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Assignment 2

MATH 236
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1

Let $\varepsilon > 0$, and let $N = \frac{1/\varepsilon + 3}{2}$. Then for any $n > N$,

$$\begin{aligned}
\left|\frac{1}{2n-3}\right| &< \left|\frac{1}{2N-3}\right| \\
&= \left|\frac{1}{2\left(\frac{1/\varepsilon+3}{2}\right)-3}\right| \\
&= |\varepsilon| \\
&= \varepsilon
\end{aligned}$$

and so, indeed, $a_n \to 0$, as desired.

2

We claim the limit is $b = \frac{3}{2}$. Note first that $\left|\frac{3n^2+4n+5}{2n^2+6n+7} - \frac{3}{2}\right| = \left|\frac{5+11/2n}{2n+6+7/n}\right|$. Let $\varepsilon > 0$, and let $N = \max\left\{7, \frac{3}{\varepsilon}\right\}$. If $n > 7$, then $5 + \frac{11}{2n} < 6$ and $2n + 6 + \frac{7}{n} < 2n + 7$. Then for any $n > N$,

$$\begin{aligned}
\left|\frac{5+11/2n}{2n+6+7/n}\right| &< \left|\frac{6}{2n+6+7/n}\right| \\
&< \left|\frac{6}{2n+7}\right| \\
&< \left|\frac{3}{n}\right| \\
&< \left|\frac{3}{N}\right| \\
&= \left|\frac{3}{3/\varepsilon}\right| \\
&= \varepsilon
\end{aligned}$$

and so $b_n \to \frac{3}{2}$, as desired.

3

Let $M > 0$. $y_n \to \infty$, so we know there is some N such that $y_n > \frac{M}{\inf\{a_n\}}$ for any $n > N$. Then $\inf\{a_n\}\, y_n > M$, so $a_n y_n > M$, and thus $a_n y_n \to \infty$, as desired.

4

We claim the limit is 0. Let $\varepsilon > 0$, and let $N = \max\left\{\frac{5}{\varepsilon}, \frac{1}{\sqrt[5]{\varepsilon}}\right\}$. Then for any $n > N$:
if n is even, then $|s_n| = \left|\frac{5}{n}\right| < \left|\frac{5}{N}\right| \leq \left|\frac{5}{5/\varepsilon}\right| = \varepsilon$, and
if n is odd, then $|s_n| = \left|\frac{1}{n^5}\right| < \left|\frac{1}{N^5}\right| \leq \left|\frac{1}{1/\sqrt[5]{\varepsilon}}\right| = \varepsilon$.
Thus $s_n \to 0$, as desired.

5

We claim the limit is 0. Let $\varepsilon > 0$, and let $N = \frac{1}{2\varepsilon}$. Then

$$\begin{aligned}
\left|\sqrt{n} - \sqrt{n-1}\right| &= \left|\frac{1}{\sqrt{n} + \sqrt{n-1}}\right| \\
&< \left|\frac{1}{2\sqrt{n}}\right| \\
&< \left|\frac{1}{2n}\right| \\
&< \left|\frac{1}{2N}\right| \\
&= \left|\frac{1}{2/2\varepsilon}\right| \\
&= \varepsilon
\end{aligned}$$

and so $u_n \to \infty$, as desired.

6

We claim the sequence converges to ∞. Let $M > 0$, and let $N = \max\left\{3, M^2\sqrt{32}\right\}$.
If $n > 3$, then $3 + \frac{3}{n} + \frac{1}{n^2} < 4$.

$$\begin{aligned}
\left|(n+1)^{3/2} - n^{3/2}\right| &= \left|\frac{(n+1)^3 - n^3}{(n+1)^{3/2} + n^{3/2}}\right| \\
&= \left|\frac{n^3 + 3n^2 + 3n + 1 - n^3}{(n+1)^{3/2} + n^{3/2}}\right| \\
&= \left|\frac{3n^2 + 3n + 1}{(n+1)^{3/2} + n^{3/2}}\right|
\end{aligned}$$

$$
\begin{aligned}
&> \left|\frac{3n^2+3n+1}{2(n+1)^{3/2}}\right| \\
&= \left|\frac{3n^2+3n+1}{2\sqrt{n^3+3n^2+3n+1}}\right| \\
&> \left|\frac{3n^2+3n+1}{2\sqrt{8n^3}}\right| \\
&= \left|\frac{3n^2+3n+1}{\sqrt{32}n^{3/2}}\right| \\
&= \left|\frac{3\sqrt{n}+3/\sqrt{n}+1/n^{3/2}}{\sqrt{32}}\right| \\
&> \left|\frac{3\sqrt{n}}{\sqrt{32}}\right| \\
&> \left|\frac{3\sqrt{N}}{\sqrt{32}}\right| \\
&= 3M \\
&> M
\end{aligned}
$$

and so $v_n \to \infty$, as desired.

7

We know that $x_n \to \infty$, so for any $\varepsilon > 0$, there is some N such that $|x_n - 15| < \varepsilon$ for all $n > N$. If we take ε to be less than 5, then all $|x_n|$, $n > N$, will lie between 10 and 20. This means that all x_n lying outside this range occur before N. All x_n with $|x_n| > 20$ will clearly be a subset of these. More formally, $\{n \mid |x_n| > 20\} \subseteq \{n \mid n \leq N\}$, and thus is finite, as desired.

8

8.a

Suppose it did, and let N be such a number that the condition holds. Let $\varepsilon = \frac{1}{N+2}$. It is clearly not the case that $|s_n| < \varepsilon$ for all $n > N$, since $N+1 > N$, but $s_{N+1} = \frac{1}{N+1} \geq \varepsilon$. Thus forms a contradiction, and so it is not the case that $s \rightsquigarrow 0$.

8.b

Let N be a number satisfying the conditions for $s_n \rightsquigarrow s$, and let $\varepsilon > 0$. By the definition of $s_n \rightsquigarrow s$, if $n > N$, then $|s_n - s| < \varepsilon$, so indeed $s_n \to s$, as desired.

8.c

Let $a_n = \begin{cases} 1 & \text{if there is a solution to } x^n + y^n = z^n \\ 0 & \text{otherwise} \end{cases}$

It is the case that $a_n \rightsquigarrow 0$. I have a truly marvelous proof this, which this homework is too narrow to contain.

8.d

The condition $s_n \rightsquigarrow s$ holds when s_n eventually becomes constant. That is, there is some N such that $s_N = s_{N+1} = \cdots = s$