EGA I

A. GROTHENDIECK

WHAT THIS IS. This is a community translation of Grothendieck's EGA I. As it is a work in progress by multiple people, it will probably have a few mistakes — if you spot any then please feel free to make a correction!

On est désolés, Grothendieck.

— Ryan Keleti, Tim Hosgood

Contents

Introduction

To Oscar Zariski and André Weil.

This memoir, and the many others that must follow, are intended to form a treatise on the foundations of algebraic geometry. They do not assume, in principle, any particular knowledge of this discipline, and it has even been that such knowledge, despite its obvious advantages, could sometimes (by the too-exclusive habit that the birational point of view it implies) to be harmful to the one who wants to become familiar with the point of view and techniques presented here. However, we assume that the reader has a good knowledge of the following topics: **5**

(a) *Commutative algebra*, as it is exhibited for example in volumes under preparation of the *Elements* of N. Bourbaki (and, pending the publication of these volumes, in Samuel-Zariski [13] and Samuel [11], [12]).
(b) *Homological algebra*, for which we refer to Cartan-Eilenberg [2] (cited as (M)) and Godement [4] (cited as (G)), as well as the recent article by A. Grothendieck [6] (cited as (T)).
(c) *Sheaf Theory*, where our main references will be (G) and (T); this theory provides the essential language for interpreting in "geometric" terms the essential notions of commutative algebra, and to "globalize" them.
(d) Finally, it will be useful for the reader to have some familiarity with *functorial language*, which will be constantly used in this Treatise, and for which the reader may consult (M), (G) and especially (T); the principles of this language and the main results of the general theory of functors will be described in more detail in a book currently in preparation by the authors of this Treatise.

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It is not the place, in this Introduction, to give a more or less summarily description from the point of view of "schemes" in algebraic geometry, nor the long list of reasons which made its adoption necessary, and in particular the systematic acceptance of nilpotent elements in the local rings of "manifolds" that we consider (which necessarily shifts the idea of rational mappings into the background, in favor of those of regular mappings or "morphisms"). This Treatise aims precisely to systematically develop the language of schemes, and will demonstrate, we hope, its necessity. Although it would be easy to do so, we will not try to **6**
give here an "intuitive" introduction to the notions developed in Chapter 1. For the reader who would like to have a glimpse of the preliminary study of the subject matter of this Treatise, we refer them to the conference by A. Grothendieck at the International Congress of Mathematicians in Edinburgh in 1958 [7], and the expose [8] of the same author. The work [14] (cited as (FAC)) of J.-P. Serre can also be considered as an intermediary exposition between the classical point of view and the point of view of schemes in algebraic geometry, and, as such, its reading may be an excellent preparation to that of our *Elements*.

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We give below the general outline planned for this Treatise, subject to later modifications, especially concerning the later chapters.

Chapter I. — The language of schemes.
— II. — Elementary global study of some classes of morphisms.
— III. — Cohomology of algebraic coherent sheaves. Applications.
— IV. — Local study of morphisms.
— V. — Elementary procedures of constructing schemes.
— VI. — Descent. General method of constructing schemes.
— VII. — Schemes of groups, principal fibre bundles.
— VIII. — Differential study of fibre bundles.
— IX. — The fundamental group.
— X. — Residues and duality.
— XI. — Theories of intersection, Chern classes, Riemann-Roch theorem.
— XII. — Abelian schemes and Picard schemes.
— XIII. — Weil cohomology.

In principal, all chapters are considered open to changes, and supplementary paragraphs can always be added later; such paragraphs would appear in separate fascicles in order to minimise the inconveniences accompanying whatever mode of publication adopted. When the writing of such a paragraph is foreseen or in progress during the publication of a chapter, it will be mentioned in the summary of the chapter in question, even if, owing to certain orders of urgency, its actual publication clearly ought to have been later. For the use of the reader, we give in "Chapter 0" the necessary tools in commutative algebra, homological algebra, and sheaf theory, that will be used throughout this Treatise, that are more or less well known but for which it was not possible to give convenient references. It is recommended for the reader to not read Chapter 0 except whilst reading the Treatise proper, when the results to which we refer seem unfamiliar. Besides, we think that in **7**
this way, the reading of this Treatise could be a good method for the beginner to familiarise themselves with commutative algebra and homological algebra, whose study, when not accompanied with tangible applications, is considered tedious, or even depressing, by many.

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It is outside of our capabilities to give a historic overview, or even a summary thereof, of the ideas and results described. The text will contain only those references considered particularly useful for comprehension, and we indicate the origin only of the most important results. Formally, at least, the subjects discussed in our work are reasonably new, which explains the scarcity of references made to the Fathers of algebraic geometry from the 19th to the beginning of the 20th century, whose works we know only by hear-say. It is suitable, however, to say some words here about the works which have most directly influenced the authors and contributed to the development of scheme-theoretic point of view. We absolutely must mention the fundamental work (FAC) of J.-P. Serre first, which has served as an introduction to algebraic geometry for more that one young student (one of the authors of this Treatise being one), deterred by the dryness of the classic *Foundations* of A. Weil [18]. It is there that it is shown, for the first time, that the "Zariski topology" of an "abstract" algebraic variety is perfectly suited to applying certain techniques from algebraic topology, and notably to be able to define a cohomology theory. Further, the definition of an algebraic variety given therein is that which translates most naturally to the idea that we develop here[1]. Serre himself had incidentally noted that the cohomology theory of affine algebraic varieties could translate without difficulty by replacing the affine algebras over a field by arbitrary commutative rings. Chapters I and II of this Treatise, and the first two paragraphs of chapter III, can thus be considered, for the most part, as easy translations, in this bigger framework, of the principal results of (FAC) and a later article of the same author [15]. We have also vastly profited from the *Séminaire de Géométrie*

[1] Just as J.-P. Serre informed us, it is right to note that the idea of defining the structure of a manifold by the data of a sheaf of rings is due to H. Cartan, who took this idea as the starting point of his theory of analytic spaces. Of course, just as in algebraic geometry, it would be important in "analytic geometry" to give the right to use nilpotent elements in local rings of analytic spaces. This extension of the definition of H. Cartan and J.-P. Serre has recently been broached by H. Grauert [5], and there is room to hope that a systematic report of analytic geometry in this setting will soon see the light of day. It is also evident that the ideas and techniques developed in this Treatise retain a sense of analytic geometry, even though one must expect more considerable technical difficulties in this latter theory. We can foresee that algebraic geometry, by the simplicity of its methods, will be able to serve as a sort of formal model for future developments in the theory of analytic spaces.

algébrique de C. Chevalley [1]; in particular, the systematic usage of "constructible sets" introduced by him has turned out to be extremely useful in the theory of schemes (cf. chap. IV). We have also borrowed from him the study of morphisms from the point of view of dimension (chap. IV), that translates with negligible change **8**
to the framework of schemes. It also merits noting that the idea of "schemes of local rings", introduced by Chevalley, naturally lends itself to being extended to algebraic geometry (not having, however, all the flexibility and generality that we intend to give it here); for the connections between this idea and our theory, see the chap. I, s. 8. One such extension has been developed by M. Nagata in a series of memoires [9], which contain many special results concerning algebraic geometry over Dedekind rings[1].

