Window Method for FIR Filter Design: Phase

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lecture notes and example questions:
https://github.com/otoolej/DSP_notes_FIR

Review: design of FIR filters using the window method

- window method: 1) derive ideal impulse response, 2) shift and truncate
- properties of different windows and how this shapes the filter
  - $H(\omega) = W(\omega) * H_d(\omega)$
- given set of design specifications, how to select window type and length M
- more precise design with Kaiser window, as extra parameter to optimise design
- pros. and cons. of window method

Learning Objectives

- linear phase and group delay
- symmetry of impulse response function $h[n]$
- generalised linear phase (with constant group delay)
  - but delay can be integer or fractional
- define four filter Types (I, II, III, IV)
  - depending on positive/negative symmetry and M even/odd
- symmetry of window important
  - both window $w[n]$ and $h_d[n]$ need to be symmetric/odd if we want $h[n]$ symmetric/odd.

Phase

- phase $\theta(\omega)$ of frequency response:

$$H(\omega) = |H(\omega)|e^{j\theta(\omega)}$$

- signal consists of several frequency components
- phase-delay is the amount of time each frequency component of the signal delayed from passing through the filter
- *A non-linear phase response causes phase distortion:*
  - different frequency components in the signal will each be delayed by a different amount, thereby altering their harmonic relationship.
  - such a distortion is undesirable in many applications, e.g. data transmission and video.

Generalised Linear Phase

$$H(\omega) = |H(\omega)|e^{-j(\omega\alpha-\beta)}$$

thus phase is

$$\theta(\omega) = -\omega\alpha + \beta$$

where α and β are constants. Linear phase when $\beta = 0$, i.e.

$$H(\omega) = |H(\omega)|e^{-j\omega\alpha}$$

Group delay (GD):

$$\begin{aligned}\mathrm{GD} &= -\frac{d\theta(\omega)}{d\omega}\\ &= \alpha\end{aligned}$$

$\Rightarrow$ group delay is constant for linear and generalised linear phase

when does impulse response $h[n]$ have (generalised) linear phase?

By imposing symmetry conditions on $h[n]$ we can enforce (generalised) linear phase:

$$\begin{aligned} h[n] &= h[M-1-n], && \text{positive symmetry}\\ h[n] &= -h[M-1-n], && \text{negative symmetry}\end{aligned}$$

for $n = 0, 1, \ldots, M-1$.

- for positive symmetry, $\alpha = \frac{M-1}{2}$ and $\beta = 0$
- for negative symmetry, $\alpha = \frac{M-1}{2}$ and $\beta = \frac{\pi}{2}$
- M even or odd?

Example: positive symmetry for $h[n]$ and M

For impulse response $h[n]$ ($n = 0, 1, \ldots, M-1$) and $M = 7$ and

$$h[n] = h[M-1-n]$$

show that filter has linear phase response.

solution:

express in terms of $H(\omega) = e^{-j\theta(\omega)}A(\omega)$, where $A(\omega)$ is real-valued (but may be negative);
also, use symmetry condition for $M = 7$:

$$h[0] = h[6];\; h[1] = h[5];\; h[2] = h[4];\; h[3]$$

frequency response $=$ Fourier transform of the impulse response:

$$\begin{aligned} H(\omega) &= \sum_{k=0}^{6} h[n]e^{-j\omega n}\\ &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} + \ldots\\ &= e^{-j3\omega}\Big(h[0]e^{j3\omega} + h[1]e^{j2\omega} + h[2]e^{j\omega} + h[3] + h[2]e^{-j\omega}\\ &\qquad + h[1]e^{-j2\omega} + h[0]e^{-j3\omega}\Big)\\ &= e^{-j3\omega}\Big(h[0](e^{j3\omega} + e^{-j3\omega}) + h[1](e^{j2\omega} + e^{-j2\omega})\\ &\qquad + h[2](e^{j\omega} + e^{-j\omega}) + h[3]\Big)\\ &= e^{-j3\omega}\left(2h[0]\cos(3\omega) + 2h[1]\cos(2\omega) + 2h[2]\cos(\omega) + h[3]\right) \qquad (1)\end{aligned}$$

thus

$$\begin{aligned} A(\omega) &= 2h[0]\cos(3\omega) + 2h[1]\cos(2\omega) + 2h[2]\cos(\omega) + h[3]\\ \theta(\omega) &= 3\omega\end{aligned}$$

Let:

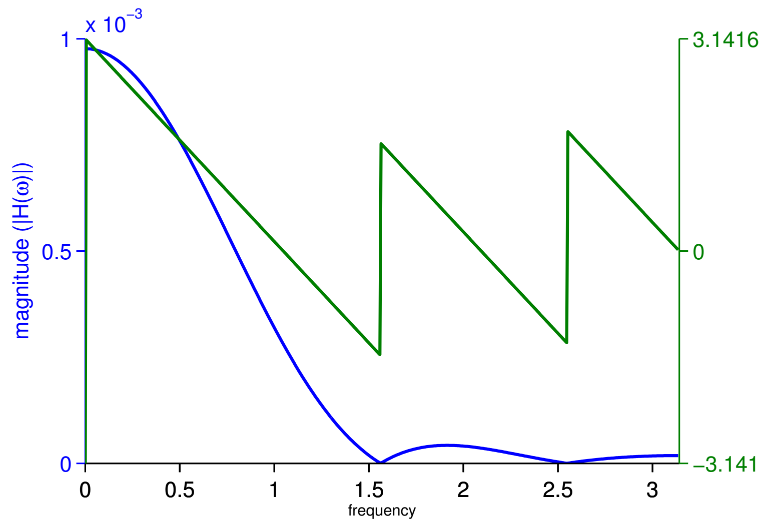
$$|H(\omega)| = |2h[0]\cos(3\omega) + 2h[1]\cos(2\omega) + 2h[2]\cos(\omega) + h[3]|$$

and

$$\theta(\omega) = \begin{cases} -3\omega & \text{when } A(\omega) \text{ is positive} \\ -3\omega + \pi & \text{when } A(\omega) \text{ is negative} \end{cases}$$

$\Rightarrow$ linear phase-response with phase jumps of π radians at frequencies where $H(\omega)$ changes sign from positive to negative and vice versa.

example frequency response for low pass filter, with $M = 7$, for $|H(w)|$ and $\theta(\omega)$:



in general, for $h[n]$ with positive symmetry:

Extrapolate for general case of a filter with arbitrary length M:

$$H(\omega) = e^{-j\omega(M-1)/2}\left[h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h(n)\cos\omega\left(\frac{M-1}{2} - n\right)\right] \tag{2}$$

The phase characteristic for this filter is:

$$\theta(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & \text{if real-valued term in eq.(2)} > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & \text{if real-valued term in eq.(2)} < 0 \end{cases}$$

NB: group delay is $(M-1)/2$.

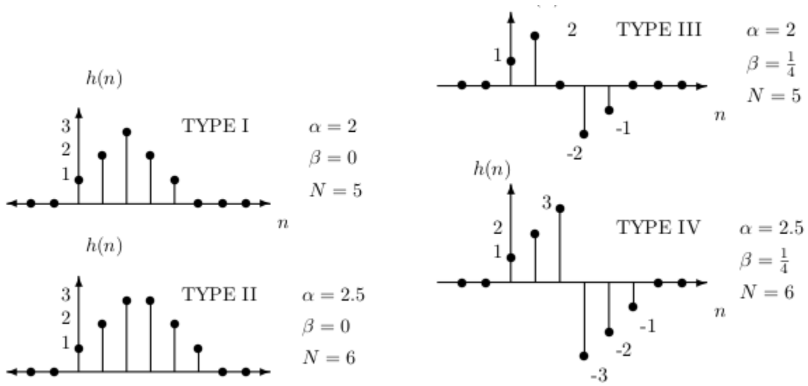
4 types of FIR filter

generalised linear phase of the form: $|H(\omega)|e^{-j(\omega\alpha-\beta)}$

type	M	symmetry	delay	β
Type I	odd	positive	integer	$0, \pi$
Type II	even	positive	fractional	$0, \pi$
Type III	odd	negative	integer	$\pi/2, 3\pi/2$
Type IV	even	negative	fractional	$\pi/2, 3\pi/2$

[for all filters $\alpha = (M-1)/2$ with is equal to group delay]

$h[n]$ has either positive or negative symmetry, and M is either even of odd $\Rightarrow$ four filter types:



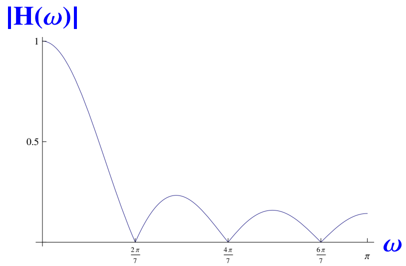
limitation of filter types:

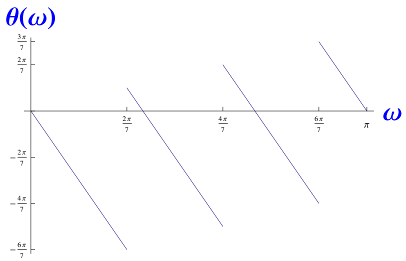
type	low-pass	high-pass	band-pass	band-stop
Type I	✓	✓	✓	✓
Type II	✓		✓	
Type III			✓	
Type IV		✓	✓	

