2) Calculate the reduced Lagrangian.

The Lagrangian for the NLS:

$$\mathcal{L} = \frac{i}{2}\left[\varphi \frac{\partial \varphi^*}{\partial t} - \varphi^* \frac{\partial \varphi}{\partial t}\right] - \alpha \left|\frac{\partial \varphi}{\partial x}\right|^2 + \frac{\kappa}{2}|\varphi|^4 \qquad (1)$$

We'll use the trial function

$$\varphi(t,x) = A(t) \exp\left[-\frac{x^2}{2a(t)^2} + i\, b(t)\, x^2\right]$$

$$(2)$$

$$\Rightarrow \varphi^*(t,x) = A^*(t) \exp\left[-\frac{x^2}{2a(t)^2} - i\, b(t)\, x^2\right]$$

Using (2) on the terms in (1):

$$\frac{\partial \varphi^*}{\partial t} = \frac{\partial}{\partial t} A^*(t) \exp\left[-\frac{x^2}{2a(t)^2} - i\, b(t)\, x^2\right]$$

$$= \frac{\partial A^*(t)}{\partial t} \exp\left[-\frac{x^2}{2a(t)^2} - i\, b(t)\, x^2\right] + A^*(t) \frac{\partial}{\partial t} \exp\left[-\frac{x^2}{2a(t)^2} - i\, b(t)\, x^2\right]$$

$$= \left[\frac{\partial A^*}{\partial t} + A^* \frac{x^2}{a^3} \frac{\partial a}{\partial t} - A^* i \frac{\partial b}{\partial t} x^2)\right] \exp\left[-\frac{x^2}{2a(t)^2} - i\, b(t)\, x^2\right]$$

$$= \left[\frac{\partial A^*}{\partial t} + A^* x^2 \left(\frac{1}{a^3}\frac{\partial a}{\partial t} - i \frac{\partial b}{\partial t}\right)\right] \exp\left[-\frac{x^2}{2a(t)^2} - i\, b(t)\, x^2\right] \qquad (3)$$

$$\frac{\partial \varphi}{\partial t} = \frac{\partial}{\partial t} A(t) \exp\left[-\frac{x^2}{2a(t)^2} + i\, b(t)\, x^2\right]$$

$$= \left[\frac{\partial A}{\partial t} + A x^2 \left(\frac{1}{a^3}\frac{\partial a}{\partial t} + i \frac{\partial b}{\partial t}\right)\right] \exp\left[-\frac{x^2}{2a(t)^2} + i\, b(t)\, x^2\right] \qquad (4)$$

And

$$\left|\frac{\partial \varphi}{\partial x}\right|^2 = \frac{\partial \varphi}{\partial x}\frac{\partial \varphi^*}{\partial x}$$

$$= \left[\frac{\partial}{\partial x} A(t) \exp\left[-\frac{x^2}{2a(t)^2} + i\,b(t)\,x^2\right]\right] \cdot \left[\frac{\partial}{\partial x} A^*(t) \exp\left[-\frac{x^2}{2a(t)^2} - i\,b(t)\,x^2\right]\right]$$

$$= A\left(-\frac{x}{a^2} + 2ibx\right) A^*\left(-\frac{x}{a^2} - 2ibx\right) \exp\left[-\frac{x^2}{2a(t)^2} + i\,b(t)\,x^2 - \frac{x^2}{2a(t)^2} - i\,b(t)\,x^2\right]$$

$$= |A|^2\left(\frac{x^2}{a^4} + 4b^2x^2\right) \exp\left[-\frac{x^2}{a^2}\right] \qquad (5)$$

