Edge Offsetting Using Least Squares

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1 1D Set Up

We have a 1D surface Γ embedded in 1D. The surface is discretized with $|V|$ vertices and $|E|$ edges connecting those vertices. Along the surface we have a desired velocity $v_p(s)$, that is determined by our physics problem (in our case the solution to the stefan problem). Due to the discretization of our surface, this desired velocity is defined at edges and ambiguous at vertices. More specifically, at each edge j, we have a constant desired velocity v_{pj} (scalar because we are operating in 1D). We want to move each edge with that desired velocity v_{pj}... however our degrees of freedom are at vertices, not edges, where the desired velocity is ambiguous. We want to determine what is the velocity v_i with which we should move vertex i so that the generated motion for any point on our surface $v(s)$ will be as close as possible to it's corresponding $v_p(s)$ (as close as possible to it's desired motion).

We attempt to solve this problem by formulating an energy we wish to minimize:

$$\min_{v_i \forall i \in |V|} \int_\Gamma ||v_p(s) - v(s)||^2 ds \tag{1}$$

In our discretization, $v_p(s)$ is piecewise constant and $v(s)$ is piecewise linear over our surface Γ, where the generated velocity for any point in edge j, $v(s)_j$, can be determined by interpolating from it's two nearest vertices:

$$v(s)_j = \begin{bmatrix} \phi_{j1} & \phi_{j2} \end{bmatrix} \begin{bmatrix} v_{j1} \\ v_{j2} \end{bmatrix} \tag{2}$$

Where $j1$ and $j2$ are used to index the local shape functions of the two nodes surrounding edge j.

Furthermore, due to the nature of shape functions,

$$\phi_{j2} = 1 - \phi_{j1} \tag{3}$$

Leading to

$$v(s)_j = \begin{bmatrix} \phi_{j1} & 1 - \phi_{j1} \end{bmatrix} \begin{bmatrix} v_{j1} \\ v_{j2} \end{bmatrix} \tag{4}$$

Our shape functions are linear, with ϕ_{j1} starting at 1 on vertex 1 (where s = s1), and linearly degrading to 0 at vertex 2 (where s = s2).

$$\phi_{j1}(s) = 1 - \frac{s - s_1}{length(\Gamma_j)} \tag{5}$$

Returning to our least squares (1) energy, we start out by expanding the integrand and Splitting the integral into three parts. For now, the velocities are scalars and 1D.

$$\int_\Gamma v_p(s)^2 - 2v_p(s)v(s) + v(s)^2 ds = \int_\Gamma v_p(s)^2 ds - 2\int_\Gamma v_p(s)v(s)ds + \int_\Gamma v(s)^2 ds \tag{6}$$

1.1 Discretizing Part 1 of Least Squares Integral : $\int_\Gamma v_p(s)^2 ds$

Let's discretize the first part of our integral, $\int_\Gamma v_p(s)^2 ds$. We start with the fact that $v_p(s)$ is piecewise constant over the domain Γ, each edge j has a constant $v_p(s) = v_{pj}$.

$$\int_\Gamma v_p(s)^2 ds = \sum_j^{|E|} \int_{\Gamma_j} v_{pj}^2 ds = \sum_j^{|E|} v_{pj}^2 \cdot l(\Gamma_j) \tag{7}$$

Where Γ_j is the domain spanning the j^{th} edge.

We can write this in vector notation if we introduce the $|E| \times 1$ vector $\boldsymbol{V_p}$, a vector that holds v_{pj} at index j, for each edge j, and $\boldsymbol{L}$ an $|E| \times |E|$ diagonal matrix that holds the length of each edge j in its diagonal.

$$\int_\Gamma v_p(s)^2 ds = \boldsymbol{V_p}^T \boldsymbol{L} \boldsymbol{V_p} \tag{8}$$

Note that this part of our integral is a little bit like the kinetic energy of the desired motion, where $\boldsymbol{L}$ serves as a diagonal mass matrix.

1.2 Discretizing Part 2 of Least Squares Integral:$-2\int_\Gamma v_p(s)v(s)ds$

Now we move on to discretizing the second component of our least squares integral, $-2\int_\Gamma v_p(s)v(s)ds$. We again note that $v_p(s)$ is piecewise constant over the domain Γ, constant over each edge j.

$$-2\int_\Gamma v_p(s)v(s)ds = -2\sum_j^{|E|} \int_\Gamma v_{pj} v(s) ds = -2\sum_j^{|E|} v_{pj} \int_{\Gamma_j} v(s) ds \tag{9}$$

Using our relation of $v(s)$ over the domain of an edge Γ_j (4):

$$\int_{\Gamma_j} v(s)ds = \int_{\Gamma_j} \begin{bmatrix}\phi_{j1} & 1-\phi_{j1}\end{bmatrix} \begin{bmatrix} v_{j1} \\ v_{j2}\end{bmatrix} ds = \int_{\Gamma_j} \begin{bmatrix}\phi_{j1} & 1-\phi_{j1}\end{bmatrix} ds \begin{bmatrix} v_{j1} \\ v_{j2}\end{bmatrix} \tag{10}$$

We can compute the resulting integral above component-wise:

$$\begin{bmatrix}\int_{\Gamma_j} \phi_{j1} ds & \int_{\Gamma_j} 1-\phi_{j1} ds\end{bmatrix} \begin{bmatrix} v_{j1} \\ v_{j2}\end{bmatrix} \tag{11}$$

Plugging in our equation for our basis function (5)

$$\int_{\Gamma_j} \phi_{j1} ds = \int_{s_1}^{s_2} 1 - \frac{s-s_1}{l(\Gamma_j)} ds \tag{12}$$