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It goes without saying that a book on algebraic geometry, and especially a book dealing with the fundamentals, is of course influenced, [...], by mathematicians such as O. Zariski and A. Weil. In particular, the *Théorie des fonctions holomorphes* de Zariski [20], properly flexible thanks to the cohomological methods and an existence theorem (chap. III, ss. 4 et 5), is (along with the method of descent described in chap. VI) one of the principal tools used in this Treatise, and it seems to us one of the most powerful at our disposal in algebraic geometry.

The general technique in which it is employed can be sketched as follows (a typical example of which will be given in chap. XI, in the study of the fundamental group). We have a proper morphism (chap. II) $f\colon X \to Y$ of algebraic varieties (more generally, of schemes) that we wish to study on the neighbourhood of a point $y \in Y$, with the aim of resolving a problem P relative to a neighbourhood of y. We follow successive steps:

(1) We can suppose that Y is affine, such that X becomes a scheme defined on the affine ring A of Y, and we can even replace A by the local ring of y. This reduction is always easy in practice (chap. V) and brings us to the case where A is a *local* ring.
(2) We study the problem in question when A is a local *artinien* ring. So that the problem P still makes sense when A is not assumed to be integral, sometimes we have to reformulate P, and it appears that we often thus obtain a better understanding of the problem during this stage, in an "infinitesimal" way.
(3) The theory of formal schemes (chap. III, ss. 3, 4, and 5) lets us pass from the case of an artinien ring to a *complete local ring*.
(4) Finally, if A is an arbitrary local ring, considering "multiform sections" of suitable schemes over X approximates the idea of a given "formal" section (chap. IV), and this will let us pass, by extension of **9**
scalars to the completion of A, from a known result of [...]to an analogous result for a finite simple (e.g. unramified) extension of A.

This sketch shows the importance of the systematic study of schemes defined over an artinien ring A. The point of view of Serre in his formulation of the theory of local class fields, and the recent works of Greenberg, seem to suggest that such a study could be undertaken by functorially attaching, to some such scheme X, a scheme X' over the residue field k of A (assumed perfect) of dimension equal (in nice cases) to $n \dim X$, where n is the height of A.

As for the influence of A. Weil, it suffices to say that it is the need to develop the tools necessary to formulate, with full generality, the definition of "Weil cohomology", and to tackle the proof[1] of all the formal properties necessary to establish the famous conjectures in diophantine geometry [19], that has been one of the principal motivations of the writing of this Treatise, as has the desire to find the natural setting of the usual ideas and methods of algebraic geometry, and to give the authors the chance to understand these ideas and methods.

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[1] Amongst the works that come close to our point of view of algebraic geometry, we pick out the work of E. Kàhler [22] and a recent note of Chow and Igusa [3], which go back over certain results of (FAC) in the context of Nagata-Chevalley theory, as well as giving a Künneth formula.

[1] To avoid any misunderstanding, we point out that this task has barely been undertaken at the moment of writing this Introduction, and still hasn't led to the proof of the Weil conjectures.

To finish, we believe it useful to warn the reader that, as was the case with all the authors themselves, they will almost certainly have difficulty before becoming accustomed to the language of schemes, and to convince themselves that the usual constructions that suggest geometric intuition can be translated, in essentially only one sensible way, to this language. As in many parts of modern mathematics, the first intuition seems further and further away, in appearance, from the correct language needed to express the mathematics in question with complete precision and the desired level of generality. In practice, the psychological difficulty comes from the need to replicate some familiar set-theoretic constructions to a category that is already quite different from that of sets (the category of preschemes, or the category of preschemes over a given prescheme): cartesian products, group laws, ring laws, module laws, fibre bundles, principal homogeneous fibre bundles, etc. It will most likely be difficult for the mathematician, in the future, to shy away from this new effort of abstraction, maybe rather negligible, on the whole, in comparison with that endowed by our fathers, to familiarise themselves with the Theory of Sets.

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The references are given following the numerical system; for example, in III, 4.9.3, the III indicates the chapter, the 4 the paragraph, the 9 the section of the paragraph. If we reference a chapter from within itself then we omit the chapter number.

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Chapter 0. — Preliminaries

1. Rings of Fractions

1.0. Rings and Algebras

(1.0.1) All the rings considered in this Treatise will have a *unit element*; all the modules on such a ring will be assumed to be *unitary*; the ring homomorphisms will always be assumed to *transform the unit element into a unit element*; unless otherwise stated, a sub-ring of a ring A will be assumed to *contain the unit element of* A. We will consider especially *commutative* rings, and when we speak of a ring without specification, it will be implied that it is commutative. If A is a ring not necessarily commutative, by A-module we will we mean a left module, unless stated otherwise. **11**

(1.0.2) Let A, B be two rings, not necessarily commutative, $\varphi : A \to B$ a homomorphism. Any left (resp. right) B-module M can be provided with a left (resp. right) A-module structure by $a \cdot m = \varphi(a) \cdot m$ (resp. $m \cdot a = m \cdot \varphi(a)$); when it will be necessary to distinguish M as an A-module or a B-module, we will denote by $M_{[\varphi]}$ the left (resp. right) A-module as defined. If L is an A-module, then a homomorphism $u : L \to M_{[\varphi]}$ is a homomorphism of commutative groups such that $u(a \cdot x) = \varphi(a) \cdot u(x)$ for $a \in A$, $x \in L$; we will also say that it is a φ-*homomorphism* $L \to M$, and that the pair (φ, u) (or, by misuse of langauge, u) is a *di-homomorphism* of (A, L) in (B, M). The pairs (A, L) formed by a ring A and an A-module L thus form a *category* for which the morphisms are di-homomorphisms.

(1.0.3) Under the hypothesis of (1.0.2), if $\mathfrak{J}$ is a left (resp. right) ideal of A, we denote by $B\mathfrak{J}$ (resp. $\mathfrak{J}B$) the left (resp. right) ideal $B\varphi(\mathfrak{J})$ (resp. $\varphi(\mathfrak{J})B$) of B generated by $\varphi(\mathfrak{J})$; it is also the image of the canonical homomorphism $B \otimes_A \mathfrak{J} \to B$ (resp. $\mathfrak{J} \otimes_A B \to B$) of left (resp. right) B-modules.

(1.0.4) If A is a (commutative) ring, B a non necessarily commutative ring, the data of a structure of an A-*algebra* on B is equivalent to the data of a ring homomorphism $\varphi : A \to B$ such that $\varphi(A)$ is contained in the center of B. For all ideals $\mathfrak{J}$ of A, $\mathfrak{J}B = B\mathfrak{J}$ is then a two-sided ideal of B, and for every B-module M, $\mathfrak{J}M$ is then a B-module equal to $(B\mathfrak{J})M$.

