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Assignment 1

MATH 236
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1

We know $0 < a < b$. a and b are nonzero, so ab is nonzero, and thus has an inverse with respect to multiplication, $b^{-1}a^{-1}$. $b^{-1}a^{-1} > 0$ since $ab > 0$, so since if $a < b$ and $c > 0$, $ac < bc$, we can conclude that since $0 < a < b$, $0 \cdot b^{-1}a^{-1} < a \cdot b^{-1}a^{-1} < b \cdot b^{-1}a^{-1}$. Multiplication is commutative over $\mathbb{R}$, so this means $0 < b^{-1} < a^{-1}$, or $0 < \frac{1}{b} < \frac{1}{a}$, as desired.

2

[Induction on n]: When $n = 2$, this is simply the triangle inequality. Suppose the claim holds for all $n \leq N$ for some N. Let $a_1, \ldots, a_{N+1} \in \mathbb{Q}$. Consider $|a_1 + \ldots + a_{N+1}|$, and denote $b = a_1 + \ldots + a_N$. Then $b \in \mathbb{Q}$, and $|a_1 + \ldots + a_{N+1}| = |b + a_{N+1}|$. By the hypothesis, $|b + a_{N+1}| \leq |b| + |a_{N+1}|$. We know $|b| = |a_1 + \ldots + a_N| \leq |a_1| + \ldots + |a_N|$, so

$$\begin{aligned} |a_1 + \ldots + a_{N+1}| &= |b + a_{N+1}| \\ &\leq |b| + |a_{N+1}| \\ &= |a_1 + \ldots + a_N| + |a_{N+1}| \\ &\leq |a_1| + \ldots + |a_N| + |a_{N+1}| \end{aligned}$$

So by induction, the claim holds.

3

3.a

(i). No. (ii). Yes, 7.

3.b

(i). Yes. (ii). Yes, 2.

3.c

(i). No. (ii). Yes, 90.

3.d

(i). No. (ii). Yes, 10.

3.e

(i). No. (ii). Yes, 1.

4

4.a

(i). No. (ii). Yes, -1.

4.b

(i). Yes. (ii). Yes, 1.

4.c

(i). No. (ii). No.

4.d

(i). No. (ii). No.

4.e

(i). No. (ii). Yes, 1.

5

Suppose not. Then there is some $x \in S \cap T$. $x \in S$, so $x \geq \inf S$. Similarly, $s \in T$, so $x \leq \sup T$. Thus we have $x \leq \sup T < \inf S \leq x$, so $x < x$, a contradiction, and so $S \cap T = \emptyset$, as desired.

6

Yes: $S = (0, 1)$, $T = \{0, 1\}$.

7

We know that if $a, b \in \mathbb{R}$ such that $a < b$, then there exists $q \in \mathbb{Q}$ with $a < q < b$.

Suppose S did have a maximum element, say s. $s \in \mathbb{Q} \subseteq \mathbb{R}$, so by the denseness of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that $q < r < \sqrt{2}$. We see by the definition of s that it contains r, so q is not the maximum element of S, a contradiction, and so S has no maximum element.

8

We consider two cases: $a < 1$, and $a \geq 1$.
Case $a < 1$: $a \neq 0$, so there must be some $n \in \mathbb{N}$ such that $an > 1$. Then $a > \frac{1}{n}$. $n \geq 1$, and $a < 1$, so we also have $n > a$. Thus $\frac{1}{n} < a < n$, as desired.
Case $a \geq 1$: Simply choose $n = \lceil (a) \rceil + 1$. $n \geq 2$, and $a \geq 1$, so $\frac{1}{n} \leq \frac{1}{2} < a$, so we have $\frac{1}{n} < a < n$, as desired.

9

$\left[\frac{1}{2}\left(a + \frac{2}{a}\right)\right]^2 > 2$:

$$\begin{aligned}
\left[\frac{1}{2}\left(a + \frac{2}{a}\right)\right]^2 &= \left[\frac{1}{2a}\left(a^2 + 2\right)\right]^2 \\
&= \frac{1}{4a^2}\left(a^2 + 2\right)^2 \\
&= \frac{1}{4a^2}\left(a^4 + 4a^2 + 4\right) \\
&= \frac{a^2}{4} + 1 + \frac{1}{a^2} \\
&> 2 + \frac{1}{a^2} \\
&> 2
\end{aligned}$$

$\frac{1}{2}\left(a + \frac{2}{a}\right) < a$:

$$\begin{aligned}
\frac{1}{2}\left(a + \frac{2}{a}\right) &= \frac{a}{2} + \frac{1}{a} \\
&= \frac{a}{2} + \frac{a}{a^2} \\
&< \frac{a}{2} + \frac{a}{2} \\
&= a
\end{aligned}$$

$\underline{\left[4/\left(a+\frac{2}{a}\right)\right]^2 < 2}$:

$$\begin{aligned}\left[4/\left(a+\frac{2}{a}\right)\right]^2 &= \left(\frac{4a}{a^2+2}\right)^2 \\ &= \frac{16a^2}{a^4+4a^2+4} \\ &< \frac{32}{4+8+4} \\ &= 2\end{aligned}$$

$\underline{a < 4/\left(a+\frac{2}{a}\right)}$:

$$\begin{aligned}\frac{4}{a+\frac{2}{a}} &= \frac{4}{a+\frac{2}{a}} \\ &= \frac{4a}{a^2+2} \\ &> \frac{4a}{2+2} \\ &= a\end{aligned}$$

If $a^2 > 2$, then $\left[\frac{1}{2}\left(a+\frac{2}{a}\right)\right]^2 > 2$, so it is not in S, a contradiction, since $0 \leq \frac{1}{2}\left(a+\frac{2}{a}\right) < a = \sup S$.
If $a^2 < 2$, then $0 \leq \left[4/\left(a+\frac{2}{a}\right)\right]^2 < 2$, so it is in S, a contradiction, since $\sup S = a < 4/\left(a+\frac{2}{a}\right)$
So it must be the case that $a^2 = 2$, as desired.