Now we do a change in variables : Introducing $s' = \frac{s-s_1}{l(\Gamma_j)}$, and adjusting the bounds of integrals to be fit for s', and using the fact that $ds = ds' l(\Gamma_j)$

$$\int_{s_1}^{s_2} 1 - \frac{s-s_1}{l(\Gamma_j)} ds = l(\Gamma_j) \int_0^1 1 - s' ds' = l(\Gamma_j)(1 - \frac{1}{2}) = \frac{l(\Gamma_j)}{2} \tag{13}$$

Following a similar procedure for the second component of our basis functions:

$$\int_{\Gamma_j} 1 - \phi_{j1} ds = \int_{s_1}^{s_2} \frac{s-s_1}{l(\Gamma_j)} ds = l(\Gamma_j) \int_0^1 s' ds' = \frac{l(\Gamma_j)}{2} \tag{14}$$

Returning to our discretization

$$-2\sum_j^{|E|} v_{pj} \int_{\Gamma_j} v(s) ds = -2\sum_j^{|E|} v_{pj} \int_{\Gamma_j} \begin{bmatrix}\phi_{j1} & 1-\phi_{j1}\end{bmatrix} ds \begin{bmatrix} v_{j1} \\ v_{j2}\end{bmatrix} = -2\sum_j^{|E|} v_{pj} \begin{bmatrix}\frac{l(\Gamma_j)}{2} & \frac{l(\Gamma_j)}{2}\end{bmatrix} \begin{bmatrix} v_{j1} \\ v_{j2}\end{bmatrix} \tag{15}$$

If we wish to write this in vector notation, we need to do a tiny bit more work. We introduce vector $\boldsymbol{V}$ a $|V| \times 1$ vector, holding the velocity at each vertex i in it's i'th entry, and we reutilize vector $\boldsymbol{V_p}$ from the previous section, holding velocity at each edge j in it's j'th entry. Finally we also need to introduce an $|E| \times |V|$ matrix $\boldsymbol{A}$ which will perform the action of multiplying the desired velocity v_{pj} at any j by a weighted sum of the velocity of both of it's nodes, just like the terms inside the sum of (15) is doing.

Finally we have our encoded vector notation for the second part of our least squares integral:

$$-2\int_{\Gamma} v_p(s)v(s)ds = -2\boldsymbol{V_p}^T\boldsymbol{A}\boldsymbol{V} \tag{16}$$

where A is given by:

$$\boldsymbol{A}_{ij} = \begin{cases} \frac{l(\Gamma_i)}{2} & \Gamma_j \in N_e(i) \\ 0 & otherwise \end{cases} \tag{17}$$

So matrix $\mathbf{A}$ is only non-zero at entries where edge i is the edge which contains vertex j. Each row will only have 2 non-zero entries, because each edge only has two incident vertices.N_v here is the condition where an edge i is incident on vertex j. Note here that the i'th row of matrix $\mathbf{A}$, represents the i'th edge in our system, and the j'th column of matrix $\mathbf{A}$ represents the j'th vertex in our system. In general throughout this paper however I index edges with j, and vertices with i, but for the purposes of building this matrix using the i=row and j=column convention, please forgive this abrupt inconsistency in notation.

1.3 Discretizing Part 3 of Least Squares Integral: $\int_\Gamma v(s)^2 ds$

Finally we move on to the last part of our least squares integral, $\int_\Gamma v(s)^2 ds$. Just like in part 2, we use our piecewise linear equation for $v(s)$ over each edge.

$$\begin{aligned}
\int_\Gamma v(s)^2 ds &= \sum_j^{|E|}\int_{\Gamma_j} (\begin{bmatrix}\phi_{j1} & 1-\phi_{j1}\end{bmatrix}\begin{bmatrix}v_{j1}\\ v_{j2}\end{bmatrix})^T(\begin{bmatrix}\phi_{j1} & 1-\phi_{j1}\end{bmatrix}\begin{bmatrix}v_{j1}\\ v_{j2}\end{bmatrix})ds && (18)\\
&= \sum_j^{|E|}\int_{\Gamma_j} \begin{bmatrix}v_{j1} & v_{j2}\end{bmatrix}\begin{bmatrix}\phi_{j1}\\ 1-\phi_{j1}\end{bmatrix}\begin{bmatrix}\phi_{j1} & 1-\phi_{j1}\end{bmatrix}\begin{bmatrix}v_{j1}\\ v_{j2}\end{bmatrix} ds && (19)\\
&= \sum_j^{|E|} \begin{bmatrix}v_{j1} & v_{j2}\end{bmatrix}\int_{\Gamma_j}\begin{bmatrix}\phi_{j1}^2 & \phi_{j1}(1-\phi_{j1})\\ (1-\phi_{j1})\phi_{j1} & (1-\phi_{j1})^2\end{bmatrix} ds \begin{bmatrix}v_{j1}\\ v_{j2}\end{bmatrix} && (20)\\
&= \sum_j^{|E|} \begin{bmatrix}v_{j1} & v_{j2}\end{bmatrix}\int_{s_0}^{s_1}\begin{bmatrix}(1-\frac{s-s_0}{l(\Gamma_j)})^2 & (1-\frac{s-s_0}{l(\Gamma_j)})\frac{s-s_0}{l(\Gamma_j)}\\ \frac{s-s_0}{l(\Gamma_j)}(1-\frac{s-s_0}{l(\Gamma_j)}) & (\frac{s-s_0}{l(\Gamma_j)})^2\end{bmatrix} ds \begin{bmatrix}v_{j1}\\ v_{j2}\end{bmatrix} && (21)
\end{aligned}$$

