\rho_\alpha \circ u$ is a morphism $T \to \mathscr{F}(U_\alpha)$, which means that the structure of type Σ on $\mathscr{F}(U)$ is the *initial structure* for the morphisms ρ_α . Conversely, suppose a presheaf $U \mapsto \mathscr{F}(U)$ on X, with values in **K**, is a *sheaf of sets* and satisfies the previous condition; it is then clear that it satisfies (F), so it is a *sheaf with values in* **K**.

(3.1.4) When Σ is a type of a group or ring structure, the fact that the presheaf $U \mapsto \mathscr{F}(U)$ with values in **K** is a sheaf of *sets* leads *ipso facto* that it is a sheaf with values in **K** (in other words, a sheaf of groups or rings within the meaning of (G))[3]. But it is not the same when, for example, **K** is the category of *topological rings* (with morphisms as continuous homomorphisms): a sheaf with values in **K** is a sheaf of rings $U \mapsto \mathscr{F}(U)$ such that for any open U and any covering of U by open sets $U_\alpha \subset U$, the topology of the ring $\mathscr{F}(U)$ is to be *the least fine*, making the homomorpisms $\mathscr{F}(U) \to \mathscr{F}(U_\alpha)$ continuous. We will say in this case that $U \mapsto \mathscr{F}(U)$, considered as a sheaf of rings (without a topology), is *underlying* the sheaf of topological rings $U \mapsto \mathscr{F}(U)$. Morphisms $u_V : \mathscr{F}(V) \to \mathscr{G}(V)$ (V an arbitrary open subset of X) of topological ring bundles are therefore homologous morphisms of the underlying sheaves of rings, such that u_V be *continuous* for all open $V \subset X$; to distinguish them from any homomorphisms of the sheaves of the underlying rings, we will call them continuous homomorphisms of sheaves of topological rings. We have similar definitions and conventions for sheaves of topological spaces or topological groups.

(3.1.5) It is clear that for any category **K**, if there is a presheaf (respectively a sheaf) $\mathscr{F}$ on X with values in **K** 25

[1] This is a special case of the more general notion of a *projective limit* (non-filtered) (*see* (T, I, 1.8) and the book in preparation announced in the Introduction).

[2] It can be proved that it also means that the canonical functor $\mathbf{K} \to (\mathrm{Ens})$ *commutes with projective limits* (not necessarily filtered).

[3] This is because in the category **K**, any morphism that is a *bijection* (as a map of sets) is an *isomorphism*. This is no longer true when **K** is the category of topological spaces, for example.

and U is an open set of X, the $\mathscr{F}(V)$ for open $V \subset U$ constitute a presheaf (or a sheaf) with values in **K**, which we call the presheaf (or sheaf) *induced* by $\mathscr{F}$ on U and that we denote $\mathscr{F}|U$.

For any morphism $u : \mathscr{F} \to \mathscr{G}$ of presheaves on X with values in **K**, we denote by $u|U$ the morphism $\mathscr{F}|U \to \mathscr{G}|U$ formed by the u_V for $V \subset U$.

(3.1.6) Suppose now that the category **K** admits *inductive limits* (T, 1.8); then, for any presheaf (and in particular any sheaf) $\mathscr{F}$ on X with values in **K** and all $x \in X$, we can define the *fiber* $\mathscr{F}_x$ as the object of **K** defined by the inductive limit of the $\mathscr{F}(U)$ with respect to the filtering set (for $\supset$) of the open neighborhoods U of x in X, and for the morphisms $\rho_U^V : \mathscr{F}(V) \to \mathscr{F}(U)$. If $u : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves with values in **K**, we define for all $x \in X$ the morphism $u_x : \mathscr{F}_x \to \mathscr{G}_x$ as the inductive limit of $u_U : \mathscr{F}(U) \to \mathscr{G}(U)$ with respect to all open neighborhoods of x; we thus define $\mathscr{F}_x$ as a covariant functor in $\mathscr{F}$, with values in **K**, for all $x \in X$.

When **K** is further defined by a kind of structure with morphisms Σ, we call *sections over* U of a *sheaf* $\mathscr{F}$ with values in **K** the elements of $\mathscr{F}(U)$, and we write $\Gamma(U, \mathscr{F})$ instead of $\mathscr{F}(U)$; for $s \in \Gamma(U, \mathscr{F})$, V an open set contained in U, we write $s|V$ instead of $\rho_V^U(s)$; for all $x \in U$, the canonical image of s in $\mathscr{F}_x$ is the *germ* of s at the point x, denoted by s_x (*we will never replace the notation* $s(x)$ *in this sense,* this notation being reserved for another notion relating to sheaves which will be considered in this Treaty (5.5.1)).

If then $u : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves with values in **K**, we will write $u(s)$ instead of $u_V(s)$ for all $s \in \Gamma(U, \mathscr{F})$.

If $\mathscr{F}$ is a sheaf of commutative groups, or rings, or modules, we say that the set of $x \in X$ such that $\mathscr{F}_x \neq \{0\}$ is the *support* of $\mathscr{F}$, denoted $\operatorname{Supp}(\mathscr{F})$; this set is not necessarily closed in X.

When **K** is defined by a type of structure with morphisms, *we systematically refrain from using the point of view of "étale spaces"* in terms of relating to sheaves with values in **K**; in other words, we will never consider a sheaf as a topological space (nor even as the whole union of its fibers), and we will not consider also a morphism $u : \mathscr{F} \to \mathscr{G}$ of such sheaves on X as a continuous map of topological spaces.

3.2. Presheaves on an open basis.

(3.2.1) We will restrict to the following categories **K** admitting *projective limits* (generalized, that is, corresponding to not necessarily filtered preordered sets, cf. (T, 1.8)). Let X be a topological space, $\mathfrak{B}$ an open basis for the topology of X. We will call a *presheaf on* $\mathfrak{B}$, *with values in* **K**, to be a family of objects $\mathscr{F}(U) \in$ **K**, corresponding to each $U \in \mathfrak{B}$, and a family of morphisms $\rho_U^V : \mathscr{F}(V) \to \mathscr{F}(U)$ defined for any pair (U, V) of elements of $\mathfrak{B}$ such that $U \subset V$, with the conditions $\rho_U^U =$ identity and $\rho_U^W = \rho_U^V \circ \rho_V^W$ if U, V, W in $\mathfrak{B}$ are such **26**
that $U \subset V \subset W$. We can associate a *presheaf with values in* **K** : $U \mapsto \mathscr{F}(U)$ in the ordinary sense, taking for all open U, $\mathscr{F}'(U) = \varprojlim \mathscr{F}(V)$, where V runs through the ordered set (for $\subset$, *not filtered* in general) of $V \in \mathfrak{B}$ sets such that $V \subset U$, since the $\mathscr{F}(V)$ form a projective system for the ρ_V^W ($V \subset W \subset U$, $V \in \mathfrak{B}$, $W \in \mathfrak{B}$). Indeed, if U, U' are two open sets of X such that $U \subset U'$, we define $\rho'^{U'}_U$ as the projective limit (for $V \subset U$) of the canonical morphisms $\mathscr{F}'(U') \to \mathscr{F}(V)$, in other words the unique morphism y $\mathscr{F}'(U') \to \mathscr{F}'(U)$, which, when composed with the canonical morphisms $\mathscr{F}'(U) \to \mathscr{F}(V)$, gives the canonical morphisms $\mathscr{F}'(U') \to \mathscr{F}(V)$; the verification of the transitivity of $\rho'^{U'}_U$ is then immediate. Moreover, if $U \in \mathfrak{B}$, the canonical morphism $\mathscr{F}'(U) \to \mathscr{F}(U)$ is an isomorphism, allowing to identify these two objects [4].

(3.2.2) For the presheaf $\mathscr{F}'$ thus defined to be a *sheaf*, it is necessary and sufficient that the presheaf $\mathscr{F}$ on $\mathfrak{B}$ satisfies the condition:

(F_0) *For any covering* (U_α) *of* $U \in \mathfrak{B}$ *by sets* $U_\alpha \in \mathfrak{B}$ *contained in* U, *and for any object* $T \in$ **K**, *the map which takes* $f \in \operatorname{Hom}(T, \mathscr{F}(U))$ *to the family* $(\rho_{U_\alpha}^U \circ f) \in \prod_\alpha \operatorname{Hom}(T, \mathscr{F}(U_\alpha))$ *is a bijection of* $\operatorname{Hom}(T, \mathscr{F}(U))$ *on the set of all*

[4] If X is a *Noetherian* space, we can still define $\mathscr{F}'(U)$ and show that it is a presheaf (in the ordinary sense) when one supposes only that **K** admits projective limits for *finite* projective systems. Indeed, if U is any open set of X, there is a *finite* covering (V_i) of U formed by sets of $\mathfrak{B}$; for every couple (i, j) of indices, let (V_{ijk}) be a finite covering of $V_i \cap V_j$ formed by sets of $\mathfrak{B}$. Let I be the set of i and triples (i, j, k), ordered only by the relations $i > (i, j, k)$, $j > (i, j, k)$; we then take $\mathscr{F}'(U)$ to be the projective limit of the system of $\mathscr{F}(V_i)$ and $\mathscr{F}(V_{ijk})$; it is easy to verify that this does not depend on the coverings (V_i) and (V_{ijk}) and that $U \mapsto \mathscr{F}'(U)$ is a presheaf.

(f_α) *such that* $\rho_V^{U_\alpha} \circ f_\alpha = \rho_V^{U_\beta} \circ f_\beta$ *for any pair of indices* (α, β) *and any* $V \in \mathfrak{B}$ *such that* $V \subset U_\alpha \cap U_\beta$ [5].

The condition is obviously necessary. To show that it is sufficient, consider first a second basis $\mathfrak{B}'$ of the topology of X, *contained in* $\mathfrak{B}$, and show that if $\mathscr{F}''$ denotes the presheaf induced by the subfamily $(\mathscr{F}(V))_{V\in\mathfrak{B}'}$, $\mathscr{F}''$ is *canonically isomorphic* to $\mathscr{F}'$. Indeed, firstly the projective limit (for $V \in \mathfrak{B}'$, $V \subset U$) canonical morphisms $\mathscr{F}'(U) \to \mathscr{F}(V)$ is a morphism $\mathscr{F}'(U) \to \mathscr{F}''(U)$ for all open U. If $U \in \mathfrak{B}$, this morphism is an isomorphism, because by hypothesis the canonical morphisms $\mathscr{F}''(U) \to \mathscr{F}(V)$ for $V \in \mathfrak{B}'$, $V \subset U$, factorize into $\mathscr{F}''(U) \to \mathscr{F}(U) \to \mathscr{F}(V)$, and it is immediate to see that the composition of morphisms $\mathscr{F}(U) \to \mathscr{F}''(U)$ and $\mathscr{F}''(U) \to \mathscr{F}(U)$ thus defined are the identities. This being so, for all open U, the morphisms $\mathscr{F}''(U) \to \mathscr{F}''(W) = \mathscr{F}(W)$ for $W \in \mathfrak{B}$ and $W \subset U$ satisfy the conditions characterizing the projective limit of $\mathscr{F}(W)$ ($W \in \mathfrak{B}$, $W \subset U$), which demonstrates our assertion given the uniqueness of a projective limit up to isomorphism.

