MATH 222 Assignment Four

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November 2, 2018

1

We define the following:

$$\begin{aligned}
\mathcal{U} &\equiv \text{Contains the pattern } LUCY \\
c_1 &\equiv \text{Contains the pattern } BROWN \\
c_2 &\equiv \text{Contains the pattern } JOHN \\
c_3 &\equiv \text{Contains the pattern } SMITH
\end{aligned}$$

Note also that the pattern $LUCY$ does not share any characters with any of $BROWN$, $JOHN$, or $SMITH$, hence our definition of $\mathcal{U}$.

We wish to find $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3})$. Using the Principle of Inclusion and Exclusion, we have

$$\begin{aligned}
N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}) = |\mathcal{U}| &- \left[N(c_1) + N(c_2) + N(c_3)\right] \\
&+ \left[N(c_1\ c_2) + N(c_1\ c_3) + N(c_2\ c_3)\right] \\
&- \left[N(c_1\ c_2\ c_3)\right]
\end{aligned}$$

We find the follwing values:

$$\begin{aligned}
|\mathcal{U}| &= 23! \\
N(c_1) &= 19! \\
N(c_2) &= 20! \\
N(c_3) &= 19! \\
N(c_1\ c_2) &= 0 \\
N(c_1\ c_3) &= 15! \\
N(c_2\ c_3) &= 0 \\
N(c_1\ c_2\ c_3) &= 0
\end{aligned}$$

So we get

$$
\begin{aligned}
N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}) &= 23! - [19! + 20! + 19!] + [0 + 15! + 0] - [0] \\
&= 23! - 20! - 2 \cdot 19! + 15!
\end{aligned}
$$

2

Let walls 1, 2, 3, and 4 be the East, North, West, and South walls, respectively.
Let corner ij be the corner connecting wall i and wall j.

We define the following:

$$
\begin{aligned}
c_1 &\equiv \text{Corner 12 is the same colour} \\
c_2 &\equiv \text{Corner 23 is the same colour} \\
c_3 &\equiv \text{Corner 34 is the same colour} \\
c_4 &\equiv \text{Corner 41 is the same colour}
\end{aligned}
$$

We wish to count $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$.

$$
\begin{aligned}
N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}) = |\mathcal{U}| &- \big[N(c_1) + N(c_2) + N(c_3) + N(c_4)\big] \\
&+ \big[N(c_1\,c_2) + N(c_1\,c_3) + N(c_1\,c_4) + N(c_2\,c_3) + N(c_2\,c_4) + N(c_3\,c_4)\big] \\
&- \big[N(c_1\,c_2\,c_3) + N(c_1\,c_2\,c_4) + N(c_1\,c_3\,c_4) + N(c_2\,c_2\,c_4)\big] \\
&+ \big[N(c_1\,c_2\,c_3\,c_4)\big] \\
= \ 6^4 &- \big[6^3 + 6^3 + 6^3 + 6^3\big] \\
&+ \big[6^2 + 6^2 + 6^2 + 6^2 + 6^2 + 6^2\big] \\
&- \big[6 + 6 + 6 + 6\big] \\
&+ \big[6\big] \\
= 630 &
\end{aligned}
$$

3

We define the following:

$$
\begin{aligned}
x_1 &= \text{Number of red marbles} \\
x_2 &= \text{Number of blue marbles} \\
x_3 &= \text{Number of white marbles} \\
x_4 &= \text{Number of green marbles}
\end{aligned}
$$

And so we can count the number of ways we can select the marbles by counting the number of solutions to the following system:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 9 \\ 0 \leq x_i \leq 3, \; i = 1, 2, 3, 4 \end{cases}$$

We know that

$$\mathcal{U} = \begin{cases} x_1 + x_2 + x_3 + x_4 = 9 \\ x_i \geq 0, \; i = 1, 2, 3, 4 \end{cases}$$

So

$$|\mathcal{U}| = \binom{4 + 9 - 1}{9} = \binom{12}{9}$$

We define $c_i \equiv x_i \geq 4$. We can now find $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4})$ using the Principle of Inclusion and Exclusion. First we notice that when three or more of the conditions are true there are 0 solutions, so we need only consider singletons and pairs. We also notice that $N(c_i) = N(c_j)$ for all valid values of i and j. Similarly, we notice that $N(c_i\ c_j) = N(c_k\ c_l)$ for all valid values of i, j, k, and l. So overall, we get

$$\begin{aligned} N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = \binom{12}{9} &- \binom{4}{1}\left[N(c_1)\right] \\ &+ \binom{4}{2}\left[N(c_1\ c_2)\right] \end{aligned}$$

We find $N(c_1)$:

$$\begin{aligned} x_1' = & x_1 - 4 \\ & \begin{cases} x_1 + x_2 + x_3 + x_4 = 9 \\ x_i \geq 0, \; i = 1, 2, 3, 4 \end{cases} \\ = & \begin{cases} x_1' + 4 + x_2 + x_3 + x_4 = 9 \\ x_1' \geq 0 \\ x_i \geq 0, \; i = 2, 3, 4 \end{cases} \\ = & \begin{cases} x_1' + x_2 + x_3 + x_4 = 5 \\ x_1' \geq 0 \\ x_i \geq 0, \; i = 2, 3, 4 \end{cases} \end{aligned}$$

So $N(c_1) = \binom{4+5-1}{5} = \binom{8}{5}$.

