CSC 320 Assignment 4

Oliver Tonnesen
V00885732

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Let $M \in \text{NP}, M \neq \emptyset, \Sigma^*$ be the language to which we wish to reduce L. $M \neq \emptyset, \Sigma^*$, so we know that both M and $\overline{M}$ are nonempty. That is, there exists at least one string in M and one string in $\overline{M}$. Let $m_0 \in M$ and $m_1 \in \overline{M}$. We give a reduction $L \leq_P M$:

$$\begin{aligned} f : \Sigma^* &\longrightarrow \Sigma^* \\ w &\longmapsto \begin{cases} m_0, & \text{if } w \in L \\ m_1, & \text{if } w \notin L \end{cases} \end{aligned}$$

Since $L \in P$, we know that there exists a polynomial time algorithm to decide whether or not a string is in L, and so our reduction does indeed run in polynomial time.

It is clear to see from the definition of f that $w \in L \iff f(w) \in M$, and so our reduction holds.

2

We know that 3-SAT is NP-complete, so we give the reduction 3-SAT $\leq_P$ 4-SAT to prove that 4-SAT is NP-complete:

$$\begin{aligned} f : \Sigma^* &\longrightarrow \Sigma^* \\ w &\longmapsto f(w) \\ X = \{x_1, x_2, \ldots, x_n\} &\longmapsto X \\ C_i = \{a, b, c\} &\longmapsto C_i' = \{a, b, c, c\} \end{aligned}$$

Our reduction simply duplicates the third variable in every clause, and so runs in polynomial time.

If $w \in$ 3-SAT, then there exists a satisfying truth assignment for w. But f(w) is simply w with redundant variables added to each clause, so any truth assignment satisfying w with automatically satisfy f(w), and vice versa. Thus $w \in$ 3-SAT $\iff f(w) \in$ 4-SAT.

3

We know that PARTITION is NP-complete, so we give the reduction PARTITION $\leq_P$ THREE SHOPPING BAG PROBLEM (TSBP) to prove that TSBP is NP-complete:

$$\begin{aligned} f : \Sigma^* &\longrightarrow \Sigma^* \\ w &\longmapsto f(w) \\ f\Big(\Big\langle \{x_1, x_2, \ldots, x_m\} \Big\rangle\Big) &\longmapsto \Big\langle \{x_1, x_2, \ldots, x_m, y\}, \frac{s}{2} \Big\rangle \end{aligned}$$

where y has weight $\frac{s}{2}$ and s is the sum of the weights of $\{x_1, x_2, \ldots, x_m\}$.

If $w \in$ PARTITION, then $x_1, x_2, \ldots, x_m$ can be partitioned into two subsets whose weight is exactly $\frac{s}{2}$. Since we added one element of weight $\frac{s}{2}$ to create $f(w)$, then the contents of w can be split into two of the bags and the new item can be put in the third, and so $f(w) \in$ TSBP. Conversely, if $f(w) \in$ TSBP, then $x_1, x_2, \ldots, x_m, y$ can be split into three bags holding at most $\frac{s}{2}$. The newest item weighs exactly $\frac{s}{2}$, and so the only choice is to place it into its own bag alone. Since the sum of the remaining elements $(x_1, x_2, \ldots, x_m)$ is s, then it must be the case that they were split evenly among the remaining two bags, and so $w \in$ PARTITION.

4

We know that CLIQUE is NP-complete, so we give the reduction CLIQUE $\leq_P$ HALF-CLIQUE to prove that HALF-CLIQUE is NP-complete. Define the reduction f as follows:

<u>Case $k = \frac{|V|}{2}$:</u> $f\Big(\big\langle G, k \big\rangle\Big) = \big\langle G \big\rangle$

It is clear to see that, in this case, $\big\langle G, k \big\rangle \in$ CLIQUE $\iff \big\langle G \big\rangle \in$ HALF-CLIQUE.

<u>Case $k > \frac{|V|}{2}$:</u> $f\Big(\big\langle G, k \big\rangle\Big) = \big\langle (V \cup Y, E) \big\rangle$

where $Y = \{y_1, y_2, \ldots, y_{2k-|V|}\}$ and $V \cap Y = \emptyset$. We add $2k - |V|$ "useless" vertices, so $|V \cup Y| = 2k$, and so $\big\langle G, k \big\rangle \in$ CLIQUE $\iff \big\langle (V \cup Y, E) \big\rangle \in$ HALF-CLIQUE is again quite clear to see.

<u>Case $k < \frac{|V|}{2}$:</u> $f\Big(\big\langle G, k \big\rangle\Big) = \big\langle (V \cup Y, E \cup F) \big\rangle$

where $Y = \{y_1, y_2, \ldots, y_{|V|-2k}\}$ and $V \cap Y = \emptyset$, and $F = \{\{v, y\} \mid v \in V, y \in Y\}$. In this case, we add $|V| - 2k$ vertices, so $|V \cup Y| = 2|V| - 2k$. If $(V \cup Y, E \cup F) \in$ HALF-CLIQUE, then $(V \cup Y, E \cup F)$ contains $K_{|V|-k}$ as a subgraph. Since we added $|V| - 2k$ vertices to this subgraph to get G', then exactly $(|V| - k) - (|V| - 2k) = k$ vertices came from G, and so G has K_k as a subgraph. Similarly, if $\langle G, k\rangle \in$ CLIQUE, then G has K_k as a subgraph, and so adding $|V| - 2k$ vertices connected to each vertex in V results in G' having $K_{k+(|V|-2k)} = K_{|V|-k}$ as a subgraph. In each case, our polynomial time reduction f guarantees that $\langle G, k\rangle \in$ CLIQUE $\iff f\Big(\langle G, k\rangle\Big) \in$ HALF-CLIQUE. CLIQUE is NP-complete, and so HALF-CLIQUE is NP-complete.