MATH 200 Assignment One

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A02 - T03

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1

The set will describe a plane orthogonal to $\vec{AB} = (7, -3, -5)$ and passing through its midpoint M.

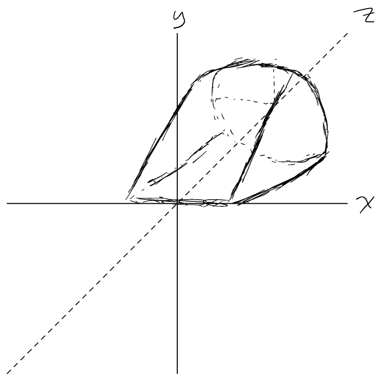
$$\begin{aligned}
M &= A + \frac{B - A}{2} \\
&= (-1, 5, 3) + \frac{(6, 2, -2) - (-1, 5, 3)}{2} \\
&= \left(-1 + \frac{7}{2}, 5 - \frac{3}{2}, 3 - \frac{5}{2} \right) \\
&= \frac{1}{2}(5, 7, 1)
\end{aligned}$$

Plane passes through M and is orthogonal to $\vec{AB}$, so the equation of the plane will be

$$\begin{aligned}
7\left(x - \frac{5}{2}\right) - 3\left(y - \frac{7}{2}\right) - 5\left(z - \frac{1}{2}\right) &= 0 \\
7x - 3y - 5z &= 7\left(\frac{5}{2}\right) - 3\left(\frac{7}{2}\right) - 5\left(\frac{1}{2}\right) \\
7x - 3y - 5z &= \frac{9}{2}
\end{aligned}$$

2

The shape is similar to a wedge with a rounded bottom.



3

The equation for a sphere is $(x-a)^2+(y-b)^2+(z-c)^2=r^2$ and describes a sphere of radius r centred at (a,b,c).

$$|\vec{r}-\vec{r_o}|=1$$
$$\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z_o)^2}=1$$
$$(x-x_o)^2+(y-y_o)^2+(z-z_o)^2=1$$

This fits the aforementioned equation for a sphere, and so the set of points (x,y,z) that satisfy $|\vec{r}-\vec{r_o}|=1$ describes a sphere of radius 1 centred at (x_o,y_o,z_o).

4

AB is the diameter of a circle centred at O, so A's x component must be the additive inverse of B's x component and the same must be true of their y components. Since A sits on the curve of a circle, let it be defined as $A=(r\sin t, r\cos t)$ where r is the radius of the circle and t is some constant in the interval $[0,2\pi]$. By our definition, B must be defined as $B=(-r\sin t,-r\cos t)$. C is a point on the same circle, so let it be defined as $C=(r\sin s, r\cos s)$, where s is some constant in the interval $[0,2\pi]$.

$$\vec{CA}=<r\sin t-r\sin s, r\cos t-r\cos s>$$
$$\vec{CB}=<-r\sin t-r\sin s,-r\cos t-r\cos s>$$

If $\vec{CA} \cdot \vec{CB}$ can be shown to be equal to 0, then $\vec{CA}$ and $\vec{CB}$ are perpendicular.

$$\begin{aligned}
\vec{CA} \cdot \vec{CB} &= (r\sin t - r\sin s)(-r\sin t - r\sin s) + (r\cos t - r\cos s)(-r\cos t - r\cos s) \\
&= (r\sin t)(-r\sin t) + (r\sin t)(-r\sin s) + (-r\sin s)(-r\sin t) + (-r\sin s)^2 + \\
&\qquad (r\cos t)(-r\cos t) + (r\cos t)(-r\cos s) + (-r\cos s)(-r\cos t) + (-r\cos s)^2 \\
&= -r^2\sin^2 t + r^2\sin^2 s - r^2\cos^2 t + r^2\cos^2 s \\
&= r^2(\sin^2 s + \cos^2 s) - r^2(\sin^2 t + \cos^2 t) \\
&= r^2(1) - r^2(1) \\
&= r^2 - r^2 \\
&= 0
\end{aligned}$$

5

$$\begin{aligned}
L_1 &= (-2, 0, -3) - (-4, -6, 1) = (2, 6, -4) \\
L_2 &= (5, 3, 14) - (10, 18, 4) = (-5, -15, 10)
\end{aligned}$$

The cross product of two vectors is a vector orthogonal to both. If the cross product of two vectors is $\vec{0}$, the two vectors are parallel, otherwise they are not parallel.

$$\begin{aligned}
L_1 \times L_2 &= \langle 60 - 60, -(20 - 20), -30 - (-30) \rangle \\
&= \langle 0, 0, 0 \rangle
\end{aligned}$$

So L_1 and L_2 are parellel.

6

If the vectors normal to the two planes are orthogonal, then the two planes themselves are orthogonal. Vectors normal to the plane can be found by taking the coefficients of x, y, and z in the plane's equation.

$$\begin{aligned}
x + 4y - 3z = 1 &\implies \langle 1, 4, -3 \rangle \\
-3x + 6y + 7z = 0 &\implies \langle -3, 6, 7 \rangle
\end{aligned}$$

$$\begin{aligned}
\langle 1, 4, -3 \rangle \cdot \langle -3, 6, 7 \rangle &= -3 + 24 - 21 \\
&= 0
\end{aligned}$$

So the planes are orthogonal.