This posed, let U be any open set of X, (U_α) a covering of U by the open sets contained in U, and $\mathfrak{B}'$ the subfamily of $\mathfrak{B}$ formed by the sets of $\mathfrak{B}$ contained in at least U_α; it is clear that $\mathfrak{B}'$ is still a basis of the topology **27**
of X, so $\mathscr{F}'(U)$ (resp. $\mathscr{F}''(U_\alpha)$) is the projective limit of $\mathscr{F}(V)$ for $V \in \mathfrak{B}'$ and $V \subset U$ (resp., $V \subset U_\alpha$), the axiom (F) is then immediately verified by virtue of the definition of the projective limit.

When (F_0) is satisfied, we will say by abuse of language that the presheaf $\mathscr{F}$ on the basis $\mathfrak{B}$ is a sheaf.

3.3. Gluing of sheaves.

3.4. Direct images of presheaves.

3.5. Inverse images of presheaves.

3.6. Constant sheaves and locally constant sheaves.

3.7. Inverse images of presheaves of groups or rings.

3.8. Sheaves on pseudo-discrete spaces.

[5] It also means that the pair formed by $\mathscr{F}(U)$ and the $\rho_\alpha = \rho^U_{U_\alpha}$ is a *solution to the universal problem* defined in (3.1.1) by the data of $A_\alpha = \mathscr{F}(U_\alpha)$, $A_{\alpha\beta} = \prod \mathscr{F}(V)$ (for $V \in \mathfrak{B}$ such that $V \subset U_\alpha \cap U_\beta$) and $\rho_{\alpha\beta} = (\rho''_V) : \mathscr{F}(U_\alpha) \to \prod \mathscr{F}(V)$ defined by the condition that for $V \in \mathfrak{B}$, $V' \in \mathfrak{B}$, $W \in \mathfrak{B}$, $V \cup V' \subset U_\alpha \cap U_\beta$, $W \subset V \cap V'$, $\rho^V_W \circ \rho''_V = \rho^{V'}_W \circ \rho''_{V'}$.

Chapter 1. — The language of schemes

Summary

The §§1-8 do little more than develop a language, which will be used in the following. It should be noted, however, that in accordance with the general spirit of this Treaty, §§7-8 will be used less than the others, and in a less essential way; we have moreover spoken of Chevalley's schemes only to make the link with the language of Chevalley [1] and Nagata [9]. The §9 gives definitions and results on quasi-coherent sheaves, some of which are no longer limited to a translation into a "geometric" language of known notions of commutative algebra, but are already of a global nature; they will be indispensable, from the following chapters, in the global study of morphisms. Finally, §10 introduces a generalization of the notion of schemes, which will be used as an intermediary in Chapter III to formulate and demonstrate in a convenient way the fundamental results of the cohomological study of the proper morphisms; moreover, it should be noted that the notion of formal schemes seems indispensable to express certain facts of the "theory of modules" (classification problems of algebraic varieties). The results of §10 will not be used before §3 of Chapter III and it is recommended to omit reading until then.

1. Affine schemes

1.1. The prime spectrum of a ring

TODO

1.2. Functorial properties of prime spectra of rings

TODO

1.3. Sheaf associated to a module

TODO

1.4. Quasi-coherent sheaves over a prime spectrum

TODO

1.5. Coherent sheaves over a prime spectrum

TODO

1.6. Functorial properties of quasi-coherent sheaves over a prime spectrum

TODO

1.7. Characterisation of morphisms of affine schemes

TODO

2. Preschemes and morphisms of preschemes

2.1. Definition of preschemes

(2.1.1) Given a ringed space $(X, \mathcal{O}_X)$, we say that an open subset V of X is an *affine open* if the ringed space $(V, \mathcal{O}_X|V)$ is an affine scheme (1.7.1).

Definition **(2.1.2)**. — *We define a prescheme to be a ringed space* $(X, \mathcal{O}_X)$ *such that every point of* X *admits an affine open neighbourhood.*

Proposition **(2.1.3)**. — *If* $(X, \mathcal{O}_X)$ *is a prescheme then the affine opens give a base for the topology of* X. **98**

In effect, if V is an arbitrary open neighbourhood of $x \in X$, then there exists by hypothesis an open neighbourhood W of x such that $(W, \mathcal{O}_X|W)$ is an affine scheme; we write A to mean its ring. In the space W, $V \cap W$ is an open neighbourhood of x; thus there exists $f \in A$ such that $D(f)$ is an open neighbourhood of x contained inside $V \cap W$ (1.1.10 (i)). The ringed space $(D(f), \mathcal{O}_X|D(f))$ is thus an affine scheme, isomorphic to A_f (1.3.6), whence the proposition.

Proposition **(2.1.4)**. — *The underlying space of a prescheme is a Kolmogoroff space.*

In effect, if x, y are two distinct points of a prescheme X then it is clear that there exists an open neighbourhood of one of these points that does not contain the other if x and y are not in the same affine open; and if they are in the same affine open, this is a result of (1.1.8).

Proposition **(2.1.5)**. — *If* $(X, \mathcal{O}_X)$ *is a prescheme then every closed irreducible subset of* X *admits exactly one generic point, and the map* $x \mapsto \overline{\{x\}}$ *is thus a bijection of* X *onto its set of closed irreducible subsets.*

In effect, if Y is a closed irreducible subset of X and $y \in Y$, and if U is an open affine neighbourhood of y in X, then $U \cap Y$ is everywhere dense in Y, as well as irreducible (**0**, 2.1.1 and 2.1.4); thus by (1.1.14), $U \cap Y$ is the closure in U of a point x, and then $Y \subset \overline{U}$ is the closure of x in X. The uniqueness of the generic point of X is a result of (2.1.4) and (**0**, 2.1.3).

(2.1.6) If Y is a closed irreducible subset of X and y its generic point then the local ring $\mathcal{O}_y$, also written $\mathcal{O}_{X/Y}$, is called the *local ring of* X *along* Y, or the *local ring of* Y *in* X.

If X itself is irreducible and x its generic point then we say that $\mathcal{O}_x$ is the *ring of rational functions on* X (cf. s. 7).

Proposition **(2.1.7)**. — *If* $(X, \mathcal{O}_X)$ *is a prescheme then the ringed space* $(U, \mathcal{O}_X|U)$ *is a prescheme for every open subset* U.

This follows directly from definition (2.1.2) and proposition (2.1.3).

We say that $(U, \mathcal{O}_X|U)$ is the prescheme *induced* on U by $(X, \mathcal{O}_X)$, or the *restriction* of $(X, \mathcal{O}_X)$ to U.

(2.1.8) We say that a prescheme $(X, \mathcal{O}_X)$ is *irreducible* (resp. *connected*) if the underlying space X is irreducible (resp. connected). We say that a prescheme is *integral* if it is *irreducible and reduced* (cf. (5.1.4)). We say that a prescheme $(X, \mathcal{O}_X)$ is *locally integral* if every $x \in X$ admits an open neighbourhood U such that the prescheme induced on U by $(X, \mathcal{O}_X)$ is integral.

2.2. Morphisms of preschemes

Definition **(2.2.1)**. — *Given two preschemes* $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, *we define a morphism (of preschemes) of* $(X, \mathcal{O}_X)$ *to* $(Y, \mathcal{O}_Y)$ *to be a morphism of ringed spaces* (ψ, θ) *such that, for all* $x \in X$, $\theta_x^\#$ *is a local homomorphism* $\mathcal{O}_{\psi(x)} \to \mathcal{O}_x$. **99**

By passing to quotients, the map $\mathcal{O}_{\psi(x)} \to \mathcal{O}_x$ gives us a monomorphism $\theta^x \colon k(\psi(x)) \to k(x)$, which lets us consider $k(x)$ as an *extension* of the field $k(\psi(x))$.

(2.2.2) The composition (ψ'', θ'') of two morphisms (ψ, θ), (ψ', θ') of preschemes is also a morphism of preschemes, since it is given by the formula $\theta''^\# = \theta^\# \circ \psi^*(\theta'^\#)$ (**0**, 3.5.5). From this we conclude that preschemes form a *category*; using the usual notation, we will write $\mathrm{Hom}(X, Y)$ to mean the set of morphisms from a prescheme X to a prescheme Y.

Example **(2.2.3)**. — *If* U *is an open subset of* X *then the canonical injection (**0**, 4.1.2) of the induced prescheme* $(U, \mathcal{O}_X|U)$ *into* $(X, \mathcal{O}_X)$ *is a morphism of preschemes; it is further a* monomorphism *of ringed spaces (and* a fortiori *a monomorphism of preschemes), which follows rapidly from (**0**, 4.1.1).*

Proposition **(2.2.4)**. — *Let* $(X, \mathcal{O}_X)$ *be a prescheme, and* $(S, \mathcal{O}_S)$ *an affine scheme associated to a ring* A. *Then there exists a canonical bijective correspondence between morphisms of preschemes from* $(X, \mathcal{O}_X)$ *to* $(S, \mathcal{O}_S)$ *and ring homomorphisms from* A *to* $\Gamma(X, \mathcal{O}_X)$. Note first of all that, if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are two arbitrary ringed spaces, a morphism (ψ, θ) from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ canonically defines a ring homomorphism $\Gamma(\theta)\colon \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y, \psi_*(\mathcal{O}_X)) = \Gamma(X, \mathcal{O}_X)$. In the case that we consider, everything boils down to showing that any homomorphism $\varphi\colon A \to \Gamma(X, \mathcal{O}_X)$ is of the form $\Gamma(\theta)$ for one and only one θ. Now, by hypothesis there is a covering (V_α) of X by affine opens; by composing of φ with the restriction homomorphism $\Gamma(X, \mathcal{O}_X) \to \Gamma(V_\alpha, \mathcal{O}_X|V_\alpha)$ we obtain a homomorphism $\varphi_\alpha\colon A \to \Gamma(V_\alpha, \mathcal{O}_X|V_\alpha)$ that corresponds to a unique morphism $(\psi_\alpha, \theta_\alpha)$ from the prescheme $(V_\alpha, \mathcal{O}_X|V_\alpha)$ to $(S, \mathcal{O}_S)$, thanks to (1.7.3). Furthermore, for every pair of indices (α, β), every point of $V_\alpha \cap V_\beta$ admits an open affine neighbourhood W contained inside $V_\alpha \cap V_\beta$ (2.1.3); it is clear that that, by composing φ_α and φ_β with the restriction homomorphisms to W, we obtain the same homomorphism $\Gamma(S, \mathcal{O}_S) \to \Gamma(W, \mathcal{O}_X|W)$, so, thanks to the relations $(\theta_\alpha^\#)_x = (\varphi_\alpha)_x$ for all $x \in V_\alpha$ and all α (1.6.1), the restriction to W of the morphisms $(\psi_\alpha, \theta_\alpha)$ and $(\psi_\beta, \theta_\beta)$ coincide. From this we conclude that there is a morphism $(\psi, \theta)\colon (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ of ringed spaces, and only one such that its restriction to each V_α is $(\psi_\alpha, \theta_\alpha)$, and it is clear that this morphism is a morphism of preschemes and such that $\Gamma(\theta) = \varphi$.

