Oliver Tonnesen
V00885732

Assignment 2

1

$G = \mathbb{Z}_1$, $A = \emptyset$. The empty set is indeed a subset of $\mathbb{Z}_1$, and the statement "for all g in G and a in A, gag^{-1} is in A" is vacuously true. A is not a normal subgroup of G since it isn't a group, as it's not nonempty.

2

Define $\varphi : G \to G$ by $\varphi(g) = 2g$. Let $g, h \in G$. $\varphi(g+h) = 2(g+h) = 2g + 2h = \varphi(g) + \varphi(h)$, so φ is a homomorphism.

$$\begin{aligned}\ker\varphi &= \varphi^{-1}(\{0\}) \\ &= \{g \in G \mid 2g = 0\} \\ &= \{g \in G \mid |g| \,\Big|\, 2\}\end{aligned}$$

G has odd order, and the order of any element of G must divide its order, so there are no elements of order 2, thus $\ker\varphi = \{0\}$. Then by the first isomorphism theorem, we have that $G \,/\, \ker\varphi \cong \operatorname{Im}\varphi$, but $G \,/\, \{0\} \cong G$, so $G \cong \operatorname{Im}\varphi$, thus φ is an isomorphism.

Most importantly, since φ is an isomorphism, it is injective, so for any $y \in G$, φ *uniquely* maps it to the elementy $2y$ (or x), as desired.

3

3.a

We proved in class that this map φ is a homomorphism. We know that for any $g \in G$, $|\varphi(g)| \,\Big|\, |g|$. We also know that for any $g \in G$, $|g| \,\Big|\, |G|$, so since $\varphi(g) \in H$, $|\varphi(g)| \,\Big|\, |H|$. Since $\gcd(|G|, |H|) = 1$, $|\varphi(g)|$ and $|g|$ share no common factors (other than 1), but $|\varphi(g)| \,\Big|\, |g|$, so $|\varphi(g)|$ must be 1, thus $\varphi(g)$ must be 1_H for any choice of g. Thus the only possible homomorphism $\varphi : G \to H$ is $\varphi(g) = 1_H$.

3.b

Suppose for a contradiction that another such φ exists. φ is a homomorphism, so we have that $\operatorname{Im}\varphi \le \mathbb{Z}$. The only subgroups of $\mathbb{Z}$ are of the form

$n\mathbb{Z}$, all of which (other than $0\mathbb{Z}$) are isomorphic to $\mathbb{Z}$. So if we assume φ is nontrivial, then its image is isomorphic to $\mathbb{Z}$. If we let $f : \operatorname{Im}\varphi \to \mathbb{Z}$ be an isomorphism, then there exists a $q \in \mathbb{Q}$ such that $f \circ \varphi(q) = 1$. But

$$\begin{aligned}
1 &= f \circ \varphi(q) \\
&= f \circ \varphi(\frac{q}{2} + \frac{q}{2}) \\
&= f \circ \varphi(\frac{q}{2}) + f \circ \varphi(\frac{q}{2}) \\
&= z + z \\
&= 2z
\end{aligned}$$

for some $z \in \mathbb{Z}$, a contradiction, since 1 cannot be written as the sum of any two integers. So the only possible homomorphism from $\mathbb{Q}$ to $\mathbb{Z}$ is the trivial homomorphism, $\varphi(z) = 0$.

4

Let $(g', h') \in G \oplus H$. $(g', h')H^* = \{(g', h'h) \mid h \in H\}$. $H^*(g', h') = \{(g', hh') \mid h \in H\}$. But $h'h$ and hh' are both already in H, so both $(g', h')H^*$ and $H^*(g', h')$ can be rewritten as $\{(g', h^*) \mid h^* \in H\}$. Thus $H^* \trianglelefteq G \oplus H$.

Define $\varphi : G \oplus H \to G$ by $\varphi((g, h)) = g$.

$$\begin{aligned}
\ker\varphi &= \{(g, h) \in G \oplus H \mid \varphi((g, h)) = 1_G\} \\
&= \{(1_G, h) \mid h \in H\} \\
&= H^*
\end{aligned}$$

By the first isomorphism theorem, we have that $G \oplus h \,/\, \ker\varphi \cong \operatorname{Im}\varphi$. Let $g \in G$. Then $\varphi((g, 1_H)) = g$, so φ is surjective and $\operatorname{Im}\varphi = G$. Thus we have the following three identities:

$$\begin{aligned}
G \oplus H \,/\, H^* &= G \oplus H \,/\, \ker\varphi \\
\operatorname{Im}\varphi &= G \\
G \oplus H \,/\, \ker\varphi &\cong \operatorname{Im}\varphi
\end{aligned}$$

and so

$$G \oplus H \,/\, H^* \cong G$$

as desired.

