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V00885732 **Assignment 4** November 7, 2019

1

We know that $\delta(G) < n(G)$, and if $\delta(G) = n(G) - 1$, then $G = K_{n(G)}$, so $\kappa(G)$ is defined to be $n(G) - 1$. So we show that if $\delta(G) = n(G) - 2$, then $\kappa(G) = n(G) - 2$:

Assume not for a contradiction that $\kappa(G) < n(G) - 2$. Then there exists $S \subseteq V(G)$ with $|S| < n(G) - 2$ such that $G - S$ is disconnected. For any $v \in V(G)$, there is at most one vertex $u \in V(G)$ not adjacent to v (not that u is adjacent to every vertex other than v, otherwise $deg(u) \le n(G) - 3 < \delta(G)$). So G can be obtained by removing at most one edge incident to any vertex in $K_{n(G)}$. Consider $v, u \in V(G)$, $v \not\sim u$. To disconnect v and u, we must remove all $n(G) - 2$ edges incident to v or u, and thus we must remove exactly $n(G) - 2$ vertices (all vertices other than v and u). So $\kappa(G) = n(G) - 2$.

2

($\Longrightarrow$): Let F be the cut $[S, \overline{S}]$. For any cycle in G, if we follow it starting from some vertex v in S, whenever it crosses to $\overline{S}$, it must cross back in order to return to v, so F contains an even number of the cycle's edges.

($\Longleftarrow$): Take $G - F$ and contract each of its connected components to a single vertex, and connect any two vertices whose original components shared an edge in F. Call this new graph G'. We show that G' is bipartite:

Let C be an arbitrary cycle in G', v a vertex in C. v is incident to two edges in F, e_1 and e_2. If e_1 and e_2 are incident to u_1 and u_2 in G, respectively, then (since u_1 and u_2 are in the same component) there is a path of the form $e_1 u_1 \sim \ldots \sim u_2 e_2$ in G. But any cycle in G has an even number of its edges in F, so the same must be true of C in G'. Since G' only has edges from F, C must then be an even cycle, and so G' is bipartite. So let S be the set of vertices in those components of $G - F$ corresponding to the vertices in one of G''s partite sets. Then $[S, \overline{S}]$ is an edge cut corresponding to F.

3

If $G - v$ is 2-connected, then we're done. So assume $G - v$ is not 2-connected. Then $G - v$ has a cut vertex, say c. Let C be a block of $G - v$ containing c. v has a neighbor in C, since if it didn't, c would still be a cut vertex, and so G would not be 2-connected. If v's neighbor in C was c, then c would again still be a cut vertex. So v has a neighbor in $C - c$, say u. C is a block, and thus would not be disconnected by the removal of u. So u is not a cut vertex in $G - v$, and so $G - v - u$ is connected, as desired.

4

4.i

We decompose G into ears. If the last ear added to make G has more than one edge, then it contains a vertex of degree 2, and we're done. If the last ear has only one edge, then $G - e$ has an ear decomposition, and thus by Whitney's theorem is 2-connected. But $G - e$ is not 2-connected by assumption, so the last ear added must have a vertex of degree 2. No vertex added in any previous ear can have degree less than 2, so $\delta(G) = 2$, as desired.

4.ii

[Induction on $n(G)$]: The only such graph on 4 vertices is C_4, for which the claim clearly holds.

Assume the claim holds for all $n(G) \leq n$. Let G be a graph on $n + 1$ vertices. Consider an ear decomposition $P_0, \ldots, P_k$ of G. If $k = 0$, then $G = C_{n+1}$, and $m(G) = n+1$. $n+1 \leq 2(n+1)-4$ is true for $n > 4$, so we're done. Otherwise, if $k > 0$, then remove P_k from G. If P_k has l vertices, then it has $l + 1$ edges. $G - P_k$ is minimally 2-connected, since it is 2-connected, and if $G - P_k - e$ was 2-connected, then $G - e$ would be 2-connected (removing P_k doesn't make G "more" connected). So by the induction hypothesis, the claim holds for $G - P_k$. That is, $m(G - P_k) \leq 2n(G - P_k) - 4$. We also have that $m(G - P_k) = m(G) - (l + 1)$ and $n(G - P_k) = n(G) - l$. Together this gives us:

$$\begin{aligned} m(G) - (l+1) &\leq 2(n(G) - l) - 4 \\ m(G) - l - 1 &\leq 2n(G) - 2l - 4 \\ m(G) &\leq 2n(G) - l - 3 \end{aligned}$$

