MATH 200 Assignment Two

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Recall the curvature formula:

$$\kappa = \frac{|f''(x)|}{\left(1+|f'(x)|^2\right)^{\frac{3}{2}}}$$

We will use this formula with $f(x) = e^x$:

$$\begin{aligned}
\kappa &= \frac{|f''(x)|}{\left(1+|f'(x)|^2\right)^{\frac{3}{2}}} \\
&= \frac{|e^x|}{\left(1+|e^x|^2\right)^{\frac{3}{2}}} \\
&= \frac{e^x}{\left(1+e^{2x}\right)^{\frac{3}{2}}} && (e^x > 0 \text{ for all } x)
\end{aligned}$$

So the curvature of $f(x)$ is $\frac{e^x}{\left(1+e^{2x}\right)^{\frac{3}{2}}}$, and we can now use basic calculus to find its maximum:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{e^x}{\left(1+e^{2x}\right)^{\frac{3}{2}}}\right) &= e^x \cdot \left(1+e^2x\right)^{-\frac{3}{2}} - 3e^{3x} \cdot \left(1+e^{2x}\right)^{-\frac{5}{2}} \\
&= \frac{e^x}{\left(1+e^{2x}\right)^{\frac{5}{2}}}\left(\left(1+e^{2x}\right) - 3e^{2x}\right) \\
&= \frac{e^x}{\left(1+e^{2x}\right)^{\frac{5}{2}}}\left(1-2e^{2x}\right) \\
&= \frac{e^x - 2e^{3x}}{\left(1+e^{2x}\right)^{\frac{5}{2}}}
\end{aligned}$$

We find the critical points:

$$\begin{aligned}
\frac{e^x - 2e^{3x}}{\left(1+e^{2x}\right)^{\frac{5}{2}}} &= 0 \\
e^x - 2e^{3x} &= 0 \\
e^x &= 2e^{3x} \\
1 &= 2e^{2x} && (e^x \neq 0 \text{ for all } x) \\
e^{2x} &= \frac{1}{2} \\
e^x &= \frac{1}{\sqrt{2}} \\
x &= \ln\left(\frac{1}{\sqrt{2}}\right)
\end{aligned}$$

So the point of maximum curvature for the curve $y = e^x$ is at $x = \frac{1}{\sqrt{2}}$. We will now find the curvature's behavior as $x \to \infty$:

$$\lim_{x\to\infty} \frac{e^x}{\left(1+e^{2x}\right)^{\frac{3}{2}}}$$

Note first that both the numerator and the denominator are always greater than zero, and so the fraction can never be less than zero.

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^x}{\left(1+e^{2x}\right)^{\frac{3}{2}}} &= \lim_{x\to\infty} \frac{e^x}{\left[\left(1+e^{2x}\right)^3\right]^{\frac{1}{2}}} \\
&= \lim_{x\to\infty} \frac{e^x}{\left(1+3e^{2x}+3e^{4x}+e^{6x}\right)^{\frac{1}{2}}} \\
&\leq \lim_{x\to\infty} \frac{e^x}{\left(e^{6x}\right)^{\frac{1}{2}}} \\
&= \lim_{x\to\infty} \frac{e^x}{e^{3x}} \\
&= \lim_{x\to\infty} \frac{1}{e^{2x}} \\
&= 0
\end{aligned}$$

So the curvature is less than or equal to a similar function that goes to zero. Recall that our curvature is always greater than zero, so the curvature goes to zero as x goes to infinity.

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$$f(x, y) = \frac{\sqrt{y - x^2}}{1 - x^2}$$

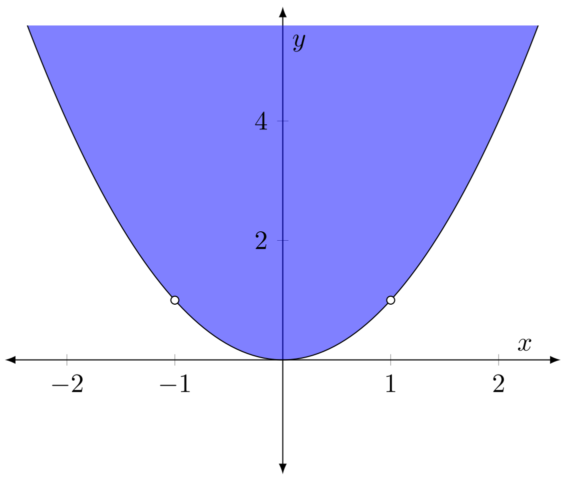
So our constraints on x and y are:

$$y - x^2 \geq 0$$
$$y \geq x^2$$

since the square root of a negative number is undefined, and

$$1 - x^2 \neq 0$$
$$x^2 \neq 1$$
$$x \neq \pm 1$$

since division by zero is undefined. So our domain is the region above and including the parabola $y = x^2$ with removable discontinuities at $x = 1$ and $x = -1$:



3

We'll sketch the isothermals at $T(x, y) = 1, 2, 3, 4$. $T(x, y) = 1$:

$$1 = \frac{100}{1 + x^2 + 2y^2}$$
$$1 + x^2 + 2y^2 = 100$$
$$x^2 + 2y^2 - 99 = 0$$

The same process can be repeated for $T(x, y) = 2, 3, 4$:

$T(x, y) = 2$:

