MATH 322 Assignment 1

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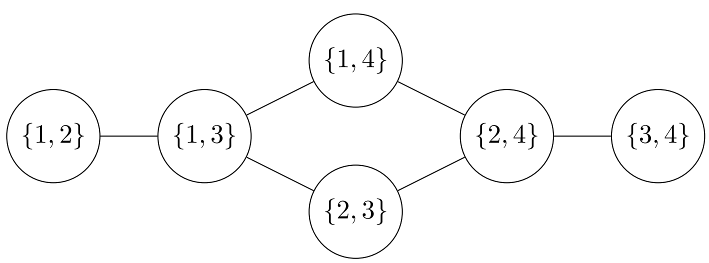
1.a

We present such a list:
$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,3,6\}, \{2,3,6\}, \{2,3,5\}, \{2,3,4\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{1,4,6\}, \{1,5,6\}, \{2,5,6\}, \{2,4,6\}, \{2,4,5\}, \{3,4,5\}, \{3,4,6\}, \{3,5,6\}, \{4,5,6\}$

1.b

We present a graph representing the adjacencies:



This graph clearly has no Hamiltonian path, and so no such list exists.

1.c

Let $A_1, A_2, ..., A_t$ be k-subsets of $[n]$. We'll use $\sim$ to represent adjacency for the remainder of the proof.

Suppose $A_1 \sim A_2$ and $A_2 \sim A_3$ and $A_1 \neq A_3$. Then either A_3 differs from A_2 in the same spot as A_1, but going in the opposite direction (for example, $A_1 = \{3,5\}$, $A_2 = \{3,6\}$, $A_3 = \{3,7\}$), or A_3 differs from A_2 in a different spot than A_1 (for example, $A_1 = \{3,5\}$, $A_2 = \{3,6\}$, $A_3 = \{2,6\}$). In any case, $A_1 \not\sim A_3$ if $A_3 \sim A_2$ and $A_2 \sim A_1$. So no neighbor of a neighbor of A_1 can be a neighbor of A_1. This is true of any $A \subseteq [n]$, $|A| = k$. So we consider the graph representing this system. It is clearly 2-colourable given its properties, and so there exists a bipartition of the k-subsets of $[n]$.

2

Let us chooose a representative for each of the n groups, and count how many options we have at the time:

A_1: We choose either of the two elements, and call it x_1.
A_2: Suppose that $A_1 \subset A_2$. Then we cannot choose x_1, so we choose either of the remaining two elements, and call it x_2. Notice that if $A_1 \not\subset A_2$, then we have more than two choices.
A_2: Suppose that $A_2 \subset A_3$. Then we cannot choose x_1 or x_2, so we choose either of the remaining two elements, and call it x_3. Notice that if $A_2 \not\subset A_3$, then we have more than two choices.
$\vdots$
A_n: Suppose that $A_{n-1} \subset A_n$. Then we cannot choose any of $x_1, x_2, \ldots, x_{n-1}$, so we choose either of the remaining two elements, and call it x_n.

Notice that at each of the n steps, we had at least two options for which element to choose. Then, by the law of product, there were at least 2^n ways we could have chosen our SDR.

3

3.a

Claim: There is an SDR which includes $a_1, a_2, \ldots, a_t$ (but not necessarily as representatives for $A_1, A_2, \ldots, A_t$).
Proof: [Induction on t]
Base: $t = 1$: Given that A_1 has an SDR, this is trivially true.
Induction Hypothesis: Suppose there exists some k such that we can construct an SDR for $A_1, A_2, \ldots, A_n$ containing $a_1, a_2, \ldots, a_l$ for all $l \leq k$.
Induction Step: We attempt to construct an SDR for $A_1, A_2, \ldots, A_n$ containing $a_1, a_2, \ldots, a_{k+1}$ given some $a_{k+1} \in A_{k+1}$:
By the induction hypothesis, we know that there exists an SDR for $A_1, A_2, \ldots, A_n$ containing $a_1, a_2, \ldots, a_k$. Given this, we can construct an SDR for $A_1, A_2, \ldots, A_{k+1}$ containing a_{k+1} as follows:
Case 1: a_{k+1} is already the representative for A_{k+1}: Done. Case 2: a_{k+1} is not the representative for A_{k+1}: Replace the current representative for A_{k+1} with a_{k+1}. If the resulting tuple is an SDR, then we're done. If not, then a_{k+1} was already used elsewhere to represent a set, and so the original SDR already satisfied the property that it contained all of $a_1, a_2, \ldots, a_{k+1}$.

Thus we have constructed the desired SDR, and by induction, the claim holds.

3.b

$A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$, $A_3 = \{3\}$
$a_1 = 1$, $a_2 = 3$, $t = 2$

4

4.a

Whenever $u \in \bigcup_{i=1}^{n} A_i$ or $v \in \bigcup_{i=1}^{n} A_i$. Consider any SDR of the family that does not contain u or v. Find a set A_l such that $u \in A_l$ or $v \in A_l$. Replace A_l's representative with u or v. The SDR remains valid.

4.b

An SDR containing both u and v exists whenever there exist at least two subsets A_i and A_j such that $u \in A_i, v \in A_j, i \neq j$, for some $1 \leq i, j \leq n$. Such an SDR can oly certainly exist when u and v are actually elements of one of $A_1, A_2, \ldots, A_n$ and when they are not found only in the same set.

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1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3
5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6

1	2	3	4	5	6	7
3	4	5	6	7	1	2
5	6	7	1	2	3	4
7	1	2	3	4	5	6
2	3	4	5	6	7	1
4	5	6	7	1	2	3
6	7	1	2	3	4	5

1	2	3	4	5	6	7
4	5	6	7	1	2	3
7	1	2	3	4	5	6
3	4	5	6	7	1	2
6	7	1	2	3	4	5
2	3	4	5	6	7	1
5	6	7	1	2	3	4

1	2	3	4	5	6	7
5	6	7	1	2	3	4
2	3	4	5	6	7	1
6	7	1	2	3	4	5
3	4	5	6	7	1	2
7	1	2	3	4	5	6
4	5	6	7	1	2	3

1	2	3	4	5	6	7
6	7	1	2	3	4	5
4	5	6	7	1	2	3
2	3	4	5	6	7	1
7	1	2	3	4	5	6
5	6	7	1	2	3	4
3	4	5	6	7	1	2

1	2	3	4	5	6	7
7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1