Contents

Introduction

The format of this document is that there will be one section per geometric concept. Within each section, we will talk about some mathematical mathematical theory and discuss how it will be visualised. There will then be an interactive visualisation where you get to play around with the concept for yourself, and see first hand whatever is being described.

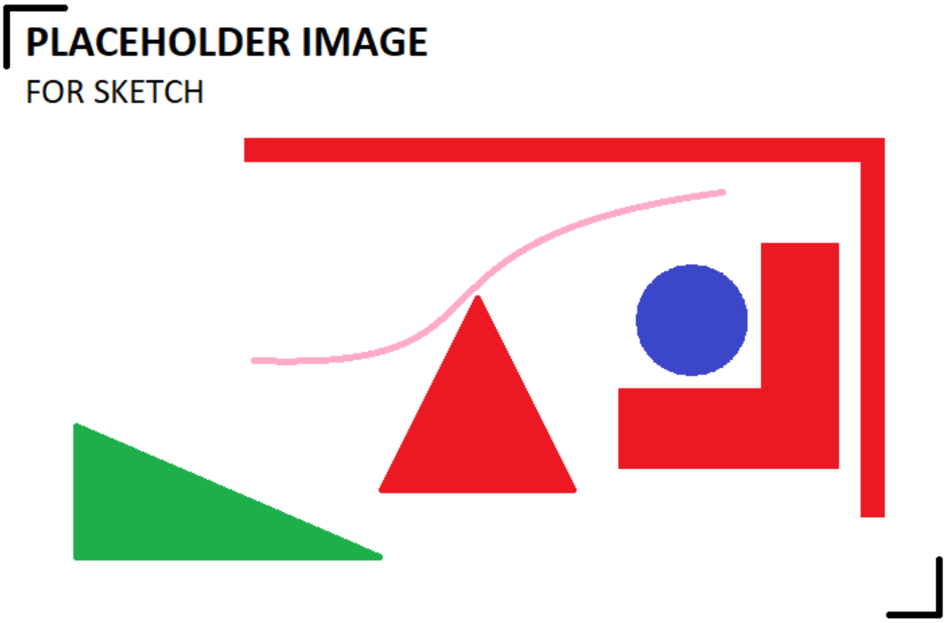
Costa's Surface

In this section, we introduce the Weierstrass functions used to parameterise costa's surface (using Gray's parameterisation). We will need both the Weierstrass zeta function and Weierstrass elliptic function. These have definitions as follows:

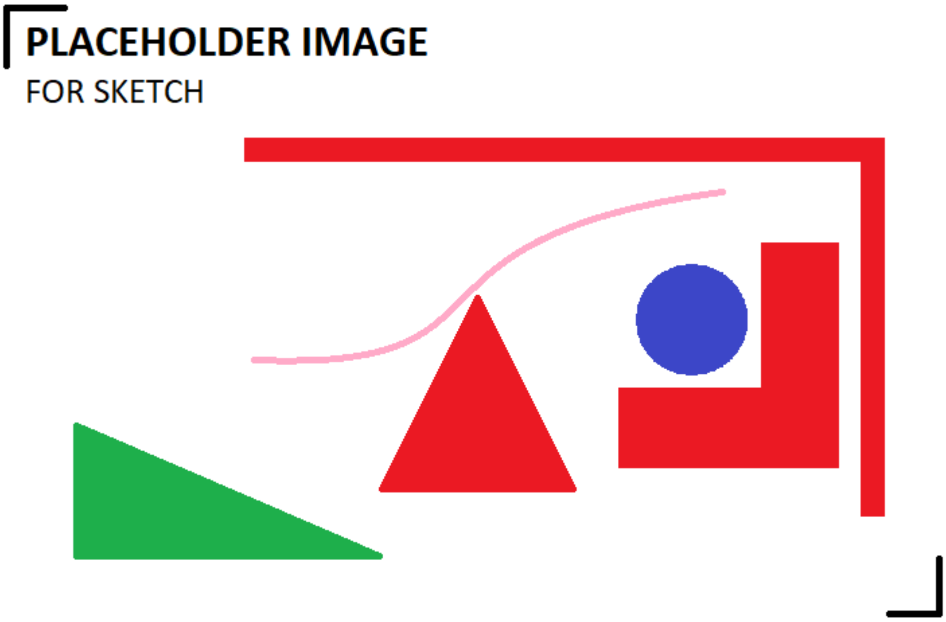
Visualisation

Below we show visualisations of approximations of each of the functions defined above. We only compute finitely many terms in order to complete the computation in finite time…

