CSC 226 Problem Set 2 Written Part

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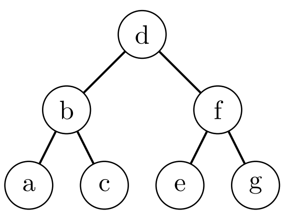
1 Number of nodes and balance

The following sequence of 7 insertions maximizes the number of nodes in a 2–3 tree:

$$d, b, f, a, c, e, g$$

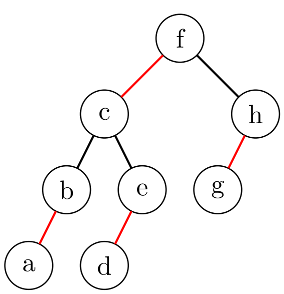
The Red-Black tree representation of the 2–3 tree:



The following sequence of 8 insertions minimizes the number of nodes in a 2–3 tree:

$$a, c, d, f, g, b, e, h$$

The Red-Black tree representation of the 2–3 tree:



In general, the larger the number of 3–nodes in a 2–3 tree, the higher the 'imbalance' of the corresponding Red-Black tree. When a 2–3 tree has only 2–nodes, it is exactly the same as its Red-Black tree representation; since a 2–3 tree is always perfectly balanced, this Red-Black tree is also perfectly balanced. In a Left-Leaning Red-Black representation of a 2–3 tree, the imbalance comes from an abundance of 3–nodes. If the 2–3 tree has a large number of 3–nodes on a single branch, the corresponding Red-Black tree will be very imbalanced due to the red links following the path of the 3–nodes in the 2–3 tree.

2 A hybrid MST algorithm

Claim: The algorithm constructs the minimum weight spanning tree of G.
Proof: [Induction]

For the first step, $A = \emptyset$. We know that both Kruskal's and Prim's algorithms are correct, so running either initially will not add an edge that does not belong to the minimum weight spanning tree, and thus the first step of our hybrid algorithm is correct.

Suppose there exists some k such that the first k iterations of the algorithm only add edges belonging to the minimum weight spanning tree of the graph.
By the k^{th} iteration of the algorithm, we have a subgraph of the minimum weight spanning tree of G. We now have the following two cases: the case that our constructed subgraph has an even number of edges, and we use a variation of Kruskal's algorithm, and the case that our constructed has an odd number of edges, and we use a variation of Prim's algorithm. For both cases, we will define a cut that satisfies the Cut Property Theorem to prove that the iteration of the algorithm will successfully add an edge that does belong to the minimum weight spanning tree of G.

In the first case, we define $S = \{v \in V; \exists u \in V, uv \in A \vee vu \in A\}$. So S contains only vertices that are part of the subgraph of the minimum weight spanning tree of G constructed so far by the algorithm. In other words, no vertex in S has an edge in A that connects it with a vertex not in S. Thus, by the Cut Property Theorem, the minimum weight edge crossing the cut $(S, V \setminus S)$ must be in the minimum weight spanning tree of G. This is the edge the algorithm now chooses, so this step is correct.

In the second case, we define $S = \{v \in V; \text{v is in the randomly selected component}\}$. We now define the cut $(S, V \setminus S)$. By the Cut Property Theorem, the minimum weight edge crossing this cut must be in the minimum weight spanning tree of G. This is the edge the algorithm now chooses, so this step is also correct.
By induction, we have now shown that our hybrid algorithm constructs the minimum weight spanning tree of G. $\square$

3 Uniqueness of MSTs

Proof: [Contradiction] Suppose G has two minimum weight spanning trees, A and B. Consider the minimum weight edge $e_1 = vu$ that is in B but that is not in A, and add it to A. Since A was a tree, adding an edge to it must create a cycle. B must not contain all of the other edges in the cycle, or it would have originally contained a cycle. So there is an edge e_2 in the cycle that is not in B but that is in A. The edge e_2 must have weight greater than that of e_1, since it is the maximum weight edge in the cycle and the graph contains only edges of unique weight. So remove this edge from A and replace it with e. A is now a spanning tree of G with lower weight. This is a contradiction, since A was defined to be a minimum weight spanning tree of G, and so G can only have a single minimum weight spanning tree if all of its edge weights are distinct. $\square$

Non-distinct edge weights

Claim: Kruskal's algorithm constructs a minimum weight spanning tree of a graph $G = (V, E)$ with non-distinct edge weights.