Let $u\colon A \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism, and $v = (\psi, \theta)$ the corresponding morphism $(X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$. For every $f \in A$ we have that

(2.2.4.1)
$$\psi^{-1}(D(f)) = X_{u(f)}$$

with the notation of (**0**, 5.5.2) relative to the locally-free sheaf $\mathcal{O}_X$. In fact, it suffices to verify this formula when X itself is affine, and then this is nothing but (1.2.2.2).

Proposition **(2.2.5)**. — *Under the hypotheses of (2.2.4), let* $\varphi\colon A \to \Gamma(X, \mathcal{O}_X)$ *be a ring homomorphism,* $f\colon (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ *the corresponding morphism of preschemes,* $\mathscr{G}$ *(resp.* $\mathscr{F}$*) an* $\mathcal{O}_X$*-module (resp.* $\mathcal{O}_S$*-module), and* $M = \Gamma(S, \mathscr{F})$. *Then there exists a canonical bijective correspondence between* f*-morphisms* $\mathscr{F} \to \mathscr{G}$ *(**0**, 4.4.1) and* A*-homomorphisms* **100**
$M \to (\Gamma(X, \mathscr{G}))_{[\varphi]}$. Indeed, reasoning as in (2.2.4), we are rapidly led to the case where X is affine, and the proposition then follows from (1.6.3) and (1.3.8).

(2.2.6) We say that a morphism of preschemes $(\psi, \theta)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is *open* (resp. *closed*) if, for all open subsets U of X (resp. all closed subsets F of X), $\psi(U)$ is open (resp. $\psi(F)$ is closed) in Y. We say that (ψ, θ) is *dominant* if $\psi(X)$ is dense in Y, and *surjective* if ψ is surjective. We will point out that these conditions rely only on the continuous map ψ.

Proposition **(2.2.7)**. — *Let*

$$f = (\psi, \theta)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y);$$
$$g = (\psi', \theta')\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$$

be two morphisms of preschemes.

(i) If f *and* g *are both open (resp. closed, dominant, surjective), then so too is* $g \circ f$.
(ii) If f *is surjective and* $g \circ f$ *closed, then* g *is closed.*
(iii) If $g \circ f$ *is surjective, then* g *is surjective.*

Claims *(i)* and *(iii)* are evident. Write $g \circ f = (\psi'', \theta'')$. If F is closed in Y then $\psi^{-1}(F)$ is closed in X, so $\psi''(\psi^{-1}(F))$ is closed in Z; but since ψ is surjective, $\psi(\psi^{-1}(F)) = F$, so $\psi''(\psi^{-1}(F)) = \psi'(F)$, which proves *(ii)*.

Proposition **(2.2.8)**. — *Let* $f = (\psi, \theta)$ *be a morphism* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$*, and* (U_α) *an open cover of* Y. *For* f *to be open (resp. closed, surjective, dominant), it is necessary and sufficient that its restriction to every induced prescheme* $(\psi^{-1}(U_\alpha), \mathcal{O}_X|\psi^{-1}(U_\alpha))$*, considered as a morphism of preschemes from this induced prescheme to the induced prescheme* $(U_\alpha, \mathcal{O}_Y|U_\alpha)$ *is open (resp. closed, surjective, dominant).* The proposition follows immediately from the definitions, taking into account the fact that a subset F of Y is closed (resp. open, dense) in Y if and only if each of the $F \cap U_\alpha$ are closed (resp. open, dense) in U_α.

(2.2.9) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two preschemes; suppose that X (resp. Y) has a finite number of irreducible components X_i (resp. Y_i) $(1 \leqslant i \leqslant n)$; let ξ_i (resp. η_i) be the generic point of X_i (resp. Y_i) (2.1.5). We say that a morphism

$$f = (\psi, \theta) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is *birational* if, for all i, $\psi^{-1}(\eta_i) = \{\xi_i\}$ and $\theta^{\#}_{\xi_i} \colon \mathcal{O}_{\eta_i} \to \mathcal{O}_{\xi_i}$ is an *isomorphism*. It is clear that a birational morphism is dominant (**0**, 2.1.8), and so is surjective if it is also closed.

Notational conventions **(2.2.10).** — *In all that follows, when there is no risk of confusion, we* supress *the structure sheaf (resp. the morphism of structure sheaves) from the notation of a prescheme (resp. morphism of preschemes). If U is an open subset of the underlying space X of a prescheme, then whenever we speak of U as a prescheme we always mean the induced prescheme on U.*

2.3. Gluing of preschemes

(2.3.1) It follows from definition (2.1.2) that every ringed space obtained by *gluing* preschemes (**0**, 4.1.6) is **101**
again a prescheme. In particular, although every prescheme admits, by definition, a cover by affine open sets, we see that every prescheme can actually be obtained by *gluing affine schemes*.

Example **(2.3.2).** — *Let K be a field, and $B = K[s]$, $C = K[t]$ be two polynomial rings in one indeterminate over K, and define $X_1 = \mathrm{Spec}(B)$, $X_2 = \mathrm{Spec}(C)$, which are two isomorphic affine schemes. In X_1 (resp. X_2), let U_{12} (resp. U_{21}) be the affine open $D(s)$ (resp. $D(t)$) where the ring B_s (resp. C_t) is formed of rational fractions of the form $f(s)/s^m$ (resp. $g(t)/t^n$) with $f \in B$ (resp. $g \in C$). Let u_{12} be the isomorphism of preschemes $U_{21} \to U_{12}$ corresponding (2.2.4) to the isomorphism from B to C that, to $f(s)/s^m$, associates the rational fraction $f(1/t)/(1/t^m)$. We can glue X_1 and X_2 along U_{12} and U_{21} by using u_{12}, because there is clearly no gluing condition. We later show that the prescheme X obtained in this manner is a particular case of a general method of construction (**II**, 2.4.3). Here we only show that X* is not an affine scheme*; this will follow from the fact that the ring $\Gamma(X, \mathcal{O}_X)$ is* isomorphic *to K, and so its spectrum reduces to a point. In effect, a section of $\mathcal{O}_X$ over X has a restriction over X_1 (resp. X_2), identified to an affine open of X, that is a polynomial $f(s)$ (resp. $g(t)$), and it follows from the definitions that we should have $g(t) = f(1/t)$, which is not possible if $f = g \in K$.*

2.4. Local schemes

(2.4.1) We say that an affine scheme is a *local scheme* if it is the affine scheme associated to a local ring A; then there exists in $X = \mathrm{Spec}(A)$ a single *closed point* $\mathfrak{a} \in X$, and for all other $\mathfrak{b} \in X$ we have that $\mathfrak{a} \in \overline{\{\mathfrak{b}\}}$ (1.1.7).

For all preschemes Y and points $y \in Y$, the local scheme $\mathrm{Spec}(\mathcal{O}_y)$ is called the *local scheme of* Y *at the point* y. Let V be an affine open of Y containing y, and B the ring of the affine scheme V; then $\mathcal{O}_y$ is canonically identified with B_y (1.3.4), and the canonical homomorphism $B \to B_y$ thus corresponds (1.6.1) to a morphism of preschemes $\mathrm{Spec}(\mathcal{O}_y) \to V$. If we compose this morphism with the canonical injection $V \to Y$, then we obtain a morphism $\mathrm{Spec}(\mathcal{O}_y) \to Y$, which is *independent* of the affine open V (containing y) that we chose: indeed, if V' is some other affine open containing y, then there exists a third affine open W containing y and such that $W \subset V \cap V'$ (2.1.3); we can thus assume that $V \subset V'$, and then if B' is the ring of V', everything comes down to remarking that the diagram

$$\begin{tikzcd} B' \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & \mathcal{O}_y & \end{tikzcd}$$

is commutative (**0**, 1.5.1). The morphism

$$\mathrm{Spec}(\mathcal{O}_y) \to Y$$

thus defined is said to be *canonical*.

Proposition **(2.4.2).** — *Let $(Y, \mathcal{O}_Y)$ be a prescheme; for all $y \in Y$, let (ψ, θ) be the canonical morphism $(\mathrm{Spec}(\mathcal{O}_y), \widetilde{\mathcal{O}}_y) \to$* **102**
$(Y, \mathcal{O}_Y)$. Then ψ is a homeomorphism from $\mathrm{Spec}(\mathcal{O}_y)$ to the subspace S_y of Y given by the z such that $y \in \overline{\{z\}}$ (or, equivalenty, the ??? of y (**0**, 2.1.2))*; furthermore, if $z = \psi(\mathfrak{p})$, then $\theta^{\#}_z \colon \mathcal{O}_z \to (\mathcal{O}_y)_{\mathfrak{p}}$ is an isomorphism; (ψ, θ) is thus a monomorphism of ringed spaces.*

As the unique closed point $\mathfrak{a}$ of $\mathrm{Spec}(\mathcal{O}_y)$ is a member of every point of this space, and since $\psi(\mathfrak{a}) = y$, the image of $\mathrm{Spec}(\mathcal{O}_y)$ under the continuous map ψ is contained in S_y. Since S_y is contained in every affine open containing y, one can consider just the case where Y is an affine scheme; but then this proposition follows from (1.6.2).

We see (2.1.5) that there is a bijective correspondence between $\mathrm{Spec}(\mathcal{O}_y)$ *and the set of closed irreducible subsets of* Y *containing* y.

Corollary **(2.4.3)**. — *For* $y \in Y$ *to be the generic point of an irreducible component of* Y*, it is necessary and sufficient that the only prime ideal of the local ring* $\mathcal{O}_y$ *is its maximal ideal (*in other words, that $\mathcal{O}_y$ is of *dimension zero).*

Proposition **(2.4.4)**. — *Let* $(X, \mathcal{O}_X)$ *be a local scheme of ring* A*,* a *its unique closed point, and* $(Y, \mathcal{O}_Y)$ *a prescheme. Every morphism* $u = (\psi, \theta) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *then factorises uniquely as* $X \to \mathrm{Spec}(\mathcal{O}_{\psi(a)}) \to Y$*, where the second arrow denotes the canonical morphism, and the first corresponds to a local homomorphism* $\mathcal{O}_{\psi(a)} \to A$*. This establishes a canonical bijective correspondence between the set of morphisms* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *and the set of local homomorphisms* $\mathcal{O}_y \to A$ *for* ($y \in Y$).