(1.0.5) We will not return to the notions of *module finite type* and *algebra* (commutative) *of finite type*; to say that an A-module M is of finite type means that there exists an exact sequence $A^p \to M \to 0$. We say that an A-module M admits a *finite presentation* if it is isomorphic to the cokernel of a homomorphism $A^p \to A^q$, in other words, there exists an exact sequence $A^p \to A^q \to M \to 0$. We note that for a *Noetherian* ring A, every A-module of finite type admits a finite presentation. **12**

Let us recall that an A-algebra B is called *integral* over A if every element in B is a root in B of a monic polynomial with coefficients in A; equivalently, every element of B is contained in a subalgebra of B which is an A-*module of finite type*. When this is so, and B is commutative, the subalgebra of B generated by a finite part of B is an A-module of finite type; for a commutative algebra B to be integral and of finite type over A, it is necessary and therefore sufficient that B be an A-module of finite type; we also say that B is an *integral* A-*algebra of finite type* (or simply *finite* if there is no confusion). It will be observed that in these definitions, it is not assumed that the homomorphism $A \to B$ defining the structure of an A-algebra is injective.

(1.0.6) An *integral domain* is a ring in which the product of a finite family of elements $\neq 0$ is $\neq 0$; equivalently, in such a ring we have $0 \neq 1$ and the product of two elements $\neq 0$ is non zero. A *prime* ideal of a ring A is an ideal $\mathfrak{p}$ such that $A/\mathfrak{p}$ is integral; this therefore entails $\mathfrak{p} \neq A$. For a ring A to have at least one prime ideal, it is necessary and sufficent that $A \neq \{0\}$.

(1.0.7) A *local* ring is a ring A in which there exists a single maximal ideal, which is then the complement of the invertible elements and contains all the ideals $\neq A$. If A and B are two local rings, $\mathfrak{m}$ and $\mathfrak{n}$ their respective maximal ideals, we say that a homomorphism $\varphi : A \to B$ is *local* if $\varphi(\mathfrak{m}) \subset \mathfrak{n}$ (or, equivalently, $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$). By passing to quotients, such a homomorphism then defines a momomorphism of the residue field $A/\mathfrak{m}$ into the residue field $B/\mathfrak{n}$. The composition of two local homomorphisms is a local homomorphism.

1.1. Root (radical) of an ideal. Nilradical and radical of a ring.

(1.1.1) Let $\mathfrak{a}$ be an ideal of a ring A; the *root* (*radical*) of $\mathfrak{a}$, denoted by $\mathfrak{r}(\mathfrak{a})$, is the set of $x \in A$ such that $x^n \in \mathfrak{a}$ for an integer $n > 0$; it is an ideal containing $\mathfrak{a}$. We have $\mathfrak{r}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a})$; the relation $\mathfrak{a} \subset \mathfrak{b}$ leads to $\mathfrak{r}(\mathfrak{a}) \subset \mathfrak{r}(\mathfrak{b})$; the root of a finite intersection of ideals is the intersection of their roots. If φ is a homomorphism of a ring A' into A, then we have $\mathfrak{r}(\varphi^{-1}(\mathfrak{a})) = \varphi^{-1}(\mathfrak{r}(\mathfrak{a}))$ for any ideal $\mathfrak{a} \subset A$. For an ideal to be the root of an ideal, it is necessary and sufficient that it be an intersection of prime ideals. The root of an ideal $\mathfrak{a}$ is the intersection of the *minimal* prime ideals among those containing $\mathfrak{a}$; if A is Noetherian, these minimal prime ideals are finite in number.

The root of the ideal (0) is also called the *nilradical* of A; it is the set $\mathfrak{N}$ of the nilpotent elements of A. It is said that the ring A is *reduced* if $\mathfrak{N} = (0)$; for every ring A, the quotient $A/\mathfrak{N}$ of A by its nilradical is a reduced ring.

(1.1.2) Recall that the *radical* $\mathfrak{R}(A)$ of a ring A (not necessarily commutative) is the intersection of the maximal left ideals of A (and also the intersection of maximal right ideals). The radical of $A/\mathfrak{R}(A)$ is (0).

1.2. Modules and rings of fractions.

(1.2.1) We say that a subset S of a ring A is *multiplicative* if $1 \in S$ and if the product of two elements of We say **13**
that a part S of a ring A is *multiplicative* if $1 \in S$ and if the product of two elements of S is in S. The examples which will be the most important for the following are: 1st the set S_f of powers f^n $(n \geqslant 0)$ of an element $f \in A$; 2nd the complement $A - \mathfrak{p}$ of a *prime* ideal $\mathfrak{p}$ of A.

(1.2.2) Let S be a multiplicative subset of a ring A, M an A-module; in the set $M \times S$, the relation between couples (m_1, s_1), (m_2, s_2):

"There exists $s \in S$ such that $s(s_1 m_2 - s_2 m_1) = 0$"

is an equivalence relation. We denote by $S^{-1}M$ the quotient set of $M \times S$ by this relation, by m/s the canonical image in $S^{-1}M$ of the pair (m, s); we call the *canonical* mapping of M in $S^{-1}M$ the mapping $i^S_M : m \mapsto m/1$ (also denoted i^S). This mapping is generally neither injective nor surjective; its kernel is the set of $m \in M$ such that there exists an $s \in S$ for which $sm = 0$.

On $S^{-1}M$ we define an additive group law by taking

$$(m_1/s_1) + (m_2/s_2) = (s_2 m_1 + s_1 m_2)/(s_1 s_2)$$

(we check that it is independent of the expressions of the elements of $S^{-1}M$ considered). On $S^{-1}A$ we further define a multiplicative law by taking $(a_1/s_1)(a_2/s_2) = (a_1 a_2)/(s_1 s_2)$, and finally an external law on $S^{-1}M$, having $S^{-1}A$ as a set of operators, by setting $(a/s)(m/s') = (am)/(ss')$. It is thus verified that $S^{-1}A$ is provided with a ring structure (called *the ring of fractions of* A, *with denominators in* S) and $S^{-1}M$ the structure of an $S^{-1}A$-module (called *the module of fractions of* M, *with denominators in* S); for all $s \in S$, $s/1$ is invertible in $S^{-1}A$, its inverse being $1/s$. The canonical mapping i^S_A (resp. i^S_M) is a homomorphism of rings (resp. a homomorphism of A-modules, $S^{-1}M$ being considered A-module by means of the homomorphism $i^S_A : A \to S^{-1}A$).

