

MATH 222 Assignment One

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1

1.a

Recall that for any graph G on ≥ 3 vertices, if $\deg(v) \geq \frac{n}{2}$ for all v , then G has a Hamiltonian cycle.

$\overline{C_n} = K_n \setminus C_n$. Each vertex in C_n has degree 2, and each vertex in K_n has degree $n - 1$, thus removing the edges in C_n from K_n leaves each vertex in K_n with 2 fewer incident edges. So each vertex in $\overline{C_n}$ has degree $n - 3$.

We will now find the smallest n for which $\deg(v) \geq \frac{n}{2}$.

$$\deg(v) \geq \frac{n}{2}$$

$$\deg(v) = n - 3$$

$$n - 3 \geq \frac{n}{2}$$

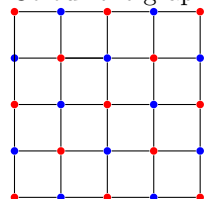
$$2n - 6 \geq n$$

$$n \geq 6$$

So $\overline{C_n}$ has a Hamiltonian cycle for all $n \geq 6$. We can manually check this statement's validity for $n < 6$, and find that it is true for $n = 5$ and false for $n < 5$. So $\overline{C_n}$ has a Hamiltonian cycle for $n \geq 5$.

1.b

Colour the graph as so:



Note that this graph forms a bipartition. Let $R = \{r \in V | r \text{ is coloured red}\}$ and $B = \{b \in V | b \text{ is coloured blue}\}$.

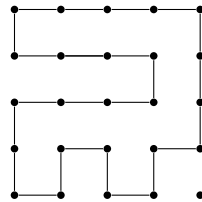
Note that $|R| = 36$ and $|B| = 40$.

Claim: G does not contain a Hamiltonian cycle.

Proof: (Contradiction)

Suppose there exists a Hamiltonian cycle in G . Let this cycle begin in some $v \in A$. Then after $2|A| = 72$ edges have been traversed, we will end up back at v . But if only 72 edges have been traversed, there must still be 4 vertices in B that have not been visited, and so the cycle is not Hamiltonian. This is a contradiction, and so there exists no Hamiltonian cycle in G .

1.c

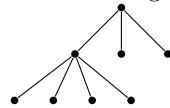


There exists no cycle on all n vertices, so this cycle on $n - 1$ vertices must be the largest.

2

2.a

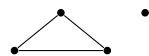
G has vertices of degree 3 and 4, so it must have at least one vertex of each. Consider, then, a tree with one vertex of degree 3, one vertex of degree 4, and the rest vertices of degree 1. Let the vertex of degree 3 be the root, and the vertex of degree 4 one of its children.



This tree has 6 children. The addition of any more non-leaf vertices will never lower the number of leaves, so this is the minimal number of leaves in a tree that satisfies the given constraints.

2.b

Counterexample: $m = 3$, $n = 4$, $k = 1$:



$0 \leq 3 \leq 4 - 1$ holds, but the graph has 2 components despite k being 1. So the claim is false.

3

Recall: For any connected plane graph, $n - m + r = 2$ and $m \leq 3n - 6$.

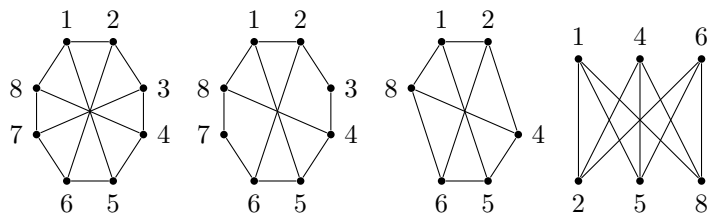
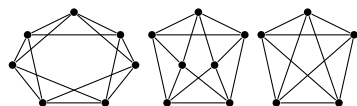
$$\begin{aligned}
 n - m + r &= 2 \\
 n - m + 96 &= 2 & (r = 96) \\
 n - m &= -94 \\
 n - (3n - 6) &\geq -94 & (m \leq 3n - 6) \\
 -2n + 6 &\geq -94 \\
 2n &\geq 100 \\
 n &\geq 50
 \end{aligned}$$

So the graph must have at least n vertices.

4



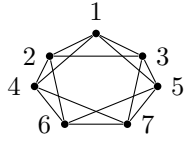
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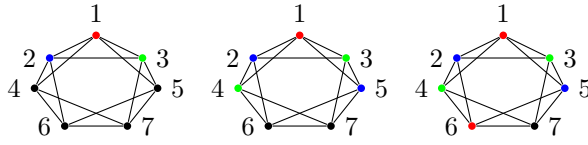
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Note that the first graph has C_7 as a subgraph, and so its chromatic number must be at least 3.

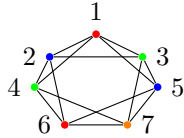
We will now attempt to construct a 4-colouring for this graph:



Consider the three copies of K_3 : $\{1, 2, 3\}$, $\{1, 3, 5\}$, and $\{2, 4, 6\}$. Note that each vertex must not share its colour within the set. We will try to colour the vertices in each triangle



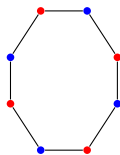
The rest of the graph cannot be coloured any differently; each copy of K_3 must have 3 distinct colours. So there exists no 3-colouring for this graph. We will now provide a 4-colouring for this graph.



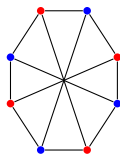
So this first graph's chromatic number is 4.

For the second graph, we shall first prove that there exists no 2-colouring, and then provide a 3-colouring to prove its existence.

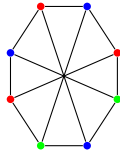
Suppose there exists a 2-colouring of the graph. Then its outer edges forming C_8 must be coloured as so:



If we now insert the rest of the edges belonging to the original graph, we find that there are edges connected red vertices to red vertices and blue vertices to blue vertices:



This is a contradiction, and so there is no 2-colouring for the graph. We now provide a 3-colouring:



Thus, the second graph's chromatic number is 3.

7 Bonus