Indeed, for all $x \in X$, we have that $a \in \overline{\{x\}}$, so $\psi(a) \in \overline{\{\psi(x)\}}$, which shows that $\psi(X)$ is contained in every affine open containing $\psi(a)$. So it suffices to consider the case where $(Y, \mathcal{O}_Y)$ is an affine scheme of ring B, and we then have that $u = ({}^a\varphi, \tilde{\varphi})$, where $\varphi \in \mathrm{Hom}(B, A)$ (1.7.3). Further, we have that $\varphi^{-1}(j_a) = j_{\psi(a)}$, and hence that the image under φ of any element of $B \setminus j_{\psi(a)}$ is invertible in the local ring A; the factorisation in the result follows from the universal property of the ring of fractions (**0**, 1.2.4). Conversely, to every local homomorphism $\mathcal{O}_y \to A$ there exists a unique corresponding morphism $(\psi, \theta) \colon X \to \mathrm{Spec}(\mathcal{O}_y)$ such that $\psi(a) = y$ (1.7.3), and, by composing with the canonical morphism $\mathrm{Spec}(\mathcal{O}_y) \to Y$, we obtain a morphism $X \to Y$, which proves the proposition.

(2.4.5) The affine schemes whose ring is a field K have an underlying space that is just a point. If A is a local ring with maximal ideal $\mathfrak{m}$, then every local homomorphism $A \to K$ has kernel equal to $\mathfrak{m}$, and so factorises as $A \to A/\mathfrak{m} \to K$, where the second arrow is a monomorphism. The morphisms $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$ thus correspond bijectively to monomorphisms of fields $A/\mathfrak{m} \to K$.

Let $(Y, \mathcal{O}_Y)$ be a prescheme; for every $y \in Y$ and every ideal $\mathfrak{a}_y$ of $\mathcal{O}_y$, the canonical homomorphism $\mathcal{O}_y \to \mathcal{O}_y/\mathfrak{a}_y$ defines a morphism $\mathrm{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \to \mathrm{Spec}(\mathcal{O}_y)$; if we compose this with the canonical morphism $\mathrm{Spec}(\mathcal{O}_y) \to Y$, then we obtain a morphism $\mathrm{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \to Y$, again said to be *canonical*. For $\mathfrak{a}_y = \mathfrak{m}_y$, this says that $\mathcal{O}_y/\mathfrak{a}_y = \mathbf{k}(y)$, and so prop. (2.4.4) says that:

Corollary **(2.4.6)**. — *Let* $(X, \mathcal{O}_X)$ *be a local scheme whose ring* K *is a field,* ξ *be the unique point of* X*, and* $(Y, \mathcal{O}_Y)$ *a prescheme. Then every morphism* $u \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *factorises uniquely as* $X \to \mathrm{Spec}(\mathbf{k}(\psi(\xi))) \to Y$*, where the second arrow denotes the canonical morphism, and the first corresponds to a monomorphism* $\mathbf{k}(\psi(\xi)) \to K$*. This establishes a canonical bijective correspondance between the set of morphisms* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *and the set of monomorphisms* $\mathbf{k}(y) \to K$ *(for* $y \in Y$*).* **103**

Corollary **(2.4.7)**. — *For all* $y \in Y$*, every canonical morphism* $\mathrm{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \to Y$ *is a monomorphism of ringed spaces.*

We have already seen this when $\mathfrak{a}_y = 0$ (2.4.2), and it suffices to apply (1.7.5).

Remark **(2.4.8)**. — *Let* X *be a local scheme, and* a *its unique closed point. Since every affine open containing* a *is necessarily in the whole of* X*, every* invertible $\mathcal{O}_X$*-module (**0**, 5.4.1) is necessarily* isomorphic to $\mathcal{O}_X$ *(or, as we say, again,* trivial*). This property doesn't hold in general, for an arbitrary affine scheme* $\mathrm{Spec}(A)$*; we will see in chap. V that if* A *is a normal ring then this is true when* A *is* factorial.

2.5. Preschemes over a prescheme

Definition **(2.5.1)**. — *Given a prescheme* S*, we say that the data of a prescheme* X *and a morphism of preschemes* $\varphi \colon X \to S$ *defines a prescheme* X over the prescheme S*, or an* S-prescheme*; we say that* S *is the* base prescheme *of the* S*-prescheme* X*. The morphism* φ *is called the* structure morphism *of the* S*-prescheme* X*. When* S *is an affine scheme of ring* A*, we also say that* X *endowed with* φ *is a prescheme* over the ring A *(or an* A-prescheme*).*

It follows from (2.2.4) that the data of a prescheme over a ring A is equivalent to the data of a prescheme $(X, \mathcal{O}_X)$ whose structure sheaf $\mathcal{O}_X$ is a sheaf of A*-algebras*. *An arbitrary prescheme can always be considered as a* $\mathbb{Z}$*-prescheme in a unique way.*

If $\varphi \colon X \to S$ is the structure morphism of an S-prescheme X, we say that a point $x \in X$ is *over a point* $s \in S$ if $\varphi(x) = s$. We say that X *dominates* S if φ is a dominant morphism (2.2.6).

(2.5.2) Let X and Y be two S-preschemes; we say that a morphism of preschemes $u\colon X \to Y$ is a *morphism of preschemes over* S (or an S-*morphism*) if the diagram

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

(where the diagonal arrows are the structure morphisms) is commutative: this ensures that, for all $s \in S$ and $x \in X$ over s, $u(x)$ is also above s.

From this definition it follows immediately that the composition of two S-morphisms is an S-morphism; S-preschemes thus form a *category*.

We denote by $\mathrm{Hom}_S(X, Y)$ the set of S-morphisms from an S-prescheme X to an S-prescheme Y; the identity morphism of an S-prescheme is denoted by 1_X.

When S is an affine scheme of ring A, we will also say A-*morphism* instead of S-morphism.

(2.5.3) If X is an S-prescheme, and $v\colon X' \to X$ a morphism of preschemes, then the composition $X' \to X \to S$ **104**
endows X' with the structure of an S-prescheme; in particular, every prescheme induced by an open set U of X can be considered as an S-prescheme by the canonical injection.

If $u\colon X \to Y$ is an S-morphism of S-preschemes, then the restriction of u to any prescheme induced by an open subset U of X is also an S-morphism $U \to Y$. Conversely, let (U_α) be an open cover of X, and for each α let $u_\alpha\colon U_\alpha \to Y$ be an S-morphism; if, for all pairs of indices (α, β), the restrictions of u_α and u_β to $U_\alpha \cap U_\beta$ agree, then there exists an S-morphism $X \to Y$, and only one such that the restriction to each U_α is u_α.

If $u\colon X \to Y$ is an S-morphism such that $u(X) \subset V$, where V is an open subset of Y, then u, considered as a morphism from X to V, is also an S-morphism.

(2.5.4) Let $S' \to S$ be a morphism of preschemes; for all S'-preschemes, the composition $X \to S' \to S$ endows X with the structure of an S-prescheme. Conversely, suppose that S' is the induced prescheme of an open subset of S; let X be an S-prescheme and suppose that the structure morphism $f\colon X \to S$ is such that $f(X) \subset S'$; then we can consider X as an S'-preschemes. In this latter case, if Y is another S-prescheme whose structure morphism sends the underlying space to S', then every S-morphism from X to Y is also an S'-morphism.

(2.5.5) If X is an S-prescheme, with structure morphism $\varphi\colon X \to S$, we define an S-*section of* X to be an S-morphism from S to X, that is to say a morphism of preschemes $\psi\colon S \to X$ such that $\varphi \circ \psi$ is the identity on S. We denote by $\Gamma(X/S)$ the set of S-sections of X.

3. Products of preschemes

4. Sub-preschemes and immersion morphisms

5. Reduced preschemes; separation conditions

6. Finiteness conditions

7. Rational maps

8. Chevalley schemes

8.1. Allied local rings

For every local ring A, we denote by $\mathfrak{m}(A)$ the maximal ideal of A.

Lemma **(8.1.1).** — *Let* A *and* B *be two local rings such that* $A \subset B$*; then the following conditions are equivalent:* (i) $\mathfrak{m}(B) \cap A = \mathfrak{m}(A)$*;* (ii) $\mathfrak{m}(A) \subset \mathfrak{m}(B)$*;* (iii) 1 *is not an element of the ideal of* B *generated by* $\mathfrak{m}(A)$.

It's evident that (i) implies (ii), and (ii) implies (iii); lastly, if (iii) is true, then $\mathfrak{m}(B) \cap A$ contains $\mathfrak{m}(A)$ and doesn't contain 1, and is thus equal to $\mathfrak{m}(A)$.

When the equivalent conditions of (8.1.1) are satisfied, we say that B *dominates* A; this is equivalent to saying that the injection $A \to B$ is a *local* homomorphism. It is clear that, in the set of local subrings of a ring R, the relation given by domination is an order.

(8.1.2) Now consider a *field* R. For all subrings A of R, we denote by $L(A)$ the set of local rings $A_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over the prime spectrum of A; they are identified with the subrings of R containing A. Since $\mathfrak{p} = (\mathfrak{p}A_{\mathfrak{p}}) \cap A$, the map $\mathfrak{p} \to A_{\mathfrak{p}}$ from $\operatorname{Spec}(A)$ into $L(A)$ is bijective.

Lemma **(8.1.3).** — *Let R be a field, and A a subring of R. For a local subring M of R to dominate a ring $A_{\mathfrak{p}} \in L(A)$ it is necessary and sufficient that $A \subset M$; the local ring $A_{\mathfrak{p}}$ dominated by M is then unique, and corresponds to $\mathfrak{p} = \mathfrak{m}(M) \cap A$.*

Indeed, if M dominates $A_{\mathfrak{p}}$, then $\mathfrak{m}(M) \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, by (8.1.1), whence the uniqueness of $\mathfrak{p}$; on the other hand, if $A \subset M$, then $\mathfrak{m}M \cap A = \mathfrak{p}$ is prime in A, and since $A \setminus \mathfrak{p} \subset M$, we have that $A_{\mathfrak{p}} \subset M$ and $\mathfrak{p}A_{\mathfrak{p}} \subset \mathfrak{m}(M)$, so M dominates $A_{\mathfrak{p}}$

Lemma **(8.1.4).** — *Let R be a field, M and N two local subrings of R, and P the subring of R generated by $M \cup N$. Then the following conditions are equivalent:* **165**

(i) *There exists a prime ideal $\mathfrak{p}$ of P such that $\mathfrak{m}(M) = \mathfrak{p} \cap M$ and $\mathfrak{m}(N) = \mathfrak{p} \cap N$.*
(ii) *The ideal $\mathfrak{a}$ generated in P by $\mathfrak{m}(M) \cup \mathfrak{m}(N)$ is distinct from P.*
(iii) *There exists a local subring Q of R simultaneously dominating both M and N.*

It is clear that (i) implies (ii); conversely, if $\mathfrak{a} \neq P$, then $\mathfrak{a}$ is contained in a maximal ideal $\mathfrak{n}$ of P, and since $1 \notin \mathfrak{n}$, $\mathfrak{n} \cap M$ contains $\mathfrak{m}(M)$ and is distinct from M, so $\mathfrak{n} \cap M = \mathfrak{m}(M)$, and similarly $\mathfrak{n} \cap N = \mathfrak{m}(N)$. It is clear that, if Q dominates both M and N, then $P \subset Q$ and $\mathfrak{m}(M) = \mathfrak{m}(Q) \cap M = (\mathfrak{m}(Q) \cap P) \cap M$, and $\mathfrak{m}(N) = (\mathfrak{m}(Q) \cap P) \cap N$, so (iii) implies (i); the reciprocal is evident when we take $Q = P_{\mathfrak{p}}$.