Example: Type 1 Linear Phase FIR Filter

Moving Average filter with $M = 7$ and $h(n) = \frac{1}{M}$

$$\begin{aligned} H(\omega) &= e^{-j3\omega}\left[h(3) + 2\sum_{n=0}^{2} h(n)\cos\omega(3-n)\right] \\ &= \frac{1}{7}e^{-j3\omega}\left[1 + 2\cos 3\omega + 2\cos 2\omega + 2\cos\omega\right] \end{aligned}$$





Phase Delay I

- Consider a sinusoid as an input $x(n)$ to a filter with Frequency response $H(\omega)$

$$x(n) = A\cos(\omega_0 n + \phi) = \frac{1}{2}Ae^{j\phi}e^{j\omega_0 n} + \frac{1}{2}Ae^{-j\phi}e^{-j\omega_0 n}$$

- The filter output $y(n)$ is

$$y(n) = \frac{1}{2}Ae^{j\phi}H(\omega_0)e^{j\omega_0 n} + \frac{1}{2}Ae^{-j\phi}H(\omega_0)e^{-j\omega_0 n}$$

- But the frequency response of the filter $H(\omega)$ can be seperated into the magnitude response $|H(\omega)|$ and the phase response $\theta(\omega)$

$$y(n) = \frac{1}{2}Ae^{j\phi}|H(\omega_0)|e^{j\theta(\omega_0)}e^{j\omega_0 n} + \frac{1}{2}Axe^{-j\phi}|H(-\omega_0)|e^{j\theta(-\omega_0)}e^{-j\omega_0 n}$$

Phase Delay II

- But for an LTI system with $h(n)$ real then $|H(\omega)|$ is an even function of ω and $\theta(\omega)$ is an odd function of ω

$$\begin{aligned} y(n) &= \frac{1}{2}Ae^{j\phi}|H(\omega_0)|e^{j\theta(\omega_0)}e^{j\omega_0 n} \\ &\quad +\frac{1}{2}Ae^{-j\phi}|H(\omega_0)|e^{-j\theta(\omega_0)}e^{-j\omega_0 n} \\ &= \frac{1}{2}A|H(\omega_0)|\left(e^{j\omega_0 n}e^{j\phi}e^{j\theta(\omega_0)} + e^{-j\omega_0 n}e^{-j\phi}e^{-j\theta(\omega_0)}\right) \\ &= A|H(\omega_0)|\cos(\omega_0 n + \theta(\omega_0) + \phi) \end{aligned}$$

- If the input is a sinusoid then the output is a sinusoid.
- All that can change between filter input and filter output is the amplitude and the phase of the sinusoid.

Phase Delay III

- Rewrite as

$$\begin{aligned} y(n) &= A|H(\omega_0)|\cos\left(\omega_0\left(n + \frac{\theta(\omega_0)}{\omega_0}\right) + \phi\right) \\ &= A|H(\omega_0)|\cos\left(\omega_0(n - T_p(\omega_0)) + \phi\right) \end{aligned}$$

- Where

$$T_p(\omega_0) = -\frac{\theta(\omega_0)}{\omega_0}$$

  is the phase delay for frequency ω_0
- The minus sign indicates a phase lag.
- Thus for a filter with a single input frequency we define phase delay as

$$T_p(\omega_0) = -\frac{\theta(\omega_0)}{\omega_0}$$

Phase Delay IV

- For a type 1 Linear Phase Filter phase response $\theta(\omega)$ is

$$\theta(\omega) = -\omega\left(\frac{M-1}{2}\right)$$

- Thus the phase delay for a single frequency input is:

$$T_p(\omega_0) = -\frac{-\omega_0\left(\frac{M-1}{2}\right)}{\omega_0} = \frac{M-1}{2}$$

  and is the same for all frequencies

Example: FIR bandpass filter

Determine impulse response $h[n]$ and group delay of filter to meet the following specifications:

- centre of passband: 50 Hz
- width of passband: 10 Hz
- Transition Width:- 5Hz
- Stopband attenuation:- > 10dB
- Sampling frequency:- 256Hz

solution using window method:

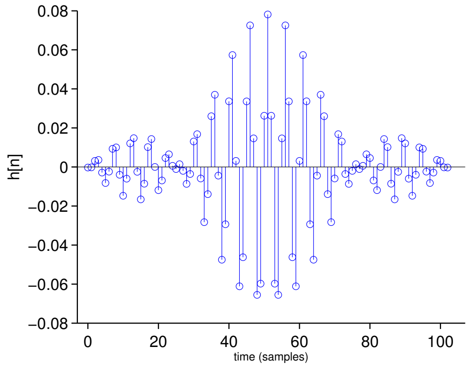
1. ideal impulse response $h_d[n]$:

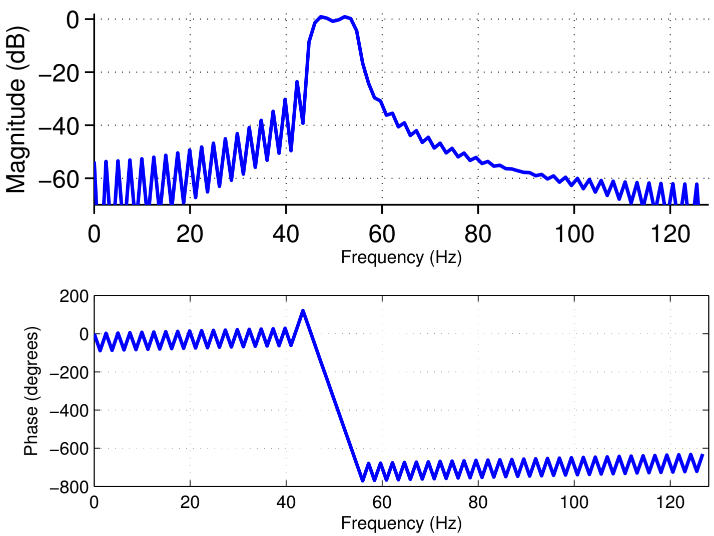
$$h_d[n] = \begin{cases} \frac{\omega_b}{\pi} & n = 0 \\ \cos(\omega_c n)\frac{\omega_b}{\pi}\frac{\sin((\omega_b/2)n)}{(\omega_b/2)n} & n \neq 0 \end{cases}$$

2. shift in time by $(M-1)/2$ and multiply by window. use rectangular window as attenuation > 10 dB:

$$h[n] = \begin{cases} \frac{\omega_b}{\pi} & n = \frac{M-1}{2} \\ \cos(\omega_c[n-(M-1)/2])\times & \\ \quad \frac{\omega_b}{\pi}\frac{\sin((\omega_b/2)[n-(M-1)/2])}{(\omega_b/2)[n-(M-1)/2]} & 0 \leq n \leq M-1 \text{ with } n \neq \frac{M-1}{2} \\ 0 & n < 0 \text{ and } n \geq M \end{cases}$$

3. but what value is M? use table, with $\Delta\omega \approx 4\pi/(M+1)$
   3.1 $M = 4\pi/(2\pi 10/256) - 1 = 101.4$
   3.2 force M odd, i.e. $M = 103$
4. $h[n]$ is symmetric as $h[n] = h[M-1-n]$ and M is odd, $\Rightarrow$ Type I filer;

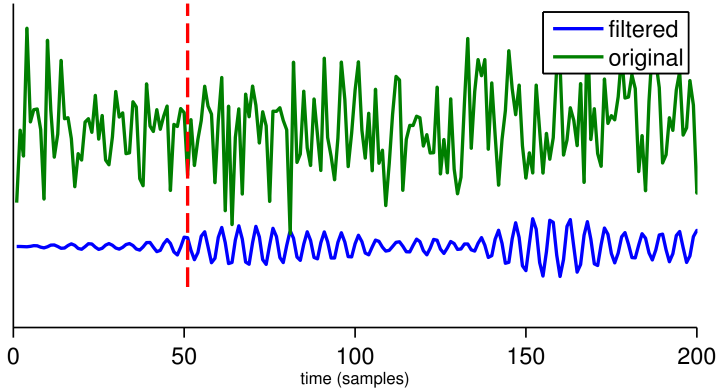




1. for Type 1 filter, generalised linear phase: $\theta(\omega) = -\omega\alpha + \beta$; $\beta = 0, \pi$ and $\alpha = (M-1)/2$
2. group delay

$$\begin{aligned} \text{GD} &= -\frac{d\theta(\omega)}{d\omega} \\ &= \alpha = \frac{M-1}{2} \\ &= 51 \quad \text{(samples)} \end{aligned}$$

example with Gaussian noise



Summary: linear phase and FIR filters

- signal filtered with a linear-phase response filter will have a constant delay (independent of individual oscillations)
- if $h[n]$ is symmetric (positive or negative), then phase response will be linear (generalised linear)
  - for either linear or generalised-linear phase, group delay is a constant
- four filter types, based on positive/negative symmetry and M even/odd
- when filter length M is even $\Rightarrow$ fractional delay
- both window $w[n]$ and $h_d[n]$ need to be symmetric/odd if we want $h[n]$ symmetric/odd.