And

$$|\varphi|^4 = |\varphi|^2|\varphi|^2$$

$$= \varphi\varphi^*\varphi\varphi^*$$

$$= A(t) \exp\left[-\frac{x^2}{2a(t)^2} + \cancel{i\,b(t)\,x^2}\right] \cdot A^*(t) \exp\left[-\frac{x^2}{2a(t)^2} - \cancel{i\,b(t)\,x^2}\right]$$

$$\cdot A(t) \exp\left[-\frac{x^2}{2a(t)^2} + \cancel{i\,b(t)\,x^2}\right] \cdot A^*(t) \exp\left[-\frac{x^2}{2a(t)^2} - \cancel{i\,b(t)\,x^2}\right]$$

$$= |A|^4 \exp\left[-\frac{2x^2}{a(t)^2}\right] \qquad (6)$$

So, (6), (5), (4) and (3) in (1)

$$\mathcal{L} = \frac{i}{2}\left[\varphi\frac{\partial\varphi^*}{\partial t} - \varphi^*\frac{\partial\varphi}{\partial t}\right] - \alpha\left|\frac{\partial\varphi}{\partial x}\right|^2 + \frac{\kappa}{2}|\varphi|^4$$

$$= \frac{i}{2}\left[A(t) \exp\left[-\frac{x^2}{2a(t)^2} + \cancel{i\,b(t)\,x^2}\right]\left[\frac{\partial A^*}{\partial t} + A^* x^2\left(\frac{1}{a^3}\frac{\partial a}{\partial t} - i\frac{\partial b}{\partial t}\right)\right] \exp\left[-\frac{x^2}{2a(t)^2} - \cancel{i\,b(t)\,x^2}\right]\right.$$

$$\left. - A^*(t) \exp\left[-\frac{x^2}{2a(t)^2} - \cancel{i\,b(t)\,x^2}\right]\left[\frac{\partial A}{\partial t} + A x^2\left(\frac{1}{a^3}\frac{\partial a}{\partial t} + i\frac{\partial b}{\partial t}\right)\right] \exp\left[-\frac{x^2}{2a(t)^2} + \cancel{i\,b(t)\,x^2}\right]\right]$$

$$-\alpha|A|^2\left(\frac{x^2}{a^4} + 4b^2x^2\right) \exp\left[-\frac{x^2}{a^2}\right]$$

$$+\frac{\kappa}{2}|A|^4 \exp\left[-\frac{2x^2}{a(t)^2}\right]$$

$$= \frac{i}{2}\left[A\frac{\partial A^*}{\partial t} + |A|^2x^2\left(\cancel{\frac{1}{a^3}\frac{\partial a}{\partial t}} - i\frac{\partial b}{\partial t}\right) - A^*\frac{\partial A}{\partial t} - |A|^2x^2\left(\cancel{\frac{1}{a^3}\frac{\partial a}{\partial t}} + i\frac{\partial b}{\partial t}\right)\right] \exp\left[-\frac{x^2}{a^2}\right]$$

$$-\alpha|A|^2\left(\frac{x^2}{a^4} + 4b^2x^2\right) \exp\left[-\frac{x^2}{a^2}\right]$$

$$+\frac{\kappa}{2}|A|^4\exp\left[-\frac{2x^2}{a(t)^2}\right]$$

$$= \frac{i}{2}\left[A\frac{\partial A^*}{\partial t} - A^*\frac{\partial A}{\partial t} - 2|A|^2x^2 i\frac{\partial b}{\partial t}\right]\cdot\exp\left[-\frac{x^2}{a^2}\right]$$

$$-\alpha|A|^2\left(\frac{1}{a^4}+4b^2\right)x^2\exp\left[-\frac{x^2}{a^2}\right]$$

$$+\frac{\kappa}{2}|A|^4\exp\left[-\frac{2x^2}{a(t)^2}\right]$$

Grouping x - terms we get (red for x^0, blue for x^2)

$$\mathcal{L} = \left(\frac{i}{2}\left(A\frac{\partial A^*}{\partial t} - A^*\frac{\partial A}{\partial t}\right)\right)\cdot\exp\left[-\frac{x^2}{a^2}\right]$$

$$+\left(|A|^2\frac{\partial b}{\partial t} - \alpha|A|^2\left(\frac{1}{a^4}+4b^2\right)\right)x^2\exp\left[-\frac{x^2}{a^2}\right]$$

$$+\frac{\kappa}{2}|A|^4\exp\left[-\frac{2x^2}{a(t)^2}\right] \qquad (7)$$

Now, we calculate the reduced Lagrangian from

$$\langle\mathcal{L}\rangle = \int_{-\infty}^{\infty}\mathcal{L}\,dx \qquad (8)$$