In the equation above, s_0 and s_1 are the 2 vertex positions of the vertices surrounded edge j. $l(\Gamma_j)$ is shorthand for the length of edge j. Continuing on like we did in part 2 with introducing a new variable $s' = \frac{s-s_1}{l(\Gamma_j)}$, and adjusting the bounds of integrals to be fit for s', and using the fact that $ds = ds' l(\Gamma_j)$.

$$\sum_{j}^{|E|} \begin{bmatrix} v_{j1} & v_{j2} \end{bmatrix} \int_{s_1}^{s_2} \begin{bmatrix} (1-\frac{s-s_1}{l(\Gamma_j)})^2 & (1-\frac{s-s_1}{l(\Gamma_j)})\frac{s-s_1}{l(\Gamma_j)} \\ \frac{s-s_1}{l(\Gamma_j)}(1-\frac{s-s_1}{l(\Gamma_j)}) & (\frac{s-s_1}{l(\Gamma_j)})^2 \end{bmatrix} ds \begin{bmatrix} v_{j1} \\ v_{j2} \end{bmatrix} \tag{22}$$

$$= \sum_{j}^{|E|} \begin{bmatrix} v_{j1} & v_{j2} \end{bmatrix} l(\Gamma_j) \int_{0}^{1} \begin{bmatrix} 1-2s'+(s')^2 & s'-(s')^2 \\ s'-(s')^2 & (s')^2 \end{bmatrix} ds' \begin{bmatrix} v_{j1} \\ v_{j2} \end{bmatrix} \tag{23}$$

$$= \sum_{j}^{|E|} \begin{bmatrix} v_{j1} & v_{j2} \end{bmatrix} l(\Gamma_j) \begin{bmatrix} 1-1+\frac{1}{3} & \frac{1}{2}-\frac{1}{3} \\ \frac{1}{2}-\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} v_{j1} \\ v_{j2} \end{bmatrix} \tag{24}$$

$$= \sum_{j}^{|E|} \begin{bmatrix} v_{j1} & v_{j2} \end{bmatrix} \left(l(\Gamma_j) \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \right) \begin{bmatrix} v_{j1} \\ v_{j2} \end{bmatrix} \tag{25}$$

$$\tag{26}$$

Where the middle matrix can be given the name of the per-element mass matrix $\boldsymbol{M_e}$. To write this equation in a general vector notation, we again need to introduce the generalized $|V| \times |V|$ mass matrix M. Each entry ij in M is the sum of the corresponding component in each of the per element mass matrices of the elements(edges) incident on vertices i and j. If i and j are not adjacent to each other, the mass matrix entry will be zero. If i and j are adjacent but are not equal to each other, the entry will be equal to the length of the edge connecting them, divided by 6. If i and j are equal to each other, there are two incident edges on this vertex, so diagonal entries of i will be the sum of the length of both these edges divided by 3.

This lets us write the vector notation of the 3rd term of our least squares integral:

$$\int_{\Gamma} v(s)^2 ds = \boldsymbol{V}^T \boldsymbol{M} \boldsymbol{V} \tag{27}$$

with

$$\boldsymbol{M}_{ij} = \begin{cases} \sum_{k}^{\Lambda} \frac{l(\Gamma_k)}{3} & \Lambda = N_e(i), i = j \\ \frac{l(\Gamma_k)}{6} & \Gamma_k = N_e(i) \cap N_e(j) \\ 0 & otherwise \end{cases} \tag{28}$$

1.4 Bringing It All Together

The energy given by equation 6 that we wish to minimize becomes.

$$\boldsymbol{V}^* = \min_{\boldsymbol{V}} \boldsymbol{V}_p^T \boldsymbol{L} \boldsymbol{V}_p - 2\boldsymbol{V}_p^T \boldsymbol{A} \boldsymbol{V} + \boldsymbol{V}^T \boldsymbol{M} \boldsymbol{V} \tag{29}$$

This energy is quadratic and convex and therefore it's minimizer will be the value $\boldsymbol{V}$ takes when the derivative of the above energy is set to zero. Taking the derivatives with respect to V and setting that to 0, and exploithing the fact that our mass matrix is symmetric we obtain:

$$\boldsymbol{V}^T \boldsymbol{M} = \boldsymbol{V}_p^T \boldsymbol{A} \tag{30}$$

$$\boldsymbol{M} \boldsymbol{V} = \boldsymbol{A}^T \boldsymbol{V}_p \tag{31}$$

2 2D Set Up

The 2D set up is a lot like the 1D set-up. The first difference we have is that all our velocities are now 2-vectors. I'll put a bar over our terms to make it clear we are dealing with a vector, and not a scalar.

$$\overline{v}_p(s) = \begin{bmatrix} v_{px}(s) \\ v_{py}(s) \end{bmatrix} \qquad \overline{v}(s) = \begin{bmatrix} v_x(s) \\ v_y(s) \end{bmatrix} \tag{32}$$

$\overline{v}_p(s)$ above describes the desired 2D velocity for any point on our surface Γ as defined by our stefan problem (constant along edges, ambiguous on vertices). $\overline{v}(s)$ describes the actual motion for any point on our surface Γ. This motion is piecewise linear and defined via linear shape functions over each edge. At edge j, this velocity is given by:

$$\overline{v}(s)_j = \begin{bmatrix} \phi_1 & 0 & \phi_2 & 0 \\ 0 & \phi_1 & 0 & \phi_2 \end{bmatrix} \begin{bmatrix} v_{x1}(s) \\ v_{y1}(s) \\ v_{x2}(s) \\ v_{y2}(s) \end{bmatrix} = \begin{bmatrix} \phi_1 \boldsymbol{I} & \phi_2 \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \overline{v}_1(s) \\ \overline{v}_2(s) \end{bmatrix} = \begin{bmatrix} \phi_1 \boldsymbol{I} & (1-\phi_1)\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \overline{v}_1(s) \\ \overline{v}_2(s) \end{bmatrix} \tag{33}$$

