Digital Signal Processing — FIR filters

1. Given the ideal frequency response for the low-pass filter in Fig. 1(a) with cut-off ω_c,

$$H(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

derive the time-domain impulse response $h[n]$. Start with $n = 0$ and then $n \neq 0$, using the identity $\sin(x) = (e^{jx} - e^{-jx})/2j$:

Answer: for $n = 0$:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\mathrm{d}\,\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \mathrm{d}\,\omega = \frac{1}{2\pi}\left[\omega\right]_{-\omega_c}^{\omega_c} \\ &= \frac{\omega_c}{\pi} \end{aligned}$$

for $n \neq 0$:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\mathrm{e}^{j\omega n}\mathrm{d}\,\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \mathrm{e}^{j\omega n}\mathrm{d}\,\omega = \frac{1}{2\pi}\left[\frac{\mathrm{e}^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} \\ &= \frac{1}{jn2\pi}\left[\mathrm{e}^{j\omega_c n} - \mathrm{e}^{-j\omega_c n}\right] \\ &= \frac{1}{\pi n}\sin(\omega_c n) \\ &= \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n} \end{aligned}$$

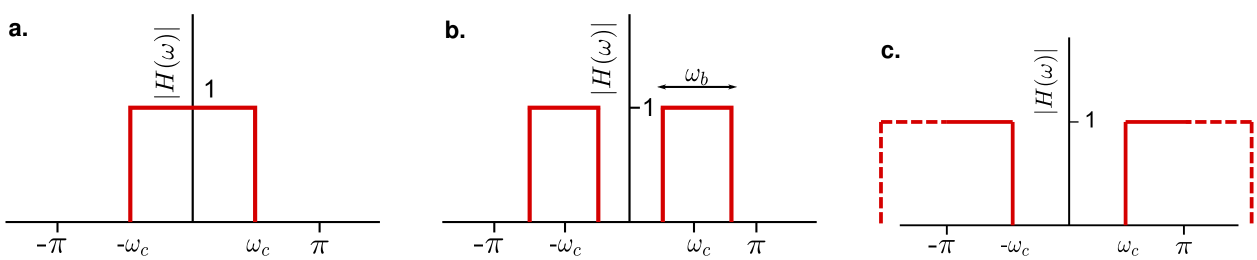

Figure 1: Ideal frequency response for (a) low-pass, (b) band-pass, and (c) and high-pass filters.

2. Derive the impulse response for high-pass FIR filter with cut-off ω_c in Fig. 1(b)

$$H_d(\omega) = \begin{cases} 1 & \omega_c - \frac{\omega_b}{2} \le \omega \le \omega_c + \frac{\omega_b}{2} \\ 1 & -\omega_c - \frac{\omega_b}{2} \ge \omega \ge -\omega_c + \frac{\omega_b}{2} \\ 0 & \text{otherwise} \end{cases}$$

Answer: Take the inverse discrete-time Fourier transform of the $H(\omega)$:

$$\begin{aligned}
h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{\omega_c-\omega_b/2}^{\omega_c+\omega_b/2} e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{-\omega_c-\omega_b/2}^{-\omega_c+\omega_b/2} e^{j\omega n}d\omega
\end{aligned}$$

Then, for $n = 0$:

$$\begin{aligned}
h_d[n] &= \frac{1}{2\pi}[\omega]_{\omega_c-\omega_b/2}^{\omega_c+\omega_b/2} + \frac{1}{2\pi}[\omega]_{-\omega_c-\omega_b/2}^{-\omega_c+\omega_b/2} \\
&= \frac{1}{2\pi}\left[-\omega_c + \frac{\omega_b}{2} + \omega_c + \frac{\omega_b}{2} + \omega_c + \frac{\omega_b}{2} - \omega_c + \frac{\omega_b}{2}\right] \\
&= \frac{\omega_b}{\pi}
\end{aligned}$$

