STABILITY ANALYSIS WITH PERTURBATIONS

Detailed Explanation of Linearization via Taylor Expansion

OVERVIEW

- Understanding equilibrium and perturbation.
- Linearization via Taylor expansion.
- Stability conditions.

INTRODUCTION TO STABILITY

- We explore the stability of equilibrium points in a general ODE:

    - $dn/dt = f(n)$

- The equilibrium point n^* satisfies $f(n^*) = 0$.
- Introducing a perturbation variable x, such that $n = n^* + x$, allows us to study deviations from equilibrium.
- Substituting $n = n^* + x$ into the ODE gives:

    - $dx/dt = f(n^* + x)$.

LINEARIZATION VIA TAYLOR EXPANSION

- Expand $f(n^* + x)$ using a first-order Taylor expansion around n^*:

  - $f(n^* + x) \approx f(n^*) + f'(n^*)x$ + higher-order terms.

- Since $f(n^*) = 0$ (equilibrium condition), this reduces to:

  - $dx/dt \approx f'(n^*)x$.

- The higher-order terms are negligible for small perturbations x.

CONDITIONS FOR STABILITY

- The equilibrium point n^* is:

- **Stable:** If $f'(n^*) < 0$, perturbations decay over time ($x \to 0$).
- **Unstable:** If $f'(n^*) > 0$, perturbations grow over time ($x \to \infty$).

- The magnitude of $f'(n^*)$ determines the rate of return to equilibrium.

RATE OF RETURN TO EQUILIBRIUM

- The magnitude of $f'(n^*)$ determines the rate of return to equilibrium.
- Larger $|f'(n^*)|$ implies faster convergence (if stable) or divergence (if unstable).
- The return time τ is inversely proportional to $|f'(n^*)|$:
- $\tau \approx 1/|f'(n^*)|$.

RELATIONSHIP BETWEEN TAYLOR EXPANSION AND THE JACOBIAN MATRIX

Understanding Local Linearization and Stability Analysis

TAYLOR EXPANSION OVERVIEW

- For a multivariate function $f(x)$ where $x = [x_1, x_2 \ldots x_n]^T$:

- $f(x) \approx f(x^*) + J(x^*)(x - x^*)$ + Higher-order terms

- - $f(x^*)$: Value of the function at x^*.
- - $J(x^*)$: Jacobian matrix containing first partial derivatives.
- - Higher-order terms capture nonlinearities.

JACOBIAN MATRIX

- The Jacobian matrix J(x) is defined as:
- $$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$
- \- J(x) represents the linear part of the Taylor expansion.

LINEARIZATION USING JACOBIAN

- By truncating higher-order terms in the Taylor expansion, we approximate the system near x^* with a linear function as:

- $f(x) \approx f(x^*) + J(x^*)(x - x^*)$

- It follows that:

$$\frac{dx}{dt} = f(x) \approx J(x^*)(x - x^*)$$

where $J(x^*)$ governs the local behavior of a perturbation $(x - x^*)$ around the equilibrium x^*

APPLICATIONS IN STABILITY ANALYSIS

- The Jacobian matrix's eigenvalues determine stability:

- - All eigenvalues have negative real parts: Stable.
- - Any eigenvalue has a positive real part: Unstable.
- - Purely imaginary eigenvalues: Oscillatory behavior.

- The Jacobian matrix provides the linear approximation to analyze the behavior of nonlinear systems near equilibrium.

KEY CONNECTION AND SUMMARY

- - Taylor Expansion: Framework to approximate nonlinear functions.
- - Jacobian Matrix: Captures the first-order derivatives in the Taylor expansion.
- - The Jacobian forms the basis for local linearization and stability analysis of nonlinear multi-D systems.