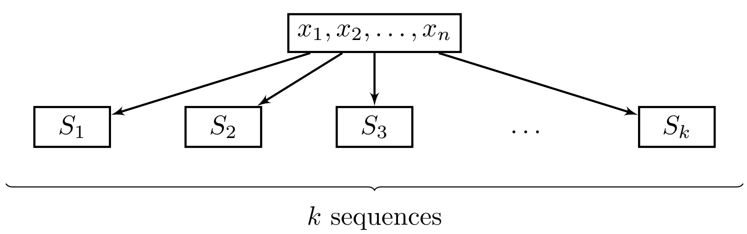
CSC 226 Problem Set 1 Written Part

Oliver Tonnesen
V00885732

September 29, 2018

1 Lower bounds for sorting

We define S_m to be the sequence $\{x_{(m-1)k+1}, x_{(m-1)k+2} \dots x_{mk}\}$.



Since each sequence is of length k, each can be sorted in $\Omega(k \log k)$ comparisons. k sequences need to be sorted, so sorting the entire sequence takes $\Omega(k^2 \log k)$ comparisons. Note that $k^2 = n$, so $\Omega(k^2 \log k)$ is equivalent to $\Omega(n \log k)$, and so the entire sequence can therefore be sorted in $\Omega(n \log k)$ time.

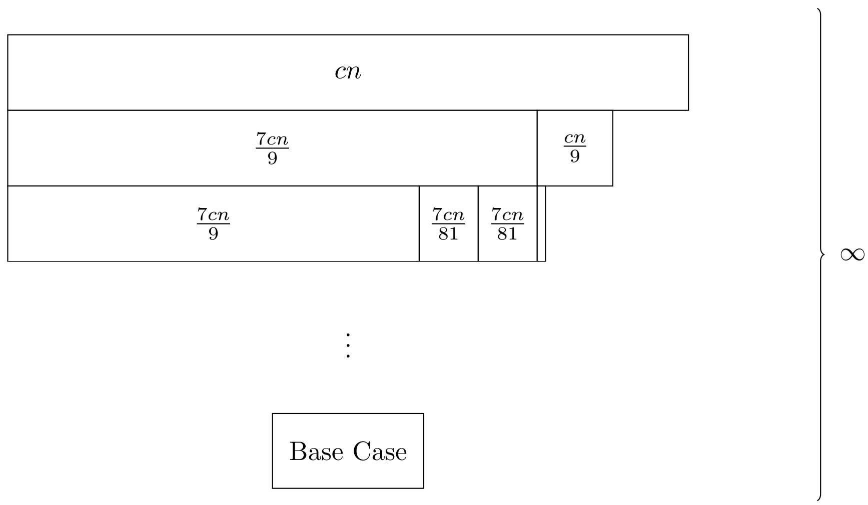
2 Quickselect with median-of-medians pivots

For each iteration of select finding the median of medians, at least half of the medians are $\geq$ the actual median of medians. So at least half of the $\frac{n}{9}$ groups contribute ≥ 5 elements that are greater than the median of medians (assuming distinct elements).

$$5 \cdot \frac{1}{2} \cdot \frac{n}{9} = \frac{5n}{18}$$

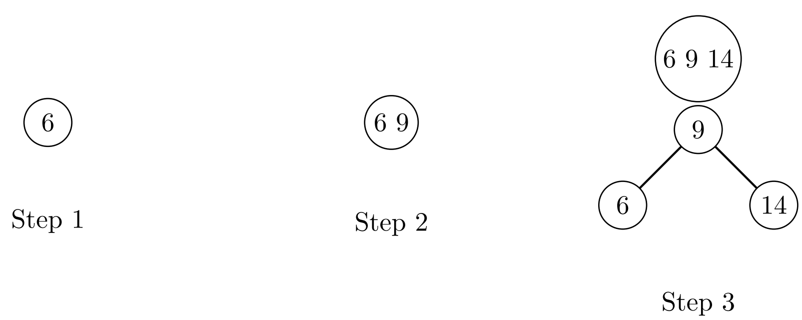
So at least $\frac{5n}{18}$ elements are greater than the median of medians. Conversely, at most $\frac{13n}{18}$ elements are lesser than the median of medians, and likewise at most $\frac{13n}{18}$ elements are greater than the median of medians. So we can set up our recurrence as follows:

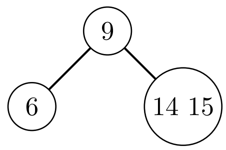
$$T(n) = T\left(\frac{13n}{18}\right) + T\left(\frac{n}{9}\right) + cn$$



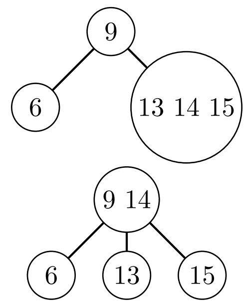
$$
\begin{aligned}
&= \sum_{i=0}^{\infty} cn\left(\frac{15}{18}\right)^i \\
&= \frac{cn}{1-\frac{15}{18}} \\
&= 6cn \\
&\in O(n)
\end{aligned}
$$