Where ϕ_1 and ϕ_2 are linear shape functions at each edge defined the exact same way as in equations 4 and 5 in the previous section. $\boldsymbol{I}$ is the identity matrix of the same dimension in which we are working. In this case, $\boldsymbol{I}$ is 2×2. The energy we wish to minimize here is:

$$\min_{\overline{v}_i \forall i \in |V|} \int_\Gamma ||\overline{v}_p(s) - \overline{v}(s)||^2 ds \tag{34}$$

Let's see what shenanigans this leads to. Expanding equation 1 just like in equation 6, we get a new expanded energy:

$$\int_\Gamma \overline{v}_p(s)^T \overline{v}_p(s) - 2\overline{v}(s)^T \overline{v}_p(s) + \overline{v}(s)^T \overline{v}(s) ds \tag{35}$$

For brevity's sake, we will omit in our notation the (s). Rest assured all of these functions within the integral however are functions of s, and will need to be evaluated at every point in the surface.

$$\int_\Gamma \overline{v}_p^T \overline{v}_p - 2\overline{v}_p^T \overline{v} + \overline{v}^T \overline{v} ds \tag{36}$$

2.1 Discretizing Part 1 of Least Squares Integral : $\int_\Gamma \overline{v}_p^T \overline{v}_p ds$

Let's start out by discretizing $\int_\Gamma \overline{v}_p^T \overline{v}_p ds$, the first part of our energy. We can speed through this just like in part 1, making sure to handle the 2 spatial dimensions of $\overline{v}_p$ as appropriate. $\overline{v}_p$ is constant over each edge.

$$\int_\Gamma \overline{v}_p^T \overline{v}_p ds = \sum_j^{|E|} \int_{\Gamma_j} \overline{v}_p^T \overline{v}_p ds = \sum_j^{|E|} \overline{v}_{pj}^T (\int_{\Gamma_j} ds) \overline{v}_{pj} = \sum_j^{|E|} \overline{v}_{pj}^T l(\Gamma_j) \overline{v}_{pj} \tag{37}$$

In vector notation, we can write this as :

$$\boldsymbol{V}_p \boldsymbol{L} \boldsymbol{V}_p \tag{38}$$

Where in this case, $\boldsymbol{V}_p$ is a $2|E| \times 1$ vector, where the 2 elements indexed at $2j$ and $2j+1$ represent the 2 x and y components of the velocity at edge j respectively. $\boldsymbol{L}$ here is still diagonal but is now $2|E| \times 2|E|$. Each 2 diagonal components at indeces $(2j, 2j)$ and $(2j+1, 2j+1)$ stores the value of the length of edge j.

2.2 Discretizing Part 2 of Least Squares Integral: $-2\int_\Gamma \overline{v}_p^T\overline{v}ds$

Starting off with being $\overline{v}_p$ constant across each edge, then plugging in our shape functions

$$-2\int_\Gamma \overline{v}_p^T\overline{v}ds = -2\sum_j^{|E|}\overline{v}_{pj}^T\int_{\Gamma_j}\overline{v}ds = -2\sum_j^{|E|}\overline{v}_{pj}^T\int_{\Gamma_j}\begin{bmatrix}\phi_1\boldsymbol{I} & (1-\phi_1)\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\overline{v}_{j1}(s)\\ \overline{v}_{j2}(s)\end{bmatrix}ds \tag{39}$$

$$= -2\sum_j^{|E|}\overline{v}_{pj}^T\int_{\Gamma_j}\begin{bmatrix}\phi_1\boldsymbol{I} & (1-\phi_1)\boldsymbol{I}\end{bmatrix}ds\begin{bmatrix}\overline{v}_{j1}(s)\\ \overline{v}_{j2}(s)\end{bmatrix} = -2\sum_j^{|E|}\overline{v}_{pj}^T\int_{\Gamma_j}\begin{bmatrix}\phi_1\boldsymbol{I} & (1-\phi_1)\boldsymbol{I}\end{bmatrix}ds\begin{bmatrix}\overline{v}_{j1}(s)\\ \overline{v}_{j2}(s)\end{bmatrix} \tag{40}$$

$$= -2\sum_j^{|E|}\overline{v}_{pj}^T\begin{bmatrix}\int_{\Gamma_j}\phi_1 ds\boldsymbol{I} & \int_{\Gamma_j}(1-\phi_1)ds\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\overline{v}_{j1}(s)\\ \overline{v}_{j2}(s)\end{bmatrix} \tag{41}$$

$$= -2\sum_j^{|E|}\overline{v}_{pj}^T\begin{bmatrix}\frac{l(\Gamma_j)}{2}\boldsymbol{I} & \frac{l(\Gamma_j)}{2}\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\overline{v}_{j1}(s)\\ \overline{v}_{j2}(s)\end{bmatrix} \tag{42}$$

Where the last equality is obtained by doing the same integration procedure/change of variables in the equivalent subsection in 1D. In vector notation, this can be written as:

$$-2\boldsymbol{V}_p\boldsymbol{A}\boldsymbol{V} \tag{43}$$

Where $\boldsymbol{V}$ is a $2|V|\times 1$ vector, and $\boldsymbol{A}$ is a $2|E|\times 2|V|$ matrix. Just like in the previous section, if an edge i is incident on a vertex j, then we will fill the entries of A with indices $(2i, 2j)$ and $(2i+1, 2j+1)$ with the length of the edge divided by 2.