(1.2.3) If $S_f = \{f^n\}_{n \geqslant 0}$ for a $f \in A$, we write A_f and M_f instead of $S_f^{-1}A$ and $S_f^{-1}M$; when A_f is considered as algebra over A, we can write $A_f = A[1/f]$. A_f is isomorphic to the quotient algebra $A[T]/(fT - 1)A[T]$. When $f = 1$, A_f and M_f identify canonically with A and M; if f is niipotent, A_f and M_f are reduced to 0. When $S = A - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A, we write $A_\mathfrak{p}$ and $M_\mathfrak{p}$ instead of $S^{-1}A$ and $S^{-1}M$; $A_\mathfrak{p}$ is a *local ring* whose maximal ideal $\mathfrak{q}$ is generated by $i^S_A(\mathfrak{p})$, and we have $(i^S_A)^{-1}(\mathfrak{q}) = \mathfrak{p}$; by passing to the quotients, i^S_A gives a monomorphism of the integral domains $A/\mathfrak{p}$ into the field $A_\mathfrak{p}/\mathfrak{q}$, which identifies with the field of fractions of $A/\mathfrak{p}$.

(1.2.4) The ring of fractions $S^{-1}A$ and the canonical homomorphism i_A^S are a solution of a *universal mapping problem*: any homomorphism u of A into a ring B such that $u(S)$ is composed of *invertible* elements in B factorizes in one way

$$u : A \xrightarrow{i_A^S} S^{-1}A \xrightarrow{u^*} B$$

where u^* is a ring homomorphism. Under the same hypotheses, let M be an A-module, N a B-module, $v : M \to N$ a homomorphism of A-modules (for the B-module structure on N defined by $u : A \to B$); then v is factorizes in a single way **14**

$$v : M \xrightarrow{i_M^S} S^{-1}M \xrightarrow{v^*} N$$

where v^* is a homomorphism of $S^{-1}A$-modules (for the $S^{-1}A$-module structure on N defined by u^*).

(1.2.5) We define a canonical isomorphism $S^{-1}A \otimes_A M \xrightarrow{\sim} S^{-1}M$ of $S^{-1}A$- modules, sending the element $(a/s) \otimes m$ to the element $(am)/s$, the isomorphism reciprocally applying m/s to $(1/s) \otimes m$.

(1.2.7) When A is an *integral domain*, for which K denotes its field of fractions, the canonical mapping $i_A^S : A \to S^{-1}A$ is injective for any multiplicative subset S not containing 0, and $S^{-1}A$ then identifies canonically with a subring of K containing A. In particular, for every prime ideal $\mathfrak{p}$ of A , $A_\mathfrak{p}$ is a local ring containing A, with maximal ideal $\mathfrak{p}A_\mathfrak{p}$, and we have $\mathfrak{p}A_\mathfrak{p} \cap A = \mathfrak{p}$.

(1.2.8) If A is a *reduced* ring (1.1.1), so is $S^{-1}A$: indeed, if $(x/s)^n = 0$ for $x \in A$, $s \in S$, it means that there exists $s' \in S$ such that $s'x^n = 0$, hence $(s'x)^n = 0$, which, by hypothesis, entails $s'x = 0$, so $x/s = 0$.

1.3. Functorial properties.

(1.3.1) Let M, N be two A-modules, u an A-homomorphism $M \to N$. If S is a multiplicative subset of A, we define a $S^{-1}A$-homomorphism $S^{-1}M \to S^{-1}N$, denoted by $S^{-1}u$, by putting $S^{-1}u(m/s) = u(m)/s$; if $S^{-1}M$ and $S^{-1}N$ are canonically identified with $S^{-1}A \otimes_A M$ and $S^{-1}A \otimes_A N$ (1.2.5), $S^{-1}u$ is considered as $1 \otimes u$. If P is a third A-module, v an A-homomorphism $N \to P$, we have $S^{-1}(v \circ u) = (S^{-1}v) \circ (S^{-1}u)$; in other words, $S^{-1}M$ is a *covariant functor in* M, of the category of A-modules into that of $S^{-1}A$-modules (A and S being fixed).

(1.3.2) The functor $S^{-1}M$ is *exact*; in other words, if the following

$$M \xrightarrow{u} N \xrightarrow{v} P$$

is exact, so is the following

$$S^{-1}M \xrightarrow{S^{-1}u} S^{-1}N \xrightarrow{S^{-1}v} S^{-1}P.$$

In particular, if $u : M \to N$ is injective (resp. surjective), the same is true for $S^{-1}u$; if N and P are two submodules of M, $S^{-1}N$ and $S^{-1}P$ identify canonically with submodules of $S^{-1}M$, and we have **15**

$$S^{-1}(N + P) = S^{-1}N + S^{-1}P \quad \text{and} \quad S^{-1}(N \cap P) = (S^{-1}N) \cap (S^{-1}P).$$

(1.3.3) Let $(M_\alpha, \varphi_{\beta\alpha})$ be an inductive system of A-modules; then $(S^{-1}M_\alpha, S^{-1}\varphi_{\beta\alpha})$ is an inductive system of $S^{-1}A$-modules. Expressing the $S^{-1}M_\alpha$ and $S^{-1}\varphi_{\beta\alpha}$ as tensor products (1.2.5 and 1.3.1), it follows from the permutability of tensor product and inductive limit operations that we have a canonical isomorphism

$$S^{-1} \varinjlim M_\alpha \xrightarrow{\sim} \varinjlim S^{-1}M_\alpha$$

which is further expressed by saying that the functor $S^{-1}M$ (in M) *commutes with inductive limits*.

(1.3.4) Let M, N be two A-modules; there is a canonical *functorial* isomorphism (in M and N)

$$(S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) \xrightarrow{\sim} S^{-1}(M \otimes_A N)$$

which transforms $(m/s) \otimes (n/t)$ into $(m \otimes n)/st$.

(1.3.5) We also have a *functorial* homomorphism (in M and N)

$$S^{-1}\operatorname{Hom}_A(M,N) \longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

which, at u/s, corresponds to the homomorphism $m/t \mapsto u(m)/st$. When M has a finite presentation, the preceding homomorphism is an *isomorphism*: it is immediate when M is of the form A^r, and goes on to the general case starting from the following exact sequence $A^p \to A^q \to M \to 0$, and using the exactness of the functor $S^{-1}M$ and the left-exactness of the functor $\operatorname{Hom}_A(M,N)$ in M. Note that this case always occurs when A is *Noetherian* and the A-module M is *of finite type*.