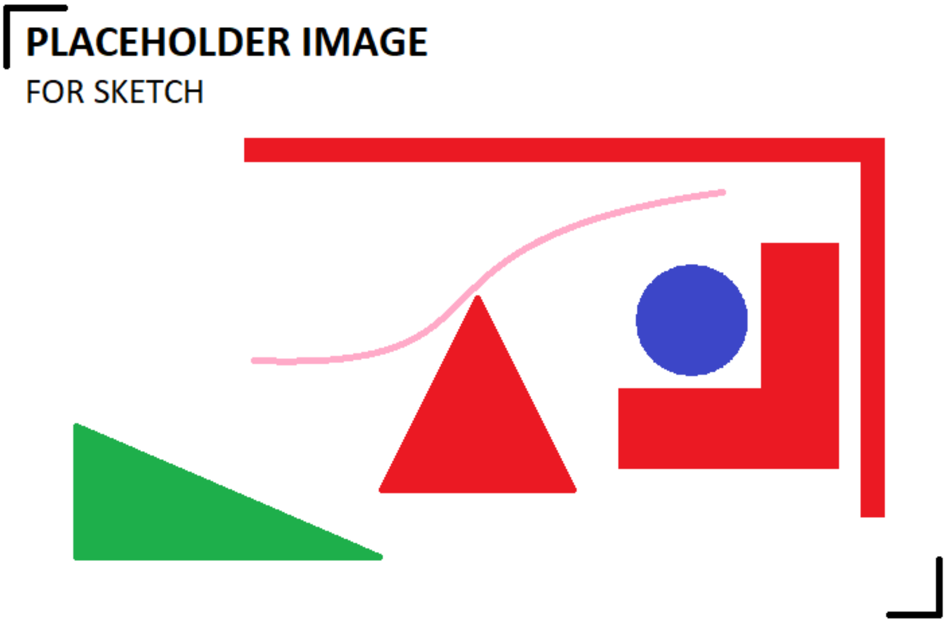
Weierstrass Sigma Function



Weierstrass P Function



Weierstrass Zeta Function

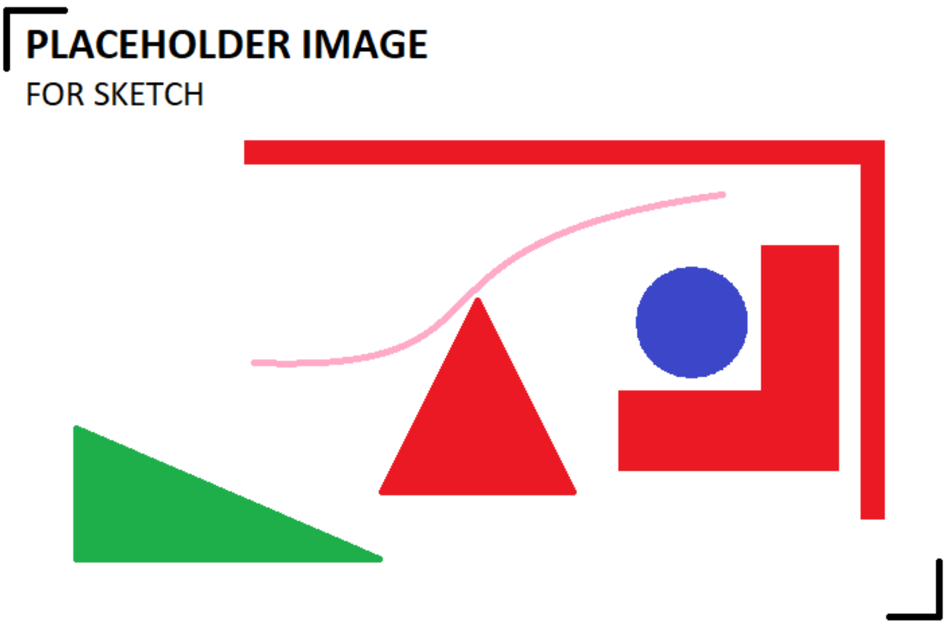


Parameterised Surface Visualisation

Using the above weierstrass functions, we can draw a parameterisation of costa's surface as follows

