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MATH 236

Assignment 3

February 28, 2020

1

Let N_1 be the first natural number such that for any $n > N_1$, $\binom{n}{4} > n^3$. Let $\varepsilon > 0$, let $a = 1 + c$, and let $N = \max\left\{N_1, \frac{1}{\varepsilon c^4}\right\}$. Then $b_n = \frac{n^2}{(1+c)^n}$, so for any $n > N$,

$$\begin{aligned}
\frac{n^2}{(1+c)^n} &= \frac{n^2}{\binom{n}{0}c^0 + \cdots + \binom{n}{n}c^n} \\
&< \frac{n^2}{\binom{n}{4}c^4} \\
&< \frac{n^2}{n^3c^4} \qquad \text{(since } n > N_1\text{)} \\
&= \frac{1}{nc^4} \\
&< \frac{1}{Nc^4} \qquad \text{(since } n > N\text{)} \\
&\le \varepsilon \qquad \text{(since } N \ge \frac{1}{\varepsilon c^4}\text{)}
\end{aligned}$$

2

Consider the kth term in the binomial expansion of a_n, $\frac{\binom{n}{k}}{n^k}$:

$$\begin{aligned}
\frac{\binom{n}{k}}{n^k} &= \frac{n!}{k!(n-k)!}\frac{1}{n^k} \\
&= \frac{(n-1)!}{k!(n-k)!}\frac{n}{n^k} \\
&= \frac{(n-1)!}{k!(n-k)!}\frac{n}{n^{k-1}} \\
&= \frac{(n-1)\cdots(n-k+1)}{k!n^{k-1}} \\
&= \frac{1}{k!}\frac{(n-1)}{n}\cdots\frac{(n-k+1)}{n} \\
&= \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)
\end{aligned}$$

From here, it is clear to see that each factor of the kth term involving n increases as n grows. Thus we can conclude that (a_n) is indeed increasing, as desired.

3

We claim $a_n > a_{n+1}$. When $n = 3$, we see that $\frac{3^2}{2^3} = \frac{9}{8} > \frac{16}{16} = \frac{4^2}{2^4}$. Suppose there exists some N such that the claim holds for any n less than N. Consider a_{N+2}:

$$\begin{aligned} a_{N+2} &= \frac{(N+2)^2}{2^{N+2}} \\ &= \frac{N^2+4N+4}{2 \cdot 2^{N+1}} \\ &= \frac{N^2/2+2N+2}{2^{N+1}} \\ &< \frac{N^2+2N+1}{2^{N+1}} \qquad \text{since } N \geq 3 \\ &= \frac{(N+1)^2}{2^{N+1}} \end{aligned}$$

So by induction, the claim holds.

4

4.i

If $0 < a < 1$, then $x_2 = \frac{2+x_1+x_1^2}{4} \in \left(\frac{1}{2}, 1\right)$. Then $x_3 \in \left(\frac{2+x_2+x_2^2}{4}, 1\right)$, and so on. The sequence of lower bounds for x_n is increasing and bounded above, and approaches 1 as $n \to 1$, so $x_n \to 1$ as well.

4.ii

If $x_1 \in (1, 2)$, say $x_1 = a$. Then $x_2 \in \left(1, \frac{2+a+a^2}{4}\right)$. Note that this is a strict subset of $(1, 2)$. It is clear to see that this time, the upper bound for the range in which x might fall approaches 1, so $x_n \to 1$.

4.iii

If $x_1 \in (2, \infty)$, say $x_1 = a$, then $\frac{2+a+a^2}{4} > 2$. Thus the lower bound is strictly larger than 2. This increasing lower bound continues and so $x_n \to \infty$.

4.iv

If $x_1 \in (-\infty, 0)$, say $|x_1| = a$, then there are two cases: if $x_1 \in (-3, 0)$, then $x_2 = \frac{2-a+a^2}{4} \in \left(\frac{1.75}{4}\right)$, so $x_n \to \infty$. if $x_1 \in (-\infty, -3)$, $x_2 = \frac{2-x+x^2}{4} > 2$, so $x_n \to \infty$.

