Digital Signal Processing — FIR filters

1. Given the ideal frequency response for the low-pass filter in Fig. 1(a) with cut-off ω_c,

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

derive the time-domain impulse response $h[n]$. Start with $n = 0$ and then $n \neq 0$, using the identity $\sin(x) = (e^{jx} - e^{-jx})/2j$:

for $n = 0$:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) \mathrm{d}\,\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \mathrm{d}\,\omega = \frac{1}{2\pi} [\omega]_{-\omega_c}^{\omega_c} \\ &= \frac{\omega_c}{\pi} \end{aligned}$$

for $n \neq 0$:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) \mathrm{e}^{j\omega n} \mathrm{d}\,\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \mathrm{e}^{j\omega n} \mathrm{d}\,\omega = \frac{1}{2\pi} \left[\frac{\mathrm{e}^{j\omega n}}{jn} \right]_{-\omega_c}^{\omega_c} \\ &= \frac{1}{jn2\pi} \left[\mathrm{e}^{j\omega_c n} - \mathrm{e}^{-j\omega_c n} \right] \\ &= \frac{1}{\pi n} \sin(\omega_c n) \\ &= \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n} \end{aligned}$$

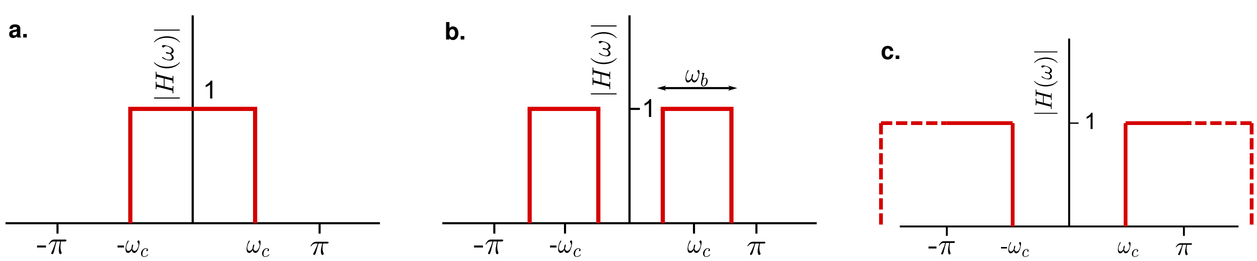

Figure 1: Ideal frequency response for (a) low-pass, (b) band-pass, and (c) and high-pass filters.

2. Derive the impulse response for high-pass FIR filter with cut-off ω_c in Fig. 1(b)

$$H_d(\omega) = \begin{cases} 1 & \omega_c - \frac{\omega_b}{2} \leq \omega \leq \omega_c + \frac{\omega_b}{2} \\ 1 & -\omega_c - \frac{\omega_b}{2} \geq \omega \geq -\omega_c + \frac{\omega_b}{2} \\ 0 & \text{otherwise} \end{cases}$$

Take the inverse discrete-time Fourier transform of the $H(\omega)$:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{\omega_c-\omega_b/2}^{\omega_c+\omega_b/2} e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{-\omega_c-\omega_b/2}^{-\omega_c+\omega_b/2} e^{j\omega n}d\omega \end{aligned}$$

Then, for $n = 0$:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi}[\omega]_{\omega_c-\omega_b/2}^{\omega_c+\omega_b/2} + \frac{1}{2\pi}[\omega]_{-\omega_c-\omega_b/2}^{-\omega_c+\omega_b/2} \\ &= \frac{1}{2\pi}\left[-\omega_c + \frac{\omega_b}{2} + \omega_c + \frac{\omega_b}{2} + \omega_c + \frac{\omega_b}{2} - \omega_c + \frac{\omega_b}{2}\right] \\ &= \frac{\omega_b}{\pi} \end{aligned}$$

And for $n \neq 0$:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi}[\frac{e^{j\omega n}}{jn}]_{\omega_c-\omega_b/2}^{\omega_c+\omega_b/2} + \frac{1}{2\pi}[\frac{e^{j\omega n}}{jn}]_{-\omega_c-\omega_b/2}^{-\omega_c+\omega_b/2} \\ &= \frac{1}{2\pi jn}\left[e^{j(\omega_c+\omega_b/2)n} - e^{j(\omega_c-\omega_b/2)n} + e^{j(-\omega_c+\omega_b/2)n} - e^{j(-\omega_c-\omega_b/2)n}\right] \\ &= \frac{1}{2\pi jn}\left[e^{j\omega_c n}e^{j(\omega_b/2)n} - e^{j\omega_c n}e^{-j(\omega_b/2)n} + e^{-j\omega_c n}e^{j(\omega_b/2)n} - e^{-j\omega_c n}e^{-j(\omega_b/2)n}\right] \\ &= \frac{e^{j\omega_c n}}{2\pi jn}\left[e^{j(\omega_b/2)n} - e^{-j(\omega_b/2)n}\right] + \frac{e^{-j\omega_c n}}{2\pi jn}\left[e^{j(\omega_b/2)n} - e^{-j(\omega_b/2)n}\right] \\ &= \frac{e^{j\omega_c n}}{\pi n}\sin((\omega_b/2)n) + \frac{e^{-j\omega_c n}}{\pi n}\sin((\omega_b/2)n) \\ &= \frac{\sin((\omega_b/2)n)}{\pi n}\left[e^{j\omega_c n} + e^{-j\omega_c n}\right] \\ &= 2\cos(\omega_c n)\frac{\sin((\omega_b/2)n)}{\pi n} \\ &= \cos(\omega_c n)\frac{\omega_b}{\pi}\frac{\sin((\omega_b/2)n)}{(\omega_b/2)n} \end{aligned}$$

and therefore

$$h_d[n] = \begin{cases} \frac{\omega_b}{\pi} & n = 0 \\ \cos(\omega_c n)\frac{\omega_b}{\pi}\frac{\sin((\omega_b/2)n)}{(\omega_b/2)n} & n \neq 0 \end{cases}$$

