Oliver Tonnesen
V00885732

Assignment 4

1

Let $\varepsilon > 0$. We show that $|b_n| < \varepsilon$.
Since $a_n \to 0$, there exists N such that for any $n > N$, $|a_n| < \varepsilon$.
If we let $n > N$, then $|b_n| \leq a_n = |a_n| < |a_N| < \varepsilon$, and so $b_n \to 0$.

2

Let $A_n = a_1 + \cdots + a_n$, $B_n = b_1 + \cdots + b_n$. Then $A_n \to \alpha$ and $B_n \to \beta$. We know that the limit of the sum of two convergent sequences is the sum of their limits, so $A_n + B_n \to \alpha + \beta$.

Similarly, we know that the limit of the product of two convergent sequences is the product of their limits, so also $A_n B_n \to \alpha\beta$.

3

($\Longrightarrow$): We prove the contrapositive. Let $a = 1$, and let $s_n = 1 + \cdots + \frac{1}{n}$ be the partial sums of the series.

$$\begin{aligned}
s_n &= 1 + \cdots + \frac{1}{n} \\
&\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{2^k} && \text{Where the terms are grouped by powers of 2} \\
&\geq \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} && \text{the constant sequence of } \frac{1}{2}
\end{aligned}$$

As $n \to \infty$, the above sequence of $\frac{1}{2}$ converges to ∞, so by the comparison test, $s_n \to \infty$. Clearly if $a < 1$, then $\frac{1}{n^a} \leq \frac{1}{n}$, so again by the comparison test $s_n = 1 + \cdots + \frac{1}{n^a} \to \infty$.

($\Longleftarrow$): Let $\varepsilon > 0$, and let s_n be the sequence of partial sums for the series. Pick N

such that $\frac{2}{2^N} < \varepsilon$, and let $2^N < m < n < 2^L$.

$$\begin{aligned}
|s_n - s_m| &= \left| \frac{1}{(m+1)^a} + \cdots + \frac{1}{n^a} \right| \\
&< \left| \frac{1}{(2^N)^a} + \frac{1}{(2^N+1)^a} + \cdots + \frac{1}{(2^L)^a} \right| \\
&= \left| \left(\frac{1}{(2^N)^a} + \cdots + \frac{1}{(2^{N+1}-1)^a} \right) + \cdots + \left(\frac{1}{(2^{L-1})^a} + \cdots + \frac{1}{(2^L-1)^a} \right) \right| \\
&< \left| \frac{1}{2^N} + \frac{1}{2^{N+1}} + \cdots + \frac{1}{2^{L-1}} \right| \\
&= \frac{2}{2^N} - \frac{1}{2^L} \\
&< \frac{2}{2^N} \\
&< \varepsilon
\end{aligned}$$

So by the Cauchy criterion for convergence, s_n converges to a real number, as desired.

4

($\Longrightarrow$): We prove the contrapositive. Let $a = 1$. For $n \geq 3$, $\log n > 1$, so it's clear to see that $\frac{1}{n \log n} \geq \frac{1}{n}$. $\sum_{n=3}^{\infty} \frac{1}{n} = \infty$, so by the comparison test, $\sum_{n=3}^{\infty} \frac{1}{n \log n} = \infty$. When $a < 1$, $\sum_{n=3}^{\infty} \frac{1}{n(\log n)^a} \geq \sum_{n=3}^{\infty} \frac{1}{n \log n}$, so again by the comparison test, $\sum_{n=3}^{\infty} \frac{1}{n(\log n)^a} = \infty$.

($\Longleftarrow$): Let $a > 1$, and s_n be the sequence of partial sums for $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^a}$. Let $2^N < m < n < 2^L$. Then there exists some c such that $cN \leq \log m < \log n \leq cL$.

So

$$\begin{aligned}
|s_n - s_m| &= \left| \frac{1}{(m+1)(\log(m+1))^a} + \cdots + \frac{1}{n(\log n)^a} \right| \\
&< \left| \left(\frac{1}{2^N(\log 2^N)^a} + \cdots + \frac{1}{(2^{N+1}-1)(\log(2^{N+1}-1))^a} \right) + \cdots \right. \\
&\quad \left. + \left(\frac{1}{2^{L-1}(\log 2^{L-1})^a} + \cdots + \frac{1}{(2^L-1)(\log(2^L-1))^a} \right) \right| \\
&< k \left| \left(\frac{1}{2^N(2^{N+1})^a} + \cdots + \frac{1}{(2^{N+1}-1)(2^{N+1})^a} \right) + \cdots \right. \\
&\quad \left. + \left(\frac{1}{2^{L-1}(2^L)^a} + \cdots + \frac{1}{(2^L-1)(2^L)^a} \right) \right| \qquad \text{where } k \text{ is a constant depending on } \varepsilon \\
&< k \left| \frac{1}{(2^{N+1})^a} + \cdots + \frac{1}{(2^{L-1})^a} \right| \\
&< k \left| \frac{1}{2^{N+1}} + \cdots + \frac{1}{2^{L-1}} \right| \\
&= k \left(\frac{2}{2^{N+1}} - \frac{1}{2^L} \right) \\
&< \frac{k}{2^N} \\
&= \frac{k}{2^{\log_2 \frac{k}{\varepsilon}}} \\
&= \varepsilon
\end{aligned}$$

So s_n is Cauchy and thus convergent to a real, so $\sum_{n=2} \frac{1}{n(\log n)^a}$ converges to a real.