We will show that every step of Kruskal's algorithm adds an edge that must belong to a minimum weight spanning tree of G.

Proof: [Induction] Let A be the partially complete spanning tree constructed by Kruskal's algorithm. Consider the case when $A = \emptyset$. The algorithm adds the lowest weight edge, which must be in the minimum weight spanning tree.

Suppose there exists some k such that the first k edges added by the algorithm to A are elements of a minimum weight spanning tree of G.

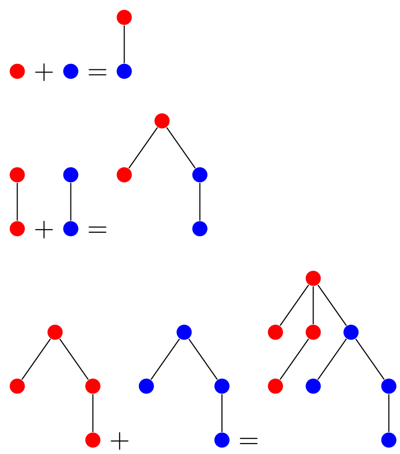
By the k^{th} addition of an edge to A, we have a subgraph of a minimum weight spanning tree of G. We now define $S = \{v \in v; \exists u \in V, uv \in A \vee vu \in A\}$. We see that S contains only vertices that are incident to edges in A. So we see that the cut $(S, V \setminus S)$ is crossed by no edges in A, and thus satisfies the Cut Property Theorem. So the minimum weight edge crossing the cut is contained in the minimum spanning tree. Kruskal's algorithm does not add edges that would create a cycle, so the non-distinct edge weights will only be picked if they are both the minimum weight and do not create a cycle.

By induction, we have now shown that Kruskal's algorithm constructs a minimum weight spanning tree of G when G has non-distinct edge weights. $\square$

4 Quick-Union with Union-by-Rank

Let $Rank(u)$ be the furthest distance from u's root to one of its leaves.
We first note that *Connected* calls *Find* twice. Since we know *Find* takes time

proportional to h, where h is the maximum height of the tree, we know that *Connected* takes time proportional to $2h$. So we aim to prove that $\log n$ is an upper bound for h, and, by extension, for *Connected*.
We first consider what exactly must occur in order for h to increase: a call to $Union(u, v)$ using Union-by-Rank increases h by one exactly when $Rank(u)$ and $Rank(v)$ are equal. Consider WLOG the case where $Rank(u) > Rank(v)$. h remains the same before and after the call to $Union(u, v)$, since adding one to $Rank(v)$ is at most equal to $Rank(u)$. So conversely when $Rank(u) = Rank(v)$, $Union(u, v)$ increases h by one.
We now consider how to maximize height with minimal nodes in our tree:



Drawing our the first few trees, the pattern becomes obvious: in order to increase h, we must at least double the number of nodes. This is due to the fact that we our trees must have the same rank in order for h to increase, and beginning with two trivial trees allows us to construct the tree with the minimum number of nodes that also has height h.
So we must at least double our nodes to increase h by one. In other words, if n is the number of nodes:

$$\begin{aligned}
h = 0 &\Leftrightarrow n \geq 1 = 2^0 \\
h = 1 &\Leftrightarrow n \geq 2 = 2^1 \\
h = 2 &\Leftrightarrow n \geq 4 = 2^2 \\
h = 3 &\Leftrightarrow n \geq 8 = 2^3 \\
&\vdots \\
h = k &\Leftrightarrow n \geq 2^k
\end{aligned}$$

So we have, in general,

$$
\begin{aligned}
n &\geq 2^h \\
\log_2 n &\geq h \\
\implies h &\in O(\log n)
\end{aligned}
$$

Recall from above that *Connected* takes time proportional to $2h$, and thus runs in time proportional to $O(2h) \in O(h) \in O(\log n)$. $\square$