1.4. Change of multiplicative subset.

(1.4.1) Let S, T be two multiplicative subsets of a ring A such that $S \subset T$; there exists a canonical homomorphism $\rho_A^{T,S}$ (or simply $\rho^{T,S}$) of $S^{-1}A$ into $T^{-1}A$, sending the element denoted a/s of $S^{-1}A$ to the element denoted a/s in $T^{-1}A$; we have $i_A^T = \rho_A^{T,S} \circ i_A^S$. For every A-module M, there exists in the same way an $S^{-1}A$-linear map of $S^{-1}M$ into $T^{-1}M$ (the latter considered as an $S^{-1}A$-module thanks to the homomorphism $\rho_A^{T,S}$), which matches the element m/s of $S^{-1}M$ to the element m/s of $T^{-1}M$; we note that the map $\rho_M^{T,S}$, or simply $\rho^{T,S}$, and we still have $i_M^T = \rho_M^{T,S} \circ i_M^S$; in canonical identification (1.2.5), $\rho_M^{T,S}$ identifies with $\rho_A^{T,S} \otimes 1$. The homomorphism $\rho_M^{T,S}$ is a *functorial morphism* (or natural transformation) of the functor $S^{-1}M$ into the functor $T^{-1}M$, in other words, the diagram

$$\begin{array}{ccc} S^{-1}M & \xrightarrow{S^{-1}u} & S^{-1}N \\ {\scriptstyle \rho_M^{T,S}}\big\downarrow & & \big\downarrow{\scriptstyle \rho_N^{T,S}} \\ T^{-1}M & \xrightarrow{T^{-1}u} & T^{-1}N \end{array}$$

is commutative, for every homomorphism $u : M \to N$; $T^{-1}u$ is entirely determined by $S^{-1}u$, because for $m \in M$ **16**
and $t \in T$, we have

$$(T^{-1}u)(m/t) = (t/1)^{-1}\rho^{T,S}((S^{-1}u)(m/1)).$$

(1.4.2) With the same notation, for two A-modules M, N, the diagrams (cf. (1.3.4) and (1.3.5))

$$\begin{array}{ccc} (S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) & \xrightarrow{\sim} & S^{-1}(M \otimes_A N) \\ \big\downarrow & & \big\downarrow \\ (T^{-1}M) \otimes_{T^{-1}A} (T^{-1}N) & \xrightarrow{\sim} & T^{-1}(M \otimes_A N) \end{array} \qquad \begin{array}{ccc} S^{-1}\operatorname{Hom}_A(M,N) & \longrightarrow & \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \\ \big\downarrow & & \big\downarrow \\ T^{-1}\operatorname{Hom}_A(M,N) & \longrightarrow & \operatorname{Hom}_{T^{-1}A}(T^{-1}M, T^{-1}N) \end{array}$$

are commutative.

(1.4.3) There is an important case in which the homomorphism $\rho^{T,S}$ is *bijective*, we know that then every element of T is divisor of an element of S; we then identify by $\rho^{T,S}$ the modules $S^{-1}M$ and $T^{-1}M$. We say that S is *saturated* if every divisor in A of an element of S is in S; by replacing S with the set T of all the divisors of the elements of S (a set which is multiplicative and saturated), we see that we can always, if we wish, be limited to the consideration of modules of fractions $S^{-1}M$, where S is saturated.

(1.4.4) If S, T, U are three multiplicative subsets of A such that $S \subset T \subset U$, we have

$$\rho^{U,S} = \rho^{U,T} \circ \rho^{T,S}.$$

(1.4.5) Consider an *increasing filtered family* (S_α) of multiplicative subsets of A (we write $\alpha \leqslant \beta$ for $S_\alpha \subset S_\beta$), and let S be the multiplicative subset $\bigcup_\alpha S_\alpha$; let us put $\rho_{\beta\alpha} = \rho_A^{S_\beta,S_\alpha}$ for $\alpha \leqslant \beta$; according to (1.4.4), the homomorphisms $\rho_{\beta\alpha}$ define a ring A' as the *inductive limit* of the inductive system of rings $(S_\alpha^{-1}A, \rho_{\beta\alpha})$. Let

ρ_α be the canonical map $S_\alpha^{-1}A \to A'$, and let $\varphi_\alpha = \rho_A^{S,S_\alpha}$; as $\varphi_\alpha = \varphi_\beta \circ \rho_{\beta\alpha}$ for $\alpha \leqslant \beta$ according to (1.4.4), we can uniquely define a homomorphism $\varphi : A' \to S^{-1}A$ such that the diagram

$$\begin{tikzcd} & S_\alpha^{-1}A \arrow[ddl, "\rho_\alpha"'] \arrow[d, "\rho_{\beta\alpha}"] \arrow[ddr, "\varphi_\alpha"] & \\ & S_\beta^{-1}A \arrow[dl, "\rho_\beta"] \arrow[dr, "\varphi_\beta"'] & \\ A' \arrow[rr, "\varphi"'] & & S^{-1}A \end{tikzcd} \qquad (\alpha \leqslant \beta)$$

is commutative. In fact, φ is an *isomorphism*; it is indeed immediate by construction that φ is surjective. On the other hand, if $\rho_\alpha(a/s_\alpha) \in A'$ is such that $\varphi(\rho_\alpha(a/s_\alpha)) = 0$, this means that $a/s_\alpha = 0$ in $S^{-1}A$, that is, to say that there exists $s \in S$ such that $sa = 0$; but there is a $\beta \geqslant \alpha$ such that $s \in S_\beta$, and consequently, as $\rho_\alpha(a/s_\alpha) = \rho_\beta(sa/ss_\alpha) = 0$, we find that φ is injective. The case for an A-module M is treated likewise, and thus we have defined canonical isomorphisms

$$\varinjlim S_\alpha^{-1}A \xrightarrow{\sim} (\varinjlim S_\alpha)^{-1}A, \qquad \varinjlim S_\alpha^{-1}M \xrightarrow{\sim} (\varinjlim S_\alpha)^{-1}M,$$

the second being *functorial* in M.

(1.4.6) Let S_1, S_2 be two multiplicative subsets of A; then S_1S_2 is also a multiplicative subset of A. Let us **17**
denote by S_2' the canonical image of S_2 in the ring $S_1^{-1}A$, which is a multiplicative subset of this ring. For every A-module M there is then a functorial isomorphism

$$S_2'^{-1}(S_1^{-1}M) \xrightarrow{\sim} (S_1S_2)^{-1}M$$

which maps $(m/s_1)/(s_2/1)$ to the element $m/(s_1s_2)$.

1.5. Change of ring.

(1.5.1) Let A, A' be two rings, φ a homomorphism $A' \to A$, S (resp. S') a multiplicative subset of A (resp. A'), such that $\varphi(S') \subset S$; the composition homomorphism $A' \xrightarrow{\varphi} A \to S^{-1}A$ factors as $A' \to S'^{-1} \xrightarrow{\varphi^{S'}} S^{-1}A$ by virtue of (1.2.4); where $\varphi^{S'}(a'/s') = \varphi(a')/\varphi(s')$. If $A = \varphi(A')$ and $S = \varphi(S')$, $\varphi^{S'}$ is *surjective*. If $A' = A$ and if φ is the identity, $\varphi^{S'}$ is none other than the homomorphism $\rho_A^{S,S'}$ defined in (1.4.1).

