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Assignment 5

CSC 349A
November 19, 2019

1

1.a

$$
\begin{aligned}
L_0(x) &= \frac{\left(x-\frac{\pi}{3}\right)\left(x-\frac{2\pi}{3}\right)(x-\pi)}{\left(0-\frac{\pi}{3}\right)\left(0-\frac{2\pi}{3}\right)(0-\pi)} \\
&= \frac{-9x^3+18\pi x^2-11\pi^2 x+2\pi^3}{2\pi^2} \\
L_1(x) &= \frac{(x-0)\left(x-\frac{2\pi}{3}\right)(x-\pi)}{\left(\frac{\pi}{3}-0\right)\left(\frac{\pi}{3}-\frac{2\pi}{3}\right)\left(\frac{\pi}{3}-\pi\right)} \\
&= \frac{27x^3-45\pi x^2+18\pi^2 x}{2\pi^2} \\
L_2(x) &= \frac{(x-0)\left(x-\frac{\pi}{3}\right)(x-\pi)}{\left(\frac{2\pi}{3}-0\right)\left(\frac{2\pi}{3}-\frac{\pi}{3}\right)\left(\frac{2\pi}{3}-\pi\right)} \\
&= \frac{-27x^3+36\pi x^2-9\pi^2 x}{2\pi^2} \\
L_0(x) &= \frac{(x-0)\left(x-\frac{\pi}{3}\right)\left(x-\frac{2\pi}{3}\right)}{(\pi-0)\left(\pi-\frac{\pi}{3}\right)\left(\pi-\frac{2\pi}{3}\right)} \\
&= \frac{9x^3-9\pi x^2+2\pi^2 x}{2\pi^2}
\end{aligned}
$$

So we get

$$
\begin{aligned}
P(x) &= \sum_{i=0}^{3} L_i(x) f(x_i) \\
&= L_0(x)\cdot 0 + L_1(x)\cdot 0.75 + L_2(x)\cdot 0.75 + L_3(x)\cdot 0 \\
&= 0.75\left(L_1(x)+L_2(x)\right) \\
&= \frac{0.75}{2\pi^3}\left(-9\pi x^2+9\pi^2 x\right) \\
&= \frac{-27\pi x^2+27\pi^2 x}{8\pi^3} \\
&= \frac{27x(\pi-x)}{8\pi^2}
\end{aligned}
$$

1.b

We execute the commands below

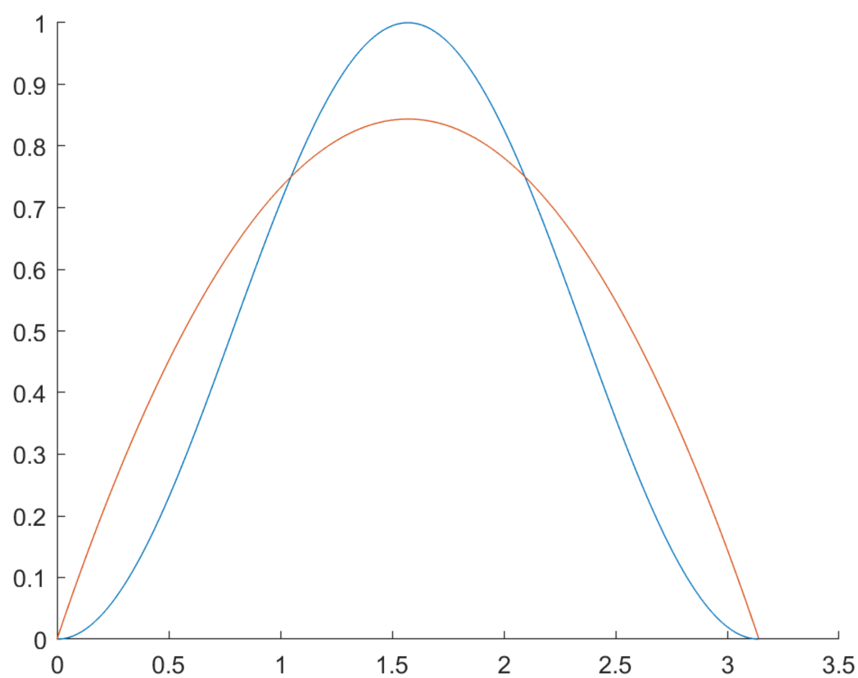
```
hold on
x=[0:0.01:pi]

y=sin(x).^2

P=27.*x.*(pi-x)/(8*pi.^2)

plot(x, y);
plot(x,P)
diary off
```

