MATH 222 Assignment Five

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1

First, we show that for any $m, n \in \mathbb{Z}$ such that $m = n + 1$ or $m = n + 2$, $\gcd(m, n) < 3$.

Recall Bézout's identity: $\gcd(m, n)$ is the smallest positive integer d such that $d = mp + nq$, $p, q \in \mathbb{Z}$.

Case 1: $m = n + 1$

$$\begin{aligned} d &= np + (n+1)q \\ &= np + nq + q \\ &= (p+q)n + q \\ &= 1 && \text{When } q = 1 \text{ and } p = -1. \end{aligned}$$

Since d is defined to be the smallest positive integer satisfying the above equation and 1 is itself the smallest positive integer, we need not justify any more.

Case 2: $m = n + 2$, where both m and n are even.
In this case $\gcd(m, n)$ is trivially 2, since both are by definition divisible by 2 and their common divisor cannot be greater than the difference of the two numbers.

Case 3: $m = n + 2$, where both m and n are odd.
In this case, we can denote $m = 2l - 1$ and $n = 2l + 1$, for some $l \in \mathbb{Z}$. We know $2l - 1$ and $2l + 1$ have 1 as a common divisor. We also know that neither has 2 as a divisor, since both are odd.
Suppose there exists some prime number $p \neq 1$ such that $p \mid 2l - 1$. Recall that p cannot be 2. So $2l - 1 = ap$ for some $a \in \mathbb{Z}$. So $2l + 1 = ap + 2$. p therefore cannot also divide $2l+1$ unless it is 1 or 2, so in this case $\gcd(m, n) = 2$.

We've shown that any two integers with a difference less than 3 have a greatest

common divisor of at most 2.
Claim: There must exist two numbers in S that differ by less than 3. Proof: We will try to construct a subset of S of size 673 containing no two pair of numbers with difference less than 3. Naturally we select every third number of the set, $1, 4, 7, 10, \ldots$ There are $\left\lfloor \frac{2018}{3} \right\rfloor = 672$ numbers in this sequence. Our subset, however, must contain 673 elements. So by the pigionhole principle, we cannot construct such a set, and so the claim holds.

2

We calculate the range of possible sums of our subsets by adding the fewest elements of low value and the most elements of high value. In our case these are $\{1\}$ and $\{108, \ldots, 117\}$ respectively, so our range is $[1, 1125]$. Notice that the extreme values of this range (1 and 1125) are only attainable by a single set, so we can throw them out. Then our new range is $[2, 1124]$. The value 2 can also only be attained by the subset $\{2\}$, so we can remove this as well. Our final range is $[3, 1124]$. This has reduced our range enough to apply the pigeonhole principle: There are exactly $2^10 - 2 = 1022$ nonempty subsets of S, and there are $1124 - 3 = 1021$ possible values, so there must exist two distinct subsets A and B with $s_A = s_B$.

3

3.a

First we analyze the relation: $x + 3y$ is odd when exactly one of x or y is odd. This knowledge will help us prove some of $\mathcal{R}$'s properties.
Reflexive: No. $(1, 1) \notin \mathcal{R}$.
Symmetric: Yes. $(x, y) \in \mathcal{R} \implies$ one of x, y is odd, so $(y, x) \in \mathcal{R}$, since one of y, x is also odd.
Antisymmetric: No. $(1, 2) \in \mathcal{R}$ and $(2, 1) \in \mathcal{R}$, but $1 \neq 2$.
Transitive: No. $(1, 2) \in \mathcal{R}$ and $(2, 3) \in \mathcal{R}$ but $(1, 3) \notin \mathcal{R}$.

3.b

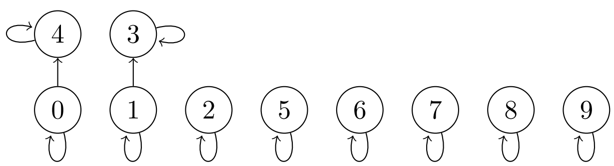
Reflexive: Yes. $X \cap \{1, 3, 6\}$ is always equal to itself.
Symmetric: Yes. The set equality operator is commutative, so if $X \cap \{1, 3, 6\} = Y \cap \{1, 3, 6\}$, then $Y \cap \{1, 3, 6\} = X \cap \{1, 3, 6\}$.
Transitive: Yes. The set equality operator is transitive, so if $X \cap \{1, 3, 6\} = Y \cap \{1, 3, 6\}$ and $Y \cap \{1, 3, 6\} = Z \cap \{1, 3, 6\}$, then $X \cap \{1, 3, 6\} = Z \cap \{1, 3, 6\}$.
Antisymmetric: No. $(\emptyset, \{2\}) \in \mathcal{T}$ and $(\{2\}, \emptyset) \in \mathcal{T}$, but $\emptyset \neq \{2\}$.

4

4.a

Reflexive: $|x-2| = |x-2|$
Symmetric: $|x-2| = |y-2| \implies |y-2| = |x-2|$
Transitive: $|x-2| = |y-2| \land |y-2| = |z-2| \implies |x-2| = |z-2|$
All hold by the definition of $=$ on $\mathbb{Z}$.

