MATH 322 Assignment 4

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1.a

Reflexive: By definition, $\preceq$ is reflexive.
Antisymmetric: Suppose

1. $x \preceq y$
2. $y \preceq x$

Then by 1., x is an odd string, and by 2., y is an odd string. We now now that either $x = y$ or x and y are adjacent in Q_n. Both x and y are odd, and can thus not be adjacent in Q_n, so $x = y$.
Transitive: Suppose $x \preceq y$ and $y \preceq z$, and that $x \neq y$. Then x is an odd string, meaning y must be an even string. The only b such that there exists an even string a with $a \preceq b$ is $b = a$, so $y \preceq z \implies y = z$. We now have that $x \preceq y$ and $y \preceq z \iff x \preceq y$ and $y \preceq y$. Thus transitivity holds when $x \neq y$, and we can see that this is trivially true when $x = y$.

1.b

Let $A = \{x \in X \mid x \text{ is an even string}\}$. Then A is a maximum antichain, since $A = \frac{|X|}{2}$ and Q_n is bipartite.

1.c

Construct our chain cover as follows:

- Take the lexicographically smallest unmarked element in X, and place it in a chain with the next element in lexicographic order.
- Mark both elements
- Repeat until no unmarked elements remain.

Each chain constructed in this fashion contains two elements, and so we end up with $\frac{|X|}{2}$ chains in our covering – the as the number of elements in our maximum antichain.

2

$$\mathcal{R} \subseteq \mathcal{R}^* \iff \nexists x, y \in X, x \neq y, \text{ s.t. } x\mathcal{T}y \text{ and } y\mathcal{T}x$$

in $\mathcal{T}$, $\mathcal{R}$'s transitive closure.
($\Longrightarrow$)
The proof of the forward direction is trivial.
($\Longleftarrow$)
We show the contrapositive is true:

$\mathcal{R} \nsubseteq \mathcal{R}^* \implies \exists x, y \in X, x \neq y$, s.t. $x\mathcal{T}y$ and $y\mathcal{T}x$ in $\mathcal{T}$, $\mathcal{R}$'s transitive closure.

$\mathcal{R}$ is not the subset of a partial order, so it must be the case that there are $x, y \in X, x \neq y$ such that $x\mathcal{R}y$ and $y\mathcal{R}x$. These x, y satisfy the above statement, and so our condition holds.

To obtain $\mathcal{R}^*$, add all elements of the form $x\mathcal{R}x$ and also ensure that if $x\mathcal{R}y$ and $x\mathcal{R}y$, then $x\mathcal{R}z$ for all $x, y, z \in X$.

3

Enumerate the subsets of size $\left\lfloor \frac{n}{2} \right\rfloor$. For each subset, pair it with the subset obtained by adding the least element that does not form a set we've already created. A partial example is given below with $n = 5$:

For $\{1, 2\}$, the least element we can add is 3, so we pair $\{1, 2\}$ with $\{1, 2, 3\}$.
For $\{1, 3\}$, the least element we can add is not 2, since we've already created the set $\{1, 2, 3\}$, so we instead add 4, pairing $\{1, 3\}$ with $\{1, 3, 4\}$. For $\{1, 4\}$, the least element we can add is 2, so we pair $\{1, 4\}$ with $\{1, 2, 4\}$.
For $\{1, 5\}$, the least element we can add is 2, so we pair $\{1, 5\}$ with $\{1, 2, 5\}$.
For $\{2, 3\}$, the least element we can add is not 1, since we've already created the set $\{1, 2, 3\}$, so we instead add 4, pairing $\{2, 3\}$ with $\{2, 3, 4\}$.
And continuing on following the same pattern.

This method works since if we fix any $x \in [n]$, there are exactly $n - 1$ subsets of size $\left\lfloor \frac{n}{2} \right\rfloor$ containing it. Similarly, there are $n - 1$ other elements of $[n]$ to add to some subset, so for each subset we consider we can create a supserset by adding a single element.

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Assume for a contradiction that no such S exists. We know that there exists a subset Y not in $\mathcal{F}$ whose complement is also not in $\mathcal{F}$. In other words, both $\mathcal{F} \cup Y$ and $\mathcal{F} \cup ([n] \setminus Y)$ are not intersecting families. This means that there are sets $U, V \in \mathcal{F}$ such that $U \cap Y = \emptyset$ and $V \cap ([n] \setminus Y) = \emptyset$, a contradiction since U and V must intersect by definition. So our assumption was false and such an S indeed exists for any $|\mathcal{F}| < 2^{n-1}$.

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Fix some $x \in [n]$, and choose every k subset of $[n]$ containing x. After fixing x, there remain $n - 1$ elements from which to choose the $k - 1$ elements of each k-subset. Thus our intersecting family $\mathcal{F}$ contains exactly $\binom{n-1}{k-1}$ subsets of $[n]$.