MATH 322 Assignment 3

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1

1.i

$$\begin{aligned}
b &= \frac{\lambda n(n-1)}{k(k-1)} \\
&= \frac{3 \cdot 25(24}{10(9)} \\
&= 20
\end{aligned}$$

$$\begin{aligned}
r &= \frac{\lambda(n-1)}{k-1} \\
&= \frac{3 \cdot 24}{9} \\
&= 8
\end{aligned}$$

By Fisher's Inequality, we see that this design cannot exist, since $b < n$.

1.ii

$$\begin{aligned}
& n(n-1) = \frac{bk(k-1)}{\lambda} \\
& \Rightarrow n^2 - n - \frac{bk(k-1)}{\lambda} = 0 \\
& \Rightarrow n^2 - n - 90 = 0 \\
& \Rightarrow n = \frac{1 \pm \sqrt{361}}{2} \\
& = \cancel{-9}, 10 \\
& = 10
\end{aligned}$$

$$\begin{aligned}
r &= \frac{\lambda(n-1)}{k-1} \\
&= \frac{2(9)}{3} \\
&= 6
\end{aligned}$$

1.iii

$$
\begin{aligned}
n &= \frac{r(k-1)}{\lambda} + 1 \\
&= \frac{13 \cdot 5}{1} + 1 \\
&= 66
\end{aligned}
$$

$$
\begin{aligned}
b &= \frac{\lambda n(n-1)}{k(k-1)} \\
&= \frac{1 \cdot 66(65)}{6(5)} = 143
\end{aligned}
$$

2

Our n parameter is clearly 16.
For the k parameter, notice that each block B_i consists of each element in the same row as i as well as each element in the same column as i, but not i. Each row and column has 4 elements, so each block thus contains 6 elements, and $k = 6$.
For the λ parameter, consider any pair of elements, say A_{ij} and A_{kl}. It's clear to see that the only two elements whose rows or columns contain both A_{ij} and A_{kl} are A_{il} and A_{kj}. Any other does not contain one or both of A_{ij} and A_{kl} in its block, so clearly each pair occurs in exactly two blocks (specifically, $B_{A_{il}}$ and $B_{A_{kj}}$).

3

Take an arbitrary pair of points, x_1 and x_2. x_1 and x_2 are contained in λ blocks, and x_1 or x_2 (inclusive) are contained in $2r$ blocks. Thus, the number of blocks containing either x_1 or x_2 (exclusive) is $2r - \lambda$. Let y be the number of blocks containing neither x_1 nor x_2. Then $b = y + 2r - \lambda \Rightarrow y = b - 2r + \lambda$. Thus if y is the number of blocks in $\mathcal{B}$ not containing x_1 or x_2, then y is the number of blocks containing both x_1 and x_2 in $\mathcal{B}'$. Thus $(X, \mathcal{B}')$ is an $(n, n-k, b-2r+\lambda)$-design.

4

4.i

$$
\begin{array}{lllll}
1-2=14 & 1-3=13 & 1-5=11 & 1-6=10 & 1-9=7 \\
1-11=5 & 2-1=1 & 2-3=14 & 2-5=12 & 2-6=11 \\
2-9=8 & 2-11=6 & 3-1=2 & 3-2=1 & 3-5=13 \\
3-6=12 & 3-9=9 & 3-11=7 & 5-1=4 & 5-2=3 \\
5-3=2 & 5-6=14 & 5-9=11 & 5-11=9 & 6-1=5 \\
6-2=4 & 6-3=3 & 6-5=1 & 6-9=12 & 6-11=10 \\
9-1=8 & 9-2=7 & 9-3=6 & 9-5=4 & 9-6=3 \\
9-11=13 & 11-1=10 & 11-2=9 & 11-3=8 & 11-5=6 \\
11-6=5 & 11-9=2 & & &
\end{array}
$$

We can clearly see that this is indeed a $(15, 7, 3)$-difference set.

