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Assignment 3

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1

We construct a new graph as follows: let $d = \max_{S \subseteq X} \operatorname{def}(S)$, and add d vertices to Y. For each $S \subseteq X$ with $\operatorname{def}(S) > 0$, connect to S $\operatorname{def}(S)$ of the d vertices we added. We now have that $\operatorname{def}(S) \leq 0$ for all $S \subseteq X$, or $0 \geq |S| - |N(S)|$, or $|N(S)| \geq |S|$ for all $S \subseteq X$. Thus by Hall's Theorem, there exists a perfex matching in this new graph. If we delete the d vertices now, we remove exactly d edges from that matching (since all d vertices were in Y), so the maximum size matching in the original graph was the number in the new graph minus d, or $|X| - d = |X| - \max_{S \subseteq X} \operatorname{def}(S)$.

2

G is 3-regular, so $n(G)$ is even. Assume for a contradiction that G has no perfect matching. Then any maximum matching misses at least two vertices, so by the Tutte-Berge theorem, $\frac{n(G)-2}{2} = \frac{1}{2}\big(n(G) - \max\{o(G \setminus S) - |S| : S \subseteq V(G)\}\big)$ so there exists some $S \subseteq V(G)$ such that $o(G \setminus S) - |S| \geq 2$. Every odd component connects to S by an odd number of edges, and all but at most two of these must connect to S by three or more edges. In other words, $3(o(G \setminus S) - 2) + 2$ edges enter S. $3(o(G \setminus S) - 2) + 2 \geq 3|S| + 2$, but due to the 3-regularity of G, only up to $3|S|$ edges can enter S, a contradiction, and so G must have a perfect matching.

3

($\Longrightarrow$): By Tutte's theorem, $o(T \setminus S) \leq |S|$ for all $S \subseteq V(T)$, so if $S = \{v\}$, $v \in V(T)$, then $o(T \setminus S) = o(T - v) \leq |S| = 1$. So $o(T - v) \leq 1$. $n(T)$ is even, so $n(T - v)$ is odd. Thus $T - v$ must have an odd component, so $o(T - v) \geq 1$, and thus $o(T - v) = 1$.

($\Longleftarrow$): [Induction on $n(T)$]: We see this holds for $n(T) = 1$. Now assume it holds for all $n(T) \leq n$. Let $n(T) = n+1$ such that $o(T-v) = 1$ for all vertices v of T. Remove an arbitrary vertex $v \in V(T)$. $T - v = T_1 \cup T_2 \cup \ldots \cup T_k \cup C$ where $T_1, T_2, \ldots, T_k, C$ are all disjoint, C is an odd component and each T_i is an even component. Clearly v's neighbor u in C is the only one it would be matched with in a perfect matching, since by the hypothesis all the T_i's and also $C - u$ have perfect matchings. Thus all of these perfect matchings along with the edge vu would again form a perfect matching, so by induction the claim holds.

Thus we've shown the forward and backward direction, and so the original claim holds.

4

Let M be a maximum matching, and assume for a contradiction that we have a minimum vertex cover C with $|C| > 2\alpha'(G)$. C covers up to two vertices for each edge in the matching, so we have at least one vertex in the cover not an endpoint of an edge in M. This vertex is in C, so it must be covering an edge, but this edge is not in M. C covers every endpoint in M, so if this last vertex were adjacent to a vertex in M, then it would not be necessary to have it in C. Thus the vertex is adjaccent to a vertex not an endpoint of an edge in M. We could add this edge to M to obtain a larger matching, a contradiction since M is maximum. Thus $\beta(G) \leq 2\alpha'(G)$.

Given $k \geq 1$, the kK_3 is an example of a graph with $\alpha'(G) = k$ and $\beta(G) = 2k$.

5

The positions of the transversal are printed in red below:

$$\begin{bmatrix} \color{red}{4} & 5 & 8 & 10 & 11 \\ 7 & 6 & \color{red}{5} & 7 & 4 \\ 8 & \color{red}{5} & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & \color{red}{7} \\ 4 & 5 & 7 & \color{red}{9} & 8 \end{bmatrix}$$

6 Bonus

Let $S \subseteq V(G)$ such that $o(G \setminus S) - |S|$ is maximized. Assume that $S \neq \emptyset$, and $s \in S$. Note that $G \setminus S$ and $G - s \setminus S - s$ refer to the same graph, so $o(G - s \setminus S - s) - |S - s|$ is still maximal. But by the Tutte-Berge theorem,

$$\begin{aligned} \alpha'(G) &= \frac{1}{2}\big(n(G) - \max\{o(G \setminus S) - |S| : S \subseteq V(G)\}\big) \\ &< \frac{1}{2}\big(n(G - s) - \max\{o(G - s \setminus S - s) - |S - s| : S \subseteq V(G)\}\big) \\ &= \alpha'(G - s) \qquad \text{(Since } o(G - s \setminus S - s) - |S - s| \text{ is maximal)} \end{aligned}$$

But $\alpha'(G) = \alpha'(G - s)$ for all $s \in V(G)$ a contradiction, so $S = \emptyset$. Then when $o(G \setminus S) - |S|$ is maximized, it is equal to 1. Thus

$$\begin{aligned}\alpha'(G) &= \frac{1}{2}\big(n(G) - \max\{o(G \setminus S) - |S| : S \subseteq V(G)\}\big)\\ &= \frac{1}{2}\big(n(G) - 1\big)\\ &= \frac{n(G) - 1}{2}\end{aligned}$$

So $\alpha'(G - v) = \frac{n(G)-1}{2}$ for all $v \in V(G)$, so every such $G - v$ has a perfect matching.