$$\boldsymbol{A}_{ij} = \begin{cases}\frac{l(\Gamma_{i'})}{2} & \Gamma_{j'}\in N_e(\Gamma_{i'}), i' = \lfloor\frac{i}{2}\rfloor, j' = \lfloor\frac{j}{2}\rfloor\\ 0 & otherwise\end{cases} \tag{44}$$

2.3 Discretizing Part 3 of Least Squares Integral: $\int_\Gamma \overline{v}^T\overline{v}ds$

Plugging in our linear basis formulation of v and proceeding through the integration by substitution like we did in the corresponding 1D section:

$$\int_\Gamma \overline{v}^T\overline{v}ds = \sum_j^{|E|}\int_{\Gamma_j}(\begin{bmatrix}\phi_1\boldsymbol{I} & (1-\phi_1)\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\overline{v}_{j1}(s)\\ \overline{v}_{j2}(s)\end{bmatrix})^T(\begin{bmatrix}\phi_1\boldsymbol{I} & (1-\phi_1)\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\overline{v}_{j1}(s)\\ \overline{v}_{j2}(s)\end{bmatrix})ds \tag{45}$$

$$= \sum_j^{|E|}\int_{\Gamma_j}\begin{bmatrix}\overline{v}_{j1}(s) & \overline{v}_{j2}(s)\end{bmatrix}\begin{bmatrix}\phi_1\boldsymbol{I}\\ (1-\phi_1)\boldsymbol{I}\end{bmatrix}(\begin{bmatrix}\phi_1\boldsymbol{I} & (1-\phi_1)\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\overline{v}_{j1}(s)\\ \overline{v}_{j2}(s)\end{bmatrix})ds \tag{46}$$

$$= \sum_j^{|E|}\begin{bmatrix}\overline{v}_{j1}(s) & \overline{v}_{j2}(s)\end{bmatrix}\int_{\Gamma_j}\begin{bmatrix}\phi_{j1}^2\boldsymbol{I} & \phi_{j1}(1-\phi_{j1})\boldsymbol{I}\\ (1-\phi_{j1})\phi_{j1}\boldsymbol{I} & (1-\phi_{j1})^2\boldsymbol{I}\end{bmatrix}ds\begin{bmatrix}\overline{v}_{j1}(s)\\ \overline{v}_{j2}(s)\end{bmatrix} \tag{47}$$

$$= \sum_j^{|E|}\begin{bmatrix}\overline{v}_{j1}(s) & \overline{v}_{j2}(s)\end{bmatrix}\begin{bmatrix}\int_{\Gamma_j}\phi_{j1}^2 ds\boldsymbol{I} & \int_{\Gamma_j}\phi_{j1}(1-\phi_{j1})ds\boldsymbol{I}\\ \int_{\Gamma_j}(1-\phi_{j1})\phi_{j1}ds\boldsymbol{I} & \int_{\Gamma_j}(1-\phi_{j1})^2 ds\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\overline{v}_{j1}(s)\\ \overline{v}_{j2}(s)\end{bmatrix} \tag{48}$$

$$= \sum_j^{|E|}\begin{bmatrix}\overline{v}_{j1}(s) & \overline{v}_{j2}(s)\end{bmatrix}\begin{bmatrix}(1/3)\boldsymbol{I} & (1/6)\boldsymbol{I}\\ (1/6)\boldsymbol{I} & (1/3)\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\overline{v}_{j1}(s)\\ \overline{v}_{j2}(s)\end{bmatrix} \tag{49}$$

Which we can write in vector notation:

$$\boldsymbol{V}^T\boldsymbol{M}\boldsymbol{V} \tag{50}$$

Where $\boldsymbol{M}$ is a sparse $2|V|\times 2|V|$ matrix that is non zero at entries $(2i, 2j)$, $(2i+1, 2j+1)$ if vertices i and j are adjacent to each other, in which case the two entries will take the value of the length of the edge connecting i and j, divided by 6. Entries $(2i, 2j)$, $(2i+1, 2j+1)$ will also be non-zero in the case that i and j are equal to each other, at which point the value at that entry will be the sum of the lengths of all incident edges on i, divided by 3.

$$\boldsymbol{M}_{ij} = \begin{cases} \sum_k^{\Lambda} \frac{l(\Gamma_k)}{3} & \Lambda = N_e(i'), i' = j' = \lfloor \frac{i}{2} \rfloor \\ \frac{l(\Gamma_k)}{6} & \Gamma_k = N_e(i') \cap N_e(j'), i' = \lfloor \frac{i}{2} \rfloor, j' = \lfloor \frac{j}{2} \rfloor \\ 0 & otherwise \end{cases} \tag{51}$$

2.4 Bringing it All Together

The energy in 2D that we wish to minimize for vertex velocities $\boldsymbol{V}$ becomes:

$$\boldsymbol{V}^* = \min_{\boldsymbol{V}} \boldsymbol{V}_p^T \boldsymbol{L} \boldsymbol{V}_p - 2\boldsymbol{V}_p^T \boldsymbol{A} \boldsymbol{V} + \boldsymbol{V}^T \boldsymbol{M} \boldsymbol{V} \tag{52}$$

This energy is quadratic and convex and therefore it's minimizer will be the value $\boldsymbol{V}$ takes when the derivative of the above energy is set to zero. Taking the derivatives with respect to V and setting that to 0, and exploithing the fact that our mass matrix is symmetric we obtain:

$$\boldsymbol{V}^T \boldsymbol{M} = \boldsymbol{V}_p^T \boldsymbol{A} \tag{53}$$

$$\boldsymbol{M} \boldsymbol{V} = \boldsymbol{A}^T \boldsymbol{V}_p \tag{54}$$

Which we can solve with a suite of linear solvers.