When the conditions of (8.1.4) are satisfied, we say, with C. Chevalley, that the local rings M and N are *allied*.

Proposition **(8.1.5).** — *Let A and B be two subrings of a field R, and C the subring of R generated by $A \cup B$. Then the following conditions are equivalent:*

(i) *For every local ring Q containing A and B, we have that $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{m}(Q) \cap A$ and $\mathfrak{q} = \mathfrak{m}(Q) \cap B$.*
(ii) *For all prime ideals $\mathfrak{r}$ of C, we have that $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$.*
(iii) *If $M \in L(A)$ and $N \in L(B)$ are allied, then they are identical.*
(iv) $L(A) \cap L(B) = L(C)$.

Lemmas (8.1.3) and (8.1.4) prove that (i) and (iii) are equivalent; it is clear that (i) implies (ii) by taking $Q = C_{\mathfrak{r}}$; conversely, (ii) implies (i), because if Q contains $A \cup B$ then it contains C, and if $\mathfrak{r} = \mathfrak{m}(Q) \cap C$ then $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$, from (8.1.3). It is immediate that (iv) implies (i), because if Q contains $A \cup B$ then it dominates a local ring $C_{\mathfrak{r}} \in L(C)$ by (8.1.3); by hypothesis we have that $C_{\mathfrak{r}} \in L(A) \cap L(B)$, and (8.1.1) and (8.1.3) prove that $C_{\mathfrak{r}} = A_{\mathfrak{p}} = B_{\mathfrak{q}}$. We prove finally that (iii) implies (iv). Let $Q \in L(C)$; Q dominates some $M \in L(A)$ and some $N \in L(B)$ (8.1.3), so M and N, being allied, are identical by hypothesis. As we then have that $C \subset M$, we know that M dominates some $Q' \in L(C)$ (8.1.3), so Q dominates Q', whence necessarily (8.1.3) $Q = Q' = M$, so $Q \in L(A) \cap L(B)$. Conversely, if $Q \in L(A) \cap L(B)$, then $C \subset Q$, so (8.1.3) Q dominates some $Q'' \in L(C) \subset L(A) \cap L(B)$; Q and Q'', being allied, are identical, so $Q'' = Q \in L(C)$, which completes the proof.

8.2. Local rings of an integral scheme

(8.2.1) Let X be an *integral* prescheme, and R its field of rational functions, identical to the local ring of the generic point a of X; for all $x \in X$, we know that $\mathcal{O}_x$ can be canonically identified with a subring of R (7.1.5), and for every rational function $f \in R$, the domain of definition $\delta(f)$ of f is the open set of $x \in X$ such that $f \in \mathcal{O}_x$. It thus follows (7.2.6) that, for every open $U \subset X$, we have

(8.2.1.1)
$$\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_x.$$

Proposition **(8.2.2).** — *Let X be an integral prescheme, and R its field of rational fractions. For X to be a scheme, it is necessary and sufficient that the relation "$\mathcal{O}_x$ and $\mathcal{O}_y$ are allied" (8.1.4), for points x, y of X, implies that $x = y$.* **166**

Suppose that this condition is verified, and aim to show that X is separated. Let U and V be two distinct affine opens of X, with rings A and B, identified with subrings of R; U (resp. V) is thus identified (8.1.2) with $L(A)$ (resp. $L(B)$), and the hypothesis tells us (8.1.5) that C is the subring of R generated by $A \cup B$, and $W = U \cap V$

is identified with $L(A) \cap L(B) = L(C)$. Further, we know ([1], p. 5-03, prop. 4 *bis*) that every subring E of R is equal to the intersection of the local rings belonging to $L(E)$; C is thus identified with the intersection of the rings $\mathcal{O}_z$ for $z \in W$, or, equivalently (8.2.1.1) with $\Gamma(W, \mathcal{O}_X)$. So consider the sub-prescheme induced by X on W; to the identity morphism $\varphi\colon C \to \Gamma(W, \mathcal{O}_X)$ there corresponds (2.2.4) a morphism $\Phi = (\psi, \theta)\colon W \to \operatorname{Spec}(C)$; we will see that Φ is an *isomorphism* of preschemes, whence W is an *affine* open. The identification of W with $L(C) = \operatorname{Spec}(C)$ shows that ψ is *bijective*. On the other hand, for all $x \in W$, $\theta_x^\#$ is the injection $C_{\mathfrak{r}} \to \mathcal{O}_x$, where $\mathfrak{r} = \mathfrak{m}_x \cap C$, and by definition $C_{\mathfrak{r}}$ is identified with $\mathcal{O}_x$, so $\theta_x^\#$ is bijective. It thus remains to show that ψ is a *homeomorphism*, i.e. that for every closed subset $F \subset W$, $\psi(F)$ is closed in $\operatorname{Spec}(C)$. But F is the trace over W of closed subspace of U, of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of A; we show that $\psi(F) = V(\mathfrak{a}C)$, which proves our claim. In fact, the prime ideals of C containing $\mathfrak{a}C$ are the prime ideals of C containing $\mathfrak{a}$, and so are the ideals of the form $\psi(x) = \mathfrak{m}_x \cap C$, where $\mathfrak{a} \subset \mathfrak{m}_x$ and $x \in W$; since $\mathfrak{a} \subset \mathfrak{m}_x$ is equivalent to $x \in V(\mathfrak{a}) = W \cap F$ for $x \in U$, we do indeed have that $\psi(F) = V(\mathfrak{a}C)$.

It follows that X is separated, because $U \cap V$ is affine and its ring C is generated by the union $A \cup B$ of the rings of U and V (5.5.6).

Conversely, suppose that X is separated, and let x, y be two points of X such that $\mathcal{O}_x$ and $\mathcal{O}_y$ are allied. Let U (resp. V) be an affine open containing x (resp. y), of ring A (resp. B); we then know that $U \cap V$ is affine and that its ring C is generated by $A \cup B$ (5.5.6). If $\mathfrak{p} = \mathfrak{m}_x \cap A$ and $\mathfrak{q} = \mathfrak{m}_y \cap B$, then $A_{\mathfrak{p}} = \mathcal{O}_x$ and $B_{\mathfrak{q}} = \mathcal{O}_y$, and since $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are allied, there exists a prime ideal $\mathfrak{r}$ of C such that $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$ (8.1.4). But then there exists a point $z \in U \cap V$ such that $\mathfrak{r} = \mathfrak{m}_z \cap C$, since $U \cap V$ is affine, and so evidently $x = z$ and $y = z$, whence $x = y$.

Corollary **(8.2.3)**. — *Let X be an integral scheme, and x, y two points of X. In order that $x \in \overline{\{y\}}$, it is necessary and sufficient that $\mathcal{O}_x \subset \mathcal{O}_y$, or, equivalently, that every rational function defined at x is also defined at y.*

The condition is evidently necessary because the domain of definition $\delta(f)$ of a rational function $f \in R$ is open; we now show that it is sufficient. If $\mathcal{O}_x \subset \mathcal{O}_y$, then there exists a prime ideal $\mathfrak{p}$ of $\mathcal{O}_x$ such that $\mathcal{O}_y$ dominates $(\mathcal{O}_x)_{\mathfrak{p}}$ (8.1.3); but (2.4.2) there exists $z \in X$ such that $x \in \overline{\{z\}}$ and $\mathcal{O}_z = (\mathcal{O}_x)_{\mathfrak{p}}$; since $\mathcal{O}_z$ and $\mathcal{O}_y$ are allied, we have that $z = y$ by (8.2.2), whence the corollary.

Corollary **(8.2.4)**. — *If X is an integral scheme then the map $x \to \mathcal{O}_x$ is injective; equivalently, if x and y are two distinct points of X, then there exists a rational function defined at one of these points but not the other.* **167**

This follows from (8.2.3) and the axiom (T_0) (2.1.4).

Corollary **(8.2.5)**. — *Let X be an integral scheme whose underlying space is Noetherian; letting f run over the field R of rational functions on X, the sets $\delta(f)$ generate the topology of X.*

In fact, every closed subset of X is thus a finite union of irreducible closed subsets, i.e. of the form $\overline{\{y\}}$ (2.1.5). But, if $x \notin \overline{\{y\}}$, then there exists a rational function f defined at x but not at y (8.2.3), or, equivalently, we have that $x \in \delta(f)$ and $\delta(f)$ is not contained in $\overline{\{y\}}$. The complement of $\overline{\{y\}}$ is thus a union of sets of the form $\delta(f)$, and by virtue of the first remark, every open subset of X is the union of finite intersections of open sets of the form $\delta(f)$.

(8.2.6) Corollary (8.2.5) shows that the topology of X is entirely characterised by the data of the local rings $(\mathcal{O}_x)_{x \in X}$ that have R as their field of fractions. It amounts to the same to say that the closed subsets of X are defined in the following manner: given a finite subset $\{x_1, \ldots, x_n\}$ of X, consider the set of $y \in X$ such that $\mathcal{O}_y \subset \mathcal{O}_{x_i}$ for at least one index i, and these sets (over all choices of $\{x_1, \ldots, x_n\}$) are the closed subsets of X. Further, once the topology on X is known, the structure sheaf $\mathcal{O}_X$ is also determined by the family of the $\mathcal{O}_x$, since $\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_x$ by (8.2.1.1). The family $(\mathcal{O}_X)_{x \in X}$ thus completely determines the prescheme X when X is an integral scheme whose underlying space is Noetherian.