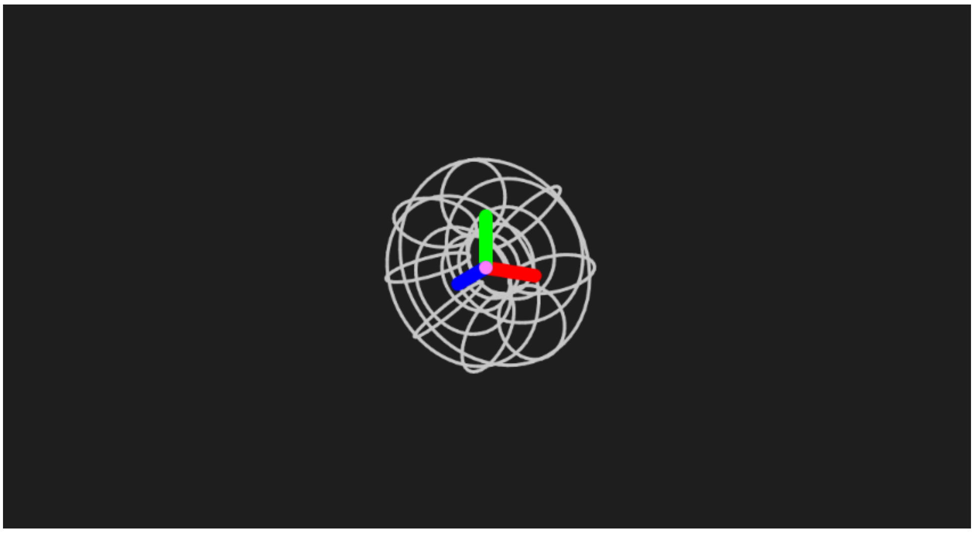
$$\begin{aligned} x &= \ldots \\ y &= \ldots \\ z &= \ldots \end{aligned} \tag{1}$$

below is a visualisation of costa's surface as parameterised above:



Parameterised Surfaces

here we test functionality of drawing parameterised surfaces in p5.js.



Fundamental Group of $\mathrm{SO}(3)$

The first geometric, or really more topological, topic that we will be showcasing is the fundamental group of $\mathrm{SO}(3)$, that is the fundamental group of the special orthogonal group in 3 dimensions. You may or may not already know that the fundamental group of $\mathrm{SO}(3)$ is $\mathbb{Z}/2\mathbb{Z}$. That is to say there are exactly 2 homotopy classes of closed paths in $\mathrm{SO}(3)$. Note that we do not need to worry about the base point of the loops since $\mathrm{SO}(3)$ is homeomorphic to projective 3-space, and so is connected.

This section will assume familiarity with the basics of
- topology
- algebraic topology
- lie groups

Mathematical Theory

We will start by showing that the topology of $\mathrm{SO}(3)$ is the same as $\mathbb{R}P^3$. This can be thought of intuitively by the fact that an element of $\mathrm{SO}(3)$, i.e. a rotation in $\mathbb{R}^3$, is determined by an axis and an angle between 0 and π. All rotations in the xy plane for example are described either by the direction $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ and some angle $\theta \in [0, \pi)$ for the rotations that map $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ to the upper half of the xy plane, or the direction $\begin{pmatrix} 0 & 0 & -1 \end{pmatrix}$ and an angle $\theta \in [0, \pi)$ for the rotations that map $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ to the lower half of the xy plane.

Proposition : Every rotation in $\mathbb{R}^3$ can be described by an angle and an axis.

Proof : TODO.

$\square$

Combining the direction, a unit vector, and angle around this direction, allows us to *almost* uniquely describe any rotation by a vector in $\mathbb{R}^3$ who's length is in $[0, \pi]$. These are exactly the vectors in the closed ball of radius π, which we denote $\overline{B_0(\pi)}$.

The caveat here is that a rotation clockwise by π radians is the same as a rotation anticlockwise by π radians. This implies that rotations described by vectors of length π describe the same rotation as their negative. So If we want to describe rotations uniquely, we need to quotient by the relation $u \sim v \Leftrightarrow \|u\| = \|v\|$ and $u = -v$. We now have that the points in the topological space $\frac{\overline{B_0(\pi)}}{\sim}$ correspond one-to-one with the elements of $\mathrm{SO}(3)$.