7

If each of the two lines' components share the same value for some t in the interval, then the particles will collide. So if we set each set of components to be equal and solve for t, then any value of t such that each set of components is equal will be a point at which the particles collide.
The first set of components:

$$\begin{aligned} t^2 &= 4t - 3 \\ t^2 - 4t + 3 &= 0 \\ t &= \frac{4 \pm \sqrt{16 - 12}}{2} \\ t &= 1, 3 \end{aligned}$$

The second set of components:

$$\begin{aligned} t^2 &= 7t - 12 \\ t^2 - 7t + 12 &= 0 \\ t &= \frac{7 \pm \sqrt{49 - 48}}{2} \\ t &= 3, 4 \end{aligned}$$

The third set of components:

$$\begin{aligned} t^2 &= 5t - 6 \\ t^2 - 5t + 6 &= 0 \\ t &= \frac{5 \pm \sqrt{25 - 24}}{2} \\ t &= 2, 3 \end{aligned}$$

So since each component intersects at $t = 3$ (and since $3 \geq 0$), the particles will collide at when $t = 3$.

8

$$\begin{aligned}
\frac{d}{dt}|\vec{r}(t)| &= \frac{d}{dt}\sqrt{\vec{r}(t)\cdot\vec{r}(t)} \\
&= \frac{1}{2}\Big(\vec{r}(t)\cdot\vec{r}(t)\Big)^{-\frac{1}{2}}\cdot\frac{d}{dt}\Big(\vec{r}(t)\cdot\vec{r}(t)\Big) && \text{(Chain Rule)} \\
&= \frac{1}{2}\cdot\frac{1}{\Big(\vec{r}(t)\cdot\vec{r}(t)\Big)^{\frac{1}{2}}}\cdot\frac{d}{dt}\Big(\vec{r}(t)\cdot\vec{r}(t)\Big) \\
&= \frac{1}{2}\cdot\frac{1}{\Big(|\vec{r}(t)|^2\Big)^{\frac{1}{2}}}\cdot\frac{d}{dt}\Big(\vec{r}(t)\cdot\vec{r}(t)\Big) \\
&= \frac{1}{2}\cdot\frac{1}{|\vec{r}(t)|}\cdot\frac{d}{dt}\Big(\vec{r}(t)\cdot\vec{r}(t)\Big) \\
&= \frac{1}{2}\cdot\frac{1}{|\vec{r}(t)|}\cdot\Big(\vec{r'}(t)\cdot\vec{r}(t)+\vec{r}(t)\cdot\vec{r'}(t)\Big) && \text{(Product Rule)} \\
&= \frac{1}{2}\cdot\frac{1}{|\vec{r}(t)|}\cdot 2\Big(\vec{r'}(t)\cdot\vec{r}(t)\Big) \\
&= \frac{2}{2}\cdot\frac{1}{|\vec{r}(t)|}\cdot\Big(\vec{r'}(t)\cdot\vec{r}(t)\Big) \\
&= \frac{\vec{r'}(t)\cdot\vec{r}(t)}{|\vec{r}(t)|}
\end{aligned}$$

Note that division by zero is undefined, and so $\vec{r}(t)$ cannot be $\vec{0}$.

9

Consider the vector function $\langle \sin t, \cos t, 2\pi\rangle$.
One full rotation of the circle (which has radius 1) on the xy-axis formed by $\sin t$ and $\cos t$ takes $2\pi \cdot t$. In other words, each helix rises $2\pi \cdot t$ for every full rotation of the circle. The function can easily be modified to gain the properties that we want as follows:

$$\begin{aligned}
\vec{r}(t) &= \left\langle 10\sin t, 10\cos t, \frac{34}{2\pi}\right\rangle \\
&= \left\langle 10\sin t, 10\cos t, \frac{17}{\pi}\right\rangle
\end{aligned}$$

Since each rotation takes $2\pi \cdot t$, 2.9×10^8 rotations will take $2\pi \cdot 2.9 \times 10^8 \cdot t$. The length of these helices can be found by calculating the arc length of one of them:

$$
\begin{aligned}
\int\limits_{0}^{2.9\times10^8\cdot2\pi} |\vec{r}(t)|dt &= \int\limits_{0}^{2.9\times10^8\cdot2\pi} \sqrt{(10\cos t)^2 + (10\sin t)^2 + (\frac{17}{\pi})^2}\,dt \\
&= \int\limits_{0}^{2.9\times10^8\cdot2\pi} \sqrt{10^2 + \left(\frac{17}{\pi}\right)^2}\,dt \\
&= \sqrt{10^2 + \left(\frac{17}{\pi}\right)^2} \cdot t \Bigg|_{0}^{2.9\times10^8\cdot2\pi} \\
&= \sqrt{10^2 + \left(\frac{17}{\pi}\right)^2} \cdot 2.9 \times 10^8 \cdot 2\pi \\
&= 20717941308 \\
&\approx 2.07 \times 10^{10}
\end{aligned}
$$

So the length of each helix is approximately 2.07×10^{10}Å.