Oliver Tonnesen
V00885732

# Take Home Final

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## 1

Yes, $f$ is uniformly continuous on $(1, 2)$.

Let $\varepsilon > 0$, and let $\delta = \varepsilon$. Then if $|x - y| < \delta$,

$$\begin{aligned}\left|\frac{1}{x} - \frac{1}{y}\right| &= \left|\frac{x-y}{xy}\right| \\ &< |x - y| \qquad \text{(Since } x, y > 1\text{)} \\ &< \delta = \varepsilon.\end{aligned}$$

So $\left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon$, as desired.

## 2

Let $\delta = 100 - \sqrt[3]{100^3 - 10^{-10}}$. Then let $|x - y| < \delta$. Note that $\left|x^3 - y^3\right|$ is maximized when one of $x$ or $y$ is equal to 100. In other words, if $|x - y| < \delta$, and $|100 - z| < \delta$, then $\left|x^3 - y^3\right| \leq \left|100^3 - z^3\right|$. $z \leq 100$, so $\left|100^3 - z^3\right| = 100^3 - z^3$. Then, since $100 - z < \delta = 100 - \sqrt[3]{100^3 - 10^{-10}}$, we have

$$\begin{aligned}&100 - z < 100 - \sqrt[3]{100^3 - 10^{-10}} \\ \implies &-z < -\sqrt[3]{100^3 - 10^{-10}} \\ \implies &z > \sqrt[3]{100^3 - 10^{-10}} \\ \implies &z^3 > 100^3 - 10^{-10} \\ \implies &100^3 - z^3 < 10^{-10} \\ \implies &\left|100^3 - z^3\right| < 10^{-10} = \varepsilon\end{aligned}$$

Since, as we saw, $\left|x^3 - y^3\right| \leq \left|100^3 - z^3\right|$, the above holds for any $x, y \in I$, $|x - y| < \delta$, as desired.

## 3

$\sum_{n=1}^{\infty} a_n$ converges to a real number, so $a_n \to 0$. This means that there exists some $N$ such taht $a_n \leq 1$ for all $n > N$. We know that $a_n^2 \leq a_n$ whenever $a_n \leq 1$, so

$$\sum_{n=N}^{\infty} a_n^2 \leq \sum_{n=N}^{\infty} a_n.$$

$\sum_{n=1}^{N-1} a_n^2$ is a finite sum, and thus is a real number. Also, since $\sum_{n=N}^{\infty} a_n^2$ is bounded above by $\sum_{n=N}^{\infty} a_n$, a real number, it must be a real number as well. Thus we have

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{N-1} a_n^2 + \sum_{n=N}^{\infty} a_n^2$$

is a sum of two real numbers, and so $\sum_{n=1}^{\infty} a_n^2$ must itself converge to a real number, as desired.

# 4

Let $\varepsilon > 0$. $g$ is uniformly continuous, so for any $\varepsilon_1 > 0$, there exists some $\delta_1 > 0$ such that if $x, y \in B$, $|x - y| < \delta_1$, then $|g(x) - g(y)| < \varepsilon_1$. Similarly, $f$ is uniformly continuous, so for any $\varepsilon_2 > 0$, there exists some $\delta_2 > 0$ such that if $w, z \in A$, $|w - z| < \delta_2$, then $|f(w) - f(z)| < \varepsilon_2$.

Choose $\varepsilon_2$ to be $\varepsilon$, and let $\delta_2$ be chosen as above. Also choose $\varepsilon_1$ to be $\delta_2$, and let $\delta_1$ be chosen as above. Let $x, y \in B$, $|x - y| < \delta_1$. Then $|g(x) - g(y)| < \varepsilon_1 = \delta_2$. $g(x), g(y) \in \operatorname{im} g \subseteq A$, so since $|g(x) - g(y)| < \delta_2$, $|f(g(x)) - f(g(y))| < \varepsilon_2 = \varepsilon$, and so $f \circ g$ is indeed uniformly continuous on $B$, as desired.

# 5

We claim $\overline{S} = [0, \infty)$. Let $x \in [0, infty)$, and let $\varepsilon > 0$. We know that for any real number $x$ and positive integer $k$, there exists an integer $a$ such that $kx \in [a, a+1)$ and $\left|x - \frac{a}{k}\right| < \frac{1}{k}$. Take $k$ to be the larger of the two following numbers: $\frac{1}{\varepsilon}$, and the smallest integer of the form $2^n$ such that $a + 1 \leq 4^n$. Although $a$ is dependent on the choice of $k$, $4^n$ grows much faster than $2^n$, so this choice of $k$ is always possible given a large enough $n$. Given this, $k = \frac{a}{2^n} \in S$. We now have

$$\begin{aligned} \left|x - \frac{a}{k}\right| &= \left|x - \frac{a}{2^n}\right| \\ &< \frac{1}{k} \\ &< \varepsilon \end{aligned}$$

So indeed, $\overline{S} = [0, \infty)$, as desired.

# 6

Let $\varepsilon = |\alpha - \beta| \neq 0$. Assume for a contradiction that the sequence $(a_n)$ is convergent. Then $(a_n)$ is Caucy.

Choose $N_1$ such that $m, n > N_1 \implies |a_m - a_n| < \frac{\varepsilon}{3}$.
Choose $N_2$ such that $p_i > N_2 \implies |a_{p_i} - \alpha| < \frac{\varepsilon}{3}$.
Choose $N_3$ such that $q_i > N_3 \implies |a_{q_i} - \beta| < \frac{\varepsilon}{3}$.
Finally, let $N = \max\{N_1, N_2, N_3\}$. Then by the triangle inequality,

$$|\alpha - a_{p_i}| + |a_{p_i} - a_{q_i}| + |a_{q_i} - \beta| \geq |\alpha - \beta|,$$

but we also have

$$\begin{aligned} |\alpha - a_{p_i}| + |a_{p_i} - a_{q_i}| + |a_{q_i} - \beta| &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

So we end up with

$$\varepsilon > |\alpha - a_{p_i}| + |a_{p_i} - a_{q_i}| + |a_{q_i} - \beta| \geq |\alpha - \beta|,$$

or $\varepsilon > |\alpha - \beta|$, but $\varepsilon = |\alpha - \beta|$, a contradiction, and so $(a_n)$ is not a convergent sequence, as desired.

## 7

Let $S' = \{|s| \mid s \in S\}$. Then $\sup S' = \max\{|\sup S|, |\inf S|\}$. We know $|\sup S|^2 = (\sup S)^2$ and $|\inf S|^2 = (\inf S)^2$, so we need only show that $\sup T = (\sup S')^2$.

Assume for a contradiction that $\sup T \neq (\sup S')^2$. Then there is some $s \in S'$ such that $s \neq \sup S'$, but $\sup T = s^2$. $s < \sup S'$, so there exists some $s' \in S'$ such that $s < s' < \sup S'$. Then $(s')^2 > s^2 = \sup T \geq (s')^2$, a contradiction, and so $\sup T = (\sup S')^2 = (\max\{|\sup S|, |\inf S|\})^2 = \max\left\{(\sup S)^2, (\inf S)^2\right\}$, as desired.

$\inf T$ need not be equal to $\min\left\{(\sup S)^2, (\inf S)^2\right\}$. Consider for example $S = (-1, 1)$. $\min\left\{(\sup S)^2, (\inf S)^2\right\} = 1$, but $\inf T = 0$.

## 8