

STABILITY ANALYSIS WITH PERTURBATIONS

Detailed Explanation of Linearization via Taylor Expansion



OVERVIEW

- Understanding equilibrium and perturbation.
- Linearization via Taylor expansion.
- Stability conditions.



INTRODUCTION TO STABILITY

- We explore the stability of equilibrium points in a general ODE:
 - $dn/dt = f(n)$
- The equilibrium point n^* satisfies $f(n^*) = 0$.
- Introducing a perturbation variable x , such that $n = n^* + x$, allows us to study deviations from equilibrium.
- Substituting $n = n^* + x$ into the ODE gives:
 - $dx/dt = f(n^* + x)$.



LINEARIZATION VIA TAYLOR EXPANSION

- Expand $f(n^* + x)$ using a first-order Taylor expansion around n^* :
 - $f(n^* + x) \approx f(n^*) + f'(n^*)x + \text{higher-order terms.}$
- Since $f(n^*) = 0$ (equilibrium condition), this reduces to:
 - $dx/dt \approx f'(n^*)x.$
- The higher-order terms are negligible for small perturbations $x.$



CONDITIONS FOR STABILITY

- The equilibrium point n^* is:
 - **Stable:** If $f'(n^*) < 0$, perturbations decay over time ($x \rightarrow 0$).
 - **Unstable:** If $f'(n^*) > 0$, perturbations grow over time ($x \rightarrow \infty$).
- The magnitude of $f'(n^*)$ determines the rate of return to equilibrium.



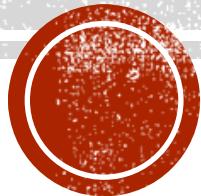
RATE OF RETURN TO EQUILIBRIUM

- The magnitude of $f(n^*)$ determines the rate of return to equilibrium.
- Larger $|f(n^*)|$ implies faster convergence (if stable) or divergence (if unstable).
- The return time τ is inversely proportional to $|f(n^*)|$:
 - $\tau \approx 1/|f(n^*)|$.



RELATIONSHIP BETWEEN TAYLOR EXPANSION AND THE JACOBIAN MATRIX

Understanding Local Linearization and Stability
Analysis



TAYLOR EXPANSION OVERVIEW

- For a multivariate function $f(x)$ where $x = [x_1, x_2 \dots x_n]^T$:
- $f(x) \approx f(x^*) + J(x^*)(x - x^*) + \text{Higher-order terms}$
- - $f(x^*)$: Value of the function at x^* .
- - $J(x^*)$: Jacobian matrix containing first partial derivatives.
- - Higher-order terms capture nonlinearities.



JACOBIAN MATRIX

- The Jacobian matrix $J(\mathbf{x})$ is defined as:

- $$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

- - $J(\mathbf{x})$ represents the linear part of the Taylor expansion.



LINEARIZATION USING JACOBIAN

- By truncating higher-order terms in the Taylor expansion, we approximate the system near x^* with a linear function as:
- $f(x) \approx f(x^*) + J(x^*)(x - x^*)$
- It follows that:

$$\frac{dx}{dt} = f(x) \approx J(x^*)(x - x^*)$$

where $J(x^*)$ governs the local behavior of a perturbation $(x - x^*)$ around the equilibrium x^*



APPLICATIONS IN STABILITY ANALYSIS

- The Jacobian matrix's eigenvalues determine stability:
 - - All eigenvalues have negative real parts: Stable.
 - - Any eigenvalue has a positive real part: Unstable.
 - - Purely imaginary eigenvalues: Oscillatory behavior.
- The Jacobian matrix provides the linear approximation to analyze the behavior of nonlinear systems near equilibrium.



KEY CONNECTION AND SUMMARY

- - Taylor Expansion: Framework to approximate nonlinear functions.
- - Jacobian Matrix: Captures the first-order derivatives in the Taylor expansion.
- - The Jacobian forms the basis for local linearization and stability analysis of nonlinear multi-D systems.