5

5.a

We use the one step subgroup test.

$\mathrm{Inn}(G) \neq \emptyset$, since if $\varphi : G \to G$, $\varphi(a) = 1a1^{-1} = a$, then $\varphi \in \mathrm{Inn}(G)$.

Let $\varphi, \psi \in \mathrm{Inn}(G)$. We show that $\varphi \circ \psi^{-1} \in \mathrm{Inn}(G)$. There are $g, h \in G$ such that $\varphi(a) = gag^{-1}$ and $\psi(a) = hah^{-1}$. Then $\psi^{-1}(a) = h^{-1}ah$, since $\psi(\psi^{-1}(a)) = h(h^{-1}ah)h^{-1} = a$, so $\psi^{-1} \in \mathrm{Inn}(G)$. Then $\varphi \circ \psi^{-1}(a) = gh^{-1}ahg^{-1}$. $(gh^{-1})^{-1} = (hg^{-1})$, so $\varphi \circ \psi^{-1} \in \mathrm{Inn}(G)$, and thus $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)$.

5.b

Let $g_1, g_2 \in G$. Then

$$\begin{aligned}
\rho(g_1 g_2)(h) &= g_1 g_2 h g_2^{-1} g_1^{-1} \\
&= g_1 (g_2 h g_2^{-1}) g_1^{-1} \\
&= g_1 (\rho(g_2)(h)) g_1^{-1} \\
&= \rho(g_1)(\rho(g_2)(h)) \\
&= \rho(g_1) \circ \rho(g_2)(h).
\end{aligned}$$

So ρ is a homomorphism.

5.c

$$\begin{aligned}
\ker \rho &= \{g \in G \mid \rho(g)(h) = ghg^{-1} = h\} \\
&= \{g \in G \mid ghg^{-1} = h\} \\
&= Z(G)
\end{aligned}$$

By the first isomorphism theorem, we have that $G \,/\, \ker \rho \cong \mathrm{Im}\, \rho$.
So $G \,/\, \ker \rho = G \,/\, Z(G) \cong \mathrm{Im}\, \rho$. We now show that $\mathrm{Im}\, \rho = \mathrm{Inn}(G)$.

$$\begin{aligned}
\mathrm{Im}\, \rho &= \{\rho(g)(h) = ghg^{-1} \mid g \in G\} \\
&= \{\varphi : G \to G \mid \text{there is some } g \in G \text{ such that for all } a \in G,\ \varphi(a) = gag^{-1}\} \\
&= \mathrm{Inn}(G)
\end{aligned}$$

So $G \,/\, Z(G) \cong \mathrm{Im}\, \rho$, and $\mathrm{Im}\, \rho = \mathrm{Inn}(G)$, thus $G \,/\, Z(G) \cong \mathrm{Inn}(G)$.

5.d

Consider $1, r^2 \in D_4$. $\rho(1)(h) = 1h1^{-1} = h$, and $\rho(r^2)(h) = r^2hr^2 = h$. So ρ is not injective, and is thus not an isomorphism.

5.e

Define $\varphi : D_4 \to \mathrm{Aut}(D_4)$ by $\varphi(g)(h) = gh$. If $g_1, g_2 \in D_4$, then

$$\begin{aligned}
\varphi(g_1g_2)(h) &= g_1g_2h \\
&= \varphi(g_1)(g_2h) \\
&= \varphi(g_1)(\varphi(g_2)(h)) \\
&= \varphi(g_2) \circ \varphi(g_2)(h)
\end{aligned}$$

so φ is a homomorphism. $\ker\varphi = \{g \in D_4 \mid \varphi(g)(h) = h\}$. Clearly $\ker\varphi = \{1\}$, so φ is injective. Note that any element of D_4 is of the form r^aj^b, for some integers a and b. Define $\psi : D_4 \to D_4$ by $\psi(r^mj^q) = r^nj^r$, so that $\psi \in \mathrm{Aut}(D_4)$. Then $\varphi(r^nj^{r+q}r^{4-m}) = \psi$, since

$$\begin{aligned}
\varphi(r^nj^{r+q}r^{4-m})(r^mj^q) &= (r^nj^{r+q}r^{4-m})(r^mj^q) \\
&= r^nj^{r+q}r^{4-m+m}j^q \\
&= r^nj^{r+q}r^4j^q \\
&= r^nj^{r+q}j^q \\
&= r^nj^{r+2q} \\
&= r^nj^r \\
&= \psi(r^mj^q)
\end{aligned}$$

So φ is also surjective, and thus φ is an isomorphism.