Similarly, we find $N(c_1\ c_2)$:

$$\begin{aligned}
x_1' =& x_1 - 4\\
x_2' =& x_2 - 4\\
&\begin{cases} x_1 + x_2 + x_3 + x_4 = 9 \\ x_i \geq 0 \end{cases}\\
=&\begin{cases} x_1' + 4 + x_2' + 4 + x_3 + x_4 = 9 \\ x_i' \geq 0,\ i = 1, 2 \\ x_i \geq 0,\ i = 3, 4 \end{cases}\\
=&\begin{cases} x_1' + x_2' + x_3 + x_4 = 1 \\ x_i' \geq 0,\ i = 1, 2 \\ x_i \geq 0,\ i = 3, 4 \end{cases}
\end{aligned}$$

So $N(c_1\ c_2) = \binom{4+1-1}{1} = \binom{4}{1}$. We can now find $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4})$ using our above equation:

$$\begin{aligned}
N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4}) = |\mathcal{U}| - \binom{4}{1}\left[\binom{8}{5}\right]&\\
+ \binom{4}{2}\left[\binom{4}{1}\right]&\\
=\ 20&
\end{aligned}$$

4

4.a

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 17 \\ x_i \geq -1,\ i = 1, 2, 3, 4 \end{cases}$$

We define $x_i' = x_i + 1$, and so

$$\begin{aligned}
&\begin{cases} x_1' - 1 + x_2' - 1 + x_3' - 1 + x_4' - 1 = 17 \\ x_i' \geq 0,\ i = 1, 2, 3, 4 \end{cases}\\
\Longrightarrow &\begin{cases} x_1' + x_2' + x_3' + x_4' = 21 \\ x_i' \geq 0,\ i = 1, 2, 3, 4 \end{cases}
\end{aligned}$$

So there are $\binom{4+21-1}{21} = \binom{24}{21}$ integer solutions.

4.b

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 17 \\ -1 \leq x_i \leq 5,\ i = 1, 2, 3, 4 \end{cases}$$

We define $x'_x - 1 + 1$, and so

$$
\begin{aligned}
&\begin{cases} x'_1 - 1 + x'_2 - 1 + x'_3 - 1 + x'_4 - 1 = 17 \\ 0 \leq x'_i \leq 6, \ i = 1,2,3,4 \end{cases} \\
\implies &\begin{cases} x'_1 + x'_2 + x'_3 + x'_4 = 21 \\ 0 \leq x'_i \leq 6, \ i = 1,2,3,4 \end{cases} \\
= &\begin{cases} x'_1 + x'_2 + x'_3 + x'_4 = 21 \\ x'_i \geq 0, \ i = 1,2,3,4 \end{cases} - \begin{cases} x'_1 + x'_2 + x'_3 + x'_4 = 21 \\ x'_i \geq 7, \ i = 1,2,3,4 \end{cases} \\
= &\binom{24}{21} - \begin{cases} x'_1 + x'_2 + x'_3 + x'_4 = 21 \\ \text{at least one } x_i \geq 7 \end{cases}
\end{aligned}
$$

We define $c_i \equiv x_i \geq 7$, and we wish to find $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4})$. From here, we use the same strategy we used for question 3, and through the Principle of Inclusion and Exclusion, we get

$$
\begin{aligned}
N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4}) = &\binom{24}{21} - \binom{4}{1}\left[\binom{17}{14}\right] \\
&+ \binom{4}{2}\left[\binom{10}{7}\right] \\
&- \binom{4}{1}\left[\binom{3}{0}\right] \\
= &\ 20
\end{aligned}
$$

5

We define the following:

$$
\begin{aligned}
c_1 &\equiv \text{divisible by } 3 \\
c_2 &\equiv \text{divisible by } 5 \\
c_3 &\equiv \text{divisible by } 7
\end{aligned}
$$

We wish to find $N(\overline{\overline{c_1}\ \overline{c_2}}\ \overline{c_3})$.