3 Normal Matching Set Up

The energy described in section 2 causes pinching behaviors whenever curvature is present. The resulting velocity at points with high curvatures (ie corners) has lower magnitude than velocities at vertices in flat neighborhoods. This is because the energy we develop previously does not explicitly enforce a magnitude constraint in the normal direction. Instead of saying we want the squared norm of the difference between the desired velocity and the interpolated velocity, we can project both velocities on the unit normal direction of each edge. Our new energy we wish to minimize is given by the following

$$\min_{\overline{v}_i \forall i \in |V|} \int_\Gamma (\overline{v}_p(s) \cdot \hat{n} - \overline{v}(s) \cdot \hat{n})^2 ds \tag{55}$$

Expanding our energy:

$$\int_\Gamma (\overline{v}_p(s) \cdot \hat{n} - \overline{v}(s) \cdot \hat{n})^2 ds = \int_\Gamma (\overline{v}_p(s) \cdot \hat{n})^2 - 2(\overline{v}_p(s) \cdot \hat{n})(\overline{v}(s) \cdot \hat{n}) + (\overline{v}(s) \cdot \hat{n})^2 ds \tag{56}$$

Before proceeding it's important to clarify a few things. First, note $\overline{v}_p(s) \cdot \hat{n}$ is piece-wise constant over each edge, because $\overline{v}_p(s)$ is constant over each edge and so is $\hat{n}$. Second, note $\overline{v}(s) \cdot \hat{n}$ is piece-wise linear over each edge, because $\overline{v}(s)$ is linear according to our shape functions and $\hat{n}$ is constant. For notational brevity we will drop the (s) and write the above equation with the scalars $v_p = \overline{v}_p(s) \cdot \hat{n}$ and $v = \overline{v}_(s) \cdot \hat{n}$.

$$\min_{\overline{v}_i \forall i \in |V|} \int_\Gamma v_p^2 - 2 v_p v + v^2 ds \tag{57}$$

3.1 Discretizing Part 1 of Least Squares Integral: $\int_\Gamma v_p^2 ds$

v_p is constant over each edge

$$\int_\Gamma v_p^2 ds = \sum_j^{|E|} \int_{\Gamma_j} v_{pj}^2 ds = \sum_j^{|E|} v_{pj}^2 \int_{\Gamma_j} ds = \sum_j^{|E|} v_{pj}^2 l(\Gamma_j) \tag{58}$$

Or in vector notation:

$$\boldsymbol{V}_{pn}^T \boldsymbol{L} \boldsymbol{V}_{pn} \tag{59}$$

Where $\boldsymbol{L}$ is the same as in the 1D case, and $\boldsymbol{V}_{pn}$ is a $|E| \times 1$ vector whose j'th index contains the normal projection of the desired velocity at of the j'th edge.

3.2 Discretizing Part 2 of Least Squares Integral $\int_\Gamma -2 v_p v ds$

$$\int_\Gamma -2 v_p v ds = -2 \sum_j^{|E|} \int_{\Gamma_j} v_{pj} v ds = -2 \sum_j^{|E|} v_{pj} \int_{\Gamma_j} v ds \tag{60}$$

$$= -2 \sum_j^{|E|} v_{pj} \int_{\Gamma_j} (\begin{bmatrix} \phi_1 \boldsymbol{I} & (1-\phi_1)\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix}) \cdot \hat{n}_j ds = -2 \sum_j^{|E|} v_{pj} \hat{n}_j^T \int_{\Gamma_j} \begin{bmatrix} \phi_1 \boldsymbol{I} & (1-\phi_1)\boldsymbol{I} \end{bmatrix} ds \begin{bmatrix} v_{j1} \\ v_{j2} \end{bmatrix} \tag{61}$$

$$= -2 \sum_j^{|E|} v_{pj} \hat{n}_j^T \begin{bmatrix} \frac{l(\Gamma_j)}{2} \boldsymbol{I} & \frac{l(\Gamma_j)}{2} \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix} \tag{62}$$

Refer to section 1.2 and 2.2 for more info on the last equality and on how to integrate our shape function row vector. At this point the only way to proceed I can think of is by explicitly evaluating the integrand in the final line to get a per element mass matrix. Using $\hat{n}_j = \begin{bmatrix} n_x & n_y \end{bmatrix}^T$ for the 2D case or $\hat{n}_j = \begin{bmatrix} n_x & n_y & n_z \end{bmatrix}^T$ for 3D, we get:

$$= -2\sum_j^{|E|} v_{pj} \begin{bmatrix} n_x & n_y \end{bmatrix} \begin{bmatrix} \frac{l(\Gamma_j)}{2}\boldsymbol{I} & \frac{l(\Gamma_j)}{2}\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix} \tag{63}$$

$$= -2\sum_j^{|E|} v_{pj} \begin{bmatrix} n_x\frac{l(\Gamma_j)}{2} & n_y\frac{l(\Gamma_j)}{2} & n_x\frac{l(\Gamma_j)}{2} & n_y\frac{l(\Gamma_j)}{2} \end{bmatrix} \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix} \tag{64}$$

$$= -2\sum_j^{|E|} v_{pj} \begin{bmatrix} n_x & n_y \end{bmatrix} \begin{bmatrix} \frac{l(\Gamma_j)}{2}\boldsymbol{I} & \frac{l(\Gamma_j)}{2}\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix} \tag{65}$$

$$= -2\sum_j^{|E|} v_{pj} \frac{l(\Gamma_j)}{2} \begin{bmatrix} n_x & n_y & n_x & n_y \end{bmatrix} \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix} \tag{66}$$

