MATH 212 Assignment 1

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1

1.a

No. Suppose $a = b \neq 0$. Then $a - b = 0$. But $0 \notin X$, so * is not a binary operation on X.

1.b

Yes. Let $A \subseteq B \subseteq \mathcal{P}(\{1, 2, 3\})$. $A \cap B$ is defined for all values of A and B, and $A \cap B \in \mathcal{P}(\{1, 2, 3\})$ for all A, B.

1.c

No. Let $A = \{1\}, B = \{1\}$. $A \times B = \{(1, 1)\}$. $\{(1, 1)\} \notin \mathcal{P}(\{1, 2, 3\})$, so $\times$ is not a binary operation.
In general, one must check that the range of * is contained in X, that * is defined on all $x \in X \times X$, and that no input $x \in X \times X$ exists such that it corresponds to two outputs.

2

Associativity:
Let $a = b = c = 2 \in \mathbb{Z}$. We show that $(a * b) * c \neq a * (b * c)$:

$$(2 * 2) * 2 = (2 \cdot 2^2) \cdot 2^2 = 32$$
$$2 * (2 * 2) = 2 \cdot (2 \cdot 2^2)^2 = 128$$

So * is not associative.

Commutativity:
Let $a = 2, b = 3, a, b \in \mathbb{Z}$. We show that $a * b \neq b * a$.

$$\begin{aligned} a * b &= 2 \cdot 3^2 = 18 \\ b * a &= 3 \cdot 2^2 = 12 \end{aligned}$$

So * is not commutative.

Identity:
Suppose e is an identity for * on $\mathbb{Z}$. Then $e*a = a = a*e$. Then $e \cdot a^2 = a = a \cdot e^2$. Consider the equation $e \cdot a^2 = a$ where $a \neq 0$:

$$\begin{aligned} e \cdot a^2 &= a \\ \iff e \cdot a &= 1 \end{aligned}$$

This clearly cannot hold for all $0 \neq a \in \mathbb{Z}$, so e cannot exist.

3

Symmetric:
All of $a, b, c, d \in \mathbb{Z}$, so then both of $ad, bc \in \mathbb{Z}$. By the properties of $=$ on $\mathbb{Z}$, we know that $ad = bc \implies bc = ad$, so $((a, b), (c, d)) \in \sim \iff ((c, d), (a, b)) \in \sim$.

Reflexive:
Similarly, we know that $=$ on $\mathbb{Z}$ is reflexive, so $ab = ab$ for all $(a, b) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$.

Transitive:
For all $(a, b), (c, d), (e, f) \in B$, suppose $ad = bc$ and $ef = de$. Suppose $c \neq 0$.

$$\begin{aligned} ad &= bc \\ adcf &= bcde \qquad (cf = de) \\ a\cancel{dc}f &= b\cancel{cd}e \qquad (c, d \neq 0) \\ af &= be \end{aligned}$$

Now suppose $c = 0$. Then $bc = cf = 0$, and consequently, $ad = de = 0$. $d \neq 0$, so $a = e = 0$, so $af = be = 0$. Thus, transitivity holds both when $c = 0$ and when $c \neq 0$.

4

4.a

For $+$, x is the identity, since $x + x = x$, and $x + y = y = y + x$, so $x := 0$.
For $\cdot$, y is the identity, since $y \cdot y = y$ and $y \cdot x = x = x \cdot y$, so $y := 1$.

4.b

For every $a \in \mathbb{Z}$, exactly one of the following holds: $a \in \mathbb{Z}^+$, $-a \in \mathbb{Z}^+$, $a = 0$. x is defined to be in the set of positive integers, but is also the additive identity. This contradicts the definition of $\mathbb{Z}$, and so $X \neq \mathbb{Z}$.

5

5.a

No. $b * a = a \neq b = a * b$.

5.b

Yes.

#	a	b	c
a	a	b	b
b	b	c	b
c	b	b	c

5.c

Yes.

*	a	b	c
a	a	b	a
b	a	c	b
c	a	b	c

5.d

No. None of a, b, c can be an identity for #:
a: $a\#c = b \neq c$
b: $b\#b = c \neq b$
c: $a\#c = b \neq a$

5.e

$\heartsuit$ is commutative on Y if and only if the table representing $\heartsuit$ is symmetric about the main diagonal. In other words, if the entry at row x and column y is the same as that at row y and column x, $\forall x, y \in Y$.

$c \in Y$ is $\heartsuit$'s identity if and only if its row and column exactly match the order of the elements along the top and left of the table, respectively. In other words,

$\heartsuit$	a	b	c	d	e
a	$\cdot$		a		$\cdot$
b		$\cdot$	b	$\cdot$	
c	a	b	c	d	e
d		$\cdot$	d	$\cdot$	
e	$\cdot$		e		$\cdot$