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Assignment 1

MATH 423
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1

($\Longrightarrow$): G decomposes into claws, so there is a collection $c_1, \ldots, c_n$ of claws such that $\bigcup c_i = G$. For any c_i, there are three leaves and a root. No root can be contained in any other claw, and no leaf can be the root of any other claw, since it is adjacent to its root which, as mentioned, is contained in exactly one claw. Thus there are no vertices in G which are both a root and a leaf in any one decomposition into claws. Thus we can bipartition G into those vertices that are a root in some claw, and those that are a leaf in some claw.

($\Longleftarrow$): Consider one of G's partite sets, X. Since every vertex in G has degree 3, deleting a vertex int X from G is like removing a claw from G. Every time we remove a vertex from X, no edges incident to any other vertices in X are affected, so after any number of such deletions, $deg(x) = 3$ for all $x \in X$. Once the final vertex in X has been deleted, we have successfully decomposed G into claws.

2

($\Longrightarrow$): Let H be a subgraph of G. H is bipartite since G is, and so the larger of its partite sets must contain at least half of its vertices.

($\Longleftarrow$): We prove the contrapositive: if G is not bipartite, then there exists $H \subseteq G$ with no independent set containing at least of of $V(H)$.

G is not bipartite, and so by the classification of bipartite graphs, contains an odd cycle. Take H to be this odd cycle, and denote it by $v_1 v_2 \ldots v_k v_1$. $\{v_1, v_3, \ldots, v_{k-2}\}$ is clearly a maximal independent set, but it only contains $\frac{k-1}{2} < \frac{k}{2}$ vertices, as desired.

3

Let $P = p_1 \ldots p_k$, $Q = q_1 \ldots q_k$. If $k > \frac{1}{2}|V(G)|$, then clearly P and Q have a common vertex, so assume $k \leq \frac{1}{2}|V(G)|$.

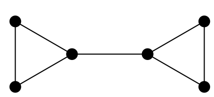
Assume for a contradiction that P and Q are disjoint. G is connected, so We know that there exists a path R from $p_{\lceil \frac{k}{2} \rceil}$ to $q_{\lceil \frac{k}{2} \rceil}$. We divide the rest of the proof into two cases:

$$\begin{aligned}
&(1): && R \cap (P \cup Q) = \{p_{\lceil \frac{k}{2} \rceil}, q_{\lceil \frac{k}{2} \rceil}\} \\
&(2): && R \cap (P \cup Q) \neq \{p_{\lceil \frac{k}{2} \rceil}, q_{\lceil \frac{k}{2} \rceil}\}
\end{aligned}$$

(1): Suppose WLOG that $p_i \in R$, where p_i is contained in the path from $p_{\lceil \frac{k}{2} \rceil}$ to p_k. Then we can construct the path $p_1 \dots p_i \dots q_{\lceil \frac{k}{2} \rceil} \dots q_k$, where the path from p_1 to p_i has length greater than $\frac{k}{2}$, and the path from $q_{\lceil \frac{k}{2} \rceil}$ to q_k has length at least $\frac{k}{2}$, leaving us with a path of length greater than k, a contradiction to the assumption that k is the length of a maximal path.

(2): R is disjoint from P and Q other than $p_{\lceil \frac{k}{2} \rceil}$ and $q_{\lceil \frac{k}{2} \rceil}$, so we can simply construct the path $p_1 \dots p_{\lceil \frac{k}{2} \rceil} \dots q_{\lceil \frac{k}{2} \rceil} \dots q_k$. If k is odd, then $p_1 \dots p_{\lceil \frac{k}{2} \rceil}$ and $q_{\lceil \frac{k}{2} \rceil} \dots q_k$ both have length $\frac{k+1}{2}$, so the total length of their combined path is at least $k+1$. If k is even, then $p_1 \dots p_{\lceil \frac{k}{2} \rceil}$ has length $\frac{k}{2}$, and $q_{\lceil \frac{k}{2} \rceil} \dots q_k$ has length $\frac{k}{2} + 1$, so the combined length of their paths is $k+1$. In both cases, this contradicts the assumption that k is the length of a maximal path.

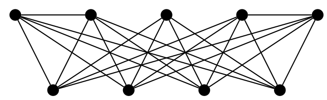
This statement is no longer true if the word "paths" is replaced by "cycles":



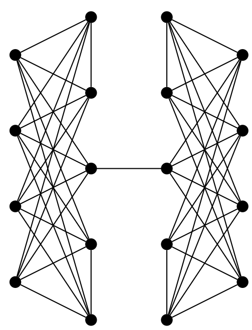
4

Suppose for a contradiction that there exists an even graph G with cut-edge e. G is even, and so each of its components has an Eulerian circuit. For any of these components, if we write this circuit such that the first edge traversed is e, like so: $v_1 e v_2 \dots v_n v_1$, then after removing e, we're left with $v_2 \dots v_n v_1$, a walk containing every vertex in that component. Since every u,v-walk contains a u,v-path, this component remains connected in $G - e$. In other words, G and $G - e$ have the same number of components, a contradiction to our assumption that e is a cut-edge. Thus every even graph has no cut-edge.

Given k, take $K_{2k+1,2k}$ and let X be the larger of its partite sets. Modify it by picking a vertex from X, and pairing off all the other vertices in X with an edge, like so:



Now every vertex except the one we chose has degree $2k+1$, so we can simply take two copies of this graph and connect the two vertices of degree $2k$:



and we're left with a $(2k+1)$-regular graph with a cut edge, as desired.

5

We know that since $G \cong \overline{G}$, for every vertex in G with degree k, there exists a vertex in G with degree $n(G)-1-k$. Define $V_k = \{v \in V(G) \mid deg(v) = k \text{ or } deg(v) = n(G)-1-k\}$. The collection of such sets forms a partition on $V(G)$. V_k is guaranteed to contain an even number of elements when $k \neq n(G)-1-k$, since each vertex of degree k must have a matching vertex of degree $n(G)-1-k$. So since there is a unique k such that $k = n(G)-1-k$, $k = \frac{n(G)-1}{2}$, and since $|V(G)|$ is odd, we have that $V_{\frac{n(G)-1}{2}}$ must contain an odd number of vertices (in other wwords, not 0). Thus there must be at least one element in $V_{\frac{n(G)-1}{2}}$, as desired.

6 Bonus

Consider a subgraph of D containing the vertices of an odd cycle in G and denote it $v_1 v_2 \ldots v_k v_1$. D is strongly connected, so there exists a path from v_i to v_{i+1} for all $i < k$, and from v_k to v_1. We can assume that every such path is either trivial (that is, it's just $v_i v_{i+1}$) or that it has odd length, as if it had even length then we could simply complete it with the edge $v_{i+1} v_i$ to obtain an odd cycle, and we would be done. So we have a collection of $k+1$ odd paths (where $k+1$ is odd). If we join these paths together end-to-end, we have an odd walk from v_1 to v_1. If v_1 is the only repeated vertex in the walk, then it is a cycle, and we're done. If there exists another repeated vertex, say v, then we can replace $v \ldots v$ with v in our $v_1 v_1$ walk. We can do this until v_1 is the only repeated vertex in our walk. If the resulting $v_1 v_1$ walk is now even, then one of the vv paths we just removed was odd, and we're done. Otherwise, our $v_1 v_1$ path is an odd cycle, as desired.