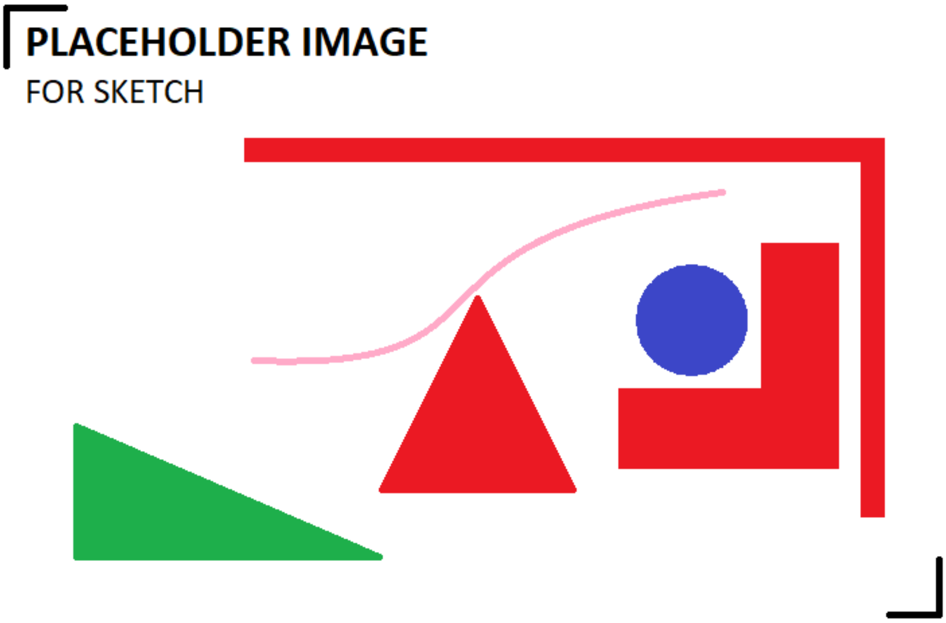
It is the case that $\overline{B_0(\pi)}/\sim \quad \cong \mathbb{R}P^3$. We know from [TODO: CITE HERE] that the first fundamental group of $\mathbb{R}P^n$ is $\frac{\mathbb{Z}}{2}\mathbb{Z}$ for all $n \geq 2$. This is the key fact that we will be visualising. That there are exactly two homotopy classes of paths of rotations in $\mathbb{R}^3$, and that for non-trivial paths of rotations, concatenating the path with itself results in a trivial path.

Visualisation Theory

Since the fundamental group consists of the set of homoptopy classes of closed paths (loops) of rotations in $\mathrm{SO}(3)$, it would be nice if we had a way of visualising both loops of rotations, and homotopies between them.

We will visualise a loop of rotations in the following way: Let $\alpha : S^1 \to \mathrm{SO}(3)$ be a loop in $\mathrm{SO}(3)$, where $S^1 = \{(\cos(\theta), \sin(\theta)) \mid 0 \leq \theta < 2\pi\}$. We will visualise this by picking some reference object, say a set of axes, or maybe something like a teapot, and placing multiple copies of it around a circle. The object at a point $(\cos(\theta), \sin(\theta))$ on the circle will have orientation $\alpha(\theta)$. As alpha is continuous, adjacent objects will have almost the same orientation, and as you go around the circle the orientation of the objects will smoothly change.

Visualisation



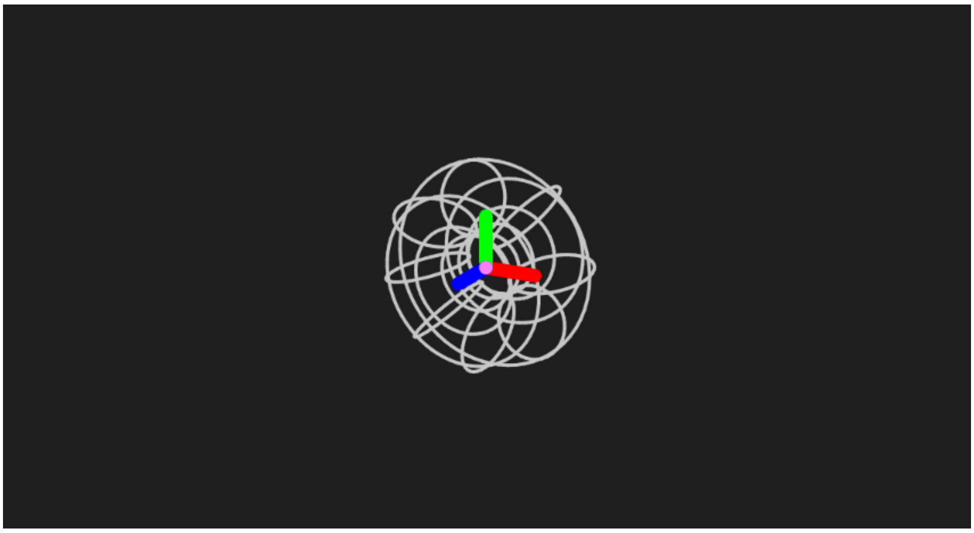
Rotations in $\mathbb{R}^4$

In this section, we will be exploring the group structure of rotations in $\mathbb{R}^4$

Mathematical Theory

Visualisation Theory

Visualisation



Curved Space

Spherical Geometry

Hyperbolic Geometry

Hyperbolic 2-Space

Hyperbolic 3-Space

How to make interactive visuals like these