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# Assignment 4



## 1

### 1.a

We know $V_i \cap \sum_{i \neq j} V_j = \{0\}$, so $\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 \leq \dim \mathbb{R}^3 = 3$, and $\dim V_\lambda \geq 1$ for any $\lambda \in \operatorname{Spec} T$, so $3 \leq \dim V_1 + \dim V_2 + \dim V_3$, thus $\dim(V_1 + V_2 + V_3) = \dim \mathbb{R}^3$, so $T$ is diagonalizeable.

### 1.b

Let $v$ be an eigenvector of $T$ with eigenvalue $\lambda$. $TSv = STv = S\lambda v = \lambda Sv$, so $Sv$ is an eigenvector of $T$. $T$ is diagonalizable, so $\dim V_1 + \dim V_2 + \dim V_3 = \dim V = 3$. That is, each eigenspace has dimension 1, so $Sv$ and $v$ must be in the same eigenspace. Thus $Sv = \lambda v$. This holds for any vectors $v_1, v_2, v_3$ in an eigenbasis for $T$, so $S$ is diagonalizable using the same eigenvectors as $T$, as desired.

### 1.c

Let $\beta = \{v_1, v_2, v_3\}$ be an eigenbasis for $T$. $[T]_\beta^\beta = [1]_\varepsilon^\beta [T]_\varepsilon^\varepsilon [1]_\beta^\varepsilon$ is clearly diagonal. We saw in part b) that $\beta$ is also an eigenbasis for $S$, so $[T]_\beta^\beta = [1]_\varepsilon^\beta [T]_\varepsilon^\varepsilon [1]_\beta^\varepsilon$ is also diagonal. Thus $Q = [1]_\varepsilon^\beta$ has the desired property.

### 1.c

Let $A_D = QAQ^{-1}$, $B_D = QBQ^{-1}$. Then $A = Q^{-1}A_DQ$, $B = Q^{-1}B_DQ$. $A + B = Q^{-1}A_DQ + Q^{-1}B_DQ = Q^{-1}(A_D + B_D)Q$. Since $A_D + B_D$ is diagonal with entries from $\operatorname{Spec} T + \operatorname{Spec} S$, and since $A + B$ corresponds to $T + S$, the eigenvalues in $\operatorname{Spec}(T + S)$ must also come from $\operatorname{Spec} T + \operatorname{Spec} S$, so $\operatorname{Spec}(T + S) \subset \operatorname{Spec} T + \operatorname{Spec} S$, as desired.

### 1.d

Using the same definition of $A_D$ and $B_D$ as above, $AB = Q^{-1}A_DQQ^{-1}B_DQ = Q^{-1}(A_DB_D)Q$. Since $A_DB_D$ is diagonal with entries from $\operatorname{Spec} T \cdot \operatorname{Spec} S$, and since $AB$ corresponds to $TS$, the eigenvalues in $\operatorname{Spec}(TS)$ must also come from $\operatorname{Spec} T \cdot \operatorname{Spec} S$, so $\operatorname{Spec}(TS) \subset \operatorname{Spec} T \cdot \operatorname{Spec} S$, as desired.

## 2

$\lambda - ST$ is invertible: $\lambda - TS$ is invertible, so $\ker(\lambda - TS) = \{0\}$. Thus there is no $v \in V$ such that $TSv = \lambda v$, so $\lambda$ is not an eigenvalue of $TS$. Let $0 \neq \lambda' \in \operatorname{Spec} ST$. Then there is some $0 \neq v \in V$ such that $STv = \lambda' v$. Then $TS(Tv) = T(STv) = T(\lambda' v) = \lambda' Tv$. Since $S(Tv) = \lambda' v \neq 0$, $Tv \neq 0$, thus $\lambda' \in \operatorname{Spec} TS$. But $\lambda \notin \operatorname{Spec} TS$, so $\lambda' \neq \lambda$. Thus $\lambda \notin \operatorname{Spec} ST$, so $\ker(\lambda - ST) = \{0\}$, and so $\lambda - ST$ is invertible, as desired.

$(\lambda - TS)^{-1} = \lambda^{-1} + \lambda^{-1}T(\lambda - ST)^{-1}S$:

$$\begin{aligned}
&(\lambda - TS)(\lambda^{-1} + \lambda^{-1}T(\lambda - ST)^{-1}S) \\
&= 1 + T(\lambda - ST)^{-1}S - \lambda^{-1}TS - \lambda^{-1}TST(\lambda - ST)^{-1}S \\
&= 1 + T((\lambda - ST)^{-1} - \lambda^{-1} - \lambda^{-1}ST(\lambda - ST)^{-1})S \\
&= 1 + T((1 - \lambda^{-1}ST)(\lambda - ST)^{-1} - \lambda^{-1})S \\
&= 1 + T(\lambda^{-1}(\lambda - ST)(\lambda - ST)^{-1} - \lambda^{-1})S \\
&= 1 + T(\lambda^{-1}1 - \lambda^{-1})S \\
&= 1 + T(0)S \\
&= 1
\end{aligned}$$

So the equality holds.

