1

($\Longrightarrow$): Suppose G is abelian. Fix $h_1, h_2 \in H$. Then since f is bijective, we know that there exist $g_1, g_2 \in G$ such that $f(g_1) = h_1$ and $f(g_2) = h_2$. We also know that $f^{-1} : H \longrightarrow G$ is an isomorphism, so $g_1 = f^{-1}(h_1)$ and $g_2 = f^{-1}(h_2)$. G is abelian, so

$$\begin{aligned}
&g_1 g_2 = g_2 g_1 \\
\Longrightarrow\ &f^{-1}(h_1)f^{-1}(h_2) = f^{-1}(h_2)f^{-1}(h_1) \\
\Longrightarrow\ &f^{-1}(h_1h_2) = f^{-1}(h_2h_1) \\
\Longrightarrow\ &f(f^{-1}(h_1h_2)) = f(f^{-1}(h_2h_1)) \\
\Longrightarrow\ &h_1h_2 = h_2h_1
\end{aligned}$$

($\Longleftarrow$): Suppose H is abelian. Fix $g_1, g_2 \in G$. Again we know that there exist $h_1, h_2 \in H$ such that $f(g_1) = h_1$ and $f(g_2) = h_2$. H is abelian, so

$$\begin{aligned}
&h_1h_2 = h_2h_1 \\
\Longrightarrow\ &f(g_1)f(g_2) = f(g_2)f(g_1) \\
\Longrightarrow\ &f(g_1g_2) = f(g_2g_1) \\
\Longrightarrow\ &f^{-1}(f(g_1g_2)) = f^{-1}(f(g_2g_1)) \\
\Longrightarrow\ &g_1g_2 = g_2g_1
\end{aligned}$$

Thus G is abelian if and only if H is abelian.

2

We first show that $(R, \oplus)$ is an abelian group:
<u>Associative</u>:

$$\begin{aligned}
((n \oplus m) \oplus k = (n + m + 1) \oplus k = (n + m - 1) + k - 1 \\
n \oplus (k \oplus m) = n + (m \oplus k) - 1 = n + (m + k - 1) - 1
\end{aligned}$$

$(n + m - 1) + k - 1 = n + (m + k - 1) - 1$, so associativity holds.
<u>Has identity</u>: $n \odot 1 = n + 1 - 1 = n$, $1 \odot n = 1 + n - 1 = n$, so 1 is the identity
<u>Has inverses</u>: $n \oplus n^{-1} = n - (n - 2) - 1 = n - n + 2 - 1 = 1$, $n^{-1} \oplus n = (n + 2) - n - 1 = n + 2 - n - 1 = 1$, so $n - 2$ is the inverse for n. We now show that $\odot$ is associative on R:
$(n \odot m) \odot k = (n + m - nm) \odot k = n + m - nm + k - (n + m - nm)k = n \odot (m \odot k)$.
Finally, we show that $\odot$ distributes over $\oplus$:

$$
\begin{aligned}
n \odot (a \oplus b) &= n \odot (a + b - 1) \\
&= n + a + b - 1 - na - nb + n \\
&= n + a - na + n + b - nb - n \\
&= 2n + a + b - na - nb - 1
\end{aligned}
$$

$$
\begin{aligned}
(n \odot a) \oplus (n \odot b) &= (n + a - na) \oplus (n + b - nb) \\
&= n + a - na + n + b - nb - 1 \\
&= 2n + a + b - na - nb - 1
\end{aligned}
$$