Proposition **(8.2.7)**. — *Let X, Y be two integral schemes, $f\colon X \to Y$ a dominant morphism (2.2.6), and K (resp. L) the field of rational functions on X (resp. Y). Then L can be identified with a sub-field of K, and for all $x \in X$, $\mathcal{O}_{f(x)}$ is the unique local ring of Y dominated by $\mathcal{O}_x$.*

In fact, if $f = (\psi, \theta)$ and a is the generic point of X, then $\psi(a)$ is the generic point of Y (**0**, 2.1.5); $\theta_a^\#$ is then a monomorphism of fields, from $L = \mathcal{O}_{\psi(a)}$ to $K = \mathcal{O}_a$. Since every non-empty affine open U of Y contains $\psi(a)$, it follows from (2.2.4) that the homomorphism $\Gamma(U, \mathcal{O}_Y) \to \Gamma(\psi^{-1}(U), \mathcal{O}_X)$ corresponding to f is the restriction of $\theta_a^\#$ to $\Gamma(U, \mathcal{O}_Y)$. So, for every $x \in X$, $\theta_x^\#$ is the restriction to $\mathcal{O}_{\psi(a)}$ of $\theta_a^\#$, and is thus a monomorphism. We also know that $\theta_x^\#$ is a local homomorphism, so, if we identify L with a sub-field of K by $\theta_a^\#$, $\mathcal{O}_{\psi(x)}$ is dominated by $\mathcal{O}_x$ (8.1.1); it is also the only local ring of Y dominated by $\mathcal{O}_x$, since two local rings of Y that are allied are identical (8.2.2).

Proposition **(8.2.8).** — *Let* X *be an* irreducible *prescheme; and* $f\colon X \to Y$ *a local immersion* (resp. *a local isomorphism); and suppose further that* f *is separated. Then* f *is an immersion* (resp. *an open immersion).*

Let $f = (\psi, \theta)$; it suffices, in both cases, to prove that ψ is a *homeomorphism* from X to $\psi(X)$ (4.5.3). Replacing f by f_{red} (5.1.6 and 5.5.1, (vi)), we can assume that X and Y are *reduced*. If Y' is the closed reduced sub-prescheme of Y having $\overline{\psi(X)}$ as its underlying space, then f factorises as $X \xrightarrow{f'} Y' \xrightarrow{j} Y$, where j is the canonical injection (5.2.2). It follows from (5.5.1, (v)) that f' is again a separated morphism; further, f' is again a local immersion **168**
(resp. a local isomorphism), because, since the condition is local on X and Y, we can reduce ourselves to the case where f is a closed immersion (resp. open immersion), and then our claim follows immediately from (4.2.2).

We can thus suppose that f is a *dominant* morphism, which leads to the fact that Y is, itself, irreducible (**0**, 2.1.5), and so X and Y are both *integral*. Further, the condition being local on Y, we can suppose that Y is an affine scheme; since f is separated, X is a scheme (5.5.1, (ii)), and we are finally at the hypotheses of (8.2.7). Then, for all $x \in X$, $\theta_x^{\#}$ is injective; but the hypothesis that F is a local immersion implies that $\theta_x^{\#}$ is surjective (4.2.2), so $\theta_x^{\#}$ is bijective, or, equivalently (with the identification of (8.2.7)) we have that $\mathcal{O}_{\psi(x)} = \mathcal{O}_x$. This implies, by (8.2.4), that ψ is an *injective* map, which already proves the proposition when f is a local isomorphism (4.5.3). When we suppose that f is only a local immersion, for all $x \in X$ there exists an open neighbourhood U of x in X and an open neighbourhood V of $\psi(x)$ in Y such that the restriction of ψ to U is a homeomorphism from U to a *closed* subset of V. But U is dense in X, so $\psi(U)$ is dense in Y and *a fortiori* in V, which proves that $\psi(U) = V$; since ψ is injective, $\psi^{-1}(V) = U$ and this proves that ψ is a homeomorphism from X to $\psi(X)$.

8.3. Chevalley schemes

(8.3.1) Let X be a *Noetherian* integral scheme, and R its field of rational functions; we denote by X' the set of local subrings $\mathcal{O}_x \subset R$, where x runs over all points of X. The set X' verifies the three following conditions:

(Sch. 1) For all $M \in X'$, R is the field of fractions of M.

(Sch. 2) There exists a finite set of Noetherian subrings A_i of R such that $X' = \bigcup_i L(A_i)$, and, for all pairs of indices i, j, the subring A_{ij} of R generated by $A_i \cup A_j$ is an algebra of finite type over A_i.

(Sch. 3) Two elements M and N of X' that are allied are identical.

We have basically seen in (8.2.1) that (Sch. 1) is satisfied, and (Sch. 3) follows from (8.2.2). To show (Sch. 2), it suffices to cover X by a finite number of affine opens U_i, whose rings are Noetherian, and to take $A_i = \Gamma(U_i, \mathcal{O}_X)$; the hypothesis that X is a scheme implies that $U_i \cap U_j$ is affine, and also that $\Gamma(U_i \cap U_j, \mathcal{O}_X) = A_{ij}$ (5.5.6); further, since the space U_i is Noetherian, the immersion $U_i \cap U_j \to U_i$ is of finite type (6.3.5), so A_{ij} is an A_i-algebra of finite type (6.3.3).

(8.3.2) The structures whose axioms are (Sch. 1), (Sch. 2), and (Sch. 3), generalise "schemes" in the sense of C. Chevalley, who supposes furthermore that R is an extension of finite type of a field K, and that the A_i are K-algebras of finite type (which renders a part of (Sch. 2) useless) [1]. Conversely, if we have such a structure on a set X', then we can associate to it an integral scheme X by using the remarks from (8.2.6): the underlying space of X is equal to X' endowed with the topology defined in (8.2.6), and with the sheaf $\mathcal{O}_X$ such that $\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_x$ for all open $U \subset X$, with the evident definition of restriction homomorphisms. We leave to the reader the task of verifying that we obtain thusly an integral scheme, whose local rings are the elements of X'; we will not use this result in what follows.

9. Supplement on quasi-coherent sheaves

9.1. Tensor product of quasi-coherent sheaves

Proposition **(9.1.1).** — *Let* X *be a prescheme* (resp. *a locally Noetherian prescheme). Let* $\mathscr{F}$ *and* $\mathscr{G}$ *be two quasi-coherent* **169**
(resp. *coherent)* $\mathcal{O}_X$*-modules; then* $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ *is quasi-coherent* (resp. *coherent) and of finite type if* $\mathscr{F}$ *and* $\mathscr{G}$ *are of finite type. If* $\mathscr{F}$ *admits a finite presentation and if* $\mathscr{G}$ *is quasi-coherent* (resp. *coherent), then* $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ *is quasi-coherent* (resp. *coherent).*

Being a local property, we can suppose that X is affine (resp. Noetherian affine); further, if $\mathscr{F}$ is coherent, then we can assume that it is the cokernel of a homomorphism $\mathcal{O}_X^m \to \mathcal{O}_X^n$. The claims pertaining to quasi-coherent sheaves then follow from (1.3.12) and (1.3.9); the claims pertaining to coherent sheaves follow from (1.5.1) and from the fact that, if M and N are modules of finite type over a Noetherian ring A, $M \otimes_A N$ and $\operatorname{Hom}_A(M, N)$ are A-modules of finite type.

Definition **(9.1.2)**. — *Let X and Y be two S-preschemes, p and q the projections of $X \times_S Y$, and $\mathscr{F}$ (resp. $\mathscr{G}$) a quasi-coherent $\mathcal{O}_X$-module (resp. quasi-coherent $\mathcal{O}_Y$-module). We define the tensor product of $\mathscr{F}$ and $\mathscr{G}$ over $\mathcal{O}_S$* (or *over S*), *denoted by $\mathscr{F} \otimes_{\mathcal{O}_S} \mathscr{G}$* (or $\mathscr{F} \otimes_S \mathscr{G}$) *to be the tensor product $p^*(\mathscr{F}) \otimes_{\mathcal{O}_{X \times_S Y}} q^*(\mathscr{G})$ over the prescheme $X \times_S Y$.*

If X_i ($1 \leqslant i \leqslant n$) are S-preschemes, and $\mathscr{F}_i$ are quasi-coherent $\mathcal{O}_{X_i}$-modules ($1 \leqslant i \leqslant n$), then we define similarly the tensor product $\mathscr{F}_1 \otimes_S \mathscr{F}_2 \otimes_S \ldots \otimes_S \mathscr{F}_n$ over the prescheme $Z = X_1 \times_S X_2 \times_s \ldots \times_S X_n$; it is a *quasi-coherent* $\mathcal{O}_Z$-module by virtue of (9.1.1) and (**0**, 5.1.4); it is *coherent* if the $\mathscr{F}_i$ are coherent and Z is *locally Noetherian*, by virtue of (9.1.1), (**0**, 5.3.11), and (6.1.1).

Note that if we take $X = Y = S$ then definition (9.1.2) gives us back the tensor product of $\mathcal{O}_S$-modules. Furthermore, as $q^*(\mathcal{O}_Y) = \mathcal{O}_{X \times_S Y}$ (**0**, 4.3.4), the product $\mathscr{F} \otimes_S \mathcal{O}_Y$ is canonically identified with $p^*(\mathscr{F})$, and, in the same way, $\mathcal{O}_X \otimes_S \mathscr{G}$ is canonically identified with $q^*(\mathscr{G})$. In particular, if we take $Y = S$ and denote by f the structure morphism $X \to Y$, we have that $\mathcal{O}_X \otimes_Y \mathscr{G} = f^*(\mathscr{G})$: the ordinary tensor product and the inverse image thus appear as particular cases of the general tensor product.

Definition (9.1.2) leads immediately to the fact that, for fixed X and Y, $\mathscr{F} \otimes_S \mathscr{G}$ is an *additive covariant bifunctor that is right-exact* in $\mathscr{F}$ and $\mathscr{G}$.

Proposition **(9.1.3)**. — *Let S, X, Y be three affine schemes of rings A, B, C (respectively), with B and C being A-algebras. Let M* (resp. N) *be a B-module* (resp. C-*module*)*, and $\mathscr{F} = \widetilde{M}$* (resp. $\mathscr{G} = \widetilde{N}$) *the associated quasi-coherent sheaf; then $\mathscr{F} \otimes_S \mathscr{G}$ is canonically isomorphic to the sheaf associated to the $(B \otimes_A C)$-module $M \otimes_A N$.* **170**

In fact, by virtue of (1.6.5), $\mathscr{F} \otimes_S \mathscr{G}$ is canonically isomorphic to the sheaf associated to the $(B \otimes_A C)$-module

$$\big(M \otimes_B (B \otimes_A C)\big) \otimes_{B \otimes_A C} \big((B \otimes_A C) \otimes_C N\big)$$

and by the canonical isomorphisms between tensor products, this latter module is isomorphic to

$$M \otimes_B (B \otimes_A C) \otimes_C N = (M \otimes_B B) \otimes_A (C \otimes_C N) = M \otimes_A N.$$

Proposition **(9.1.4)**. — *Let $f\colon T \to X$, and $g\colon T \to Y$ be two S-morphisms, and $\mathscr{F}$* (resp. $\mathscr{G}$) *a quasi-coherent $\mathcal{O}_X$-module* (resp. *quasi-coherent $\mathcal{O}_Y$-module*)*. Then*

$$(f, g)_S^*(\mathscr{F} \otimes_S \mathscr{G}) = f^*(\mathscr{F}) \otimes_{\mathcal{O}_T} g^*(\mathscr{G}).$$

If p, q are the projections of $X \times_S Y$, then the formula in fact follows from the relations $(f, g)_S^* \circ p^* = f^*$ and $(f, g)_S^* \circ q^* = g^*$ (**0**, 3.5.5), and the fact that the inverse image of a tensor product of algebraic sheaves is the tensor product of their inverse images (**0**, 4.3.3).

