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Assignment 1

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1

1.a

Let $gh_1g^{-1}, gh_2g^{-1} \in gHg^{-1}$. We wish to show the following:

$$gHg^{-1} \neq \emptyset \tag{1}$$
$$gh_1g^{-1} \cdot (gh_2g^{-1})^{-1} \in gHg^{-1} \tag{2}$$

proving that gHg^{-1} is a subgroup of G.

(1): We have that $H \leq G$, so $e \in H$. Thus $geg^{-1} = gg^{-1} = e \in gHg^{-1}$, and so $gHg^{-1} \neq \emptyset$.

(2): $h_2 \in H$, so $h_2^{-1} \in H$, and thus $(gh_2g^{-1})^{-1} = gh_2^{-1}g^{-1} \in gHg^{-1}$.

$$\begin{aligned} gh_1g^{-1} \cdot gh_2^{-1}g^{-1} &= gh_1g^{-1}gh_2^{-1}g^{-1} \\ &= gh_1h_2^{-1}g^{-1} \end{aligned}$$

$h_1h_2^{-1}$ is an element of H, so $gh_1h_2^{-1}g^{-1}$ is an element of gHg^{-1}, and therefore gHg^{-1} is a subgroup of G.

1.b

Let $g \in G$. We know that $gHg^{-1} \leq G$. $|gHg^{-1}| = |H|$, and since H is the only subgroup of G with order n, it must be the case that $gHg^{-1} = H$, and so $H \trianglelefteq G$.

2

Let $H_1 = \langle (r^2, e) \rangle = \{(e,e), (r^2, e)\}$, and $H_2 = \langle (j, e) \rangle = \{(e,e), (j,e)\}$.

Notice that r^2 commutes with every element in D_4, so $H_1 \trianglelefteq G$.

We have that $(r,e)H_2 = \{(r,e), (rj,e)\} \neq \{(r,e), (r^3j, e)\} = H_2(r,e)$, so $H_2 \ntrianglelefteq G$.

Finally, both groups have order 2, and are therefore isomorphic since there only exists a single group (up to isomorphism) of order 2.

3

3.a

Let $g \in G$, and $z \in Z(G)$. We wish to show that $gzg^{-1} \in Z(G)$.

$$\begin{aligned} gzg^{-1} &= gg^{-1}z \\ &= z \in Z(G) \end{aligned}$$

Thus $gZg^{-1} \subseteq Z(G)$, and so $Z(G) \trianglelefteq G$.

3.b

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{R})$ and $Z = \left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right) \in Z(GL_2(\mathbb{R}))$. We want Z such that $AZ = ZA$ for any A, so we have the following equation:

$$\begin{gathered} AZ = ZA \\ \left(\begin{smallmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{smallmatrix}\right) = \left(\begin{smallmatrix} wa+xc & wb+xd \\ ya+zc & yb+zd \end{smallmatrix}\right) \end{gathered}$$

Which gives us the following system of equations:

$$\begin{aligned} aw + by &= wa + xc \\ ax + bz &= wb + xd \\ cw + dy &= ya + zc \\ cx + dz &= yb + zd \end{aligned}$$

Examine the equation $aw + by = wa + xc$. This gives us $by = cx$, but b and c are arbitrary (and in particular, it could be the case that $b \neq c$, where $b \neq 0 \neq c$), so it must be the case that $x = y = 0$. Now substituting this back into the equation $ax + bz = wb + xd$, we get $bz = wb$, so we now have that $w = z$. Thus our general Z is of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$. More specifically,

$$Z(GL_2(\mathbb{R})) = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \mid a \in \mathbb{R} \setminus \{0\}\}.$$

3.c

$1331 = 11^3$, so by Lagrange's Theorem, the order of $Z(G)$ can be 1, 11, 11^2, or 11^3. $Z(G)$ has at least one element other than the identity, so it cannot have order 1. Since G is nonabelian, there are at least two elements in G which do not commute with one another, and thus both cannot cannot be in $Z(G)$, so it also cannot have order 11^3.

Suppose for a contradiction that $|Z(G)| = 11^2$. Then $|G/Z(G)| = 11$. Since the order of $G/Z(G)$ is prime, then by Lagrange's Theorem, it has no non-trivial subgroups, meaning $G/Z(G)$ is cyclic. Let $\langle aZ(G)\rangle = G/Z(G)$. Then any $g \in G$ can be rewritten as $a^k z$ from some integer k and some $z \in Z(G)$. z is in the centre, so by definition $a^k z = za^k$. This means that

$$\begin{aligned} gh &= a^m z_1 a^n z_2 \\ &= a^{m+n} z_1 z_2 \\ &= a^{n+m} z_2 z_1 \\ &= a^n z_2 a^m z_1 \\ &= gh, \end{aligned}$$

which implies that G is abelian, a contradiction, and so $|Z(G)| = 11$.

4

Let $re^{i\theta} \in H$ and $se^{i\phi} \in G$. Of course $r = 1$ by definition of H, so $re^{i\theta} \cdot se^{i\phi} = se^{i(\theta+\phi)}$. Since any two elements of H differ only by θ, they form a circle centered on the origin. Any coset of H will simply be all of these points rotated (if $\phi \neq 0$) and/or lengthened (if $s \neq 1$), and will remain a circle centered on the origin. This mental image leads naturally to the following isomorphism:

$$\begin{aligned} f : G/H &\longrightarrow \mathbb{R}^+ \\ se^{i\phi}H &\longmapsto s \end{aligned}$$

5

5.a

Let $m = |gH|$, and $n = |g|$. Since $g^n = e$, we have that $(gH)^n = g^n H = eH = H$. Thus m is no greater than n. This means that we can rewrite n as $n = qm + r$, where q and r are integers and $0 \leq r < m$.

We rewrite H as $(gH)^n = (gH)^{qm+r}$.qm is a multiple of the order of gH, so $(gH)^{qm} = H$ and $(gH)^{qm+r} = (gH)^r$, and we're left with $H = (gH)^r$. It cannot be the case that $r \neq 0$, since it would contradict the definition of m as the smallest positive integer m such that $(gH)^m = H$. Thus $r = 0$, and we have $n = qm$ for some integer q.

5.b

We consider a single element of $\langle [6]_{42} \rangle$, $[0]_{42}$, and see how many times we must add $[4]_{42}$ to it until it is once again an element in $\langle [6]_{42} \rangle$.

$$1 \cdot [4]_{42} + [0]_{42} = [4]_{42} \notin \langle [6]_{42} \rangle$$
$$2 \cdot [4]_{42} + [0]_{42} = [8]_{42} \notin \langle [6]_{42} \rangle$$
$$3 \cdot [4]_{42} + [0]_{42} = [12]_{42} \in \langle [6]_{42} \rangle$$

Thus the coset $[4]_{42} + \langle [6]_{42} \rangle$ has order 3 in $\mathbb{Z}_{42} \langle [6]_{42} \rangle$.

5.c

Let $G = \mathbb{Z}_{12}$, $H = \langle [2]_{12} \rangle$, $g_1 = [2]_{12}$, and $g_2 = [6]_{12}$.

$g_1 H$ and $g_2 H$ are clearly the same coset, and $|g_1| = 6 \neq 2 = |g_2|$.