(1.5.2) Under the hypothesis of (1.5.1), let M be an A-module. There exists a canonical functorial morphism

$$\sigma : S'^{-1}(M_{[\varphi]}) \longrightarrow (S^{-1}M)_{[\varphi^{S'}]}$$

of $S'^{-1}A'$-modules, sending each element m/s' of $S'^{-1}(M_{[\varphi]})$ to the element $m/\varphi(s')$ of $(S^{-1}M)_{[\varphi^{S'}]}$; in fact, we verify immediately that this definition does not depend on the expression m/s' of the element considered. *When* $S = \varphi(S')$, the homomorphism σ is *bijective*. When $A' = A$ and φ is the identity, σ is none other that the homomorphism $\rho_M^{S,S'}$ defined in (1.4.1).

When $M = A$ is taken in particular, the homomorphism φ defines on A an A'-algebra structure; $S'^{-1}(A_{[\varphi]})$ is then provided with a ring structure, for which it identifies with $(\varphi(S'))^{-1}A$, and the homomorphism $\sigma : S'^{-1}(A_{[\varphi]}) \to S^{-1}A$ is a homomorphism of $S'^{-1}A'$-algebras.

(1.5.3) Let M and N be two A-modules; by composing the homomorphisms defined in (1.3.4) and (1.5.2), we obtain a homomorphism

$$(S^{-1}M \otimes_{S^{-1}A} S^{-1}N)_{[\varphi^{S'}]} \longleftarrow S'^{-1}((M \otimes A)_{[\varphi]})$$

which is an isomorphism when $\varphi(S') = S$. Similarly, by composing the homorphisms in (1.3.5) and (1.5.2), we obtain a homomorphism

$$S'^{-1}((\mathrm{Hom}_A(M,N))_{[\varphi]}) \longrightarrow (\mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N))_{[\varphi^{S'}]}$$

which is an isomorphism when $\varphi(S') = S$ and M admits a finite presentation.

(1.5.4) Let us now consider an A'-module N', and form the tensor product $N' \otimes_{A'} A_{[\varphi]}$, which can be considered as an A-module by setting $a \cdot (n' \otimes b) = n' \otimes (ab)$. There is a functorial isomorphism of $S^{-1}A$-modules

$$\tau : (S'^{-1}N') \otimes_{S'^{-1}A'} (S^{-1}A)_{[\varphi^{S'}]} \xrightarrow{\sim} S^{-1}(N' \otimes_{A'} A_{[\varphi]})$$

which maps the element $(n'/s') \otimes (a/s)$ to the element $(n' \otimes a)/(\varphi(s')s)$; indeed, we verify separately that **18**
when we replace n'/s' (resp. a/s) by another expression of the same element, $(n' \otimes a)/(\varphi(s')s)$ does not change; on the other hand, we can define a reciprocal homomorphism of τ by sending $(n' \otimes a)/s$ to the element $(n'/1) \otimes (a/s)$: we use the fact that $S^{-1}(N' \otimes_{A'} A_{[\varphi]})$ is canonically isomorphic to $(N' \otimes_{A'} A_{[\varphi]}) \otimes_A S^{-1}A$ (1.2.5), so also to $N' \otimes_{A'} (S^{-1}A)_{[\varphi]}$, by designating by ψ the composite homomorphism $a' \mapsto \varphi(a')/1$ of A' into $S^{-1}A$.

(1.5.5) If M' and N' are two A'-modules, by composing the isomorphisms (1.3.4) and (1.5.4), we obtain an isomorphism

$$S'^{-1}M \otimes_{S'^{-1}A'} S'^{-1}N' \otimes_{S'^{-1}A'} S^{-1}A \xrightarrow{\sim} S^{-1}(M' \otimes_{A'} N' \otimes_{A'} A).$$

Likewise, if M' admits a finite presentation, we have by (1.3.5) and (1.5.4) an isomorphism

$$\mathrm{Hom}_{S'^{-1}A'}(S'^{-1}M', S'^{-1}N') \otimes_{S'^{-1}A'} S^{-1}A \xrightarrow{\sim} S^{-1}(\mathrm{Hom}_{A'}(M', N') \otimes_{A'} A).$$

(1.5.6) Under the hypothesis of (1.5.1), let T (resp. T') be a second multiplicative subset of A (resp. A') such that $S \subset T$ (resp. $S' \subset T'$) and $\varphi(T') \subset T$. Then the diagram

$$\begin{tikzcd} S'^{-1}A' \arrow[r, "\varphi^{S'}"] \arrow[d, "\rho^{T',S'}"'] & S^{-1}A \arrow[d, "\rho^{T,S}"] \\ T'^{-1}A' \arrow[r, "\varphi^{T'}"'] & T^{-1}A \end{tikzcd}$$

is commutative. If M is an A-module, the diagram

$$\begin{tikzcd} S'^{-1}(M_{[\varphi]}) \arrow[r, "\sigma"] \arrow[d, "\rho^{T',S'}"'] & (S^{-1}M)_{[\varphi^{S'}]} \arrow[d, "\rho^{T,S}"] \\ T'^{-1}(M_{[\varphi]}) \arrow[r, "\sigma"'] & (T^{-1}M)_{[\varphi^{T'}]} \end{tikzcd}$$

is commutative. Finally, if N' is an A'-module, the diagram

$$\begin{tikzcd} (S'^{-1}N') \otimes_{S'^{-1}A'} (S^{-1}A)_{[\varphi^{S'}]} \arrow[r, "\sim", "\tau"'] \arrow[d] & S^{-1}(N' \otimes_{A'} A_{[\varphi]}) \arrow[d, "\rho^{T,S}"] \\ (T'^{-1}N') \otimes_{T'^{-1}A'} (T^{-1}A)_{[\varphi^{T'}]} \arrow[r, "\sim", "\tau"'] & T^{-1}(N' \otimes_{A'} A_{[\varphi]}) \end{tikzcd}$$

is commutative, the left vertical arrow obtained by applying $\rho_{N'}^{T',S'}$ to $S'^{-1}N'$ and $\rho_A^{T,S}$ to $S^{-1}A$.

(1.5.7) Let A'' be a third ring, $\varphi' : A'' \to A'$ a ring homomorphism, S'' a multiplicative subset of A'' such **19**
that $\varphi'(S'') \subset S'$. Set $\varphi'' = \varphi \circ \varphi'$; then we have

$$\varphi''^{S''} = \varphi^{S'} \circ \varphi'^{S''}.$$

Let M be an A-module; evidently we have $M_{[\varphi'']} = (M_{[\varphi]})_{[\varphi']}$; if σ' and σ'' are the homomorphisms defined by φ' and φ'' as σ is defined in (1.5.2) by φ, we have the transitivity formula

$$\sigma'' = \sigma \circ \sigma'.$$

Finally, let N'' be an A''-module; the A-module $N'' \otimes_{A''} A_{[\varphi'']}$ identifies canonically with $(N'' \otimes_{A''} A'_{[\varphi']}) \otimes_{A'} A_{[\varphi]}$, and likewise the $S^{-1}A$-module $(S''^{-1}N'') \otimes_{S''^{-1}A''} (S^{-1}A)_{[\varphi''^{S''}]}$ identifies canonically with $((S''^{-1}N'') \otimes_{S''^{-1}A''} (S'^{-1}A')_{[\varphi'^{S''}]}) \otimes_{S'^{-1}A'} (S^{-1}A)_{[\varphi^{S'}]}$. With these identifications, if τ' and τ'' are the isomorphisms defined by φ' and φ'' as τ is defined in (1.5.4) by φ, we have the transitivity formula