5

Suppose for a contradiction that both $a_n \to a \in \mathbb{R}$ and $a_n \to \infty$. $a_n \to a \in \mathbb{R}$, so for any $\varepsilon > 0$ there exists some N_1 such that for any $n > N_1$, $|a_n - 1| < \varepsilon$. We also have that $a_n \to \infty$, so for any $M > 0$, there exists some N_2 such that for any $n > N_2$, $a_n > M$.

Let $M = a + \varepsilon$. Then there exists some N such that for any $n > N$, $a_n > a + \varepsilon$. WLOG let $a_n > a$, then $a_n - a = |a_n - a| > \varepsilon$, a contradiction, and so a_n cannot converge both to $a \in \mathbb{R}$ and to ∞.

6

We know $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$, so $A_n \in (-1, 1)$ for any n. (Note that $A_n \to \pm 1$ only when the an always moves in the same direction).

Note that $\sum_{i=k+1}^{\infty} \frac{1}{2^i} = 1 - \sum_{i=1}^{k} \frac{1}{2^i} = \frac{1}{2^k}$, so if $\varepsilon > 0$, we can simply choose $N = -\log_2 \frac{\varepsilon}{2}$, since after N steps, the ant can travel no further than $\frac{1}{2^N} = 2^{\log_2 \frac{\varepsilon}{2}} = \frac{\varepsilon}{2}$ away from its position at time N. That is, for any $n > N$, $A_n \in \left(A_N - \frac{\varepsilon}{2}, A_N + \frac{\varepsilon}{2}\right)$, and so for any $n, m > N$,

$$
\begin{aligned}
|A_n - A_m| &\leq |A_n - A_N| + |A_N - A_m| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

so (A_n) is Cauchy and thus converges to a real number.

7

If the ant moves in the same (say positive) direction at each step, it will end up distance $\sum_{k=1}^{\infty} \frac{1}{k}$ from 0. We know that $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$, so A_n must not necessarily converge to a real number.

8

$a_n \to a$, so there exists some N_1 such that for any $n > N_1$, $|a_n - a| < \frac{\varepsilon}{2}$. Similarly, $b_n \to b$, so there exists some N_2 such that for any $n > N_2$, $|b_n - b| < \frac{\varepsilon}{2}$. Let $\varepsilon > 0$, and let $N = \max\{N_1, N_2\}$. Then consider $|(a_n - b_n) - (a - b)|$:

$$\begin{aligned}|(a_n - b_n) - (a - b)| &= |(a_n - a) - (b_n - b)| \\ &= |(a_n - a) + (b - b_n)| \\ &\leq |a_n - a| + |b - b_n| \\ &= |a_n - a| + |b_n - b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

9

9.i

We know by the denseness of $\mathbb{Q}$ in $\mathbb{R}$ that for any $a < b \in \mathbb{R}$, there exists $r_1 \in \mathbb{Q} \subseteq \mathbb{R}$ such that $a < r_1 < b$. For any $i \in \mathbb{N}$, let $r_{i+1} \in (r_i, b)$. Since each r_i is a real number less than b, we can always find r_{i+1}, and so there are infinitely many such rational numbers.

9.ii

Define $A_i = \left(\alpha - \frac{1}{i}, \alpha\right)$. As we saw in part i), there lie infinitely many rational numbers in A_i for any i. Choose $q_{n_1} \in A_1$. For any $i > 1$, if we let $q_{n_i} \in A_i$, then given any $\varepsilon > 0$, $q_{n_i} < \varepsilon$ for all $n_i > n_{\frac{1}{\varepsilon}}$.

9.iii

Define $A_i = (i, i+1)$. Again, A_i contains infinitely many rational numbers for any i. If we let $q_{n_i} \in A_i$, then given any $M > 0$, $q_{n_i} > M$ for all $n_i > n_M$.

10

Let $b_n = \frac{1}{\sqrt{|a_n|}}$. If $M > 0$, then we can pick N such that $|a_N| < \frac{1}{M^2}$, meaning $b_n = \frac{1}{\sqrt{|a_n|}} > M$, and so $b_n \to \infty$.

Then $a_n b_n = \frac{a_n}{\sqrt{|a_n|}} = \begin{cases} \sqrt{a_n} & \text{if } a_n \geq 0 \\ -\sqrt{a_n} & \text{otherwise} \end{cases}$, which clearly converges to 0.