MATH 236 - TAKE HOME FINAL

(due on CourseSpaces by April 6th at 11:59pm)

1) [15 points] Consider the function $f(x) = \frac{1}{x}$ on the interval $(1, 2)$. Is f uniformly continuous on that interval?

2) [15 points] The function $f(x) = x^3$ is uniformly continuous on the interval $I = [10, 100]$. Find a δ such that if $|x - y| < \delta$ and $x, y \in I$, then $|f(x) - f(y)| < 10^{-10}$, proving that your δ is sufficient.

3) [15 points] Let $\sum_{n=1}^{\infty} a_n$ be a convergent series (to a real number) with $a_n \geq 0$ for each n. Show that $\sum_{n=1}^{\infty} a_n^2$ converges to a real number. [Hint: when is x^2 smaller than x?]

4) [15 points] Let $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ be uniformly continuous functions such that the range of g is contained in A. Prove that $f \circ g$ defined by $f \circ g(x) = f(g(x))$ is uniformly continuous on B.

5) [15 points] Define the set S by

$$S = \bigcup_{n=1}^{\infty} \left\{ \frac{a}{2^n} : 0 \leq a \leq 4^n, a \in \mathbb{Z} \right\}.$$

Identify $\bar{S}$, showing that $\bar{S}$ is what you claim. [Hint: for any real number x and any positive integer k, there exists $a \in \mathbb{Z}$ such that $kx \in [a, a+1)$, so that $|x - \frac{a}{k}| < \frac{1}{k}$.]

6) [10 points] Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Show that if there are subsequences $(a_{p_i})_{i=1}^{\infty}$ and $(a_{q_i})_{i=1}^{\infty}$ with $a_{p_i} \to \alpha$ and $a_{q_i} \to \beta$ with $\alpha \neq \beta$, then (a_n) is not a convergent sequence.

7) [10 points] Let S be a bounded subset of $\mathbb{R}$ (above and below) and let $T = \{x^2 \colon x \in S\}$. Prove that $\sup T = \max((\sup S)^2, (\inf S)^2)$.

Must $\inf T$ be equal to $\min((\sup S)^2, (\inf S)^2)$?

8) [5 points] Let (a_n) be a sequence of real numbers with $a_n \to \alpha$. Show that $b_n \to \frac{\alpha}{2}$, where $b_n = \frac{1}{n^2}(a_1 + 2a_2 + \ldots + na_n)$.