Note that all coefficients in (7) are independent of x. Thus, we just have to calculate the Gaussian integrals. From Physics Handbook we have that

$$\int_{-\infty}^{\infty}x^m e^{-cx^2}dx = 2\int_0^{\infty}x^m e^{-cx^2}dx = \frac{1}{c^{(m+1)/2}}\Gamma\left(\frac{m+1}{2}\right) \qquad (9)$$

We have 3 different Gaussians in (7) which are

$$\int_{-\infty}^{\infty}e^{-\frac{x^2}{a^2}}dx \overset{(9)}{=} (a^2)^{1/2}\Gamma\left(\frac{1}{2}\right) = a\sqrt{\pi} \qquad (10)$$

$$\int_{-\infty}^{\infty}x^2 e^{-\frac{x^2}{a^2}}dx = (a^2)^{3/2}\Gamma\left(\frac{3}{2}\right) = a^3\frac{\sqrt{\pi}}{2} \qquad (11)$$

$$\int_{-\infty}^{\infty}e^{-\frac{2x^2}{a^2}}dx = \left(\frac{a^2}{2}\right)^{1/2}\Gamma\left(\frac{1}{2}\right) = a\sqrt{\frac{\pi}{2}} \qquad (12)$$

(10), (11) and (12) in (8)

$$\langle \mathcal{L} \rangle = \frac{i}{2}\left(A\frac{\partial A^*}{\partial t} - A^*\frac{\partial A}{\partial t}\right) a\sqrt{\pi}$$

$$+ \left(|A|^2\frac{\partial b}{\partial t} - \alpha |A|^2\left(\frac{1}{a^4} + 4b^2\right)\right) a^3 \frac{\sqrt{\pi}}{2}$$

$$+ \frac{\kappa}{2} |A|^4 a \sqrt{\frac{\pi}{2}}$$

$$= \frac{\sqrt{\pi}}{2}\left[i\left(A\frac{\partial A^*}{\partial t} - A^*\frac{\partial A}{\partial t}\right) + |A|^2 a^3\left(\frac{\partial b}{\partial t} - \frac{\alpha}{a^4} - 4\alpha b^2\right) + \frac{\kappa a}{\sqrt{2}} |A|^4\right] \qquad (8)'$$

1.)

Here we want to show that the NLS can be derived from the Lagrangian

$$\mathcal{L} = \frac{i}{2}\left[\varphi \frac{\partial \varphi^*}{\partial t} - \varphi^* \frac{\partial \varphi}{\partial t}\right] - \alpha \left|\frac{\partial \varphi}{\partial x}\right|^2 + \frac{K}{2}|\varphi|^4$$

First we must find the Euler-Lagrange equations for a Lagrangian on the form $\mathcal{L} = \mathcal{L}\left(\varphi, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}\right)$.

We want to find an expression for φ which extremizes the functional

$$J = \int_{t_a}^{t_b}\int_{x_a}^{x_b} \mathcal{L}\left(\varphi, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}\right) dx\, dt$$

If φ extremizes the functional then it follows that

$$\phi = \varphi(x,t) + \varepsilon \overset{\text{Arbitrary}}{\eta(x,t)}, \qquad \eta(x_b,t) = \eta(x_a,t) = \eta(x,t_b) = \eta(x,t_a) = 0$$

must either decrease or increase J. Substituting into our original J

$$J = \int_{t_a}^{t_b}\int_{x_a}^{x_b} \mathcal{L}\left(\phi, \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}\right) dx\, dt$$

