Oliver Tonnesen
V00885732

Assignment 4

1

$H \cap K \leq G$, so since $H \cap K \subseteq H$, $H \cap K \leq H$. By Lagrange's theorem, we have that $|H \cap K| \mid |H|$, but $|H|$ is prime, so either $|H \cap K| = 1$ or p. In the first case, $H \cap K = \{1_G\}$, and in the second case $H \cap K = H = K$, as desired.

2

($\Longrightarrow$): Assume for a contradiction that $|G|$ is not prime. Let $p \mid |G|$, with $p \neq 1$ or $|G|$. Then there exists some subgroup $P \leq G$ with $|P| = p$, but G is abelian, so $P \trianglelefteq G$, a contradiction. Thus $|G|$ must be prime.

($\Longleftarrow$): Let $H \leq G$. $|H| \mid |G|$, but $|G|$ is prime, so $|H| = 1$ or $|G|$, thus any subgroup (hence any normal subgroup) of G is trivial, so G is simple.

3

3.a

By the third Sylow theorem, $n_{11} \equiv 1 \pmod{11}$ and $n_{11} \mid |G|$, so $n_{11} = 1, 11, 11^2, 17, 17^2, 11 \cdot 17, 11^2 \cdot 17, 11 \cdot 17^2$, or $11^2 \cdot 17^2$. Obviously any number divisible by 11 is not congruent to $1 \pmod{11}$, so $n_{11} = 1, 17$, or 17^2. $17 \equiv 6 \pmod{11}$ and $17^2 \equiv 3 \pmod{11}$, so it must be the case that $n_{11} = 1$. Thus there is only one Sylow 11-subgroup of G, and it follows then from the second Sylow theorem that this subgroup must be normal. By a similar argument, $n_{17} = 1, 11$, or 11^2, but $11 \equiv 11 \pmod{17}$ and $11^2 \equiv 2 \pmod{17}$, so $n_{17} = 1$, and therefore the unique Sylow 17-subgroup is also normal in G.

3.b

We know that any group with order p^2, p a prime is abelian, so both the Sylow 11-subgroup and the Sylow 17-subgroup are abelian (since they have orders 11^2 and 17^2, respectively). Let $N_{11}, N_{17} \trianglelefteq G$ be the Sylow 11- and Sylow 17-subgroups, respectively. We show that $G = N_{11} \times N_{17}$:

$\underline{N_{11}, N_{17} \trianglelefteq G}$: Showed in part 3.a).

$\underline{N_{11} \cap N_{17} = \{0\}}$: Let $g \in N_{11} \cap N_{17}$. $|g| \mid |N_{11}|$ and $|g| \mid |N_{17}|$, so $|g| \mid 11^2$ and $|g| \mid 17^2$, thus $|g| = 1$, so $g = 0$.

<u>$G = N_{11}N_{17}$</u>: $N_{11} \cap N_{17} = \{0\}$, so since $|G| = 11^2 \cdot 17^2 = |N_{11}||N_{17}|$, $G = N_{11}N_{17}$.

Thus $G = N_{11} \times N_{17}$, so $G \cong N_{11} \oplus N_{17}$. N_{11} and N_{17} are abelian, so their direct product is abelian, and thus G is abelian, as desired.

3.c

$N_{11} \cong \mathbb{Z}_{11^2}$, or $\mathbb{Z}_{11} \oplus \mathbb{Z}_{11}$, and $N_{17} \cong \mathbb{Z}_{17^2}$, or $\mathbb{Z}_{17} \oplus \mathbb{Z}_{17}$, so since $G \cong N_{11} \oplus N_{17}$, any group of order 34969 is isomorphic to exactly one of the following:

$$\begin{gathered}
\mathbb{Z}_{11^2} \oplus \mathbb{Z}_{17^2} \\
\mathbb{Z}_{11^2} \oplus \mathbb{Z}_{17} \oplus \mathbb{Z}_{17} \\
\mathbb{Z}_{11} \oplus \mathbb{Z}_{11} \oplus \mathbb{Z}_{17^2} \\
\mathbb{Z}_{11} \oplus \mathbb{Z}_{11} \oplus \mathbb{Z}_{17} \oplus \mathbb{Z}_{17}
\end{gathered}$$

4

$N_1 = 4$. We know that there are exactly two groups of order 4: $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Z_4 is abelian and $\langle [2]_4 \rangle \leq \mathbb{Z}_4$, so $\langle [2]_4 \rangle \trianglelefteq \mathbb{Z}_4$. $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is also abelian, and $\langle ([1]_2, [0]_2 \rangle \leq \mathbb{Z}_2 \oplus \mathbb{Z}_2$, so $\langle ([1]_2, [0]_2 \rangle \trianglelefteq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Thus, neither group of order 4 is simple.

5

$N_2 = 46$. $46 = 2 \cdot 23$. Let G be a group of order 46. We know by the first Sylow theorem that there exists a subgroup of G with order 23. This subgroup has order half that of G, so H is normal, and hence G is not simple.

6

$N_3 = 30$. $30 = 2 \cdot 3 \cdot 5$. By the third Sylow theorem, $n_5 \equiv 1 \pmod 5$, and $n_5 \mid |G|$, so $n_5 = 1$ or 6. Similarly, $n_3 \equiv 1 \pmod 3$, and $n_3 \mid |G|$, so $n_3 = 1$ or 10. If $n_5 = 1$, or $n_3 = 1$, then G has a normal Sylow 5- or Sylow 3-subgroup, and hence is not simple. So assume $n_5 = 6$ and $n_3 = 10$. 5

and 3 are prime, so all of these subgroups are cyclic and every nonidentity element generates the entire subgroup. This means that any pair of these 16 subgroups intersects only at the identity. So G is the union of 6 5-subgroups and 10 3-subgroups, and this union is disjoint when we remove the identity from each subset. So we have

$$|G| = 1 + 6(5-1) + 10(3-1) = 45 > 30 = |G|,$$

a contradiction, and so our assumption that $n_5 = 6$ and $n_3 = 10$ was false, so $n_5 = 1$ or $n_3 = 1$, meaning G has a normal subgroup, and is therefore not simple.