3. Derive the impulse response for high-pass FIR filter with cut-off ω_c [Fig. 1(c)] using the window method with a rectangular window of length M (assume M is odd).

   (a) First derive the ideal impulse response $h_d[n]$ from the ideal frequency response:

$$H_d(\omega) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & \omega_c \leq |\omega| \leq \pi \end{cases}$$

for $n = 0$:

$$\begin{aligned}
h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\mathrm{d}\,\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} \mathrm{d}\,\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} \mathrm{d}\,\omega \\
&= \frac{1}{2\pi}\left[\omega\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\omega\right]_{\omega_c}^{\pi} \\
&= \frac{1}{2\pi}[-\omega_c + \pi] + \frac{1}{2\pi}[\pi - \omega_c] \\
&= 1 - \frac{\omega_c}{\pi}
\end{aligned}$$

for $n \neq 0$:

$$\begin{aligned}
h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\mathrm{e}^{j\omega n}\mathrm{d}\,\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} \mathrm{e}^{j\omega n}\mathrm{d}\,\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} \mathrm{e}^{j\omega n}\mathrm{d}\,\omega \\
&= \frac{1}{2\pi}\left[\frac{\mathrm{e}^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{\mathrm{e}^{j\omega n}}{jn}\right]_{\omega_c}^{\pi} \\
&= \frac{1}{jn2\pi}\left[\mathrm{e}^{-j\omega_c n} - \mathrm{e}^{-j\pi n}\right] + \frac{1}{jn2\pi}\left[\mathrm{e}^{j\pi n} - \mathrm{e}^{j\omega_c n}\right] \\
&= \frac{1}{jn2\pi}\left[\mathrm{e}^{j\pi n} - \mathrm{e}^{-j\pi n}\right] - \frac{1}{jn2\pi}\left[\mathrm{e}^{j\omega_c n} - \mathrm{e}^{-j\omega_c n}\right] \\
&= \frac{\sin(\pi n)}{\pi n} - \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n} \\
&= -\frac{\sin(\omega_c n)}{\pi n}
\end{aligned}$$

and therefore

$$h_d[n] = \begin{cases} 1 - \frac{\omega_c}{\pi} & n = 0 \\ -\frac{\sin(\omega_c n)}{\pi n} & n \neq 0 \end{cases}$$

(b) Next, shift the impulse response function in time by $M_h = (M-1)/2$:

$$h_d[n] \rightarrow h_d[n - M_h]$$

(c) truncate: multiple by rectangular window $w[n]$, where $w[n] = 1$ for $0 \leq n \leq M-1$.

Thus, the impulse response is:

$$\begin{aligned}
h[n] &= h_d\left[n - M_h\right] w[n] \\
&= \begin{cases} \frac{-\sin[\omega_c(n-M_h)]}{\pi(n-M_h)} & 0 \leq n \leq M-1 \,\text{and}\, n \neq M_h \\ 1 - \frac{\omega_c}{\pi} & n = M_h \\ 0 & n < 0 \,\text{and}\, n > M-1 \end{cases}
\end{aligned}$$

1 Appendix: identities

$$\sin(\omega) = \frac{\mathrm{e}^{j\omega} - \mathrm{e}^{-j\omega}}{2j}$$
$$\cos(\omega) = \frac{\mathrm{e}^{j\omega} + \mathrm{e}^{-j\omega}}{2}$$

window type	side-lobe amplitude (dB)	approximate width of main lobe ($\approx \Delta\omega$)	approximate stopband $20\log_{10}(\delta)$ (dB)	approximate passband $20\log_{10}(1+\delta)$ (dB)
rectangular	-13	$4\pi/(M+1)$	-21	0.7416
Hanning	-31	$8\pi/M$	-44	0.0546
Hamming	-41	$8\pi/M$	-53	0.0194
Blackman	-57	$12\pi/M$	-74	0.0017

Table 1: Window parameters for length M window with relation to filter transition width $\Delta\omega$.

1.1 Kaiser Window

Kaiser window with parameter β and length M is defined as:

$$w[n] = \begin{cases} \frac{I_0[\beta(1-[(n-M_h)/M_h]^2)]^{1/2}}{I_0(\beta)} & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$$

where $M_h = (M-1)/2$, I_0 is zero-order modified Bessel function of the first kind.

For filter transition $\Delta\omega$ with pass-band ripple $A = -20\log_{10}\delta$,

$$\beta = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases}$$

and

$$M = \frac{A-8}{2.285\Delta\omega}$$