Oliver Tonnesen
V00885732

Assignment 3

MATH 311
November 8, 2019

1

<u>T is linear</u>: Let $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in \mathbb{R}^3$.

$$\begin{aligned}
Tv + Tw &= T(v_1, v_2, v_3) + T(w_1, w_2, w_3) \\
&= (v_1 + v_2 + 2v_3x + v_1x^2) + (w_1 + w_2 + 2w_3x + w_1x^2) \\
&= (v_1 + w_1) + (v_2 + w_2) + 2(v_3 + w_3)x + (v_1 + w_1)x^2 \\
&= T(v + w)
\end{aligned}$$

Let $v \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$.

$$\begin{aligned}
T(\lambda v) &= T(\lambda v_1, \lambda v_2, \lambda v_3) \\
&= \lambda v_1 + \lambda v_2 + 2\lambda v_3 x + \lambda v_1 x^2 \\
&= \lambda(v_1 + v_2 + 2v_3x + v_1x^2) \\
&= \lambda Tv
\end{aligned}$$

So T is linear.

<u>α is a basis for $\mathbb{R}^3$</u>: First we show that α is independent in $\mathbb{R}^3$, then we show that it spans $\mathbb{R}^3$. Let $a, b, c \in \mathbb{R}^3$, such that $a(1,1,0) + b(1,1,1) + c(0,1,1) = (0,0,0)$. Then $a + b = 0$, $a + b + c = 0$, and $b + c = 0$. So $a = -b$, and $c = -b$. Then we have

$$\begin{aligned}
0 &= a + b + c \\
&= -b + b - b \\
&= -b
\end{aligned}$$

so $b = 0$. Then since $a + b = 0$ and $b + c = 0$, $a = 0$ and $c = 0$, so $a = b = c = 0$, thus α is independent in $\mathbb{R}^3$. Let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. A straightforward computation gives us $v = (v_2 - v_3)(1,1,0) + (v_1 - v_2 + v_3)(1,1,1) + (v_2 - v_1)(0,1,1)$, so α spans $\mathbb{R}^3$, and thus α is an ordered basis for $\mathbb{R}^3$.

<u>Compute $[T]_\alpha^\beta$ and $[T^{-1}]_\beta^\alpha$:</u>

$$\begin{aligned}
T(1,1,0) &= 2 + x^2 \\
T(1,1,1) &= 2 + 2x + x^2 \\
T(0,1,1) &= 1 + 2x
\end{aligned}$$

$$[T(1,1,0)]_\beta = (2,0,1)$$
$$[T(1,1,1)]_\beta = (2,2,1)$$
$$[T(0,1,1)]_\beta = (1,2,0)$$

So $[T]_\alpha^\beta = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix}$. We know that $[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}$, so $[T^{-1}]_\beta^\alpha = \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ -1 & \frac{1}{2} & 2 \\ 1 & 0 & -2 \end{pmatrix}$

2

Let $\dim W = k$. $W \subset V$, so $k < n$. Let $\gamma = \{w_1, \dots, w_k\}$ be an ordered basis for W, $\alpha = \{v_1, \dots, v_n\}$ a basis for V. By the replacement theorem, there are $n-k$ vectors in α which, when added to γ, make it a basis for V. Suppose WLOG that these vectors are $v_{k+1}, \dots, v_n$, and let $\beta = \{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$. $T(W) \subset W$, so we can write any vector $T(w)$, where $w \in W$, as a linear combination of vectors in γ. Specifically, we can write it <u>without</u> any vectors we added from α. Thus we construct $[T]_\beta^\beta$.

The first k columns are $[T(w_i)]_\beta$, $1 \le i \le k$. As we mentioned, $T(w_i)$ can be written as a linear combination of vectors in γ without any of the vectors we added from α. So the bottom $n-k$ entries of each $[T(w_i)]_\beta$ are all 0. In other words, $[T]_\beta^\beta$ has a $n-k$-by-$n-k$ matrix of zeros in the bottom left. $[T]_\beta^\beta$ is n-by-n, so the rest of A, B, and C are k-by-k, k-by-$n-k$, and $n-k$-by-$n-k$ matrices, as desired.

3

V is finite dimensional, so we have that $\operatorname{Nullity} T + \operatorname{Rank} T = \dim V$.

Note that $\operatorname{ran}(T^{k+1}) \subseteq \operatorname{ran}(T^k)$, since $T^{k+1}(v) = T^k(T(v))$. If we take k to be at least n, we see that $\operatorname{ran}(T^{k+1}) = \operatorname{ran}(T^k)$ (note that this can be the case for smaller values of k). Let $v \in \operatorname{ran}(T^k) \cap \ker(T^k)$. Then $v = T^k(w)$ for some $w \in V$. $v \in \operatorname{ran}(T^k) \cap \ker(T^k)$, so $T^k(v) = 0$. $T^k(v) = T^k(T^k(w)) = T^{2k}(w) = 0$. $k \ge n$, so $\operatorname{ran}(T^k) = \operatorname{ran}(T^{2k})$, and so $T^{2k}(w) \in \operatorname{ran}(T^k)$, and $T^{2k}(w) = 0$, so $w \in \ker(T^{2k})$. $\ker(T^{2k}) = \ker(T^k)$ by a similar argument, so $w \in \ker(T^k)$. Thus since $v = T^k(w)$, $v = 0$. v was arbitrarily chosen from $\operatorname{ran}(T^k) \cap \ker(T^k)$, and so $\operatorname{ran}(T^k) \cap \ker(T^k) = \{0\}$.

By the dimension theorem, we have $\dim \operatorname{ran}(T^k) + \dim \ker(T^k) = \dim V$, and $\operatorname{ran}(T^k) \cap \ker(T^k) = \{0\}$, so since $\operatorname{ran}(T^k) + \ker(T^k) = V$, we have that

$V = \text{ran}(T^k) \oplus \ker(T^k)$, as desired.

4

4.a

ST is invertible, so ST is bijective, thus S is surjective and T is injective. Similarly, TS is bijective, and so T is surjective and S is injective. Thus S and T are both bijective and hence invertible.

4.b

Let $V = P(\mathbb{R})$. Define $T, S : P(\mathbb{R}) \longrightarrow P(\mathbb{R})$ by $Tf(x) \longmapsto \int_0^x f(x)dx$, $Sf(x) \longmapsto \frac{d}{dx}f(x)$.

T is not surjective, since $1 \in P(\mathbb{R})$ is not in the range of T. S is not injective, since $0 \neq 1 \in P(\mathbb{R})$ is in the kernel of S. Hence neither S nor T is invertible.

Let $f(x) = \sum_i a_i x^i \in P(\mathbb{R})$. $T(\sum_i a_i x^i) = \sum_i \frac{a_i x^{i+1}}{i+1}$, and $S(\sum_i \frac{a_i x^{i+1}}{i+1}) = \sum_i (i+1)\frac{a_i x^i}{i+1} = f(x)$, so $ST = 1_{P(\mathbb{R})}$, and is thus invertible, as desired.