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MATH 423

Assignment 2

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1

($\Longrightarrow$): We know that the number of edges in a tree is $n-1$. By the handshaking lemma, $\sum_{v \in V(G)} d(v) = 2e(G)$. Since $e(G) = n-1$, and $\sum_{v \in V(G)} d(v) = \sum_{i=1}^{n} d_i$, we have $\sum_{i=1}^{n} d_i = 2n-2$.

($\Longleftarrow$): [Induction on n]: When $n = 2$, we have $\sum_{i=1}^{2} d_i = 2(2) - 2 = 2$, so the only graphic sequence is $(1,1)$, corresponding to the tree on 2 vertices. Assume this holds for n. Let $d_1, \ldots, d_{n+1}$ be integers such that $\sum_{i=1}^{n+1} d_i = 2(n+1) - 2$. Suppose WLOG that $d_{n+1} \leq \ldots \leq d_1$. Then $d_{n+1} = 1$, otherwise $d_i \geq 2$ for all $1 \leq i \leq k$, and $\sum_{i=1}^{n+1} d_i \geq 2(n+1) > 2(n+1) - 2$. So if we remove the vertex corresponding to d_{n+1} from our list (note that we must also subtract 1 from some other degree, say d_j, so that our sequence remains graphic) then $\sum_{i=1}^{n} d_i = 2(n+1) - 4 = 2n-2$. By the induction hypothesis, there exists a tree with degrees $d_1, \ldots, d_j - 1, \ldots, d_n$. Take this tree, and add a new leaf to it, adjacent to a vertex with degree $d_j - 1$. Then we have a tree with degrees $d_1, \ldots, d_{n+1}$. Thus by induction, the claim holds.

2

We claim that for each $m < n$, if G is a graph with n vertices and more than $n(m-1) - \binom{m}{2}$ edges, then G contains each tree with m edges.

[Induction on n]: When $n = 1$, the only choice of m is 0, so the claim is clearly true. Assume the above claim holds for n. Let G be a graph with n vertices and $n(m-1) - \binom{m}{2}$ edges. We want to add a vertex v to G such that $e(G+v) > (n+1)m - \binom{m+1}{2}$.

$$
\begin{aligned}
e(G+v) &> (n+1)m - \binom{m+1}{2} \\
&= (n+1)(m-1+1) - \frac{(m+1)m}{2} \\
&= n(m-1+1) + (m-1+1) - \frac{(m-1+2)m}{2} \\
&= n(m-1) + n + m - \frac{m(m-1)+2m}{2} \\
&= n(m-1) - \frac{m(m-1)}{2} + n + m - \frac{2m}{2} \\
&= n(m-1) - \binom{m}{2} + n
\end{aligned}
$$

We see that to obtain the desired inequality, the vertex we add must have degree n. By the induction hypothesis, G contains every tree with m edges. Since v is adjacent to every vertex in G, $G+v$ must therefore contain every tree with $m+1$ edges. So by induction, the claim holds.

3

Suppose for a contradiction that X has no leaf. $d(x) \geq 2$ for any $x \in X$. Each edge has exactly one endpoint in X, so $\sum d(x) = e(T)$. But $\sum d(x) \geq 2|X| \geq 2\left(\frac{n}{2}\right) \geq n$, so $e(T) \geq n$, a contradiction since T is a tree. Thus X must contain a leaf.

4

Consider a vertex cover for G. Any vertex in this cover can cover at most $\Delta(G)$ edges, so it must have size at most $\frac{e(G)}{\Delta(G)}$. Thus, by the Kőnig-Egerváry Theorem, since a minimum vertex cover for G has size at most $\frac{e(G)}{\Delta(G)}$, a maximum matching for G must have size at least $\frac{e(G)}{\Delta(G)}$.

$\Delta(K_{n,n}) = n-1$, so a vertex cover S of a subgraph of $K_{n,n}$ with at least $(k-1)n$ edges has $|S| \geq \frac{(k-1)n}{n-1} > \frac{(k-1)n}{n} = k-1$, thus $|S| > k-1$, or $|S| \geq k$. By the Kőnig-Egerváry Theorem, there exists a maximum matching M with $|M| = k$.

5

($\Longrightarrow$): Let $S \subseteq X$, and $S' \subseteq S$ be the smallest subset such that $N(S') = N(S)$. Suppose for a contradiction that $|S'| > k$. Then for any $s' \in S'$, there exists $y \in N(S')$ that is uniquely covered by s'. If we pair off each such set of s' and y, we end up with a copy of $|S'|K_2$, where $|S'| > k$, a contradiction, so $|S'| \leq k$.

($\Longleftarrow$): Let $S \subseteq X$ such that $N(S) = Y$ and $|S|$ is as small as possible, and let $S = \{s_1, \ldots, s_n\}$. Since S is minimal, we have that $N(s_i) \not\subseteq \bigcup_{j \neq i} N(s_j)$, for all $1 \leq i \leq n$. So the only $S' \subseteq S$ with $N(S') = N(S)$ is $S' = S$, and $k = |S'|$. For any $s_i \in S$, there is an element in $N(s_i)$ which is not in any $N(s_j)$, $j \neq i$; call this element y_i. Then $\{s_i y_i \mid 1 \leq i \leq n\}$ is in fact a (maximum) set of copies of K_2, since y_i is not adjacent to s_j for any $j \neq i$. Since the set of copies of K_2 we constructed is maximum, and since it contains exactly k copies, G contains kK_2, but more importantly, it does *not* contain $(k+1)K_2$, as desired.

6 Bonus

Let S be a maximal independent set in D. In the underlying graph, each vertex not in S is adjacent to one or more vertices in S.

[Induction on $n(D)$]: When $n(D) = 1$, any vertex is already in the only indpendent set, so we're done. Suppose there exists some k such that our claim holds for all $n(D) \leq k$. Let $v \in V \setminus S$, either there exists an edge from v into S, or all edges between S and v point towards v. Let X be the set of all latter such vertices. By the induction hypothesis, there exists an independent set $X' \subseteq X$ in $D[X]$ such that any vertex in X can reach X' in at most two steps. Let Y be the set of vertices in S that have edges leading to a vertex in X', and let $S' = (S \setminus Y) \cup X'$. We claim S' is an independent set reachable by any vertex in V in at most two steps. It is clear from its definition that S' is independent, so we must show that any vertex in V reaches it in at most two steps. We have five cases to check:

<u>Case 1: $v \in S'$</u>
We're done.
<u>Case 2: $v \in X \setminus X'$</u>
v reaches $x' \in X' \subseteq S'$ in at most two steps by definition, so we're done.

Case 3: $v \in Y$
v is adjacent to a vertex in $X' \subseteq S'$, so it reaches S' in one step, and we're done.
Case 4: v is adjacent to a vertex in $S \setminus Y$
v reaches $S \setminus Y \subseteq S'$ in one step, so we're done.
Case 5: v is adjacent to $y \in Y$
y is adjacent to a vertex in $X' \subseteq S'$, so v reaches S' in two steps, and we're done.

So by induction, the claim holds.