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Project 5

MATH 423
December 2, 2019

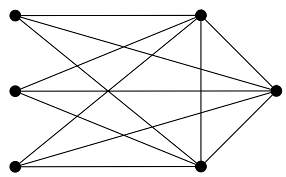
1 Introduction

1.1 Definition of split graphs

A split graph is one that can be partitioned into a clique and an independent set.

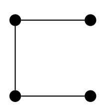
1.2 Examples

$K_n \vee \overline{K_m}$:

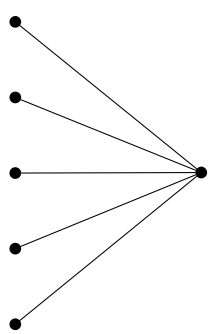


K_2:

P_4:



$K_{1,n}$:



1.3 Basic properties

- If G is a split graph, then $\overline{G}$ is a split graph.

- If G is a split graph with partition (S, K), S an independent set and K a clique, then exactly one of the following holds:
  1. $|S| = \alpha(G)$ and $|K| = \omega(G)$.
  2. $|S| = \alpha(G)$ and $|K| = \omega(G) - 1$, and there exists a vertex $x \in S$ such that $K + x$ is a complete graph.
  3. $|S| = \alpha(G) - 1$ and $|K| = \omega(G)$, and there exists a vertex $y \in K$ such that $S + y$ is independent.
- G is a split graph if and only if both G and $\overline{G}$ are chordal. [1]

2 Characterizations

2.1 Vertex ordering

Let G be a graph with degree sequence $d_1 \leq \cdots \leq d_n$, and let m be the largest i with $d_i \geq i - 1$. Then G is a split graph if and only if

$$\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i.$$

Furthermore, if the above equality holds, then $\omega(G) = m$. [2]

2.2 Forbidden subgraphs

G is a split graph if and only if G contains no induced subgraph isomorphic to any of C_4, $\overline{C_4}$, or C_5. [2]

3 Optimization problems

3.1 Colouring

Split graphs are in particular chordal graphs. [1] gives an algorithm to colour chordal graphs in linear time, so split graphs can also be coloured in linear time. This is a very hard problem in general, and colouring an arbitrary graph takes exponential time.

3.2 Maximum clique

m as defined in the vertex ordering characterization of split graphs allows us to find a maximum clique: the m vertices of largest degree form a maximum clique on G. This process can be done in linear time.

3.3 Maximum independent set

Let K be the maximum clique obtained using the above method. $V(G) \setminus K$ is an independent set, so we know from the basic properties that exactly one of the three cases is possible, and $\alpha(G)$ is either $|V(G) \setminus K|$ or $|V(G) \setminus K|+1$. If it is $|V(G) \setminus K|+1$, then there exists a vertex $y \in K$ such that $V(G) \setminus K + y$ is still independent. We can search K for such a vertex in linear time, and we're done.

3.4 Minimum clique covering

We know that $\overline{G}$ is a split graph, so we can colour $\overline{G}$ using the algorithm given in [1] to get a set of colour classes, all of which are independent sets in $\overline{G}$, hence cliques in G. Since the colouring is optimal, the corresponding clique cover is minimal.

References

[1] Martin C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs.* Elsevier, 2nd edition, 2004.

[2] P. L. Hammer and B. Simeone. The splittance of a graph. *Combinatorica*, 1(3):275–284, 1977.