Taking the derivative w.r.t. ε at $\varepsilon = 0$ gives

$$\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_{t_a}^{t_b}\int_{x_a}^{x_b} \left(\frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial \phi}{\partial \varepsilon} + \frac{\partial \mathcal{L}}{\partial(\partial \phi/\partial t)}\frac{\partial(\partial \phi/\partial t)}{\partial \varepsilon} + \frac{\partial \mathcal{L}}{\partial(\partial \phi/\partial x)}\frac{\partial(\partial \phi/\partial x)}{\partial \varepsilon}\right) dx\, dt \,\Bigg|_{\varepsilon=0}$$

$$= \int_{t_a}^{t_b}\int_{x_a}^{x_b} \left(\frac{\partial \mathcal{L}}{\partial \phi}\eta(x,t) + \frac{\partial \mathcal{L}}{\partial(\partial \phi/\partial t)}\frac{\partial \eta(x,t)}{\partial t} + \frac{\partial \mathcal{L}}{\partial(\partial \phi/\partial x)}\frac{\partial \eta(x,t)}{\partial x}\right) dx\, dt \,\Bigg|_{\varepsilon=0}$$

We would like this in terms of $\eta(x,t)$. To do this we'll use integration by parts. We have

$$\int_{t_a}^{t_b}\int_{x_a}^{x_b}\left(\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial t)}\frac{\partial \eta(x,t)}{\partial t}\right)dx\,dt$$

$$= \int_{x_a}^{x_b} \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial t)}\eta\Big|_{t=t_a}^{t=t_b}}_{\text{Vanishes by assumption}} dx - \int_{t_a}^{t_b}\int_{x_a}^{x_b}\frac{\partial^2 \mathcal{L}}{\partial(\partial\phi/\partial t)\,\partial t}\eta\,dx\,dt$$

$$= -\int_{t_a}^{t_b}\int_{x_a}^{x_b}\frac{\partial^2 \mathcal{L}}{\partial(\partial\phi/\partial t)\,\partial t}\eta\,dx\,dt$$

Similarly

$$\int_{t_a}^{t_b}\int_{x_a}^{x_b}\left(\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)}\frac{\partial \eta(x,t)}{\partial x}\right)dx\,dt$$

$$= \int_{t_a}^{t_b} \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)}\eta\Big|_{x=x_a}^{x=x_b}}_{\text{Vanishes by assumption}} dt - \int_{t_a}^{t_b}\int_{x_a}^{x_b}\frac{\partial^2 \mathcal{L}}{\partial(\partial\phi/\partial x)\,\partial x}\eta\,dx\,dt$$

$$= -\int_{t_a}^{t_b}\int_{x_a}^{x_b}\frac{\partial^2 \mathcal{L}}{\partial(\partial\phi/\partial x)\,\partial x}\eta\,dx\,dt$$

Putting this bach in our original equation we get

$$\frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} = \int_{t_a}^{t_b}\int_{x_a}^{x_b}\left(\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial^2 \mathcal{L}}{\partial(\partial\phi/\partial x)\,\partial x} - \frac{\partial^2 \mathcal{L}}{\partial(\partial\phi/\partial t)\,\partial t}\right)\eta(x,t)\,dx\,dt\Big|_{\varepsilon=0}$$