$$\begin{aligned}
N(c_1) &= \left\lfloor \frac{2018}{3} \right\rfloor \\
N(c_2) &= \left\lfloor \frac{2018}{5} \right\rfloor \\
N(c_1\ c_2) &= \left\lfloor \frac{2018}{15} \right\rfloor \\
N(c_1\ c_3) &= \left\lfloor \frac{2018}{21} \right\rfloor \\
N(c_2\ c_3) &= \left\lfloor \frac{2018}{35} \right\rfloor \\
N(c_1\ c_2\ c_3) &= \left\lfloor \frac{2018}{105} \right\rfloor
\end{aligned}$$

So

$$\begin{aligned}
N(\overline{\overline{c_1}\ \overline{c_2}}\ \overline{c_3}) =& N(c_1) + N(c_2) \\
& - \Big[N(c_1\ c_2) + N(c_1\ c_3) + N(c_2\ c_3) \Big] \\
& + N(c_1\ c_2\ c_3) \\
=& \left\lfloor \frac{2018}{3} \right\rfloor + \left\lfloor \frac{2018}{5} \right\rfloor \\
& - \left[\left\lfloor \frac{2018}{15} \right\rfloor + \left\lfloor \frac{2018}{21} \right\rfloor + \left\lfloor \frac{2018}{35} \right\rfloor \right] \\
& + \left\lfloor \frac{2018}{105} \right\rfloor
\end{aligned}$$

6

6.a

Recall the following:

$$\begin{aligned}
\phi(n) &= n - \sum_{1 \le i \le t} \frac{n}{p_i} + \sum_{1 \le i \le j \le t} \frac{n}{p_i p_j} - \sum_{1 \le i \le j \le k \le t} \frac{n}{p_i p_j p_k} + \cdots + (-1)^n \frac{n}{p_1 p_2 \ldots p_t} \\
&= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_t}\right)
\end{aligned}$$

We now note the following:

$$\phi(n^m) = n^m - \sum_{1\leq i\leq t} \frac{n^m}{p_i} + \sum_{1\leq i\leq j\leq t} \frac{n^m}{p_i p_j} - \sum_{1\leq i\leq j\leq m\leq t} \frac{n^m}{p_i p_j p_m} + \cdots + (-1)^{n^m} \frac{n^m}{p_1 p_2 \ldots p_t}$$

Since n^m is composed strictly of multiples of n, it will share all of its primes with n. In other words, n and n^m will have the same number of distinct prime factors. So

$$\begin{aligned}
\phi(n^m) &= n^m \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_t}\right) \\
&= n^{m-1}\left[n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_t}\right)\right] \\
&= n^{m-1}\left[\phi(n)\right] \\
&= n^{m-1}\phi(n)
\end{aligned}$$

6.b

When considering the following two equations:

$$n = \frac{32}{1 - \frac{1}{p_1}}$$

and

$$n = \frac{32}{(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})}$$

the first two solutions found were $n = 64$ and $n = 80$. Upon closer inspection, 64 works, since $\lfloor \frac{64}{2} \rfloor = 32$ and 80 works, since $80 - \left[\lfloor \frac{80}{2} \rfloor + \lfloor \frac{80}{5} \rfloor - \lfloor \frac{80}{10} \rfloor\right] = 32$.

7

To prove this identity, we will count the number of ways we can paint m walls with n colours.

First, we will select one of n colours, for the first wall, then do the same for the second and onward to the m^{th}. We count n^m possible ways.

$S(m, k)$ is the number of ways each of m colours can be distributed into k nonempty subsets.

$\binom{n}{k} n! = P(n, k)$ is the number of ways k elements can be selected from n elements and assign each to one of the k subsets.

For the first iteration of our sum (where $k = 1$ we make one subset of all of the walls. Out of n colours, we choose one to assign to the one subset.
We continue this for every value of k. Since we use a different number of colours each iteration, we know that we are not overcounting. So we end up with all possible permutations of m walls being painted with n colours. We count $\sum_{k=1}^{n} \binom{n}{k}(k!)S(m,k)$ possible ways.

Thus, $n^m = \sum_{k=1}^{n} \binom{n}{k}(k!)S(m,k)$.

8 Bonus

We will distribute distinct marbles numbered $1, 2, \ldots, n$ into distinct urns numbered $1, 2, \ldots, n$ without placing marble i into urn i.

Place marble 1 into an urn other than urn 1 and note its number, i. There are $(n-1)$ ways to do this. Now take the i^{th} marble. You have two options: place it into urn 1, or place it into some other urn. If you place it into urn 1, then marble 1, urn 1, marble i and urn i are now no longer being considered, and so the remaining problem is d_{n-2}. If you place it into some other urn, then there are now $n-1$ marbles and $n-1$ urns, including marble i and urn 1. Since we specifically chose not to put marble i into urn 1, this situation is the exact same as d_{n-1}, in that each marble has exactly one urn into which it cannot be put. Thus $d_n = (n-1)(d_{n-1} + d_{n-2})$.