MATH 200 Assignment Three

Oliver Tonnesen
V00885732
A02 - T03

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We define the volume, length, width, and height as follows:

$$
\begin{aligned}
l &= l(t) \\
w &= w(t) \\
h &= h(t)
\end{aligned}
$$

We are given the following:

$$
\begin{aligned}
l(t_0) &= 1 \\
w(t_0) &= 2 \\
h(t_0) &= 2 \\
l'(t_0) &= 2 \\
w'(t_0) &= 2 \\
h'(t_0) &= -3
\end{aligned}
$$

1.a

$$
\begin{aligned}
V &= lwh \\
\frac{\partial V}{\partial t} &= \frac{\partial V}{\partial l} \cdot \frac{dl}{dt} + \frac{\partial V}{\partial w} \cdot \frac{dw}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt} \\
&= 2wh + 2lh - 3lw \\
&= 2(2)(2) + 2(1)(2) - 3(1)(2) \\
&= 6m^3/s
\end{aligned}
$$

1.b

$$A = 2lw + 2lh + 2wh$$

$$\begin{aligned}
\frac{\partial A}{\partial t} &= \frac{\partial A}{\partial l} \cdot \frac{dl}{dt} + \frac{\partial A}{\partial w} \cdot \frac{dw}{dt} + \frac{\partial A}{\partial h} \cdot \frac{dh}{dt} \\
&= (2w + 2h) \cdot 2 + (2l + 2h) \cdot 2 + (2l + 2w) \cdot (-3) \\
&= (4 + 4)(2) + (2 + 4)(2) + (2 + 4)(-3) \\
&= 10m^2/s
\end{aligned}$$

1.c

We will use the diagonal formed by the length (l) and width (w) of the box.

$$D = \sqrt{l^2 + w^2}$$

$$\begin{aligned}
\frac{\partial D}{\partial t} &= \frac{\partial D}{\partial l} \cdot \frac{dl}{dt} + \frac{\partial D}{\partial w} \cdot \frac{dw}{dt} + \frac{\partial D}{\partial h} \cdot \frac{dh}{dt} \\
&= \frac{l}{\sqrt{l^2 + w^2}} \cdot 2 + \frac{w}{\sqrt{l^2 + w^2}} \cdot 2 \\
&= \frac{1}{\sqrt{5}} \cdot 2 + \frac{2}{\sqrt{5}} \cdot 2 \\
&= \frac{2}{\sqrt{5}} \big(1 + 2\big) \\
&= \frac{6}{\sqrt{5}}
\end{aligned}$$

2

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial f} \cdot \frac{df}{dx} + \frac{\partial z}{\partial g} \cdot \frac{dg}{dx} \\
&= \frac{\partial z}{\partial f} + \frac{\partial z}{\partial g}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial^2 z}{\partial f^2} \cdot \frac{df}{dx} + \frac{\partial^2 z}{\partial g^2} \cdot \frac{dg}{dx} \\
&= \frac{\partial^2 z}{\partial f^2} + \frac{\partial^2 z}{\partial g^2}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial f} \cdot \frac{df}{dt} + \frac{\partial z}{\partial g} \cdot \frac{dg}{dt} \\
&= a\frac{\partial z}{\partial f} - a\frac{\partial z}{\partial g}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 z}{\partial t^2} &= a\frac{\partial^2 z}{\partial f^2} \cdot \frac{df}{dt} - a\frac{\partial^2 z}{\partial g^2} \cdot \frac{dg}{dt} \\
&= a^2\frac{\partial^2 z}{\partial f^2} + a^2\frac{\partial^2 z}{\partial g^2} \\
&= a^2\left(\frac{\partial^2 z}{\partial f^2} + \frac{\partial^2 z}{\partial g^2}\right) \\
&= a^2\left(\frac{\partial^2 z}{\partial x^2}\right)
\end{aligned}
$$

$$
\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}
$$

3

$$
\begin{aligned}
f(x, y, z) &= 3(x-1)^2 + 2(y+3)^2 - z + 7 \\
\nabla f &= \big\langle 6(x-1), 4(y+3), -1 \big\rangle \\
\nabla f(2, -2, 12) &= \big\langle 6, 4, -1 \big\rangle
\end{aligned}
$$

$$
\begin{aligned}
P_0 &= \big(3, -2, 18\big) \text{ is a point on the tangent plane} \\
xf_x + yf_y + zf_z &= 3f_x - 2f_y + 18f_z \\
6x + 4y - z &= -8
\end{aligned}
$$

4

$$
\begin{aligned}
D_{\vec{PQ}} f(2,8) &= \nabla f \cdot \vec{PQ} \\
\nabla f &= \left\langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \right\rangle \\
\vec{PQ} &= \langle 3, 4 \rangle \\
\hat{PQ} &= \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
\nabla f \cdot \hat{PQ} &= \frac{y}{2\sqrt{xy}} \cdot \frac{3}{5} + \frac{x}{2\sqrt{xy}} \cdot \frac{4}{5} \\
\nabla f(2,8) \cdot \hat{PQ} &= \frac{2}{5}
\end{aligned}
$$

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Recall that ∇f goes in the direction of maximum change.

$$
\begin{aligned}
\nabla f &= \langle y \cos xy, x \cos xy \rangle \\
\nabla f(1,0) &= \langle 0, 1 \rangle \\
|\nabla f(1,0)| &= 1
\end{aligned}
$$

So the maximum rate of change is 1 and occurs in the direction of $\langle 0, 1 \rangle$.

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$$
\begin{aligned}
D_{\vec{u}} f(0,2) &= 1 \\
\nabla f(0,2) \cdot \vec{u} &= 1 \\
\nabla f &= \left\langle -y^2 e^{-xy}, e^{-xy}, -xye^{-xy} \right\rangle \\
\nabla f(0,2) &= \left\langle -4, 1 \right\rangle
\end{aligned}
$$

So we have the following system of equations:

$$
\begin{aligned}
-4u_1 + u_2 &= 1 \\
u_1^2 + u_2^2 &= 1
\end{aligned}
$$

and must simply solve for u_1 and u_2.

$$\begin{aligned} u_2 &= 4u_1 + 1 \\ u_1^2 + (4u_1+1)^2 &= 1 \\ 17u_1^2 + 8u_1 &= 0 \\ u_1 &= \frac{-8 \pm \sqrt{64}}{34} \\ u_1 &= 0, -\frac{8}{17} \end{aligned}$$

So thus

$$\Big\langle 0, 1 \Big\rangle$$

and

$$\Big\langle -\frac{8}{17}, -\frac{15}{17} \Big\rangle$$

are the two directions in which the directional derivative is 1 at (0,2).

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$$\begin{aligned} f_x &= e^x \cos y \\ f_y &= -e^x \sin y \end{aligned}$$

Critical points are wherever the partial derivatives are 0 or undefined, so

$$\begin{aligned} f_x = 0 &: \left(k, \frac{\pi}{2} + k\pi\right), k \in \mathbb{Z} \\ f_y = 0 &: \left(k, k\pi\right), k \in \mathbb{Z} \end{aligned}$$

8

$$\begin{aligned} f_x &= 2x - 4y \\ f_y &= 8x - 4y \\ 0 = 2x - 4y \text{ and } 0 &= 8y - 4x \end{aligned}$$

Both of these equations are lines, and thus each has infinitely many solutions.

$$\begin{aligned} f_{xx} &= 2 \\ f_{yy} &= 8 \\ f_{xy} &= -4 \\ D = 16 - 16 = 0 \text{ for all } x &\text{ and } y \end{aligned}$$