4.b



4.c

$\{0,4\}, \{1,3\}, \{2\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}$

5

5.a

There are $|A \times A| = 49$ elements that can be in any relation on A. Of these 49, the 7 elements of the form (x, x), $x \in A$ and (a, b) and (b, c) are already chosen, so there are $49 - 7 - 1 - 1$ elements to decide to include or exclude. So $|\mathcal{R}| = 2^{49-7-1-1} = 2^{40}$ such relations are possible.

5.b

Each element not of the form (x, x), $x \in A$ included in the relation must be accompanied by its "pair" (i.e. $(1, 2)$ must be with included with $(2, 1)$). So we must make decisions for only one of each pair of non-reflexive elements. Additionally, (a, b) must be contained, so there are $2^{20-1} = 2^{19}$ such relations.

5.c

We will count the number of symmetric relations on A and the number of symmetric and reflexive relations on A. We will subtract the latter from the former to calculate the number of symmetric but not reflexive relations on A.
Symmetric: 2^{27}
Symmetric and reflexive: 2^{20}
So there are $2^{27} - 2^{20}$ such relations.

5.d

For each pair of non-reflexive elements in A, either or neither, but not both elements can be in the relation. In other words, there are 3 for each pair instead of 4. Note that we still have two options for each reflexive element. Since the relation must contain (a, b), one of the 20 decisions for the pairs of non-reflexive elements is removed. Since the relation cannot contain (b, c), another of the 20 decisions for the pairs of non-reflexive elements loses one of its options. Overall, then, there are $3^{18} \cdot 2^7 \cdot 2$ such relations.

6

6.a

Suppose $\mathcal{R}_1 \cup \mathcal{R}_2$ is not symmetric. Then there exists some $(x, y) \in \mathcal{R}_1 \cup \mathcal{R}_2$ such that $(y, x) \notin \mathcal{R}_1$ and $(y, x) \notin \mathcal{R}_2$. Since $(x, y) \in \mathcal{R}_1 \cup \mathcal{R}_2$, (x, y) must be in one of $\mathcal{R}_1$ or $\mathcal{R}_2$, and therefore (y, x) must be in one of $\mathcal{R}_1$ or $\mathcal{R}_2$, and therefore must be in $\mathcal{R}_1 \cup \mathcal{R}_2$. This is a contradiction, and so $\mathcal{R}_1 \cup \mathcal{R}_2$ is symmetric.

6.b

Suppose $(x, y) \in \mathcal{R}_1 \cap \mathcal{R}_2$ and $(x, y) \in \mathcal{R}_1 \cap \mathcal{R}_2$, but $x \neq y$. Then $(x, y) \in \mathcal{R}_1$ and $(y, x) \in \mathcal{R}_1$. This is a contradiction, since $x \neq y$ and $\mathcal{R}_1$ is defined to be antisymmetric, so $\mathcal{R}_1 \cap \mathcal{R}_2$ is antisymmetric.

6.c

Suppose WLOG that there exists some $x \in A$ such that $(x, x) \notin \mathcal{R}_1$. Then $(x, x) \notin \mathcal{R}_1 \cap \mathcal{R}_2$. This is a contradiction, since $\mathcal{R}_1 \cap \mathcal{R}_2$ is reflexive.

6.d

False. $A = \{1, 2, 3\}$, $\mathcal{R}_1 = \{(1, 2), (1, 3)\}$, $\mathcal{R}_2 = \{(2, 3)\}$.

7

7.a

Reflexive: Let $a = a_1 a_2 \dots a_n \in A$. For any $1 \leq i \leq n$, $a_i \leq a_i$, so $\mathcal{R}$ is reflexive.
Antisymmetric: Let $a = a_1 a_2 \dots a_n, b = b_1 b_2 \dots b_n \in A$. If $a\mathcal{R}b$ and $a \neq b$, then there exists some $1 \leq k \leq n$ such that $a_k = 0$ and $b_k = 1$. Thus, $b \not\mathcal{R} a$, and so $\mathcal{R}$ is antisymmetric.
Transitive: Let $a, b, c \in A$. Suppose $a\mathcal{R}b$ and $b\mathcal{R}c$. Then for every $1 \leq k \leq n$, $a_k \leq b_k$ and $b_k \leq c_k$, thus $a_k \leq c_k$, and so $a\mathcal{R}c$. So $\mathcal{R}$ is transitive.

7.b

8 Bonus

We first note that the chess player cannot play more than $11 \cdot 12 = 132$ games during the 77 day period. Let c_i be the number of games played by the i^{th} day. Note that $1 \leq c_1 \leq \ldots \leq c_{77} \leq 132$. We now must prove that there must exist some $1 \leq i \leq j \leq 77$ such that $c_j = c_i + 21$. So we extend our original sequence of numbers $c_1, c_2, \ldots, c_{77}$ with the sequence $c_1 + 21, c_2 + 21, \ldots, c_{77} + 21$ to get the new sequence $c_1, \ldots, c_{77}, c_1 + 21, \ldots, c_{77} + 21$. This sequence contains 154 elements. The value of any element in the sequence is in the range [1,132+21], so there are 153 distinct values that any element in the sequence can have. By the pigeonhole principle, there must exist $i < j$ such that $c_j = c_i + 21$, and therefore there exists a sequence of days over the course of which the chess player plays exactly 21 games.