$$\tau'' = \tau \circ (\tau' \otimes 1).$$

(1.5.8) Let A be a subring of a ring B; for every *minimal* prime ideal $\mathfrak{p}$ of A, there exists a minimal prime ideal $\mathfrak{q}$ of B such that $\mathfrak{p} = A \cap \mathfrak{q}$. Indeed, $A_\mathfrak{p}$ is a subring of $B_\mathfrak{p}$ (1.3.2) and has *a single* prime ideal $\mathfrak{p}'$ (1.2.6); since $B_\mathfrak{p}$ is not reduced to 0, it has at least one prime ideal $\mathfrak{q}'$ and we have necessarily $\mathfrak{q}' \cap A_\mathfrak{p} = \mathfrak{p}'$; the prime ideal $\mathfrak{q}_1$ of B, a reciprocal image of $\mathfrak{q}'$ is thus such that $\mathfrak{q}_1 \cap A = \mathfrak{p}$, and *a fortiori* we have $\mathfrak{q} \cap A = \mathfrak{p}$ for every minimal prime ideal $\mathfrak{q}$ of B contained in $\mathfrak{q}_1$.

1.6. Indentification of the module M_f as an inductive limit.

(1.6.1) Let M be an A-module, f an element of A. Consider a sequence (M_n) of A-modules, all identical to M, and for each pair of integers $m \leqslant n$, let φ_{nm} be the homomorphism $z \mapsto f^{n-m}z$ of M_m into M_n; it is immediate that $((M_n), (\varphi_{nm}))$ is an *inductive system* of A-modules; let $N = \varinjlim M_n$ be the inductive limit of this system. We define a canonical A-isomorphism, *functorial* of N on M_f. For this reason, let us note that, for all n, $\theta_n : z \mapsto z/f^n$ is an A-homomorphism of $M = M_n$ into M_f, and it follows from the definitions that we have $\theta_n \circ \varphi nm = \theta_m$ for $m \leqslant n$. There exists therefore an A-homomorphism $\theta : N \to M_f$ such that, if φ_n denotes the canonical homomorphism $M_n \to N$, we have $\theta_n = \theta \circ \varphi_n$ for all n. Since, by hypothesis, every element of M_f is of the form z/f^n for at least n, it is clear that θ is surjective. On the other hand, if $\theta(\varphi_n(z)) = 0$, in other words $z/f^n = 0$, there exists an integer $k > 0$ such that $f^k z = 0$, so $\varphi_{n+k,n}(z) = 0$, which results in $\varphi_n(z) = 0$. We can therefore identify M_f and $\varinjlim M_n$ by means of θ.

(1.6.2) Now write $M_{f,n}$, φ^f_{nm} and φ^f_n instead of M_n, φ_{nm} and φ_n. Let g be a second element of A. As f^n divides $f^n g^n$, we have a functorial homomorphism

$$\rho_{fg,f} : M_f \longrightarrow M_{fg} \quad \text{(1.4.1 and 1.4.3)};$$

if we indentify M_f and M_{fg} with $\varinjlim M_{f,n}$ and $\varinjlim M_{fg,n}$ respectively, $\rho_{fg,f}$ identifies with the *inductive limit* 20
of the maps $\rho^n_{fg,f} : M_{f,n} \to M_{fg,n}$, defined by $\rho^n_{fg,f}(z) = g^n z$. Indeed, this follows immediately from the commutivity of the diagram

$$\begin{tikzcd}
M_{f,n} \arrow[r, "\rho^n_{fg,f}"] \arrow[d, "\varphi^f_n"'] & M_{fg,n} \arrow[d, "\varphi^{fg}_n"] \\
M_f \arrow[r, "\rho_{fg,f}"] & M_{fg}.
\end{tikzcd}$$

1.7. Support of a module.

(1.7.1) Given an A-module M, we call the *support* of M and denote by $\mathrm{Supp}(M)$ the set of prime ideals $\mathfrak{p}$ of A such that $M_\mathfrak{p} \neq 0$. For $M = 0$, it is necessary and sufficient that $\mathrm{Supp}(M) = \varnothing$, because if $M_\mathfrak{p} = 0$ for all $\mathfrak{p}$, the annihilator of an element $x \in M$ cannot be contained in any prime ideal of A, so A is total.

(1.7.2) If $0 \to N \to M \to P \to 0$ is an exact sequence of A-modules, we have

$$\mathrm{Supp}(M) = \mathrm{Supp}(N) \cup \mathrm{Supp}(P)$$

because for every prime ideal $\mathfrak{p}$ of A, the sequence $0 \to N_\mathfrak{p} \to M_\mathfrak{p} \to P_\mathfrak{p} \to 0$ is exact (1.3.2) and for $M_\mathfrak{p} = 0$, it is necessary and sufficient that $N_\mathfrak{p} = P_\mathfrak{p} = 0$.

(1.7.3) If M is the sum of a family (M_λ) of submodules, $M_\mathfrak{p}$ is the sum of $(M_\lambda)_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of A (1.3.3 and 1.3.2), so $\mathrm{Supp}(M) = \bigcup_\lambda \mathrm{Supp}(M_\lambda)$.

(1.7.4) If M is an A-module *of finite type*, $\mathrm{Supp}(M)$ is the set of prime ideals *containing the annihilator of* M. Indeed, if M is cyclic and generated by x, say that $M_\mathfrak{p} = 0$ means that there exists $s \notin \mathfrak{p}$ such that $s \cdot x = 0$, so that $\mathfrak{p}$ does not contain the annihilator of x. If now M admits a finite system $(x_i)_{1 \leqslant i \leqslant n}$ of generators and if $\mathfrak{a}_i$ is the annihilator of x_i, it follows from (1.7.3) that $\mathrm{Supp}(M)$ is th set of $\mathfrak{p}$ containing one of $\mathfrak{a}_i$, or, equivalently, the set of $\mathfrak{p}$ containing $\mathfrak{a} = \bigcap_i \mathfrak{a}_i$, which is the annihilator of M.

(1.7.5) If M and N are two A-modules *of finite type*, we have

$$\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N).$$

It can be seen that if $\mathfrak{p}$ is a prime ideal of A, the condition $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} \neq 0$ is equivalent to "$M_\mathfrak{p} \neq 0$ and $N_\mathfrak{p} \neq 0$" (taking into account (1.3.4)). In other words, it is about seeing that if P, Q are two modules of finite type over a *local* ring B, not reduced to 0, then $P \otimes_B Q \neq 0$. Let $\mathfrak{m}$ be the maximal ideal of B. By virtue of Nakayama's lemma, the vector spaces $P/\mathfrak{m}P$ and $Q/\mathfrak{m}Q$ are not reduced to 0, so it is the same with the tensor product $(P/\mathfrak{m}P) \otimes_{B/\mathfrak{m}} (Q/\mathfrak{m}Q) = (P \otimes_B Q) \otimes_B (B/\mathfrak{m})$, hence the conclusion.