to get the following plot (the blue curve is $\sin^2 x$, the red curve is the polynomial interpolation):



2

We have the following twelve conditions:

$$
\begin{aligned}
&1\colon\ S_0(0) = \sin^2(0) && 2\colon\ S_1\left(\frac{\pi}{3}\right) = \sin^2\left(\frac{\pi}{3}\right) \\
&3\colon\ S_2\left(\frac{2\pi}{3}\right) = \sin^2\left(\frac{2\pi}{3}\right) && 4\colon\ S_2(\pi) = \sin^2(\pi) \\
&5\colon\ S_1\left(\frac{\pi}{3}\right) = S_0\left(\frac{\pi}{3}\right) && 6\colon\ S_2\left(\frac{2\pi}{3}\right) = S_1\left(\frac{2\pi}{3}\right) \\
&7\colon\ S_1'\left(\frac{\pi}{3}\right) = S_0'\left(\frac{\pi}{3}\right) && 8\colon\ S_2'\left(\frac{2\pi}{3}\right) = S_1'\left(\frac{2\pi}{3}\right) \\
&9\colon\ S_1''\left(\frac{\pi}{3}\right) = S_0''\left(\frac{\pi}{3}\right) && 10\colon\ S_2''\left(\frac{\pi}{3}\right) = S_1''\left(\frac{\pi}{3}\right) \\
&11\colon\ S_0'(0) = 2\sin 0 \cos 0 && 12\colon\ S_2'(\pi) = 2\sin\pi\cos\pi
\end{aligned}
$$

giving us twelve equations:

$$
\begin{aligned}
&1\colon\ a_0 = 0 && 2\colon\ a_1 = \frac{3}{4} \\
&3\colon\ a_2 = \frac{3}{4} && 4\colon\ a_2 + b_2\left(\frac{\pi}{3}\right) + c_2\left(\frac{\pi}{3}\right)^2 + d_2\left(\frac{\pi}{3}\right)^3 \\
&5\colon\ a_1 - a_0 - b_0\left(\frac{\pi}{3}\right) - c_0\left(\frac{\pi}{3}\right)^2 - d_0\left(\frac{\pi}{3}\right)^3 = 0 && 6\colon\ a_2 - a_1 - b_1\left(\frac{\pi}{3}\right) - c_1\left(\frac{\pi}{3}\right)^2 - d_1\left(\frac{\pi}{3}\right)^3 = 0 \\
&7\colon\ b_1 - b_0 - 2c_0\left(\frac{\pi}{3}\right) - 3d_0\left(\frac{\pi}{3}\right)^2 = 0 && 8\colon\ b_2 - b_1 - 2c_1\left(\frac{\pi}{3}\right) - 3d_1\left(\frac{\pi}{3}\right)^2 = 0 \\
&9\colon\ 2c_1 - 2c_0 - 3d_0\left(\frac{\pi}{3}\right) = 0 && 10\colon\ 2c_2 - 2c_1 - 3d_1\left(\frac{\pi}{3}\right) = 0 \\
&11\colon\ b_0 = 0 && 12\colon\ b_2 + 2c_2\left(\frac{\pi}{3}\right) + 3d_2\left(\frac{\pi}{3}\right)^2 = 0
\end{aligned}
$$