As we saw in question 4.ii), $\delta(G) = 2$, so if we simply use a decomposition for G whose final ear P_k is just a vertex of degree 2, then we get $l = 1$, giving us $m(G) \leq 2n(G) - l - 3 = 2n(G) - 4$. Thus by induction the claim holds.

5

5.i

We prove the contrapositive: if $k > 5$, then there exists a pair of nonintersecting odd cycles. Any colouring of G has at least 6 colours. Consider the vertices of G coloured 4, 5, or 6. If the subgraph induced by these vertices was 2-colourable, then that colouring along with the original colouring for the vertices coloured 1, 2, or 3 would be a 5-colouring for G, so the subgraph is not 2-colourable, hence not bipartite, and so it contains an odd cycle. No vertex in this set is coloured 1, 2, or 3, so in particular no vertex in the odd cycle is coloured 1, 2, or 3. We can use the same argument to show the subgraph induced by the set of vertices coloured 1, 2, or 3 contains an odd cycle. Since these two sets of vertices are disjoint, the two odd cycles are disjoint, as desired.

5.ii

Consider some k-colouring of G. Let C_i, C_j be the subgraphs of G containing those vertices coloured i and j, respectively The graph induced by C_i and C_j is bipartite, since no two vertices sharing a colour are adjacent. No vertex in this graph has degree 3 or more, otherwise it would contain a claw (and hence so would G). So any vertex in this graph has degree 1 or 2, and is thus either in a path or an even cycle. If the number of vertices in C_i and C_j differ by more than one, then there exist more odd paths beginning and ending in WLOG C_i than there are beginning and ending in C_j. If we recolour them such that the number of odd paths beginning and ending in C_i and C_j differs by at most one (by choosing a number of odd paths and simply colouring all vertices coloured i with j and colouring all vertices coloured j with i), then the number of vertices coloured i and j in the new colouring will differ by at most one. This recolouring does not affect any other pairs of colours, and so this process can be done between all pairs of colour classes to obtain a colouring in which the size of any two colour classes differs by at most one, as desired.

6 Bonus

Let $v_1, \dots, v_n$ be some ordering of the vertices of G, and let k be the number of colours used in a greedy colouring of G. We want to find a clique of size k in G. We know G has a clique, and that it has highest colour k. So let

C be a maximum clique in G, and let its vertices be $\{c_1, \ldots, c_l\}$. We show that $l = k$. If $l < k$, then since c_l has colour k, every vertex in C must be adjacent to a vertex with colour $k - l - 1$. If each vertex in C were adjacent to the same vertex of this colour, we could add it to C to get a larger clique, but C is maximum so no such vertex exists. Let a have colour $k - l - 1$ be adjacent to as many vertices in C as possible, and let c_i be some vertex in C that is not adjacent to a. c_i is adjacent to some vertex with colour $k - l - 1$, say b. Note that a and b are coloured the same, and hence are nonadjacent. Neither a nor b are adjacent to all vertices in C, but a is adjacent to more vertices in C than b is, so choose some vertex in C that is adjacent to a but not b and that is not c_i, call it c_j. Then we have four vertices: a, c_j, c_i, b. a and b are the same colour and thus not adjacent, c_i was chosen to be not adjacent to a, c_j was chosen to be not adjacent to b, and $a \sim c_j \sim c_i \sim b$, so G contains P_4 as an induced subgraph, a contradiction, and so our assumption that $l < k$ was false, so $l = k$, and thus the largest clique in G has the same size as the number of colours used in a greedy colouring of, hence the greedy colouring was optimal.