In particular, if M is an A-module of finite type, $\mathfrak{a}$ an ideal of A, $\mathrm{Supp}(M/\mathfrak{a}M)$ is the set of prime ideals containing both $\mathfrak{a}$ and the annihilator $\mathfrak{n}$ of M (1.7.4), that is, the set of prime ideals containing $\mathfrak{a} + \mathfrak{n}$.

2. Irreducible spaces. Noetherian spaces.

2.0. Irreducible spaces.

(2.1.1) We say that a topological space X is *irreducible* if it is nonempty and if it is not a union of two distinct closed subspaces of X. It is the same to say that $X \neq \varnothing$ and that the intersection of two nonempty open sets (and consequently of a finite number of open sets) of X is nonempty, or that every nonempty open set is everywhere dense, or that any closed subset is *rare*, or finally that all open sets of X are *connected*.

(2.1.2) For a subspace Y of a topological spave X to be irreducible, it is necessary and sufficient that its closure $\overline{Y}$ be irreducible. In particular, any subspace which is the closure $\overline{\{x\}}$ of a reduced subspace to a point is irreducible; we will express the relation $y \in \overline{\{x\}}$ (equivalent to $\overline{\{y\}} \subset \overline{\{x\}}$) by saying that there is a *specialization of* x or that there is a *generalization of* y. When there exists in an irreducible space X a point x such that $X = \overline{\{x\}}$, we will say that x is a *generic point* of X. Any nonempty open subset of X then contains x, and any subspace containing x admits x for a generic point.

(2.1.3) Recall that a *Kolmogoroff space* is a topological space X satisfying the axiom of separation:

(T_0) If $x \neq y$ are any two points of X, there is an open set containing one of the points x, y and not the other.

If an irreducible Kolmogoroff space admits a generic point, it admits *only* one since a nonempty open set contains any generic point.

Recall that a topological space X is said to be *quasi-compact* if, from any collection of open sets of X, one can extract a finite cover of X (or, equivalently, if any decreasing filter family of nonempty closed sets has a nonempty intersection). If X is a quasi-compact space, then any nonempty closed subset A of X contains a *minimal* nonempty closed set M, because the set of nonempty closed subsets of A is inductive for the relation $\supset$; if in addition X is a Kolmogoroff space, M is necessarily reduced to a single point (or, as we say by abuse of language, is a *closed point*).

2.1. Noetherian spaces.

Chapter 1. — The language of schemes

Summary

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Sections 1 to 8 intend only to develop a language, which will be used in all that follows. We note, however, that following the general spirit of this Treatise, sections 7 and 8 will be less used than the others, and in a less essential manner; we speak of Chevalley schemes only in order to be able to link to the language of Chevalley [1] and Nagata [9]. Section 9 gives some definitions and results about quasi-coherent sheaves

TODO

1. Affine schemes

1.0. The prime spectrum of a ring

2. Preschemes and morphisms of preschemes

2.0. Definition of preschemes

(2.1.1) Given a ringed space $(X, \mathcal{O}_X)$, we say that an open subset V of X is an *affine open* if the ringed space $(V, \mathcal{O}_X|V)$ is an affine scheme (1.7.1).

Definition **(2.1.2)**.— *We define a prescheme to be a ringed space* $(X, \mathcal{O}_X)$ *such that every point of* X *admits an affine open neighbourhood.*

Proposition **(2.1.3)**.— *If* $(X, \mathcal{O}_X)$ *is a prescheme then the open affines give a base for the topology of* X. **98**

In effect, if V is an arbitrary open neighbourhood of $x \in X$, then there exists by hypothesis an open neighbourhood W of x such that $(W, \mathcal{O}_X|W)$ is an affine scheme; we write A to mean its ring. In the space W, $V \cap W$ is an open neighbourhood of x; thus there exists $f \in A$ such that $D(f)$ is an open neighbourhood of x contained inside $V \cap W$ (1.1.10 (i)). The ringed space $(D(f), \mathcal{O}_X|D(f))$ is thus an affine scheme, isomorphic to A_f (1.3.6), whence the proposition.

Proposition **(2.1.4)**.— *The underlying space of a prescheme is a Kolmogoroff space.*

In effect, if x, y are two distinct points of a prescheme X then it is clear that there exists an open neighbourhood of one of these points that does not contain the other if x and y are not in the same open affine; and if they are in the same open affine, this is a result of (1.1.8).

Proposition **(2.1.5)**.— *If* $(X, \mathcal{O}_X)$ *is a prescheme then every closed irreducible subset of* X *admits exactly one generic point, and the map* $x \mapsto \overline{\{x\}}$ *is thus a bijection of* X *onto its set of closed irreducible subsets.*

In effect, if Y is a closed irreducible subset of X and $y \in Y$, and if U is an open affine neighbourhood of y in X, then $U \cap Y$ is everywhere dense in Y, as well as irreducible (**0**, 2.1.1 and 2.1.4); thus by (1.1.14), $U \cap Y$ is the closure in U of a point x, and then $Y \subset \overline{U}$ is the closure of x in X. The uniqueness of the generic point of X is a result of (2.1.4) and (**0**, 2.1.3).

(2.1.6) If Y is a closed irreducible subset of X and y its generic point then the local ring $\mathcal{O}_y$, also written $\mathcal{O}_{X/Y}$, is called the *local ring of* X *along* Y, or the *local ring of* Y *in* X.

If X itself is irreducible and x its generic point then we say that $\mathcal{O}_x$ is the *ring of rational functions on* X (cf. s. 7).

Proposition **(2.1.7).** — *If* $(X, \mathcal{O}_X)$ *is a prescheme then the ringed space* $(U, \mathcal{O}_X|U)$ *is a prescheme for every open subset* U.

This follows directly from definition (2.1.2) and proposition (2.1.3).

We say that $(U, \mathcal{O}_X|U)$ is the prescheme *induced* on U by $(X, \mathcal{O}_X)$, or the *restriction* of $(X, \mathcal{O}_X)$ to U.

(2.1.8) We say that a prescheme $(X, \mathcal{O}_X)$ is *irreducible* (resp. *connected*) if the underlying space X is irreducible (resp. connected). We say that a prescheme is *integral* if it is *irreducible and reduced* (cf. (5.1.4)). We say that a prescheme $(X, \mathcal{O}_X)$ is *locally integral* if every $x \in X$ admits an open neighbourhood U such that the prescheme induced on U by $(X, \mathcal{O}_X)$ is integral.

2.1. Morphisms of preschemes

TODO