4.ii

$D = \{1, 4, 5, 6, 7, 9, 11, 16, 17\}$
$\mathbb{Z}_{19} \setminus D = \{0, 2, 3, 8, 10, 12, 13, 14, 15, 18\}$

$0-2=17$	$0-3=16$	$0-8=11$	$0-10=9$	$0-12=7$
$0-13=6$	$0-14=5$	$0-15=4$	$0-18=1$	$2-0=2$
$2-3=18$	$2-8=13$	$2-10=11$	$2-12=9$	$2-13=8$
$2-14=7$	$2-15=6$	$2-18=3$	$3-0=3$	$3-2=1$
$3-8=14$	$3-10=12$	$3-12=10$	$3-13=9$	$3-14=8$
$3-15=7$	$3-18=4$	$8-0=8$	$8-2=6$	$8-3=5$
$8-10=17$	$8-12=15$	$8-13=14$	$8-14=13$	$8-15=12$
$8-18=9$	$10-0=10$	$10-2=8$	$10-3=7$	$10-8=2$
$10-12=17$	$10-13=16$	$10-14=15$	$10-15=14$	$10-18=11$
$12-0=12$	$12-2=10$	$12-3=9$	$12-8=4$	$12-10=2$
$12-13=18$	$12-14=17$	$12-15=16$	$12-18=13$	$13-0=13$
$13-2=11$	$13-3=10$	$13-8=5$	$13-10=3$	$13-12=1$
$13-14=18$	$13-15=17$	$13-18=14$	$14-0=14$	$14-2=12$
$14-3=11$	$14-8=6$	$14-10=4$	$14-12=2$	$14-13=1$
$14-15=18$	$14-18=15$	$15-0=15$	$15-2=13$	$15-3=12$
$15-8=7$	$15-10=5$	$15-12=3$	$15-13=2$	$15-14=1$
$15-18=16$	$18-0=18$	$18-2=16$	$18-3=15$	$18-8=10$
$18-10=8$	$18-12=6$	$18-13=5$	$18-14=4$	$18-15=3$

Thus $\mathbb{Z}_{19} \setminus D$ is a $(19, 10, 5)$-difference set.

5

Fix $x \in \mathbb{Z}_n$. x can be written as the difference of two elements in $\mathbb{Z}_n$ in exactly n ways, as the difference of two elements in D in λ ways, and as the difference of two elements in $\overline{D}$ in $n - 2k + \lambda$ ways. Thus, we have that x can be written as the difference of two elements, one in D and the other in $\overline{D}$ in

$$n - \lambda - (n - 2k + \lambda) = 2k - 2\lambda = 2(k - \lambda)$$

ways.

6

6.i

$q = 7$ is the only multiplier satisfying the Multiplier Theorem. Let D be a difference set fixed by q, i.e. $7D = D$. D must be some union of the following sets:

$$\{0\}$$
$$\{1, 7, 9, 10, 12, 16, 26, 33, 34\}$$
$$\{2, 14, 15, 18, 20, 24, 29, 31, 32\}$$
$$\{3, 4, 11, 21, 25, 27, 28, 30, 36\}$$
$$\{5, 6, 8, 13, 17, 19, 22, 23, 35\}$$

We see that $\{1, 7, 9, 10, 12, 16, 26, 33, 34\}$ is a $(37, 9, 2)$-difference set.

6.ii

$q = 3$ and $q = 9$ are the possible multipliers satisfying the Multiplier Theorem. Let D be a difference set fixed by q, i.e. $3D = D$ or $9D = D$. D must be some union of the either these sets ($q = 3$):

$\{0\}$	$\{1, 3, 9, 27, 25, 19\}$	$\{2, 6, 18, 54, 50, 38\}$
$\{4, 12, 36, 52, 44, 20\}$	$\{5, 15, 45, 23, 13, 39\}$	$\{7, 21\}$
$\{8, 24, 16, 48, 32, 40\}$	$\{10, 30, 34, 46, 26, 22\}$	$\{11, 33, 43, 17, 51, 41\}$
$\{14, 42\}$	$\{28\}$	$\{29, 31, 37, 55, 53, 47\}$

or of these sets ($q = 9$):

$\{0\}$	$\{1, 9, 25\}$	$\{2, 18, 50\}$	$\{3, 27, 19\}$
$\{4, 36, 44\}$	$\{5, 45, 13\}$	$\{6, 54, 38\}$	$\{7\}$
$\{8, 16, 32\}$	$\{10, 34, 26\}$	$\{11, 43, 51\}$	$\{12, 52, 20\}$
$\{14\}$	$\{15, 23, 39\}$	$\{17, 41, 33\}$	$\{21\}$
$\{22, 30, 46\}$	$\{24, 48, 40\}$	$\{28\}$	$\{29, 37, 53\}$
$\{31, 55, 47\}$	$\{35\}$	$\{42\}$	$\{49\}$

We see that no such union gives a difference set, and so no difference set exists.