And for $n \neq 0$:

$$\begin{aligned}
h_d[n] &= \frac{1}{2\pi}[\frac{e^{j\omega n}}{jn}]_{\omega_c-\omega_b/2}^{\omega_c+\omega_b/2} + \frac{1}{2\pi}[\frac{e^{j\omega n}}{jn}]_{-\omega_c-\omega_b/2}^{-\omega_c+\omega_b/2} \\
&= \frac{1}{2\pi jn}\left[e^{j(\omega_c+\omega_b/2)n} - e^{j(\omega_c-\omega_b/2)n} + e^{j(-\omega_c+\omega_b/2)n} - e^{j(-\omega_c-\omega_b/2)n}\right] \\
&= \frac{1}{2\pi jn}\left[e^{j\omega_c n}e^{j(\omega_b/2)n} - e^{j\omega_c n}e^{-j(\omega_b/2)n} + e^{-j\omega_c n}e^{j(\omega_b/2)n} - e^{-j\omega_c n}e^{-j(\omega_b/2)n}\right] \\
&= \frac{e^{j\omega_c n}}{2\pi jn}\left[e^{j(\omega_b/2)n} - e^{-j(\omega_b/2)n}\right] + \frac{e^{-j\omega_c n}}{2\pi jn}\left[e^{j(\omega_b/2)n} - e^{-j(\omega_b/2)n}\right] \\
&= \frac{e^{j\omega_c n}}{\pi n}\sin((\omega_b/2)n) + \frac{e^{-j\omega_c n}}{\pi n}\sin((\omega_b/2)n) \\
&= \frac{\sin((\omega_b/2)n)}{\pi n}\left[e^{j\omega_c n} + e^{-j\omega_c n}\right] \\
&= 2\cos(\omega_c n)\frac{\sin((\omega_b/2)n)}{\pi n} \\
&= \cos(\omega_c n)\frac{\omega_b}{\pi}\frac{\sin((\omega_b/2)n)}{(\omega_b/2)n}
\end{aligned}$$

and therefore

$$h_d[n] = \begin{cases} \frac{\omega_b}{\pi} & n = 0 \\ \cos(\omega_c n)\frac{\omega_b}{\pi}\frac{\sin((\omega_b/2)n)}{(\omega_b/2)n} & n \neq 0 \end{cases}$$

3. Derive the impulse response for high-pass FIR filter with cut-off ω_c [Fig. 1(c)] using the window method with a rectangular window of length M (assume M is odd).

Answer:

(a) First derive the ideal impulse response $h_d[n]$ from the ideal frequency response:

$$H_d(\omega) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & \omega_c \leq |\omega| \leq \pi \end{cases}$$

for $n = 0$:

$$\begin{aligned}
h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\mathrm{d}\,\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} \mathrm{d}\,\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} \mathrm{d}\,\omega \\
&= \frac{1}{2\pi}\left[\omega\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\omega\right]_{\omega_c}^{\pi} \\
&= \frac{1}{2\pi}[-\omega_c + \pi] + \frac{1}{2\pi}[\pi - \omega_c] \\
&= 1 - \frac{\omega_c}{\pi}
\end{aligned}$$

for $n \neq 0$:

$$\begin{aligned}
h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\mathrm{e}^{j\omega n}\mathrm{d}\,\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} \mathrm{e}^{j\omega n}\mathrm{d}\,\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} \mathrm{e}^{j\omega n}\mathrm{d}\,\omega \\
&= \frac{1}{2\pi}\left[\frac{\mathrm{e}^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{\mathrm{e}^{j\omega n}}{jn}\right]_{\omega_c}^{\pi} \\
&= \frac{1}{jn2\pi}\left[\mathrm{e}^{-j\omega_c n} - \mathrm{e}^{-j\pi n}\right] + \frac{1}{jn2\pi}\left[\mathrm{e}^{j\pi n} - \mathrm{e}^{j\omega_c n}\right] \\
&= \frac{1}{jn2\pi}\left[\mathrm{e}^{j\pi n} - \mathrm{e}^{-j\pi n}\right] - \frac{1}{jn2\pi}\left[\mathrm{e}^{j\omega_c n} - \mathrm{e}^{-j\omega_c n}\right] \\
&= \frac{\sin(\pi n)}{\pi n} - \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n} \\
&= -\frac{\sin(\omega_c n)}{\pi n}
\end{aligned}$$

and therefore

$$h_d[n] = \begin{cases} 1 - \frac{\omega_c}{\pi} & n = 0 \\ -\frac{\sin(\omega_c n)}{\pi n} & n \neq 0 \end{cases}$$

(b) Next, shift the impulse response function in time by $M_h = (M-1)/2$:

$$h_d[n] \to h_d[n - M_h]$$

(c) truncate: multiple by rectangular window $w[n]$, where $w[n] = 1$ for $0 \leq n \leq M-1$.