Corollary **(9.1.5)**. — *Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be S-morphisms, and $\mathscr{F}'$* (resp. $\mathscr{G}'$) *a quasi-coherent $\mathcal{O}_{X'}$-module* (resp. *quasi-coherent $\mathcal{O}_{Y'}$-module*)*. Then*

$$(f, g)_S^*(\mathscr{F}' \otimes_S \mathscr{G}') = f^*(\mathscr{F}') \otimes_S g^*(\mathscr{G}')$$

This follows from (9.1.4) and the fact that $f \times_S g = (f \circ p, g \circ q)_S$, where p, q are the projections of $X \times_S Y$.

Corollary **(9.1.6)**. — *Let X, Y, Z be three S-preschemes, and $\mathscr{F}$* (resp. $\mathscr{G}, \mathscr{H}$) *a quasi-coherent $\mathcal{O}_X$-module* (resp. *quasi-coherent $\mathcal{O}_Y$-module, quasi-coherent $\mathcal{O}_Z$-module*)*; then the sheaf $\mathscr{F} \otimes_S \mathscr{G} \otimes_S \mathscr{H}$ is the inverse image of $(\mathscr{F} \otimes_S \mathscr{G}) \otimes_S \mathscr{H}$ by the canonical isomorphism from $X \times_S Y \times_S Z$ to $(X \times_S Y) \times_S Z$.*

In fact, this isomorphism is given by $(p_1, p_2)_S \times_S p_3$, where p_1, p_2, p_3 are the projections of $X \times_S Y \times_S Z$.

Similarly, the inverse image of $\mathscr{G} \otimes_S \mathscr{F}$ by the canonical isomorphism from $X \times_S Y$ to $Y \times_S X$ is $\mathscr{F} \otimes_S \mathscr{G}$.

Corollary **(9.1.7)**. — *If X is an S-prescheme, then every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ is the inverse image of $\mathscr{F} \otimes_S \mathcal{O}_S$ by the canonical isomorphism from X to $X \times_S S$ (3.3.3).*

In fact, this isomorphism is $(1_X, \varphi)_S$, where φ is the structure morphism $X \to S$, and the corollary follows from (9.1.4) and the fact that $\varphi^*(\mathcal{O}_S) = \mathcal{O}_X$.

(9.1.8) Let X be an S-prescheme, $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-module, and $\varphi\colon S' \to S$ a morphism; we denote by $\mathscr{F}_{(\varphi)}$ or $\mathscr{F}_{(S')}$ the quasi-coherent sheaf $\mathscr{F} \otimes_S \mathcal{O}_{S'}$ over $X \times_S S' = X_{(\varphi)} = X_{(S')}$; so $\mathscr{F}_{(S')} = p^*(\mathscr{F})$, where p is the projection $X_{(S')} \to X$.

Proposition **(9.1.9)**. — *Let $\varphi''\colon S'' \to S'$ be a morphism. For every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ on the S-prescheme X, $(\mathscr{F}_{(\varphi)})_{(\varphi')}$ is the inverse image of $\mathscr{F}_{(\varphi\circ\varphi')}$ by the canonical isomorphism $(X_{(\varphi)})_{(\varphi')} \xrightarrow{\sim} X_{(\varphi\circ\varphi')}$* (3.3.9).

This follows immediately from the definitions and from (3.3.9), and is written

(9.1.9.1) $$(\mathscr{F} \otimes_S \mathcal{O}_{S'}) \otimes_{S'} \mathcal{O}_{S''} = \mathscr{F} \otimes_S \mathcal{O}_{S''}.$$

Proposition **(9.1.10)**. — *Let Y be an S-prescheme, and $f\colon X \to Y$ an S-morphism. For every quasi-coherent $\mathcal{O}_Y$-module and every morphism $S' \to S$, we have that $(f_{(S')})^*(\mathscr{G}_{(S')}) = (f^*(\mathscr{G}))_{(S')}$.*

This follows immediately from the commutativity of the diagram **171**

$$\begin{array}{ccc} X_{(S')} & \xrightarrow{f_{(S')}} & Y_{(S')} \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Corollary **(9.1.11)**. — *Let X and Y be S-preschemes, and $\mathscr{F}$* (resp. *$\mathscr{G}$) a quasi-coherent $\mathcal{O}_X$-module (*resp. *quasi-coherent $\mathcal{O}_Y$-module). Then the inverse image of the sheaf $(\mathscr{F}_{(S')}) \otimes_{(S')} (\mathscr{G}_{(S')})$ by the canonical isomorphism $(X \times_S Y)_{(S')} \xrightarrow{\sim} (X_{(S')}) \times_{S'} (Y_{(S')})$* (3.3.10) *is equal to $(\mathscr{F} \otimes_S \mathscr{G})_{(S')}$.*

If p, q are the projections of $X \times_S Y$, then the isomorphism in question is nothing but $(p_{(S')}, q_{(S')})_{S'}$; the corollary follows from propositions (9.1.4) and (9.1.10).

Proposition **(9.1.12)**. — *With the notation from (9.1.2), let z be a point of $X \times_S Y$, $x = p(z)$, and $y = q(z)$; the fibre $(\mathscr{F} \otimes_S \mathscr{G})_z$ is isomorphic to $(\mathscr{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z) \otimes_{\mathcal{O}_z} (\mathscr{G}_y \otimes_{\mathcal{O}_y} \mathcal{O}_z) = \mathscr{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z \otimes_{\mathcal{O}_y} \otimes \mathscr{G}_y$.*

As we can reduce ourselves to the affine case, the proposition follows from equation (1.6.5.1).

Corollary **(9.1.13)**. — *If $\mathscr{F}$ and $\mathscr{G}$ are of finite type, then we have that*

$$\mathrm{Supp}(\mathscr{F} \otimes_S \mathscr{G}) = p^{-1}(\mathrm{Supp}(\mathscr{F})) \cap q^{-1}(\mathrm{Supp}(\mathscr{G})).$$

Since $p^*(\mathscr{F})$ and $q^*(\mathscr{G})$ are both of finite type over $\mathcal{O}_{X\times_S Y}$, we are reduced, by (9.1.12) and (**0**, 1.7.5), to the case where $\mathscr{G} = \mathcal{O}_Y$, that is, it remains to prove the following equation:

(9.1.13.1) $$\mathrm{Supp}(p^{-1}(\mathscr{F})) = p^{-1}(\mathrm{Supp}(\mathscr{F})).$$

The same reasoning as in (**0**, 1.7.5) leads us to prove that, for all $z \in X \times_S Y$, we have $\mathcal{O}_z/\mathfrak{m}_x\mathcal{O}_z \neq 0$ (with $x = p(z)$), which follows from the fact that the homomorphism $\mathcal{O}_x \to \mathcal{O}_z$ is *local*, by hypothesis.

We leave it to the reader to extend the results in this section to the more general case of arbitrarily (but finitely) many factors, instead of just two.

9.2. Direct image of a quasi-coherent sheaf

Proposition **(9.2.1)**. — *Let $f\colon X \to Y$ be a morphism of preschemes. We suppose that there exists a cover (Y_α) of Y by affine opens having the following property: every $f^{-1}(Y_\alpha)$ admits a* finite *cover $(X_{\alpha i})$ by affine opens contained in $f^{-1}(Y_\alpha)$ such that every intersection $X_{\alpha i} \cap X_{\alpha j}$ is itself a* finite *union of affine opens. With these hypotheses, for every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a quasi-coherent $\mathcal{O}_Y$-module.*

Since this is a local condition on Y, we can assume that Y is equal to one of the Y_α, and thus omit the indices α.

a) First, suppose that the $X_i \cap X_j$ are themselves *affine* opens. We set $\mathscr{F}_i = \mathscr{F}|X_i$ and $\mathscr{F}_{ij} = \mathscr{F}|(X_i \cap X_j)$, and let $\mathscr{F}'_i$ and $\mathscr{F}'_{ij}$ be the images of $\mathscr{F}_i$ and $\mathscr{F}_{ij}$ (respectively) by the restriction of f to X_i and $X_i \cap X_j$ (respectively); we know that the $\mathscr{F}'_i$ and $\mathscr{F}'_{ij}$ are quasi-coherent (1.6.3). Set $\mathscr{G} = \bigoplus_i \mathscr{F}'_i$ and $\mathscr{H} = \bigoplus_{i,j} \mathscr{F}'_{ij}$; $\mathscr{G}$ and $\mathscr{H}$ are quasi-coherent $\mathcal{O}_Y$-modules; we will define a homomorphism $u\colon \mathscr{G} \to \mathscr{H}$ such that $f_*(\mathscr{F})$ is the *kernel* of u; it will follow from this that $f_*(\mathscr{F})$ is quasi-coherent (1.3.9). It suffices to define u as a homomorphism of presheaves; taking into account the definitions of $\mathscr{G}$ and $\mathscr{H}$, it thus **172** suffices, for every open subset $W \subset Y$, to define a homomorphism

$$u_W\colon \bigoplus_i \Gamma(f^{-1}(W) \cap X_i, \mathscr{F}) \to \bigoplus_{i,j} \Gamma(f^{-1}(W) \cap X_i \cap X_j, \mathscr{F})$$

in such a way that it satisfies the usual compatibility conditions when W varies. If, for every section $s_i \in \Gamma(f^{-1}(W) \cap X_i, \mathscr{F})$, we denote by $s_{i|j}$ the restriction to $f^{-1}(W) \cap X_i \cap X_j$, then we set

$$u_W\big((s_i)\big) = (s_{i|j} - s_{j|i})$$

and the compatibility conditions are clearly satisfied. To prove that the kernel $\mathscr{R}$ of u is $f_*(\mathscr{F})$, we define a homomorphism from $f_*(\mathscr{F})$ to $\mathscr{R}$ by sending each section $s \in \Gamma(f^{-1}(W), \mathscr{F})$ to the family (s_i), where s_i is the restriction of s to $f^{-1}(W) \cap X_i$; the axioms (F1) and (F2) of sheaves (G, II, 1.1) tell us that this homomorphism is *bijective*, which finishes the proof in this case.

b) In the general case, the same reasoning applies once we have established that the $\mathscr{F}_{ij}$ are quasi-coherent. But, by hypothesis, $X_i \cap X_j$ is a finite union of affine opens X_{ijk}; and since the X_{ijk} are affine opens *in a scheme*, the intersection of any two of them is again an affine open (5.5.6). We are thus led to the first case, and so we have proved (9.2.1).

