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Assignment 5

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1

Let $G = (X, Y)$. We construct H:

WLOG let $|X| \leq |Y|$. Add $|Y| - |X|$ vertices to X so that $|X| = |Y|$. G is bipartite, so we have that $\sum_{x \in X} deg(x) = \sum_{y \in Y} deg(y)$. So choose $x \in X$, $y \in Y$ with $deg(x), deg(y) < \Delta(G)$, and join them with an edge.

After adding an edge, the equality $\sum_{x \in X} deg(x) = \sum_{y \in Y} deg(y)$ still holds, so H remains bipartite. So if we continue this process until no such x and y can any longer be chosen, then the resulting graph is $\Delta(G)$-regular, and was constructed from – and therefore contains – G, as desired.

2

Let H be an odd connected component in G. Assume for a contradiction that H is bipartite, with bipartition (X, Y). WLOG let $|X| < |Y|$ (we know that $|X| \neq |Y|$ since H is odd). H is k-regular, so since $\sum_{x \in X} deg(x) = \sum_{y \in Y} deg(y)$, we have $k|X| = k|Y|$, a contradiction since $|X| \neq |Y|$. Thus H is not bipartite, and thus contains an odd cycle, and is class 2. Thus G must be class 2, and $\chi'(G) = \Delta(G) + 1$, as desired.

3

Let G, H be Hamiltonian, and let $g_1 \ldots g_l g_1$ and $h_1 \ldots h_k h_1$ be Hamilton cycles in G and H, respectively. Then

$$(g_1, h_1)(g_2, h_1) \ldots (g_l, h_1)(g_l, h_2) \ldots (g_l, h_k)(g_l, h_1)(g_1, h_1)$$

is a Hamilton path in $G \,\square\, H$, so $G \,\square\, H$ is Hamiltonian, as desired.

$Q_k = Q_{k-1} \,\square\, Q_{k-1}$, so Q_k is Hamiltonian when $k \geq 2$ (Q_1 is not Hamiltonian).

4

($\Longrightarrow$): Choose a vertex $v \in V(G)$, and let $u_1, \ldots, u_k$ be its neighbors. Suppose WLOG that $u_1 \sim u_2, \ldots, u_{k-1} \sim u_k, u_k \sim u_1$. Then $vu_1u_2, \ldots, vu_{k-1}u_k, vu_ku_1$ are all triangles. If v is coloured 1 and u_i is coloured 2, then u_{i-1} and u_{i+1} must be coloured 4. This is clearly only the case when v has an even number of neighbors, otherwise we would need a fourth colour. G is 3-colourable, so it must be the case that for any choice of v, v has even degree. G is a plane triangulation, and thus connected, so it is therefore Eulerian.

($\Longleftarrow$): Choose a face F, and 3-colour its vertices arbitrarily. We claim this determines the colour of each other vertex, and that it gives a 3-colouring. Suppose for a contradiction that a vertex v is the first to be assigned two different colours from two different faces. That is, there are two sequences of faces, not identical, which assign to v two different colours. Let these sequences be $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$. We know $A_k \neq B_l$, so let $C = A_i = B_i$ be the first face for which A_{i+1} and B_{i+1} differ. Consider the cycle bouding the region bounded by $C, A_{i+1}, \ldots, A_k, B_l, \ldots, B_{i+1}, C$. Let this bounded region be as small as possible. Let x be a vertex in the bounded region, but not on the cycle, and which is contained in the faces X_1 and X_m. Createa new region $X_1, \ldots, X_m, B_j, \ldots, B_l, A_k, \ldots, A_j, X_1$, where $X_1, \ldots, X_m$ are the faces forming a sequence that all contain x. x was assigned only one colour, since v was the first to be assigned two different colours, and so if we begin our sequential colouring as before but now starting at x, the rest of the graph will be coloured in the same way. But this means that the procedure will still assign two colours to v, but the new bounding region is smaller, contradiction the minimality of the original region. Thus no vertex is assigned two different colours, and so G is 3-colourable.

5

Suppose G is a plane graph and $G \cong G^*$. The faces of G correspond to the vertices of G^*, so $n(G) = f(G)$. Then by Euler's formula, we have $n(g) - m(G) + f(G) = 2$, so $n(G) - m(g) + n(G) = 2$, and so $m = 2n - 2$, as desired.

Consider the Wheel graph, $W_n = W_{n-1} \vee K_1$. W_n^* has a cycle of $n - 1$ vertices corresponding to the faces of each of the $n - 1$ triangles, and each of these faces is adjacent to the face surrounding the graph, resulting in another copy of W_n, as desired.