Rewriting this in vector notation:

$$-2\boldsymbol{V}_{pn}\boldsymbol{A}\boldsymbol{V} \tag{67}$$

Where $\boldsymbol{V}_{pn}$ is the same as in the previous section, the vector $\boldsymbol{V}_n$ is also the same in the section 2.2. However, our $\boldsymbol{A}$ matrix is very different now! In this case $\boldsymbol{A}$ is a $|E| \times 2|V|$ matrix where each row represents an edge, and each column is nonzero if the edge is adjacent to the vertex corresponding to that column. For explanation's sake we will refer to $\frac{l(\Gamma_j)}{2}\begin{bmatrix} n_x & n_y & n_x & n_y \end{bmatrix}$ as our per element Ae matrix. We will fill out the general $\boldsymbol{A}$ matrix by first initializing it to zero, then for each edge k connecting vertices i and j:

$$A(k, 2i : 2i+1) = Ae(0, 0 : 1) \tag{68}$$

$$A(k, 2j : 2j+1) = Ae(0, 2 : 3) \tag{69}$$

3.3 Discretizing Part 3 of Least Squares Integral $\int_\Gamma v^2 ds$

v is piecewise linear along each edge.

$$\int_\Gamma v^2 ds = \sum_j^{|E|} \int_{\Gamma_j} v^2 ds = \sum_j^{|E|} \int_{\Gamma_j} (\begin{bmatrix} \phi_1\boldsymbol{I} & (1-\phi_1)\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix}) \cdot \hat{n}_j)^2 ds \tag{70}$$

$$= \sum_j^{|E|} \int_{\Gamma_j} (\hat{n}_j^T \begin{bmatrix} \phi_1\boldsymbol{I} & (1-\phi_1)\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix})^T (\hat{n}_j^T \begin{bmatrix} \phi_1\boldsymbol{I} & (1-\phi_1)\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix}) ds \tag{71}$$

$$= \sum_j^{|E|} \int_{\Gamma_j} ((\begin{bmatrix} \phi_1\boldsymbol{I} & (1-\phi_1)\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix})^T \hat{n}_j) \hat{n}_j^T \begin{bmatrix} \phi_1\boldsymbol{I} & (1-\phi_1)\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix} ds \tag{72}$$

$$= \sum_j^{|E|} \int_{\Gamma_j} \begin{bmatrix} \overline{v}_{j1} & \overline{v}_{j2} \end{bmatrix} \begin{bmatrix} \phi_1\boldsymbol{I} \\ (1-\phi_1)\boldsymbol{I} \end{bmatrix} \hat{n}_j \hat{n}_j^T \begin{bmatrix} \phi_1\boldsymbol{I} & (1-\phi_1)\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix} ds \tag{73}$$

$$= \sum_j^{|E|} \begin{bmatrix} \overline{v}_{j1} & \overline{v}_{j2} \end{bmatrix} \int_{\Gamma_j} \begin{bmatrix} \phi_1\boldsymbol{I} \\ (1-\phi_1)\boldsymbol{I} \end{bmatrix} \hat{n}_j \hat{n}_j^T \begin{bmatrix} \phi_1\boldsymbol{I} & (1-\phi_1)\boldsymbol{I} \end{bmatrix} ds \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix} \tag{74}$$

Using $\hat{n}_j = \begin{bmatrix} n_x & n_y \end{bmatrix}^T$ for the 2D case or $\hat{n}_j = \begin{bmatrix} n_x & n_y & n_z \end{bmatrix}^T$ for 3D, we get:

$$= \sum_{j}^{|E|} \begin{bmatrix} \overline{v}_{j1} & \overline{v}_{j2} \end{bmatrix} \int_{\Gamma_j} \begin{bmatrix} \phi_1 \boldsymbol{I} \\ (1-\phi_1)\boldsymbol{I} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} \begin{bmatrix} n_x & n_y \end{bmatrix} \begin{bmatrix} \phi_1 \boldsymbol{I} & (1-\phi_1)\boldsymbol{I} \end{bmatrix} ds \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix} \tag{75}$$

$$= \sum_{j}^{|E|} \begin{bmatrix} \overline{v}_{j1} & \overline{v}_{j2} \end{bmatrix} \int_{\Gamma_j} \begin{bmatrix} \phi_1 \boldsymbol{I} \\ (1-\phi_1)\boldsymbol{I} \end{bmatrix} \begin{bmatrix} n_x^2 & n_x n_y \\ n_y n_x & n_y^2 \end{bmatrix} \begin{bmatrix} \phi_1 \boldsymbol{I} & (1-\phi_1)\boldsymbol{I} \end{bmatrix} ds \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix} \tag{76}$$

$$= \sum_{j}^{|E|} \begin{bmatrix} \overline{v}_{j1} & \overline{v}_{j2} \end{bmatrix} \int_{\Gamma_j} \begin{bmatrix} n_x^2\phi_1^2 & n_x n_y \phi_1^2 & n_x^2 \phi_1(1-\phi_1) & n_x n_y \phi_1 (1-\phi_1) \\ n_x n_y \phi_1^2 & n_y^2 \phi_1^2 & n_x n_y \phi_1 (1-\phi_1) & n_y^2 \phi_1 (1-\phi_1) \\ n_x^2 \phi_1 (1-\phi_1) & n_x n_y \phi_1 (1-\phi_1) & n_x^2 (1-\phi_1)^2 & n_x n_y (1-\phi_1)^2 \\ n_x n_y \phi_1 (1-\phi_1) & n_y^2 \phi_1 (1-\phi_1) & n_x n_y (1-\phi_1)^2 & n_y^2 (1-\phi_1)^2 \end{bmatrix} ds \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix} \tag{77}$$