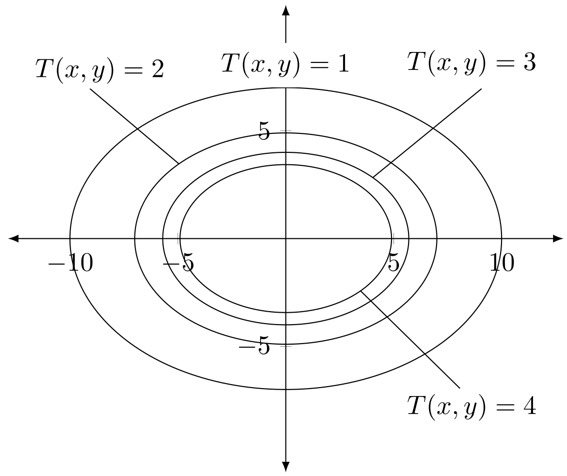
$$2x^2 + 4y^2 - 98 = 0$$

$T(x, y) = 3$:

$$3x^2 + 6y^2 - 97 = 0$$

$T(x,y) = 4$:

$$4x^2 + 8y^2 - 96 = 0$$



4

We consider two lines: $\{(x,y)|x = z^2, y = z^2\}$ and $\{(x,y)|x = z^2, y = 0\}$.

$\{(x,y)|x = z^2, y = z^2\}$:

$$\begin{aligned}
&\lim_{z\to 0} \frac{(z^2)(z^2) + (z^2)z^2 + (z^2)z^2}{(z^2)^2 + (z^2)^2 + z^4} \\
&= \lim_{z\to 0} \frac{3z^4}{3z^4} \\
&= \lim_{z\to 0} \frac{1}{1} \\
&= 1
\end{aligned}$$

$\{(x, y)|x = z^2, y = 0\}$:

$$\begin{aligned}
&\lim_{z\to 0} \frac{(z^2)(0) + (0)z^2 + (z^2)z^2}{(z^2)^2 + (0)^2 + z^4} \\
&= \lim_{z\to 0} \frac{0 + 0 + z^4}{z^4 + 0 + z^4} \\
&= \lim_{z\to 0} \frac{z^4}{2z^4} \\
&= \lim_{z\to 0} \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

$1 \neq \frac{1}{2}$, so the limit does not exist.

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First we define the function $z(x, y) = x^2 + y^2$. (Note that $\lim_{(x,y)\to(0,0)} z(x, y) = 0$) Then

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2} &= \lim_{z\to 0} \frac{e^{-z} - 1}{z} \\
&= \lim_{z\to 0} \frac{-e^{-z}}{1} && \text{(L'Hôpital's Rule)} \\
&= -e^{-0} \\
&= -e^{0} \\
&= -1
\end{aligned}$$

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$$\frac{\partial w}{\partial x} = \frac{1}{x + 2y + 3z}$$

$$\frac{\partial w}{\partial y} = \frac{2}{x + 2y + 3z}$$

$$\frac{\partial w}{\partial z} = \frac{3}{x + 2y + 3z}$$

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Recall Clairaut's Theorem:
Suppose f is defined on a disk D that contains (a, b). If f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

$$\frac{\partial f}{\partial x} = ye^{xy} \sin y$$
$$\frac{\partial^2 f}{\partial x \partial y} = e^{xy} \sin y + yxe^{xy} \sin y + ye^{xy} \cos y$$
$$\frac{\partial f}{\partial y} = xe^{xy} \sin y + e^{xy} \cos y$$
$$\frac{\partial^2 f}{\partial y \partial x} = xye^{xy} \sin y + e^{xy} \sin y + ye^{xy} \cos y$$

We find that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, and so Clairaut's Theorem holds.

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$$\frac{\partial z}{\partial u} = (v - w)^{\frac{1}{2}}$$
$$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{2}(v - w)^{-\frac{1}{2}}$$

$$\frac{\partial^3 z}{\partial u \partial v \partial w} = \frac{1}{4}(v - w)^{-\frac{3}{2}}$$

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Recall the Law of Cosines:

$$a^2 = b^2 + c^2 + 2bc \cos\left(A\right)$$
$$A = \arccos\left(\frac{a^2 - b^2 - c^2}{2bc}\right)$$

$$
\begin{aligned}
\frac{\partial A}{\partial a} &= -\frac{\frac{2a}{2bc}}{\sqrt{1-\left(\frac{a^2-b^2-c^2}{2bc}\right)^2}} \\
&= -\frac{a}{bc\sqrt{1-\left(\frac{a^2-b^2-c^2}{2bc}\right)^2}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial A}{\partial b} &= -\frac{\frac{-a^2-b^2+c^2}{2cb^2}}{\sqrt{1-\left(\frac{a^2-b^2-c^2}{bc}\right)^2}} \\
&= \frac{a^2+b^2-c^2}{2cb^2\sqrt{1-\left(\frac{a^2-b^2-c^2}{bc}\right)^2}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial A}{\partial c} &= -\frac{\frac{-a^2+b^2-c^2}{2bc^2}}{\sqrt{1-\left(\frac{a^2-b^2-c^2}{bc}\right)^2}} \\
&= \frac{a^2-b^2+c^2}{2bc^2\sqrt{1-\left(\frac{a^2-b^2-c^2}{bc}\right)^2}}
\end{aligned}
$$