But note that at $\varepsilon = 0$

$$\phi(x,t)\Big|_{\varepsilon=0} = \left(\varphi(x,t) + \varepsilon\,\eta(x,t)\right)\Big|_{\varepsilon=0} = \varphi(x,t)$$

$$\Rightarrow \quad \mathcal{L}\left(\phi, \frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}\right)\Big|_{\varepsilon=0} = \mathcal{L}\left(\varphi, \frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial x}\right)$$

and since we assume that $\mathcal{L}\left(\varphi, \frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial x}\right)$ extremizes the functional it follows that

$$\frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} = \int_{t_a}^{t_b}\int_{x_a}^{x_b}\left(\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial^2\mathcal{L}}{\partial(\partial\phi/\partial x)\partial x} - \frac{\partial^2\mathcal{L}}{\partial(\partial\phi/\partial t)\,\partial t}\right)\eta(x,t)\,dx\,dt\Big|_{\varepsilon=0}$$

$$= \int_{t_a}^{t_b}\int_{x_a}^{x_b}\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \frac{\partial^2\mathcal{L}}{\partial(\partial\varphi/\partial x)\partial x} - \frac{\partial^2\mathcal{L}}{\partial(\partial\varphi/\partial t)\,\partial t}\right)\eta(x,t)\,dx\,dt = 0$$

Since $\eta(x,t)$ is arbitrary it follows that the kernel must be 0 and

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial x)} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial t)} \quad (+\ c.c) = 0$$

So, with our given Lagrangian

$$\mathcal{L} = \frac{i}{2}\left[\varphi\frac{\partial\varphi^*}{\partial t} - \varphi^*\frac{\partial\varphi}{\partial t}\right] - \alpha\left|\frac{\partial\varphi}{\partial x}\right|^2 + \frac{\kappa}{2}|\varphi|^4$$

We have (with corresponding treatment to c.c.)

$$\frac{\partial\mathcal{L}}{\partial\varphi} = \frac{i}{2}\frac{\partial\varphi^*}{\partial t} + \kappa\,\varphi^*|\varphi|^2$$

$$\frac{\partial\mathcal{L}}{\partial\varphi^*} = -\frac{i}{2}\frac{\partial\varphi}{\partial t} + \kappa\,\varphi|\varphi|^2$$

$$\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial x)} = \frac{\partial}{\partial x}\left(-\alpha\frac{\partial\varphi^*}{\partial x}\right) = -\alpha\frac{\partial^2\varphi^*}{\partial x^2}$$

$$\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial(\partial\varphi^*/\partial x)} = \frac{\partial}{\partial x}\left(-\alpha\frac{\partial\varphi}{\partial x}\right) = -\alpha\frac{\partial^2\varphi}{\partial x^2}$$

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial t)} = \frac{\partial}{\partial t}\left(-\frac{i}{2}\varphi^*\right) = -\frac{i}{2}\frac{\partial\varphi^*}{\partial t}$$

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial\varphi^*/\partial t)} = \frac{\partial}{\partial t}\left(\frac{i}{2}\varphi\right) = \frac{i}{2}\frac{\partial\varphi}{\partial t}$$

$$\therefore\quad 0 = \frac{\partial \mathcal{L}}{\partial\varphi} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x)} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial t)} \quad (+\ c.c)$$

$$= \frac{i}{2}\frac{\partial\varphi^*}{\partial t} + \kappa\,\varphi^*|\varphi|^2 - \frac{i}{2}\frac{\partial\varphi}{\partial t} + \kappa\,\varphi|\varphi|^2 + \alpha\frac{\partial^2\varphi^*}{\partial x^2} + \alpha\frac{\partial^2\varphi}{\partial x^2}$$

$$-\frac{i}{2}\frac{\partial\varphi}{\partial t} + \frac{i}{2}\frac{\partial\varphi^*}{\partial t}$$

$$= i\left(\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi^*}{\partial t}\right) + \alpha\left(\frac{\partial^2\varphi^*}{\partial x^2} + \frac{\partial^2\varphi}{\partial x^2}\right) + \kappa|\varphi|^2(\varphi + \varphi^*)$$

$$= i\frac{\partial\varphi}{\partial t} + \alpha\frac{\partial^2\varphi^*}{\partial x^2} + \kappa|\varphi|^2\varphi + c.c. = 0$$

$$= \mathrm{Re}(NLS) + \mathrm{Im}(NLS)$$

A