Oliver Tonnesen
V00885732

Assignment 2

1

Suppose that $\{f, f', \ldots, f^{(n)}\}$ is linearly dependent, and let $f(x) = \sum_{i=0}^{n} a_n x^n$. Then there exist $b_0, \ldots, b_n$, not all 0, such that $b_0 f(x) + \cdots + b_n f^{(n)}(x) = 0$. Equivalently, $b_0 f(x) + \cdots + b_n f^{(n)}(x) = \sum_{i=0}^{n} 0x^i$. f has degree exactly n, and $f', \ldots, f^{(n)}$ have degree strictly less than n, so in $b_0 f(x) + \cdots + b_n f^{(n)}(x)$, the degree n term is $b_0 a_n x^n$. $a_n \neq 0$, since $f(x)$ has degree n, so since the degree n term in $b_0 f(x) + \cdots + b_n f^{(n)}(x)$ is also $0x^i$, we can conclude that $b_0 a_n = 0$. $\mathbb{R}$ is a field, so since $a_n \neq 0$, $b_0 = 0$. Then $b_0 f(x) + \cdots + b_n f^{(n)}(x) = b_1 f'(x) + \cdots + b_n f^{(n)}(x)$, and as before, the degree $n-1$ term in this expression is $(b_0 a_{n-1} + b_1 n a_n)x^{n-1} = b_1 n a_n x^{n-1}$. Again, $b_1 n a_n x^{n-1} = 0$. Since $a_n \neq 0$ and $n \neq 0$, we can conclude that $b_1 = 0$. This argument can be continued recursively to prove that $b_2 = \cdots = b_n = 0$, a contradiction since we assumed that $b_0, \ldots, b_n$ were not all 0. So $\{f, \ldots, f^{(n)}\}$ is linearly independent in $P_n(\mathbb{R})$. We know that $P_n(\mathbb{R})$ has dimension $n+1$, so since $|\{f, \ldots, f^{(n)}\}| = n+1$, it must be a basis for $P_n(\mathbb{R})$.

Let $\beta = \{f, \ldots, f^{(n)}\} = \{v_1, \ldots, v_n\}$. β is an ordered basis for $P_n(\mathbb{R})$. We calculate $[T]_\beta$.

$A = [T]_\beta$ if $Tv_j = \sum_{i=1}^{n} a_{ij} v_i$, $1 \leq j \leq n$. $Tv_i = Tf^{(j)} = f^{(j-1)} = v_{j+1}$, so $a_{ij} = \begin{cases} 1 & \text{if } i = j+1 \\ 0 & \text{otherwise} \end{cases}$, meaning that

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Or an identity matrix with an extra 0 row vector on top and 0 column vector on the right.

2

Let $v_1 + v_2, u_1 + u_2 \in V_1 + V_2$, $\lambda \in F$.

$$\begin{aligned} v_1 + v_2 + \lambda(u_1 + u_2) &= v_1 + v_2 + \lambda u_1 + \lambda u_2 \\ &= v_1 + \lambda u_1 + v_2 + \lambda u_2 \\ &= (v_1 + \lambda u_1) + (v_2 + \lambda u_2) \end{aligned}$$

$v_1 + \lambda u_1 \in V_1$ and $v_2 + \lambda u_2 \in V_2$, since V_1 and V_2 are both vector spaces over F, so $(v_1+\lambda u_1)+(v_2+\lambda u_2) \in V_1+V_2$, and thus V_1+V_2 is a subspace of V.

Let β, γ be bases for V_1, and V_2, respectively. Let $v_1 + v_2 \in V_1 + V_2$. Then v_1 and v_2 can be written as linear combinations of vectors in β and γ, respectively, so $V_1 + V_2 \subseteq \operatorname{Span} \beta \cup \gamma$, thus any basis for $V_1 + V_2$ has at most $|\beta \cup \gamma|$ vectors, thus $\dim(V_1 + V_2) \le |\beta \cup \gamma| \le |\beta| + |\gamma| = \dim(V_1) + \dim(V_2)$.

We show the necessity and sufficiency of $V_1 \cap V_2 = \{0\}$ for equality to hold:

($\Longrightarrow$): Assume $\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2)$. Let β_0 be a basis for $V_1 \cap V_2$. By the Replacement Theorem, β_0 can be extended to β_1, β_2, bases for V_1 and V_2, respectively. Then $\dim(V_1 + V_2) = |\beta_1 \cup \beta_2|$, so $|\beta_1 \cup \beta_2| = \dim(V_1) + \dim(V_2) = |\beta_1| + |\beta_2|$. Thus $\beta_1 \cap \beta_2 = \emptyset$. But $\beta_1 \cap \beta_2 = \beta_0$, so $\beta_0 = \emptyset$, thus $\operatorname{Span} \beta_0 = V_1 \cap V_2 = \{0\}$.

($\Longleftarrow$): Assume $V_1 \cap V_2 = \{0\}$. Let β_1 and β_2 be bases for V_1 and V_2, respectively. $\beta_1 \cap \beta_2 = \emptyset$, so $\dim(V_1) + \dim(V_2) = |\beta_1| + |\beta_2| = |\beta_1 \cup \beta_2| = \dim(V_1 + V_2)$.

Thus $\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) \Longleftrightarrow V_1 \cap V_2 = \{0\}$, as desired.

3

Assume $\operatorname{ran}(S) \cap \operatorname{ran}(T) = \{0\}$.

Assume for a contradiction that $S = \lambda T$ for some λ. Let $w \in \operatorname{ran}(S)$, and $v \in V$ such that $Sv = w$. Then $\lambda T v = w$. By linearity, $\lambda^{-1}\lambda T v = \lambda^{-1} w$, but $\lambda^{-1}\lambda T v = Tv$, so $\lambda^{-1} w \in \operatorname{ran}(T)$. Similarly, since $Sv = w$, $S(\lambda^{-1} w) = \lambda^{-1}(Sv) = \lambda^{-1} w \in \operatorname{ran}(S)$. So $\lambda^{-1} w \in \operatorname{ran}(S) \cap \operatorname{ran}(T)$, a contradiction, so there exists no λ such that $S \neq \lambda T$, and thus $\{S, T\}$ is linearly independent in $\mathcal{L}(V, W)$.

4

$\ker(T) = \{0\}$, so T is injective.

Let $\beta = \{v_1, \ldots, v_n\}$ be an ordered basis for V. T is injective, so $\{Tv_1, \ldots, Tv_n\}$ is linearly independent in V. Thus $\{Tv_1, \ldots, Tv_n\}$ is an ordered basis for V. Let $\gamma = \{Tv_1, \ldots, Tv_n\}$. We find $[T]_\beta^\gamma$:

$A = [T]_\beta^\gamma$ if $A_{ij} = a_{ij}$ such that $Tv_j = \sum_{i=1}^n a_{ij} Tv_i$, $1 \leq i \leq n$. Clearly $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$, so $A = \mathbb{1}$, as desired.