Thus, the impulse response is:

$$\begin{aligned}
h[n] &= h_d\left[n - M_h\right] w[n] \\
&= \begin{cases} \frac{-\sin[\omega_c(n - M_h)]}{\pi(n - M_h)} & 0 \leq n \leq M-1 \,\text{and}\, n \neq M_h \\ 1 - \frac{\omega_c}{\pi} & n = M_h \\ 0 & n < 0 \,\text{and}\, n > M-1 \end{cases}
\end{aligned}$$

4. Find the phase and magnitude of the frequency response for an FIR filter with a positive symmetrical impulse response and $M = 5$. What is the filter Type (I, II, III, or IV) and why? Determine the group delay for this filter.

Answer: Filter is Type I because positive symmetrical and because M is odd.

For $M = 5$ and with positive symmetry,

$$h[0] = h[4];\ h[1] = h[3];\ h[2]$$

Take the discrete-time Fourier transform for $h[n]$,

$$\begin{aligned}
H(\omega) &= \sum_{n=0}^{4} h[n]\mathrm{e}^{-j\omega n} \\
&= h[0] + h[1]\mathrm{e}^{-j\omega} + h[2]\mathrm{e}^{-j2\omega} + h[3]\mathrm{e}^{-j3\omega} + h[4]\mathrm{e}^{-j4\omega} \\
&= \mathrm{e}^{-j2\omega}\left(h[2] + h[0]\mathrm{e}^{j2\omega} + h[1]\mathrm{e}^{j\omega} + h[3]\mathrm{e}^{-j\omega} + h[4]\mathrm{e}^{-j2\omega}\right) \\
&= \mathrm{e}^{-j2\omega}\left(h[2] + h[0](\mathrm{e}^{j2\omega} + \mathrm{e}^{-j2\omega}) + h[1](\mathrm{e}^{j\omega} + \mathrm{e}^{-j\omega})\right) \\
&= \mathrm{e}^{-j2\omega}\left(h[2] + 2h[0]\cos(2\omega) + 2h[1]\cos(\omega)\right)
\end{aligned}$$

and thus

$$\begin{aligned}
\angle H(\omega) &= -2\omega + \beta \\
|H(\omega)| &= |h[2] + 2h[0]\cos(2\omega) + 2h[1]\cos(\omega)|
\end{aligned}$$

where $\beta = 0$ or $\beta = \pi$ if expression $h[2] + 2h[0]\cos(2\omega) + 2h[1]\cos(\omega)$ is positive of negative, respectively.

Group delay:

$$-\frac{d\angle H(\omega)}{d\omega} = 2.$$

5. Derive the phase and magnitude of the frequency response for an FIR filter with a an impulse response with positive symmetry and M odd (a Type I filter).

Answer: When M is odd and $h[n] = h[M-1-n]$, then

$$h[\tfrac{M-1}{2} - n] = h[\tfrac{M-1}{2} + n] \qquad \text{for } n = 1, 2, \ldots, \tfrac{M-1}{2}$$

Take the discrete-time Fourier transform of the impulse function $h[n]$ to find frequency response:

$$\begin{aligned}
H(\omega) &= \sum_{n=0}^{M-1} h[n]e^{-j\omega n} \\
&= h[\tfrac{M-1}{2}]e^{-j\omega(M-1)/2} + \sum_{n=1}^{(M-1)/2} h[\tfrac{M-1}{2} - n]e^{-j\omega[(M-1)/2-n]} \\
&\quad + \sum_{n=1}^{(M-1)/2} h[\tfrac{M-1}{2} + n]e^{-j\omega[(M-1)/2+n]} \\
&= h[\tfrac{M-1}{2}]e^{-j\omega(M-1)/2} + e^{-j\omega(M-1)/2}\left\{\sum_{n=1}^{(M-1)/2} h[\tfrac{M-1}{2} - n]e^{j\omega n} + \sum_{n=1}^{(M-1)/2} h[\tfrac{M-1}{2} - n]e^{-j\omega n}\right\} \\
&= h[\tfrac{M-1}{2}]e^{-j\omega(M-1)/2} + e^{-j\omega(M-1)/2}\left\{\sum_{n=1}^{(M-1)/2} h[\tfrac{M-1}{2} - n](e^{j\omega n} + e^{-j\omega n}\right\} \\
&= e^{-j\omega(M-1)/2}\left\{h[\tfrac{M-1}{2}] + 2\sum_{n=1}^{(M-1)/2} h[\tfrac{M-1}{2} - n]\cos(\omega n)\right\}
\end{aligned}$$

6. Derive the phase and magnitude of the frequency response for an FIR filter $h[n]$ of length M, where $h[n]$ has negative symmetry and M is even (a Type IV filter). Determine the group delay for this filter.

