1

G is a finite abelian group, so it is isomorphic to the direct product of finite cyclic groups of prime power order, and can thus be written as $\bigoplus_i \mathbb{Z}_{p_i^{k_i}}$, where each p_i is prime and each k_i is an integer.

We know that every group of the form $\mathbb{Z}_n$ is a ring when equipped with multiplication modulo n, so the direct product of finite cyclic groups above is also a direct product of rings, giving us a ring whose additive group is isomorphic to G, as desired.

2

Let $\frac{a}{b}+\mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, and suppose WLOG that $\gcd(a,b)=1$. Then $\left|\frac{a}{b}+\mathbb{Z}\right| = b$, since $b \cdot \frac{a}{b} \in \mathbb{Z}$, and a and b are coprime. Thus any element in $\mathbb{Q}/\mathbb{Z}$ has finite order b, but since $b \in \mathbb{Z}$ and $\mathbb{Z}$ has no maximum element, this order can be arbitrarily large.

Suppose for a contradiction that R is a commutative unital ring whose additive group is isomorphic to $\mathbb{Q}/\mathbb{Z}$. Let 1 be R's multiplicative identity. $(R,+) \cong \mathbb{Q}/\mathbb{Z}$, so 1 has finite order, say n. Then for any $r \in R$, we have

$$\begin{aligned} n \cdot r &= n \cdot (1 \cdot r) \\ &= (n \cdot 1) \cdot r \\ &= 0 \cdot r \\ &= 0 \end{aligned}$$

so $|r| \leq n$, a contradiction, since the order of an element in $\mathbb{Q}/\mathbb{Z}$ (and hence $(R,+)$) can be arbitrarily large, thus greater than n, and so no such ring R exists, as desired.

3

By the first isomorphism theorem, we know that $F/\ker\varphi \cong \operatorname{im}\varphi$. We also know that the kernel of a homomorphism is an ideal. F is a field, so its only ideals are $\{0\}$ and itself. Suppose $\ker\varphi = F$. Then $\operatorname{im}\varphi \cong F/F \cong \{0\}$. This is not possible since φ is unital, so it must be the case that $\ker\varphi = \{0\}$, and thus φ is injective, as desired.