1

1.a

We consider two cases: Case $a \neq 0$:
Let $[a, b] \in Q$ with $0 \neq a = p + q\sqrt{3}$ and $b = n + m\sqrt{3}$. Then suppose there exists $c \in R, k \in \mathbb{Z}^{\times}$ such that $[a, b] = [c, k]$. Let $c = x + y\sqrt{3}$. It must be the case that $ak = bc$.

$$\begin{aligned} bc &= (n + m\sqrt{3})(x + y\sqrt{3}) \\ &= (nx + 3my) + (ny + mx)\sqrt{3} \end{aligned}$$

$$\begin{aligned} ak &= (p + q\sqrt{3})(k) \\ &= (pk) + (qk)\sqrt{3} \end{aligned}$$

Thus $nx + 3my = pk$ and $qk = ny + mx$. $x = \frac{pk - 3my}{n}$, so $qk = ny + m\left(\frac{pk-3my}{n}\right)$.

$$\begin{aligned} qkn &= n^2 y + mpk - 3m^2 y \\ \implies qkn - mpk &= y(n^2 - 3m^2) \\ \implies y &= \frac{k(qn - mp)}{n^2 - 3m^2} \\ \implies y &= \frac{qk - mx}{n} \end{aligned}$$

$$\begin{aligned} nx + 3m\left(\frac{qk - mx}{n}\right) &= pk \\ \implies n^2 x + 3mqk - 3m^2 x &= pkn \\ \implies x(n^2 - 3m^2) &= pkn - 3mqk \\ \implies x &= \frac{k(pn - 3mq)}{n^2 - 3m^2} \end{aligned}$$

Then let $c = x + y\sqrt{3}$ such that $x, y \in \mathbb{Z}$ and $c = R$, and let $k = n^2 - 3m^2$. Then there exists $c \in R, k \in \mathbb{Z}$ such that $[a, b] = [c, k], b \neq 0$. Case $a = 0$:
$a = 0$ and $ak = bc$, so $bc = 0$, by definition, $b \neq 0$, and so $c = 0$ and thus $[a, b] = [c, k]$ for all $k \in \mathbb{Z}^{\times}$.

1.b

Let $m \in F$ where $m = x + y\sqrt{3}$, $x = \frac{a}{b}$, $y = \frac{p}{q}$, with $a, p \in \mathbb{Z}$ and $b, q \in \mathbb{Z}^{\times}$. Then $m = x + \sqrt{3}y = \frac{a}{b} + \sqrt{3}\frac{p}{q} = \frac{aq + \sqrt{3}pb}{bq}$.

Finally, we define our isomorphism:

$$f : F \longrightarrow Q$$
$$\frac{aq + \sqrt{3}pb}{bq} \longmapsto [aq + \sqrt{3}bp, bq]$$

Surjective: take any $[c, u] \in Q$, then $c = n + \sqrt{3}m$, $k = i + \sqrt{3}j$ then by (a), $[c, k] = [n + \sqrt{3}m, i + \sqrt{3}j] = [v + w\sqrt{3}, k]$.
****Thus WTS $[v + w\sqrt{3}, k] = [aq + \sqrt{3}bp, bq]$ such that $v = aq$, $w = bp$, $k = bq$****.