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Assignment 5

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1

We already know by its definition that f is continuous at each point in $[a, c] \setminus \{b\}$, so we need only show that f is also continuous at b. Let $\varepsilon > 0$, and WLOG let $x \in [a, b]$. We know that f is continuous over $[a, b]$, so there is some δ such that if $|x - b| < \varepsilon$, then $|f(x) - f(b)| < \delta$. So f is also continuous at b, and is thus continuous on all of $[a, c]$.

2

f is continuous at $x = 0$, and discontinuous elsewhere.

<u>f continuous at 0</u>: Let $\varepsilon > 0$, and let $\delta = \varepsilon$. If $x \in \mathbb{R} \setminus \mathbb{Q}$, then $f(x) = x$, and so $|f(x)| = |x| < \varepsilon = \delta$, so $|f(x)| < \delta$. If $x \in \mathbb{Q}$, then $|f(x)| = 0 < \varepsilon = \delta$, so again $f(x) < \delta$, as desired, and so f is continuous at $x = 0$.

<u>f discontinuous elsewhere</u>: Let $0 \neq \alpha \in \mathbb{Q}$, and let $x_n \to \alpha$ be a sequence of irrational numbers. Then $f(x_n) = 0$ for all n, so $lim_{n\to\infty} f(x_n) = 0$, but $f(\alpha) = \alpha \neq 0$, defying the limit definition of continuity, so f is discontinuous at $x \neq 0$.

3

Let $\varepsilon > 0$. S_n converges to $L \in \mathbb{R}$, so there exists some N such that for all $n > N$, $|S_n - L| < \varepsilon$. Note that the sequence of partial sums of a_n, say T_n, corresponds to $S_{\frac{n}{2}}$. Let $N' = 2N$. Then for any $n > N'$, we have

$$\begin{aligned}
|T_n - L| &= \left|S_{\frac{n}{2}} - L\right| \\
&< \left|S_{\frac{N'}{2}} - L\right| \\
&= \left|S_{\frac{2N}{2}} - L\right| \\
&= |S_N - L| \\
&< \varepsilon
\end{aligned}$$

So $T_n \to L$, and thus $\sum_{k=1}^{\infty} a_k = L$, as desired.

4

4.a

We show that $a_{n+1} - a_n > 0$ for any n.

$$\begin{aligned} a_{n+1} - a_n &= \left(1 - \frac{1}{2} + \cdots + \frac{1}{2n+1} - \frac{1}{2n+2}\right) - \left(1 - \frac{1}{2} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}\right) \\ &= \frac{1}{2n+1} - \frac{1}{2n+2} \\ &> 0 \end{aligned}$$

So (a_n) is monotonic increasing.

4.b

We show that $b_{n+1} - b_n < 0$ for any n.

$$\begin{aligned} b_{n+1} - b_n &= \left(1 - \cdots - \frac{1}{2n} + \frac{1}{2n+1}\right) - \left(1 - \cdots - \frac{1}{2n-2} + \frac{1}{2n-1}\right) \\ &= \frac{1}{2n+1} - \frac{1}{2n+2} \\ &> 0 \end{aligned}$$

So (b_n) is monotonic decreasing.

4.c

It's clear to see that $a_m < a_N$ and $b_N < b_n$, so we need only show that $a_N < b_N$. Note first that by definition, $a_N = b_N + \frac{(-1)^{2N+1}}{2N}$. $2N+1$ is always odd, so we have $a_N = b_N - \frac{1}{2N}$. $\frac{1}{2N} > 0$, so $a_N < b_N$, as desired.

4.d

(a_n) is monotonic increasing and as we saw in part (c), it is bounded above by all terms of (b_n), so it converges to a real number a. Similarly, (b_n) is monotinic decreasing and is bounded below by all terms of (a_n), so it converges to a real number b.

4.e

Let $\varepsilon > 0$, and let $N = \frac{1}{2\varepsilon}$. Then for any $n > N$,

$$\begin{aligned}|b_n - a_n| &= \frac{1}{2n} \\ &< \frac{1}{2N} \\ &= \frac{1}{2\frac{1}{2\varepsilon}} \\ &= \varepsilon\end{aligned}$$

The terms of (a_n) and (b_n) get arbitrarily close, so indeed $\lim_{n\to\infty} a_n = a = b = \lim_{n\to\infty} b_n$, as desired.

4.f

Let $s_n = \sum_{k=1}^{n} \frac{(-1)^k+1}{k}$. (a_n) and (b_n) represent the even and odd terms of (s_n), respectively. Both converge to a, and so the whole sequence (s_n) must converge to a.

4.g

Every $3n$th partial sum of (z_n) is simply the $3(n-1)$th partial sum of (z_n) added to $\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}$. In other words, the $3n$th partial sum of (z_n) is $\sum_{j=1}^{n}\left(\frac{1}{4j-3}+\frac{1}{4j-1}-\frac{1}{2j}\right)$.

$$\begin{aligned}&\sum_{j=1}^{n}\left(\frac{1}{4j-3}-\frac{1}{4j-2}+\frac{1}{4j-1}-\frac{1}{4j}\right)+\sum_{j=1}^{n}\left(\frac{1}{4j-2}-\frac{1}{4j}\right)\\ =&\sum_{j=1}^{n}\left(\frac{1}{4j-3}-\frac{1}{4j-2}+\frac{1}{4j-1}-\frac{1}{4j}+\frac{1}{4j-2}-\frac{1}{4j}\right)\\ =&\sum_{j=1}^{n}\left(\frac{1}{4j-3}+\frac{1}{4j-1}-\frac{2}{4j}\right)\\ =&\sum_{j=1}^{n}\left(\frac{1}{4j-3}+\frac{1}{4j-1}-\frac{1}{2j}\right)\end{aligned}$$

So indeed the partial sums of (z_n) are equal to the given expression.

4.h

Note first that the first term of the expression is just the $(3n)$th partial sum of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$, and that the second term of the expression is half that. So, since $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$, as we saw in (f), it must be the case that $\sum_{n=1}^{\infty} z_n = \frac{3a}{2}$.

5

Let $a_n = \frac{1}{2n}$. We know that this subsequence consisting of the even terms $\frac{(-1)^n}{n}$ converges to ∞, so for any $M > 0$, there exists N such that $s_n = \sum_{i=1}^{n} a_n > M$ for all $n > N$. We construct our sequence as follows:

Beginning with an empty sequence, we choose N_1 such that $s_n > 2$ for all $n > N_1$. We append the first N_1 terms of a_n to our sequence, followed by the first odd term. Now our sequence sums to at least 1.

Next, we choose N_2 such that $s_n > 3$ for all $n > N_2$. We append the next even terms of a_n up to N_2 to our sequence, followed by the second odd term.

At each step i, we choose N_i such that $s_n > i$ for all $n > N_i$, and append to our sequence from the N_{i-1} to N_ith even terms, followed by the ith odd term. In the end, we end up with the sequence

$$\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2N_1} - 1\right) + \left(\cdots\right) + \cdots + \left(\cdots\right) + \left(\frac{1}{2N_{i-1}+2} + \frac{1}{2N_{i-1}+3} + \cdots + \frac{1}{2N_i} - \frac{1}{i}\right) + \cdots .$$

At each step in this process, our sequence sums to at least i, so it is clear to see that it indeed converges to ∞ as n grows large.

6