Oliver Tonnesen
V00885732

Assignment 1

MATH 311
September 24, 2019

1

1.a

$$F^3 = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

In general, F^n has 2^n elements.

1.b

If (x, y, z) is a general vector in $\text{Span}(S)$, then we have $a(1,1,0)+b(1,0,1)+c(0,1,1) = (x,y,z)$ for some $a,b,c \in F$, not all zero. So we have $x = a+b$, $y = a+c$, and $z = b+c$:

$$\begin{aligned}
z &= b + c \\
b &= x - a \\
c &= y - a \\
\implies z &= (x-a) + (y-a) \\
\implies 2a &= x + y - z \\
\implies 0 &= x + y - z \qquad\qquad \text{Since } F = \mathbb{Z}_2
\end{aligned}$$

So S does not span F^3, and $x + y - z = 0$ is a non-trivial condition for a vector $(x,y,z) \in F^3$ to be in S.

1.c

$\text{Span}(S) = \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}$

2

Consider the vector $1 \in P_3(\mathbb{R})$. We show that there do not exist $a,b,c \in \mathbb{R}$ such that $a(x^3 - 2x^2 + 1) + b(4x^2 - x + 3) + c(3x - 2) = 1$. We have:

$$\begin{aligned}
a &= 0 && (1)\\
-2a + 4b &= 0 && (2)\\
-b + 3c &= 0 && (3)\\
a + 3b - 2c &= 1 && (4)
\end{aligned}$$

If we take (2) + 2 · (1), we have $4b = 0$, so $b = 0$. Now we can plug in $b = 0$ to $-b + 3c = 0$, obtaining $3c = 0$, or $c = 0$. So $a = b = c = 0$, but $a + 3b - 2c = 1$, so $0 = 1$, thus there exist no such a, b, c, and 1 is not in the span of the three given vectors, and therefore they do not span $P_3(\mathbb{R})$.

3

We prove of S that it satisfies the three conditions sufficient to show that it is a subspace: (1): $0 \in S$, (2): $v, w \in S \implies (v+w) \in S$, and (3): $a \in F, v \in S \implies av \in S$. (For clarity, we denote F's identity by 0, and V's by $\vec{0}$ for the remainder of this proof.)

(1): $\lambda f(v) = f(\lambda v) \implies 0f(v) = f(0v) = f(\vec{0}) = 0$, so $\vec{0} \in S$.
(2): let $v, w \in S$. $f(v+w) = f(v) + f(w) = 0 + 0 = 0$, so $v + w \in S$.
(3): let $v \in S, a \in F$. $f(av) = af(v) = a0 = 0$, so $av \in S$.

Thus S is a subspace of V.

4

($\Longrightarrow$): Suppose for a contradiction that $\{u+v, u+w, v+w\}$ is linearly dependent. Then $u+v = a(u+w) + b(v+w)$ for some $a, b \in F$ not both zero. Then we have $(1-a)u + (1-b)v = (a+b)w$.

Case 1 ($a+b=0$): $a+b=0$, so $a=-b$. F is not of characteristic 2, so if $a=1$, then $b \neq 1$. Thus $(1-a)u+(1-b)v \neq 0$. If $(1-a) \neq 0 \neq (1-b)$, then we have $v = \frac{1-a}{1-b}u$, a contradiction since $\{u, v, w\}$ is linearly independent. If one of a or b is 1, say WLOG a=1, then we have $(1-b)u = 0$, but $1-b \neq 0$, so $u = 0$, a contradiction since $\{u, v, w\}$ is linearly independent.

Case 2 ($a+b \neq 0$): $a+b \neq 0$, so $w = \frac{1-a}{a+b}u + \frac{1-b}{a+b}$. Again, not both of $1-a$ and $1-b$ can be 0, so we have a nontrivial linear combination, a contradiction since $\{u, v, w\}$ is linearly independent.

($\Longleftarrow$): Suppose for a contradiction that $\{u, v, w\}$ is linearly dependent. Then $u = av + bw$ for some $a, b \in F$ not both 0. So $\{u+v, u+w, v+w\} = \{av+bw+v, av+bw+w, v+w\} = \{(a+1)v+bw, av+(b+1)w, v+w\}$. Each vector in the set is a linear combination of v and w, so it is clearly linearly dependent, a contradiction since $\{u+v, u+w, v+w\}$ is linearly dependent.

Thus we've shown both the forward and backward direction, and so $\{u, v, w\}$ is linearly independent if and only if $\{u+v, u+w, v+w\}$ is, as desired.