3 Insertion in 2-3 trees

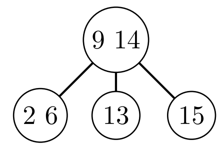




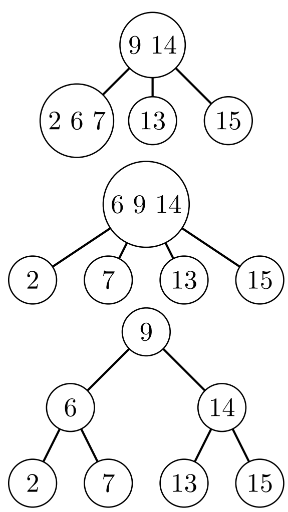
Step 4



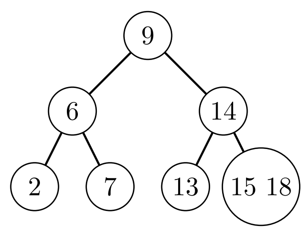
Step 5



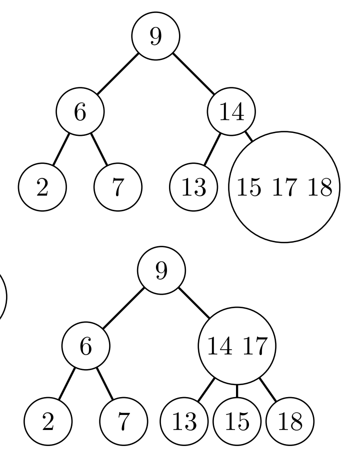
Step 6



Step 7



Step 8



Step 9

4 Relaxed AVL trees

First, we define $N(h)$ as a recurrence relation and prove its validity.
Claim:

$$N(h) = \begin{cases} 1 & h = 0 \\ 2 & h = 1 \\ 3 & h = 2 \\ N(h-1) + N(h-3) + 1 & h > 2 \end{cases}$$

Proof: (Induction)
$N(h)$ is defined as the minimum number of vertices in a Relaxed AVL tree. It is clear to see from this definition that the the three base cases hold.

Suppose the claim holds for all $l < h$.
We will now show that the claim also holds for h by constructing the Relaxed AVL tree of height h with the minimum number of vertices. Take one vertex and let it be the root of our new tree. Our tree must have height h, so at least one of its children must have height $h-1$. By our hypothesis, $N(h-1)$ is the minimum number of vertices in a Relaxed AVL tree of height $h-1$. In order to satisfy the Relaxed height-balance property, our tree's children's heights must not differ by more than two, so to minimize our vertices we shall choose our next child to have height $n-3$. Again, by our hypothesis we know that the minimum number of vertices in a Relaxed AVL tree of height $h-3$ is $N(h-3)$. If we now consider our tree with one root vertex and two children, both of which are also Relaxed AVL trees with $N(h-1)$ and $N(h-2)$ vertices, our number of vertices will be $N(h-1)+N(h-3)+1$. Thus our definition of $N(h)$ holds for all h.

We will now show that $N(h) \in \Omega(k^h)$ for some $k > 1$.
It is sufficient to show that $N(h) \geq ck^h$ for some $c > 0, k > 1$. Claim: There exists some $c > 0, k > 1$ such that $N(h) \geq ck^h$ for all h.
Proof: (Induction)

$$N(0) = 1: \quad 1 \geq ck^0 \quad \text{if} \quad \frac{1}{k^0} \geq c$$

$$N(1) = 2: \quad 2 \geq ck^1 \quad \text{if} \quad \frac{1}{k^1} \geq c$$

$$N(2) = 3: \quad 3 \geq ck^2 \quad \text{if} \quad \frac{1}{k^2} \geq c$$

Note that the cases for $h = 0$ and $h = 1$ also hold when $c \leq \frac{1}{k^2}$, so let $c \leq \frac{1}{k^2}$.
Suppose $N(l) \geq ck^l$ for all $l < h$.
We want to show that $N(h) \geq ck^h$ for some $c > 0, k > 1$. Recall the definition for $N(h)$:

$$N(h) = N(h-1) + N(h-3) + 1$$

By our hypothesis, we have

$$\begin{aligned} N(h) &\geq ck^{h-1} + ck^{h-3} + 1 \\ &\geq ck^{h-1} + ck^{h-3} \end{aligned}$$

So if we can show that $ck^{h-1} + ck^{h-3} \geq ck^h$, it would be sufficient to show that $N(h) \geq ck^h$.

$$\begin{aligned} &ck^{h-1} + ck^{h-3} \geq ck^h \\ &\implies \frac{ck^{h-1} + ck^{h-3}}{ck^{h-3}} \geq \frac{ck^h}{ck^{h-3}} \\ &\implies k^2 + 1 \geq k^3 \\ &\implies k^3 - k^2 - 1 \leq 0 \end{aligned}$$

The real root of the polynomial $k^3 - k^2 - 1 = 0$ is approximately 1.4655, so any value for k less than this value will do. Recall that we defined $c \leq \frac{1}{k^2}$. So the claim holds for $k = 1.4655$ and $c = \frac{1}{1.4655^2}$, and therefore $N(h) \geq \frac{1}{1.4655^2} 1.4655^h$ for any value of h.
Thus we have proven the claim that $N(h) \geq ck^h$ for some $c > 0, k > 1$, and therefore $N(h) \in \Omega(1.4655^h)$.