Corollary **(9.2.2)**. — *The conclusion of* (9.2.1) *holds true in each of the following cases:*

a) f *is separated and quasi-compact.*
b) f *is separated and of finite type.*
c) f *is quasi-compact and the underlying space of* X *is locally Noetherian.*

In case *a)*, the $X_{\alpha i} \cap X_{\alpha j}$ are affine (5.5.6). Case *b)* is a particular case of *a)* (6.6.3). Finally, in case *c)*, we can reduce to the case where Y is affine and the underlying space of X is Noetherian; then X admits a finite cover of affine opens (X_i), and the $X_i \cap X_j$, being quasi-compact, are finite unions of affine opens (2.1.3).

9.3. Extension of sections of quasi-coherent sheaves

Theorem **(9.3.1)**. — *Let* X *be a prescheme whose underlying space is Noetherian, or a scheme whose underlying space is quasi-compact. Let* $\mathscr{L}$ *be an invertible* $\mathcal{O}_X$*-module* (**0**, 5.4.1)*,* f *a section of* $\mathscr{L}$ *over* X*,* X_f *the open set of* $x \in X$ *such that* $f(x) \neq 0$ (**0**, 5.5.1)*, and* $\mathscr{F}$ *a quasi-coherent* $\mathcal{O}_X$*-module.*

(i) *If* $s \in \Gamma(X, \mathscr{F})$ *is such that* $s|X_f = 0$*, then there exists a whole number* $n > 0$ *such that* $s \otimes f^{\otimes n} = 0$*.*
(ii) *For every section* $s \in \Gamma(X_f, \mathscr{F})$*, there exists a whole number* $n > 0$ *such that* $s \otimes f^{\otimes n}$ *extends to a section of* $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ *over* X*.*

(i) Since the underlying space of X is quasi-compact, and thus the union of finitely-many affine opens U_i with $\mathscr{L}|U_i$ is isomorphic to $\mathcal{O}_X|U_i$, we can reduce to the case where X is affine and $\mathscr{L} = \mathcal{O}_X$. In this case, f is identified with an element of $A(X)$, and we have that $X_f = D(f)$; s is identified with an element of an $A(X)$-module M, and $s|X_f$ to the corresponding element of M_f, and the result is then trivial, recalling the definition of a module of fractions.

(ii) Again, X is a finite union of affine opens U_i ($1 \leqslant i \leqslant r$) such that $\mathscr{L}|U_i \cong \mathcal{O}_X|U_i$, and for every i, **173**
$(s \otimes f^{\otimes n})|(U_i \cap X_f)$ is identified (by the aforementioned isomorphism) with $(f|(U_i \cap X_f))^n (s|(U_i \cap X_f))$. We then know (1.4.1) that there exists a whole number $n > 0$ such that, for all i, $(s \otimes f^{\otimes n})|(U_i \cap X_f)$ extends to a section s_i of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ over U_i. Let $s_{i|j}$ be the restriction of s_i to $U_i \cap U_j$; by definition we have that $s_{i|j} - s_{j|i} = 0$ in $X_f \cap U_i \cap U_j$. But, if X is a Noetherian space, then $U_i \cap U_j$ is quasi-compact; if X is a scheme, then $U_i \cap U_j$ is an affine open (5.5.6), and so again quasi-compact. By virtue of (i), there thus exists a whole number m (independent of i and j) such that $(s_{i|j} - s_{j|i}) \otimes f^{\otimes m} = 0$. It immediately follows that there exists a section s' of $\mathscr{F} \otimes \mathscr{L}^{\otimes(n+m)}$ over X, restricting to $s_i \otimes f^{\otimes m}$ over each U_i, and restricting to $s \otimes f^{\otimes(n+m)}$ over X_f.

The following corollaries give an interpretation of theorem (9.3.1) in a more algebraic language:

Corollary **(9.3.2)**. — *With the hypotheses of* (9.3.1)*, consider the graded ring* $A_* = \Gamma_*(\mathscr{L})$ *and the graded* A_**-module* $M_* = \Gamma_*(\mathscr{L}, \mathscr{F})$ (**0**, 5.4.6)*. If* $f \in A_n$*, where* $n \in \mathbb{Z}$*, then there is a canonical isomorphism* $\Gamma(X_f, \mathscr{F}) \xrightarrow{\sim} ((M_*)_f)_0$ (the subgroup of the module of fractions $(M_*)_f$ consisting of elements of degree 0*).*

Corollary **(9.3.3)**. — *Suppose that the hypotheses of* (9.3.1) *are satisfied, and suppose further that* $\mathscr{L} = \mathcal{O}_X$*. Then, setting* $A = \Gamma(X, \mathcal{O}_X)$ *and* $M = \Gamma(X, \mathscr{F})$*, the* A_f*-module* $\Gamma(X_f, \mathscr{F})$ *is canonically isomorphic to* M_f*.*

Proposition **(9.3.4)**. — *Let* X *be a Noetherian prescheme,* $\mathscr{F}$ *a coherent* $\mathcal{O}_X$*-module, and* $\mathscr{J}$ *a coherent sheaf of ideals in* $\mathcal{O}_X$*, such that the support of* $\mathscr{F}$ *is contained in that of* $\mathcal{O}_X|\mathscr{J}$*. Then there exists a whole number* $n > 0$ *such that* $\mathscr{J}^n\mathscr{F} = 0$*.*

Since X is a union of finitely-many affine opens whose rings are Noetherian, we can suppose that X is affine of Noetherian ring A; then $\mathscr{F} = \widetilde{M}$, where $M = \Gamma(X, \mathscr{F})$ is an A-module of finite type, and $\mathscr{J} = \widetilde{\mathfrak{J}}$, where $\mathfrak{J} = \Gamma(X, \mathscr{J})$ is an ideal of A (1.4.1 and 1.5.1). Since A is Noetherian, $\mathfrak{J}$ admits a finite system of generators f_i ($1 \leqslant i \leqslant m$). By hypothesis, every section of $\mathscr{F}$ over X is zero in each of the $D(f_i)$; if s_j ($1 \leqslant j \leqslant q$) are sections of $\mathscr{F}$ generating M, then there exists a whole number h, independent of i and j, such that $f_i^h s_j = 0$. (1.4.1), whence $f_i^h s = 0$ for all $s \in M$. We thus conclude that if $n = mh$ then $\mathfrak{J}^n M = 0$, and so the corresponding $\mathcal{O}_X$-module $\mathscr{J}^n\mathscr{F} = \widetilde{\mathfrak{J}^n M}$ (1.3.13) is zero.

Corollary **(9.3.5)**. — *With the hypotheses of* (9.3.4)*, there exists a closed sub-prescheme* Y *of* X*, whose underlying space is the support of* $\mathcal{O}_X/\mathscr{J}$*, such that, if* $j: Y \to X$ *is the canonical injection, then* $\mathscr{F} = j_*(j^*(\mathscr{F}))$*.*

First of all, note that the supports of $\mathcal{O}_X/\mathscr{J}$ and $\mathcal{O}_X/\mathscr{J}^n$ are the same, since, if $\mathscr{J}_x = \mathcal{O}_x$, then $\mathscr{J}_x^n = \mathcal{O}_x$, and we also have that $\mathscr{J}_x^n \subset \mathscr{J}_x$ for all $x \in X$. We can, thanks to (9.3.4), thus suppose that $\mathscr{J}\mathscr{F} = 0$; we can then take Y to be the closed sub-prescheme of X defined by $\mathscr{J}$, and since $\mathscr{F}$ is then an $(\mathcal{O}_X/\mathscr{J})$-module, the conclusion follows immediately.

9.4. Extension of quasi-coherent sheaves

(9.4.1) Let X be a topological space, $\mathscr{F}$ a sheaf of sets (resp. of groups, of rings) on X, U an open subset of X, $\psi: U \to X$ the canonical injection, and $\mathscr{G}$ a sub-sheaf of $\mathscr{F}|U = \psi^*(\mathscr{F})$. Since ψ_* is left exact, $\psi_*(\mathscr{G})$ is a sub-sheaf of $\psi_*(\psi^*(\mathscr{F}))$; if we denote by ρ the canonical homomorphism $\mathscr{F} \to \psi_*(\psi^*(\mathscr{F}))$ (**0**, 3.5.3), then we denote by $\overline{\mathscr{G}}$ the sub-sheaf $\rho^{-1}(\psi_*(\mathscr{G}))$ of $\mathscr{F}$. It follows immediately from the definitions that, for every open subset V of X, $\Gamma(V, \overline{\mathscr{G}})$ consists of sections $s \in \Gamma(V, \mathscr{F})$ whose restriction to $V \cap U$ is a section of $\mathscr{G}$ over $V \cap U$. We thus have that $\overline{\mathscr{G}}|U = \psi^*(\overline{\mathscr{G}}) = \mathscr{G}$, and that $\overline{\mathscr{G}}$ is the *biggest* sub-sheaf of $\mathscr{F}$ that restricts to $\mathscr{G}$ over U; we say that $\overline{\mathscr{G}}$ is the *canonical extension* of the sub-sheaf $\mathscr{G}$ of $\mathscr{F}|U$ to a sub-sheaf of $\mathscr{F}$. **174**

Proposition **(9.4.2)**. — *Let* X *be a prescheme,* U *an open subset of* X *such that the canonical injection* $j: U \to X$ *is a quasi-compact morphism* (which will be the case for *all* U if the underlying space of X is *locally Noetherian* (6.6.4, (i))). *Then:*

- (i) *For every quasi-coherent* $(\mathcal{O}_X|U)$*-module* $\mathscr{G}$*,* $j_*(\mathscr{G})$ *is a quasi-coherent* $\mathcal{O}_X$*-module, and* $j_*(\mathscr{G})|U = j^*(j_*(\mathscr{G})) = \mathscr{G}$*.*
- (ii) *For every quasi-coherent* $\mathcal{O}_X$*-module* $\mathscr{F}$ *and every quasi-coherent sub-*$(\mathcal{O}_X|U)$*-module* $\mathscr{G}$*, the canonical extension* $\overline{\mathscr{G}}$ *of* $\mathscr{G}$ (9.4.1) *is a quasi-coherent sub-*$\mathcal{O}_X$*-module of* $\mathscr{F}$*.*

9.5. Closed image of a prescheme; closure of a sub-prescheme

9.6. Quasi-coherent sheaves of algebras; change of structure sheaf

10. Formal schemes