5

($\Longrightarrow$): Let $a = 1$. When $n > 321$, $\log n \log\log n \geq 1$, so by the same argument as above, $\sum_{n=3} \frac{1}{n \log n \log\log n}$ converges, and so does $\sum_{n=3} \frac{1}{n \log n (\log\log n)^a}$ when $a < 1$.

($\Longleftarrow$):

6

Let s_n be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$. We s_n converges, say $\lim_{n\to\infty} s_n = \alpha$. Then

$$\begin{aligned}\sum_{n=k}^{\infty} a_n &= \left(\lim_{n\to\infty} s_n\right) - s_{k-1} \\ &= \alpha - s_{k-1}\end{aligned}$$

and so

$$\begin{aligned}b_k &= \lim_{k\to\infty} \sum_{n=k}^{\infty} a_n \\ &= \lim_{k\to\infty} (\alpha - s_{k-1}) \\ &= \alpha - \alpha \\ &= 0\end{aligned}$$

7

Let $x_0 \in \mathbb{R}$. Let $\varepsilon > 0$, and let $\delta = \varepsilon$. Then if $|x - x_0| < \delta$, $|x - x_0| < \varepsilon$. It's clear to see from the definition of $g(x)$ that $|g(x) - g(x_0)|$ can be no greater than $|x - x_0|$, so we have

$$|g(x) - g(x_0)| \leq |x - x_0| < \varepsilon$$

, and so $g(x)$ is continuous.

8

8.a

We first show by induction that $\prod_{i=1}^{m}(1-a_i) \geq 1 - \sum_{i=1}^{m} a_i$:
When $m = 1$, equality holds. Assume the inequality holds for some N.

$$\begin{aligned}
\prod_{i=1}^{N+1}(1-a_i) &= (1-a_{N+1})\prod_{i=1}^{N}(1-a_i) \\
&\geq (1-a_{N+1})\left(1-\sum_{i=1}^{N} a_i\right) && \text{By the hypothesis} \\
&= 1 - a_{N+1} - \sum_{i=1}^{N} a_i + a_{N+1}\sum_{i=1}^{N} a_i \\
&\geq 1 - \sum_{i=1}^{N+1} a_i
\end{aligned}$$

So by induction, the inequality holds.

Now,

$$\begin{aligned}
\lim_{n\to\infty}\prod_{i=1}^{n}(1-a_i) &\geq \lim_{n\to\infty}\left(1-\sum_{i=1}^{n} a_i\right) \\
&= 1 - \lim_{n\to\infty}\sum_{i=1}^{n} a_i,
\end{aligned}$$

so $\prod_{i=1}^{\infty}(1-a_i) \geq 1 - \sum_{i=1}^{\infty} a_i$, as desired.

8.b

Let a_i be non-negative. Then $1 - a_i^2 \leq 1$. Dividing both sides by $1 - a_i$, we get $1 + a_i \leq \frac{1}{1-a_i}$. We know, since $\sum_{i=1}^{\infty} a_i < \infty$, that $\sum_{i=k}^{\infty} a_i \to 0$. Consider $\prod_{i=k}^{\infty}(1+a_i) \leq \prod_{i=k}^{\infty}\frac{1}{1-a_i}$. $a_n \leq a_k$ for all $n > k$, so $\prod_{i=k}^{\infty}\frac{1}{1-a_i} \leq \left(\frac{1}{1-a_k}\right)\left(\frac{1}{1-a_k}\right)\cdots$. As $k \to \infty$, $1 - a_k \to 1$, so $\left(\frac{1}{1-a_k}\right)\left(\frac{1}{1-a_k}\right)\cdots \to 1$. $\prod_{i=1}^{k}$ is finite, so $\prod_{i=1}^{\infty}\frac{1}{1-a_i}$ is too, and so $\prod_{i=1}^{\infty}(1+a_i)$ exists and is finite.

8.c

Each of $a_1, a_2, \ldots$ appears as a term in the expansion of $(1+a_1)(1+a_2)\cdots$, so $\sum_{i=1}^{\infty} a_i \leq \prod_{i=1}^{\infty}(1+a_i)$. Thus by the comparison test, since $\prod_{i=1}^{\infty}(1+a_i)$ converges, so does $\sum_{i=1}^{\infty} a_i$.