$ST$ and $TS$ have the same nonzero eigenvalues: Let $0 \neq \lambda \in \operatorname{Spec} TS$. Then there is some $v \in V$ such that $TSv = \lambda v$. $ST(Sv) = S(TSv) = \lambda Sv$, so $ST(Sv) = \lambda Sv$, thus $\lambda \in \operatorname{Spec} ST$. So $\operatorname{Spec} TS \setminus \{0\} \subseteq \operatorname{Spec} ST \setminus \{0\}$.

Now let $0 \neq \lambda \in \operatorname{Spec} ST$. Then there is some $v \in V$ such that $STv = \lambda v$. $TS(Tv) = T(STv) = \lambda Tv$, so $TS(Tv) = \lambda Tv$, thus $\lambda \in \operatorname{Spec} TS$. So $\operatorname{Spec} ST \setminus \{0\} \subseteq \operatorname{Spec} TS \setminus \{0\}$.

Thus $\operatorname{Spec} ST \setminus \{0\} = \operatorname{Spec} TS \setminus \{0\}$, so $TS$ and $ST$ have the same nonzero eigenvalues, as desired.

Example of $S$,$T$ such that 0 is an eigenvalue of $ST$ but 0 is not an eigenvalue of $TS$: Let $V = \{(a_n)_{n=1}^{\infty} \mid a_n \in \mathbb{N}\}$, and let $T$ and $S$ be the right and left shift operators, respectively. Then $TS = I$, so $0 - TS = -I = I$ is invertible, thus $0 \notin \operatorname{Spec} TS$, but $ST$ is not invertible, since $(1, 0, \ldots) \notin \operatorname{ran} ST$, and so $0 - ST$ is not invertible, thus $0 \notin \operatorname{Spec} ST$, as desired.

## 3

<u>$T^*$ is linear</u>: Let $f, g \in W^*$, $\lambda \in F$.

$$\begin{aligned} T^*(f + \lambda g) &= (f + \lambda g) \circ T \\ &= f \circ T + \lambda g \circ T \\ &= T^*(f) + \lambda T^*(g) \end{aligned}$$

So $T^*$ is linear.

<u>Transpose map is linear</u>: Let $S, T \in \mathcal{L}(V, W)$, $\lambda \in F$. Let $f \in W^*$.

$$\begin{aligned} (S + \lambda T)^*(f) &= f \circ (S + \lambda T) \\ &= f \circ S + f \circ \lambda T \\ &= S^*(f) + \lambda T^*(f) \end{aligned}$$

So the transpose map is linear.

### 3.a

<u>$i$ is linear</u>: Let $w_1, w_2 \in W$, $\lambda \in F$.

$$\begin{aligned} i(w_1 + \lambda w_2) &= w_1 + \lambda w_2 \\ &= i(w_1) + \lambda i(w_2) \end{aligned}$$

So $i$ is linear.

<u>$r$ is linear</u>: Let $f, g \in V^*$, $\lambda \in F$. If $f \in V^*$, let $f_W$ be $f$ restricted to $W$.

$$\begin{aligned} r(f + \lambda g) &= f_W + \lambda g_W \\ &= r(f) + \lambda r(g) \end{aligned}$$

So $r$ is linear.

### 3.b

$$\begin{aligned} \ker r &= \{f \in V^* \mid r(f) = 0\} \\ &= \{f \in V^* \mid f_W = 0\} \\ &= \{f \in V^* \mid f(w) = 0 \text{ for any } w \in W\} \\ &= W^\perp \end{aligned}$$

### 3.c

Let $f \in V^*$. $r(f) = f_W = f \circ i$, since $\operatorname{Im} i = W$, so $r = i^*$.

## 4

<u>$f$ is surjective $\implies$ $f^*$ is injective</u>: Let $g \in \ker f^*$. Then $f^*(g) = 0$, so $g \circ f(v) = 0$ for any $v \in V$. $f$ is surjective, so $g(w) = 0$ for any $w \in W$, so $g = 0$. Thus $\ker f^* = \{0\}$, so $f^*$ is injective.

<u>$f*$ is surjective $\implies$ $f$ is injective</u>: $f$ is injective, so it has an inverse map $f^{-1} \colon \operatorname{Im} f \longrightarrow V$. Fix some $v \in V^*$. If we define $g \in W^*$ by $g(x) = v(f^{-1}(x))$, then $f^*(g) = g \circ f$. $g \circ f(x) = v(f^{-1}(f(x))) = v(x)$ for any $x \in V$, so $f^*(g) = v$, thus $f^*$ is surjective.