We carry out the inner integral component wise. Luckily from section 1.3 we know how to evaluate the integral of many of the terms, namely:

$$\int_{\Gamma_j} \phi_1^2 ds = \int_{\Gamma_j} (1-\phi_1)^2 ds = \frac{l(\Gamma_j)}{3} \tag{78}$$

$$\int_{\Gamma_j} \phi_1(1-\phi_1) ds = \frac{l(\Gamma_j)}{6} \tag{79}$$

Plugging the above into our big integral matrix equation:

$$\sum_{j}^{|E|} \begin{bmatrix} \overline{v}_{j1} & \overline{v}_{j2} \end{bmatrix} \frac{l(\Gamma_j)}{6} \begin{bmatrix} 2n_x^2 & 2n_x n_y & n_x^2 & n_x n_y \\ 2n_x n_y & 2n_y^2 & n_x n_y & n_y^2 \\ n_x^2 & n_x n_y & 2n_x^2 & 2n_x n_y \\ n_x n_y & n_y^2 & 2n_x n_y & 2n_y^2 \end{bmatrix} \begin{bmatrix} \overline{v}_{j1} \\ \overline{v}_{j2} \end{bmatrix} \tag{80}$$

It might be useful to compare the element mass matrix above to our element Mass matrix from section 2.3.

$$\frac{l(\Gamma_j)}{6} \begin{bmatrix} 2n_x^2 & 2n_x n_y & n_x^2 & n_x n_y \\ 2n_x n_y & 2n_y^2 & n_x n_y & n_y^2 \\ n_x^2 & n_x n_y & 2n_x^2 & 2n_x n_y \\ n_x n_y & n_y^2 & 2n_x n_y & 2n_y^2 \end{bmatrix} \qquad \frac{l(\Gamma_j)}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \tag{81}$$

Note that both are symmetric, but this new one is Denser and experiences cross-talk between the x and the y dimensions. I need more intuition still about what this means. Our equation can be written in vector notation:

$$\boldsymbol{V}^T \boldsymbol{M} \boldsymbol{V} \tag{82}$$

where $\boldsymbol{M}$ refers to the contribution of the element mass matrix from each edge as described above. The way we fill this general mass matrix out is by intializing it to zero, and then for each edge, calculating the elmeent mass matrix, and adding it on to the general mass matrix in the appropriate indices. The equations below show how to add the contribution of one edge connecting i and j with element mass matrix M_e to the general mass matrix.

$$M(2i : 2i+1, 2i : 2i+1) \mathrel{+}= Me(0:1, 0:1) \tag{83}$$

$$M(2i : 2i+1, 2j : 2j+1) \mathrel{+}= Me(0:1, 2:3) \tag{84}$$

$$M(2j : 2j+1, 2i : 2i+1) \mathrel{+}= Me(2:3, 0:1) \tag{85}$$

$$M(2j : 2j+1, 2j : 2j+1) \mathrel{+}= Me(2:3, 2:3) \tag{86}$$

3.4 Bringing it All Together

Our energy in vector notation now becomes:

$$\min_{\overline{v}_i \forall i \in |V|} \int_\Gamma v_p^2 - 2v_p v + v^2 ds = \min_{\boldsymbol{V}} \boldsymbol{V}_{pn}^T \boldsymbol{L} \boldsymbol{V}_{pn} - 2\boldsymbol{V}_{pn}^T \boldsymbol{A} \boldsymbol{V} + \boldsymbol{V}^T \boldsymbol{M} \boldsymbol{V} \tag{87}$$

Taking the derivative with respect to$\boldsymbol{V}$ and setting it to zero leads to:

$$-2\boldsymbol{V}_{pn}^T \boldsymbol{A} + 2\boldsymbol{V}^T \boldsymbol{M} = 0 \tag{88}$$

$$2\boldsymbol{V}^T \boldsymbol{M} = 2\boldsymbol{V}_{pn}^T \boldsymbol{A} \tag{89}$$

$$\boldsymbol{M}\boldsymbol{V} = \boldsymbol{A}^T \boldsymbol{V}_{pn} \tag{90}$$

However we notices that this energy can be minimized by some degenerate pick of V. V can be as large in magnitude as it wants, so long as when it is projected to the normal, it is as close to Vp as possible, there is no limit to how big V can get if it just keeps changing it's direction. For this reason we also add a regularization term to our energy that makes the optimization seek node velocities of small magnitudes:

$$\min_{\boldsymbol{V}} \boldsymbol{V}_{pn}^T \boldsymbol{L} \boldsymbol{V}_{pn} - 2\boldsymbol{V}_{pn}^T \boldsymbol{A} \boldsymbol{V} + \boldsymbol{V}^T \boldsymbol{M} \boldsymbol{V} + \lambda \boldsymbol{V}^T \boldsymbol{V} \tag{91}$$

Creating the modified system we wish to solve

$$(\boldsymbol{M} + \lambda \boldsymbol{I})\boldsymbol{V} = \boldsymbol{A}^T \boldsymbol{V}_{pn} \tag{92}$$