Answer: When M is even and $h[n] = -h[M-1-n]$, then

$$h[\tfrac{M}{2} - n] = -h[\tfrac{M}{2} + n - 1] \qquad \text{for } n = 1, 2, \ldots, \tfrac{M}{2}$$

Discrete-time Fourier transform of the impulse function $h[n]$:

$$\begin{aligned}
H(\omega) &= \sum_{n=0}^{M-1} h[n]e^{-j\omega n} \\
&= \sum_{n=1}^{M/2} h[\tfrac{M}{2} - n]e^{-j\omega(\frac{M}{2}-n)} + \sum_{n=1}^{M/2} h[\tfrac{M}{2} + n - 1]e^{-j\omega(\frac{M}{2}+n-1)} \\
&= e^{-j\omega M/2}\left\{\sum_{n=1}^{M/2} h[\tfrac{M}{2} - n]e^{j\omega n} - \sum_{n=1}^{M/2} h[\tfrac{M}{2} - n]e^{-j\omega(n-1)}\right\} \\
&= e^{-j\omega M/2}e^{j\omega/2}\left\{\sum_{n=1}^{M/2} h[\tfrac{M}{2} - n](e^{j\omega n} - e^{-j\omega(n-1)})e^{-j\omega/2}\right\} \\
&= e^{-j\omega(M-1)/2}\left\{\sum_{n=1}^{M/2} h[\tfrac{M}{2} - n](e^{j\omega(n-1/2)} - e^{-j\omega(n-1/2)})\right\} \\
&= e^{-j\omega(M-1)/2}\left\{2j\sum_{n=1}^{M/2} h[\tfrac{M}{2} - n]\sin(\omega[n - \tfrac{1}{2}])\right\} \\
&= je^{-j\omega(M-1)/2}\left\{2\sum_{n=1}^{M/2} h[\tfrac{M}{2} - n]\sin(\omega[n - \tfrac{1}{2}])\right\}
\end{aligned}$$

Magnitude and phase of filter:

$$|H(\omega)| = \left|2\sum_{n=1}^{M/2} h[\tfrac{M}{2}-n]\sin(\omega[n-\frac{1}{2}])\right|$$

$$\angle H(\omega) = \begin{cases} -\omega(\frac{M-1}{2})+\frac{\pi}{2} & \text{for } A(\omega)\geq 0 \\ -\omega(\frac{M-1}{2})+\frac{3\pi}{2} & \text{for } A(\omega)<0 \end{cases}$$

Group delay is:

$$-\frac{d\angle H(\omega)}{d\omega} = \frac{M-1}{2}.$$

1 Appendix: identities

$$\sin(\omega) = \frac{e^{j\omega} - e^{-j\omega}}{2j}$$
$$\cos(\omega) = \frac{e^{j\omega} + e^{-j\omega}}{2}$$

window type	side-lobe amplitude (dB)	approximate width of main lobe ($\approx \Delta\omega$)	approximate stopband $20\log_{10}(\delta)$ (dB)	approximate passband $20\log_{10}(1+\delta)$ (dB)
rectangular	-13	$4\pi/(M+1)$	-21	0.7416
Hanning	-31	$8\pi/M$	-44	0.0546
Hamming	-41	$8\pi/M$	-53	0.0194
Blackman	-57	$12\pi/M$	-74	0.0017

Table 1: Window parameters for length M window with relation to filter transition width $\Delta\omega$.

1.1 Kaiser Window

Kaiser window with parameter β and length M is defined as:

$$w[n] = \begin{cases} \frac{I_0[\beta(1-[(n-M_h)/M_h]^2)^{1/2}]}{I_0(\beta)} & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$$

where $M_h = (M-1)/2$, I_0 is zero-order modified Bessel function of the first kind.

For filter transition $\Delta\omega$ with pass-band ripple $A = -20\log_{10}\delta$,

$$\beta = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases}$$

and

$$M = \frac{A-8}{2.285\Delta\omega}$$