We put these twelve equations together to get the below matrix, where the columns, in order, represent $a_0, b_0, c_0, d_0, a_1, \ldots, d_2$:

$$\left(\begin{array}{cccccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{3}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{\pi}{3} & \left(\frac{\pi}{3}\right)^2 & \left(\frac{\pi}{3}\right)^3 & 0 \\
-1 & -\frac{\pi}{3} & -\left(\frac{\pi}{3}\right)^2 & -\left(\frac{\pi}{3}\right)^3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -\frac{\pi}{3} & -\left(\frac{\pi}{3}\right)^2 & -\left(\frac{\pi}{3}\right)^3 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & -2\frac{\pi}{3} & -3\left(\frac{\pi}{3}\right)^2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -2\frac{\pi}{3} & -3\left(\frac{\pi}{3}\right)^2 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -2 & -3\frac{\pi}{3} & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & -3\frac{\pi}{3} & 0 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\frac{\pi}{3} & 3\left(\frac{\pi}{3}\right) & 0
\end{array}\right)$$

3

3.a

We use MATLAB to find the coefficients for the three cubic polynomials for our spline interpolant:

```
format short
X=[0,pi/3,2*pi/3,pi]

X =

         0    1.0472    2.0944    3.1416

Y=[0,0,3/4,3/4,0,0]

Y =

         0         0    0.7500    0.7500         0
         0

pp=spline(X,Y)

pp =

      form: 'pp'
    breaks: [0 1.0472 2.0944 3.1416]
```

```
       coefs: [3x4 double]
      pieces: 3
       order: 4
         dim: 1

[b,c]=unmkpp(pp)

b =

         0    1.0472    2.0944    3.1416


c =

   -0.6531    1.3678         0         0
    0.0000   -0.6839    0.7162    0.7500
    0.6531   -0.6839   -0.7162    0.7500

diary off
```

and so our polynomials are:

$$
\begin{aligned}
S_0(x) &\approx 1.3678(x-0)^2 - 0.6531(x-0)^3 \\
S_1(x) &\approx 0.75 + 0.7162\left(x - \frac{\pi}{3}\right) - 0.6839\left(x - \frac{\pi}{3}\right)^2 \\
S_2(x) &\approx 0.75 - 0.7162\left(x - \frac{2\pi}{3}\right) - 0.6839\left(x - \frac{2\pi}{3}\right)^2 + 0.6531\left(x - \frac{2\pi}{3}\right)^3
\end{aligned}
$$

3.b

We use the following MATLAB statements

```
X=[0,pi/3,2*pi/3,pi]

X =

         0    1.0472    2.0944    3.1416

Y=[0,0,3/4,3/4,0,0]
```

```

Y =

         0         0    0.7500    0.7500         0         0

pp=spline(X,Y)

pp =

      form: 'pp'
    breaks: [0 1.0472 2.0944 3.1416]
     coefs: [3x4 double]
    pieces: 3
     order: 4
       dim: 1

[b,c]=unmkpp(pp)

b =

         0    1.0472    2.0944    3.1416


c =

   -0.6531    1.3678         0         0
    0.0000   -0.6839    0.7162    0.7500
    0.6531   -0.6839   -0.7162    0.7500

X1=linspace(0,pi/3,100)

X2=linspace(pi/3,2*pi/3,100)

X3=linspace(2*pi/3,pi,100)

Y1=c(1,1)*(X1-0).^3+c(1,2)*(X1-0).^2+c(1,3)*(X1-0)+c(1,4)

```

```
Y2=c(2,1)*(X2-pi/3).^3+c(2,2)*(X2-pi/3).^2+c(2,3)*(X2-pi/3)+c(2,4)

Y3=c(3,1)*(X3-2*pi/3).^3+c(3,2)*(X3-2*pi/3).^2+c(3,3)*(X3-2*pi/3)+c(3,4)

Xf=linspace(0,pi,300)

Yf=sin(X0).^2

plot(X1,Y1,'--r',X2,Y2,'-r',X3,Y3,'--r',Xf,Yf,'-b')
diary off
```

to create the plot below, where the blue line is $\sin^2 x$ and the red line is the spline function: