FIR Filter Design: Windows Method

Dr John O' Toole

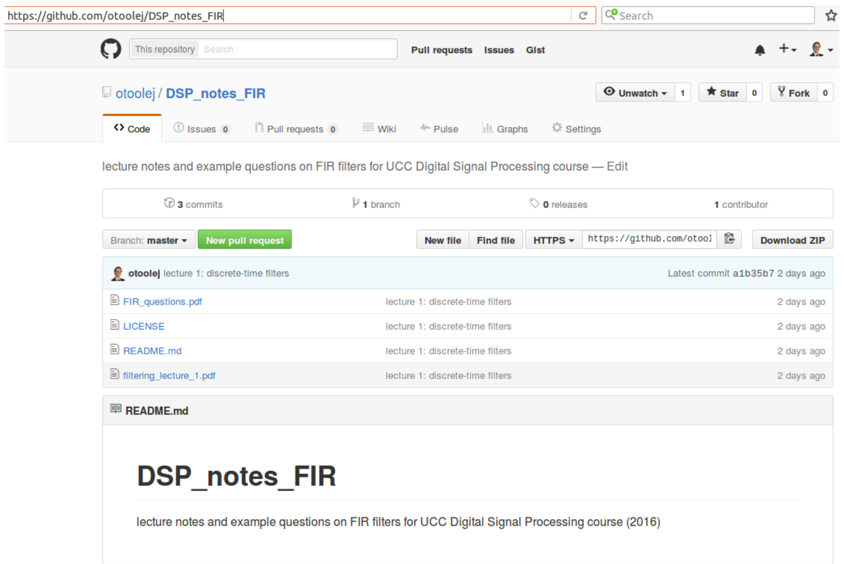
Irish Centre for Fetal and Neonatal Translational Research (INFANT)
Department of Paediatrics and Child Health,
University College Cork.
jotoole@ucc.ie
Tel: 021-4205599

February 15, 2016

lecture notes and example questions

https://github.com/otoolej/DSP_notes_FIR

click on 'Download ZIP' on right-hand side



Learning Objectives

procedure to design non-ideal (FIR) filters using the window method

- review: FIR and IIR filters, importance of phase, ideal filters
- ideal filter not possible $\Rightarrow$ use non-ideal filter.
- Window method:
  1. define ideal in the frequency domain
  2. find impulse response function in time domain ($H_d(\omega) \Rightarrow h_d[n]$)
  3. shift impulse response function in time
  4. truncate impulse response (infinite $\Rightarrow$ finite) using a window
- effect of window shape and window length on filter

Review: discrete-time filters

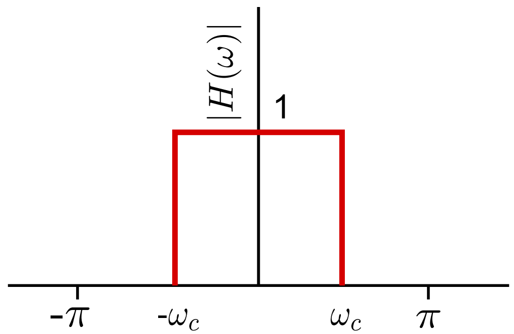
- convolution and modulation: $x[n] * h[n] \leftrightarrow X(\omega)H(\omega)$
- frequency response: magnitude and phase
  - magnitude scales frequency components (e.g. low-pass filter)
  - phase provides information on time-delay
  - linear phase: constant time-delay
  - non-linear phase: different time-delays for different frequencies
- comparing FIR and IIR:
  - stability, phase response, design complexity, sensitivity to numerical precision
- ideal filter: non-causal, unstable, and not realisable
  - $\Rightarrow$ need non-ideal filter

Ideal Low-Pass Filter

Frequency Response

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

Magnitude response:

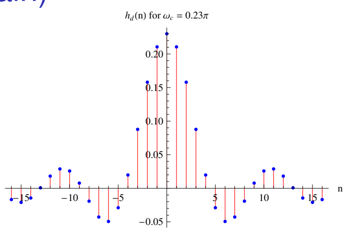


Phase response:
$\angle H(\omega) = 0$

Impulse Response (time-domain)

Inverse DTFT

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\ &= \begin{cases} \frac{\omega_c}{\pi} & n = 0 \\ \frac{\omega_c}{\pi}\frac{\sin \omega_c n}{\omega_c n} & n \neq 0 \end{cases} \end{aligned}$$



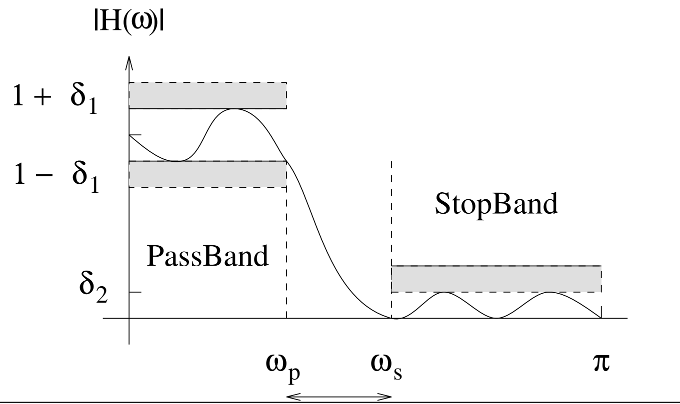
Problems:

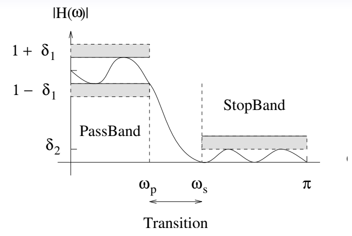
- ideal filter is non-causal
- not realisable for stored data ($n = -\infty$)
- unstable

Non-Ideal Filters

Compromise:

- let $|H(\omega)| \approx 0$ in stopband
- let $|H(\omega)| \approx 1$ in passband
- smooth transition from passband to stopband





- ω_p defines *bandwidth* with passband ripple (in dB):

$$A_p = 20\log_{10}(1+\delta_1)$$

- ω_s start of the stopband with Stopband attenuation:

$$A_s = -20\log_{10}\delta_2$$

- Width of the transition band $\omega_s - \omega_p$.

FIR Filters

- Force to be causal and finite:

$$h[n] = 0, \qquad \text{for } n < 0 \text{ and } n \geq M$$

- impulse response is then FIR:

$$y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]$$

The output at n is the weighted linear combination input samples $x[n], x[n-1], \ldots x[n-M+1]$

FIR Frequency Response

$$\begin{aligned} H(\omega) &= \text{DTFT}\{h[n]\} \\ &= \sum_{k=0}^{M-1} h[n]\mathrm{e}^{-j\omega k} \end{aligned}$$

- The FIR Digital filter design problem:
  - **determine** $h[n]$
  - to satisfy limits set for δ_1, δ_2 and $\omega_s - \omega_p$
- Accuracy depends on $h[n]$ and on M.

Windows Method: shift and truncate

1. Start with ideal frequency response: $H_d(\omega)$
2. calculate impulse response $h_d[n]$ from inverse DTFT:

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\mathrm{e}^{j\omega n}\mathrm{d}\omega$$

3. But is not casual ($h_d[n] \neq 0$ for $n < 0$) and has infinite duration. Solution:
   3.1 shift $h_d[n]$ by $(M-1)/2$ (M is odd)
   3.2 truncate $h_d[n]$ to finite duration M

- order is important: shift first then truncate, i.e. truncate $h_d[n-(M-1)/2]$ and not $h_d[n]$
- finite impulse response:

$$h[n] = 0 \qquad \text{for } n < 0 \text{ and } n \geq M$$

- $h[n]$ now equal to $h_d[n-(M-1)/2]w[n]$, where $w[n]$ is *rectangular* window defined as

$$w[n] = \begin{cases} 1 & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$$

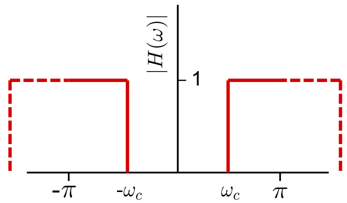
- notation:
  - $h_d[n]$: infinite ideal impulse response
  - $h[n]$: finite non-ideal impulse response

Relation of $h[n]$ to Ideal Impulse Response $h_d[n]$

Let $M_h = \frac{M-1}{2}$ (M odd):

$$\begin{aligned} h[n] &= h_d\left[n - M_h\right] w[n] \\ &= \begin{cases} h_d\left[n - M_h\right] & 0 \leq n \leq M-1 \\ 0 & n < 0 \quad \text{and} \quad n \geq M \end{cases} \end{aligned}$$

High-Pass FIR Filter



Question: Derive the impulse response for high-pass FIR filter with cut-off ω_c using the window method with a rectangular window of length M (assume M is odd) Ideal Impulse

Response (with cutoff ω_c):

$$h_d[n] = \begin{cases} 1 - \frac{\omega_c}{\pi} & n = 0 \\ -\frac{\sin(\omega_c n)}{\pi n} & n \neq 0 \end{cases}$$

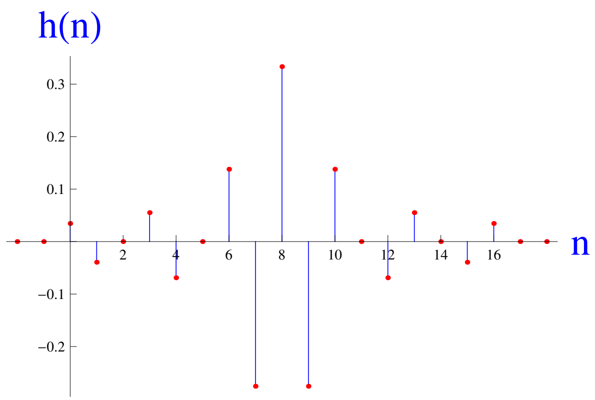
shift and truncate

- shift: $h_d[n] \rightarrow h_d[n - M_h]$
- truncate: multiple by rectangular window $w[n]$

$$\begin{aligned} h[n] &= h_d\left[n - M_h\right] w[n] \\ &= \begin{cases} \frac{-\sin[\omega_c(n-M_h)]}{\pi(n-M_h)} & 0 \leq n \leq M-1 \text{ and } n \neq M_h \\ 1 - \frac{\omega_c}{\pi} & n = M_h \\ 0 & n < 0 \text{ and } n > M-1 \end{cases} \end{aligned}$$

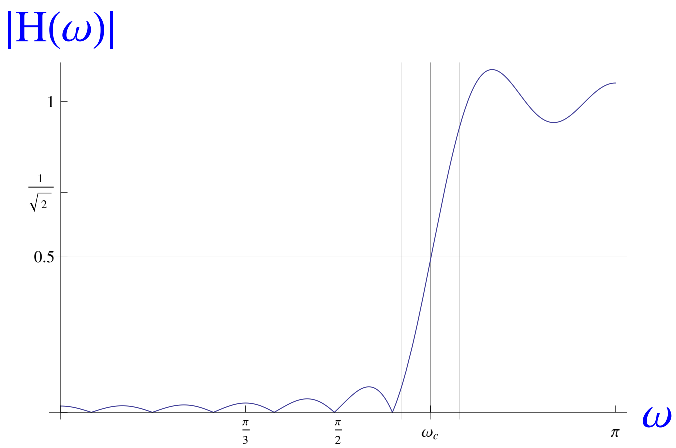
Example impulse response (FIR high-pass filter):

$$\omega_c = \frac{2\pi}{3} \quad \text{and} \quad M = 17$$

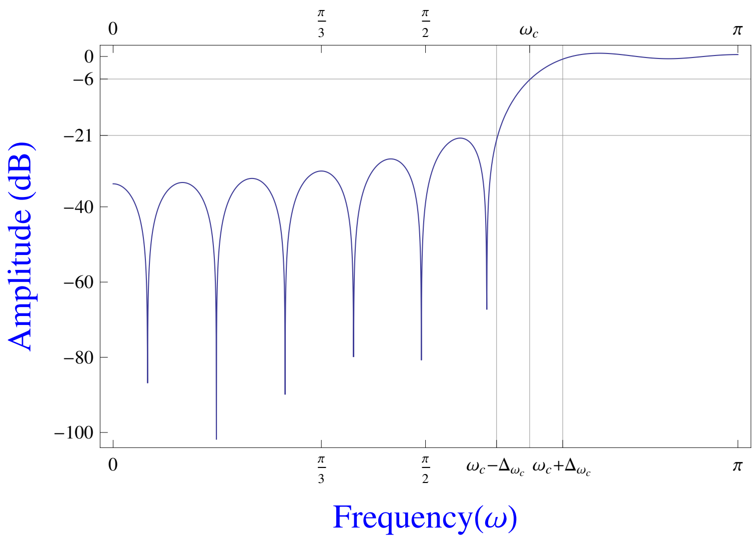


... and frequency response (magnitude):

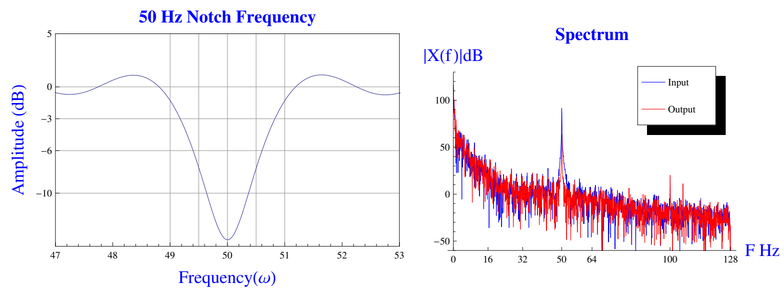
$$\omega_c = \frac{2\pi}{3}$$



... and in dB:



Effect of Windowing



FIR notch filter for EEG.

- How to calculate transition bandwidth?
- What factors affect the Stopband Attenuation?

- multiplication in the time-domain (i.e. window ideal impulse response):

$$h[n] = h_d[n]w[n]$$

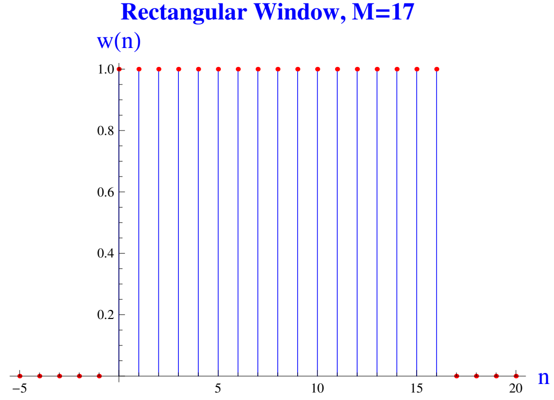
- equivalent to convolution in frequency domain:

$$\begin{aligned} H(\omega) &= H_d(\omega) * W(\omega) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\nu)W(\omega-\nu)d\nu \end{aligned}$$

- *main lobe* of window $\Rightarrow$ (discontinuity in) transition bandwidth
- *side lobe* of window $\Rightarrow$ (level of) passband ripple and stopband attenuation

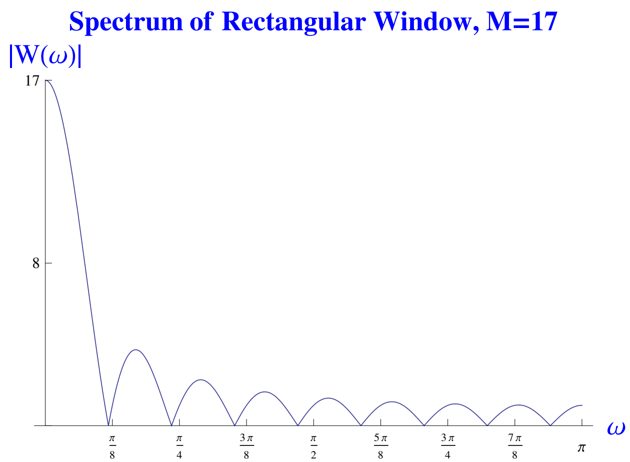
Rectangular Window

$$w[n] = \begin{cases} 1 & 0 \le n \le M-1 \\ 0 & n < 0, n > M-1 \end{cases}$$



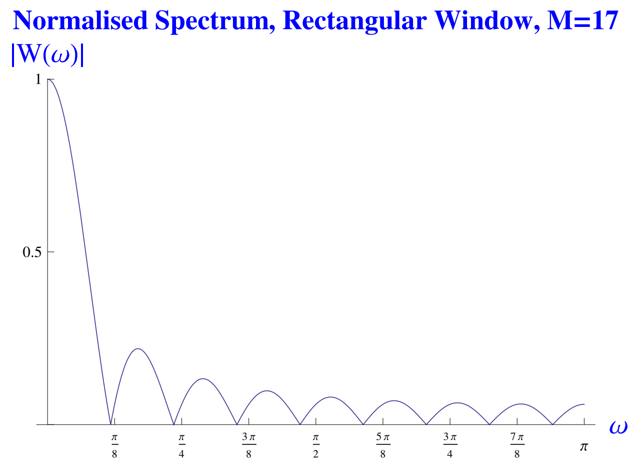
spectrum:

$$W(\omega) = \text{DTFT}\{w[n]\} = \mathrm{e}^{-j\omega(M-1)/2}\frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

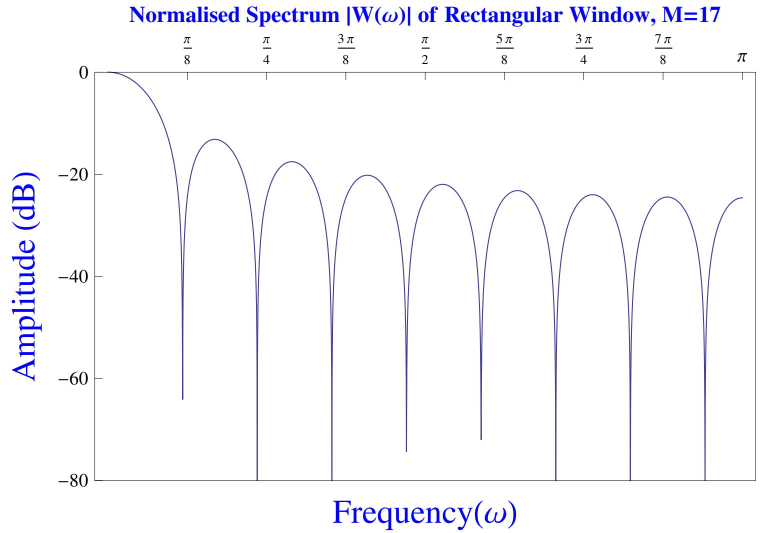


normalise spectrum

$$W(\omega) \leftarrow \frac{W(\omega)}{|W(0)|}$$



... in dB:

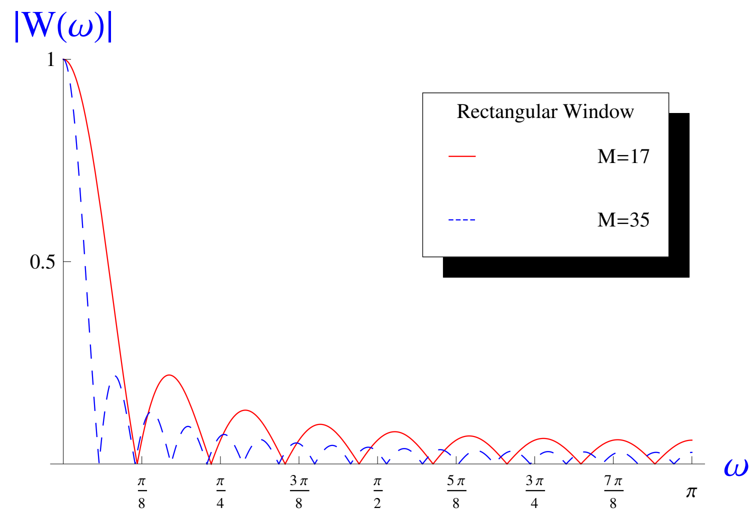


Width of Main Lobe

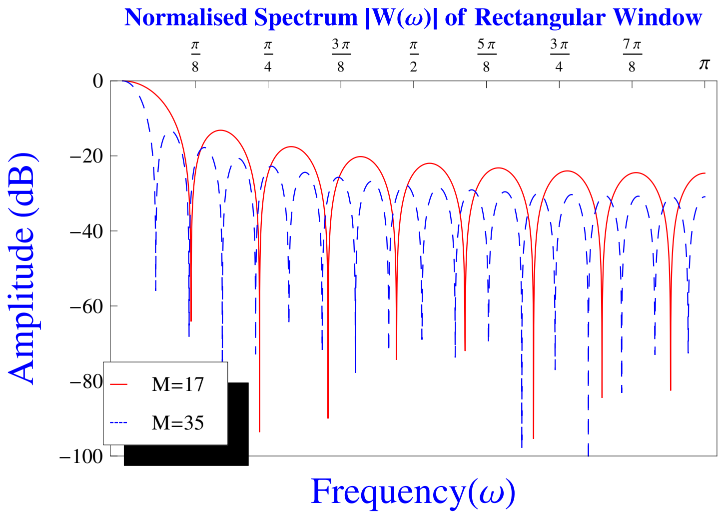
- The main lobe width is defined as the distance from $\omega = 0$ to the first zero of $W(\omega)$
- The width of the main lobe is $2\pi/M$
  - hence as M increases the main lobe becomes narrower.
- The main lobe of $W(\omega)$ affects the transition region from the passband to the stopband.
  - Recall: our desired frequency response $H_d(\omega)$ has a discontinuity or infinitely sharp transition from stop to pass band.

- The wider the main lobe of the window function the more gradual the transition.
- Therefore by increasing the window length M we reduce the main lobe width and thus get a reduced transition region.
- But increasing M also increases computational load of the filter.

Increasing M

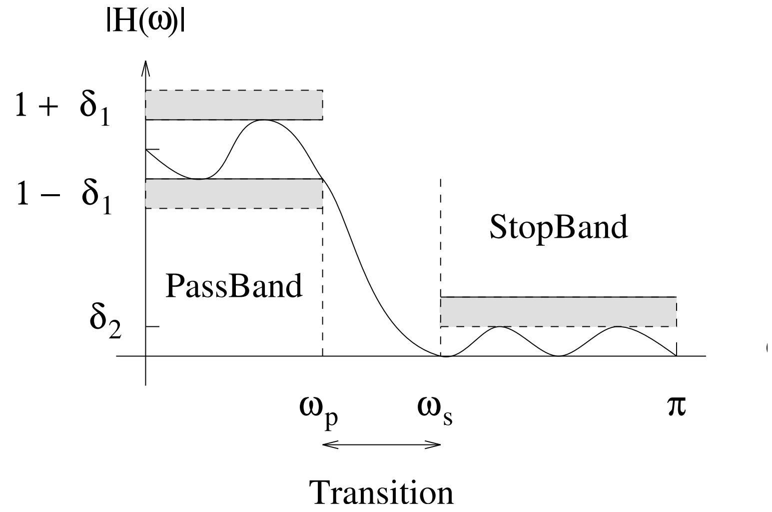


Increasing M (in dB)



---

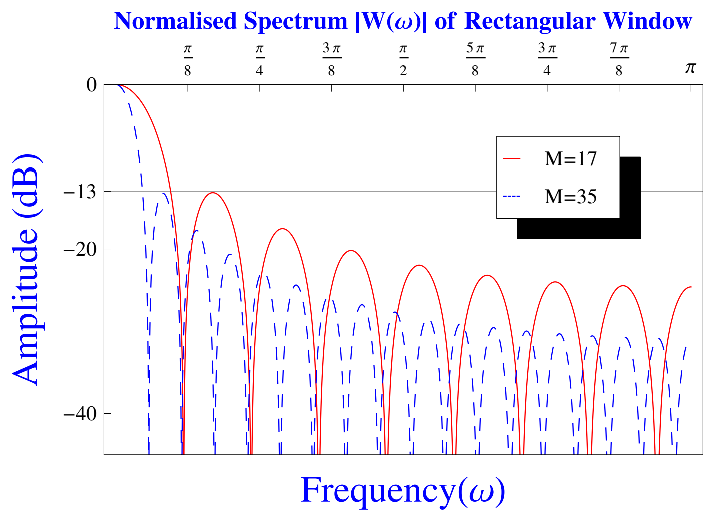
recall:



Side Lobe

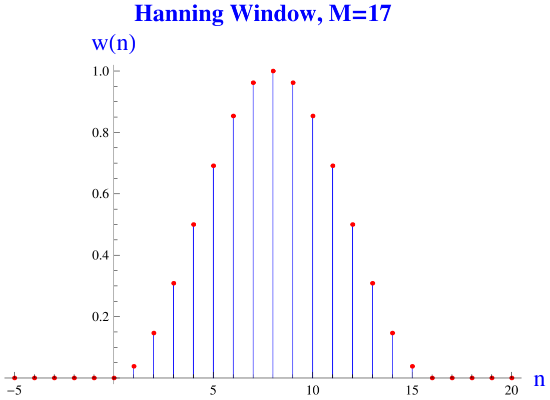
- The sidelobes of $W(\omega)$ cause a different distortion.
- The effect is most evident in the frequency ranges in which $H_d(\omega)$ is constant:
  - i.e. passband and stopband.
- For passband, these sidelobe effects appear both as overshoots and undershoots to the desired response.
- For stopband they appear as a nonzero response.
- For rectangular window sidelobes remain unaffected by increase M

Sidelobe at -13dB



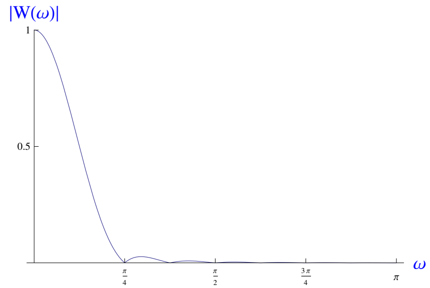
Hanning Window

$$w[n] = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) & 0 \le n \le M-1 \\ 0 & n < 0, n > M-1 \end{cases}$$

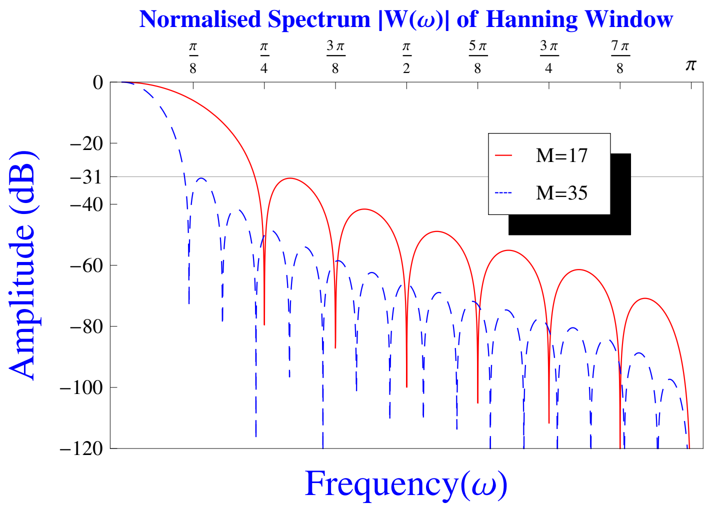


normalised spectrum for $M = 17$:

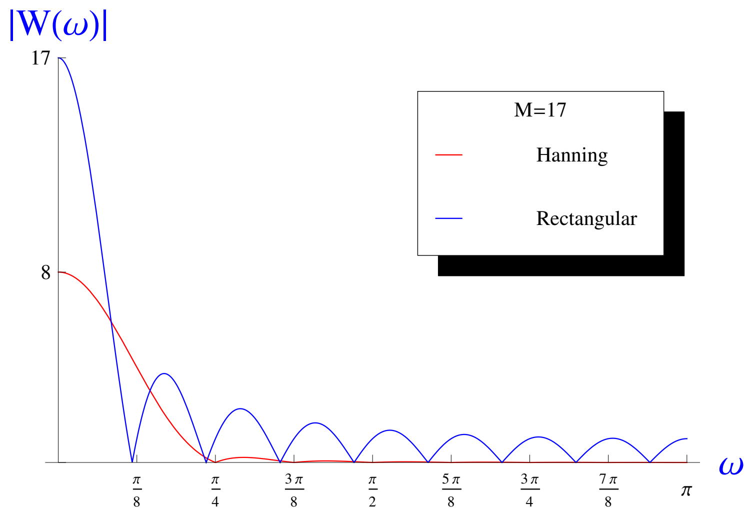
$$\begin{aligned} W(\omega) &= e^{-j8\omega}\left[w(8) + 2\sum_{n=0}^{7} w[n]\cos\omega\,(8-n)\right] \\ &= e^{-j8\omega}\left[1 + 2\sum_{n=0}^{7}\left(0.5 - 0.5\cos\left(\frac{\pi n}{8}\right)\right)\cos\left(\omega\,(8-n)\right)\right] \end{aligned}$$



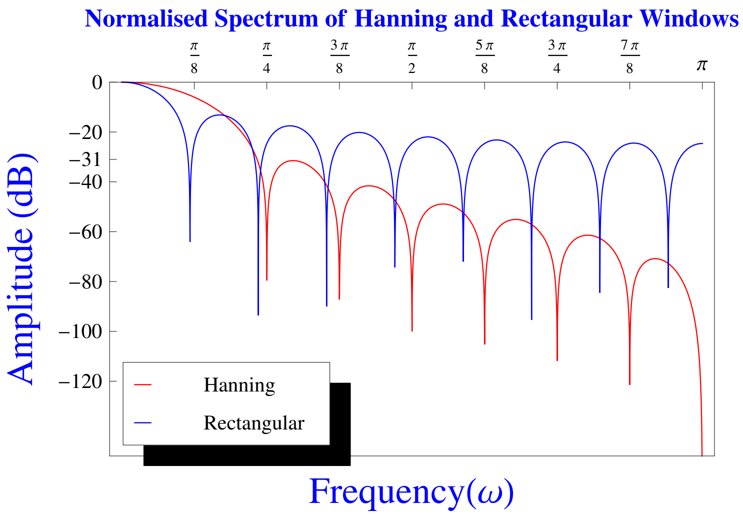
... in dB



Rectangular Vs Hanning



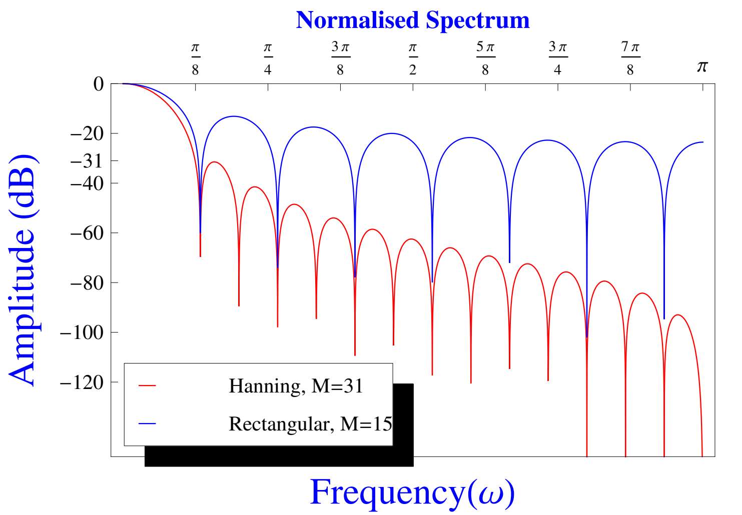
Comparing windows in dB:



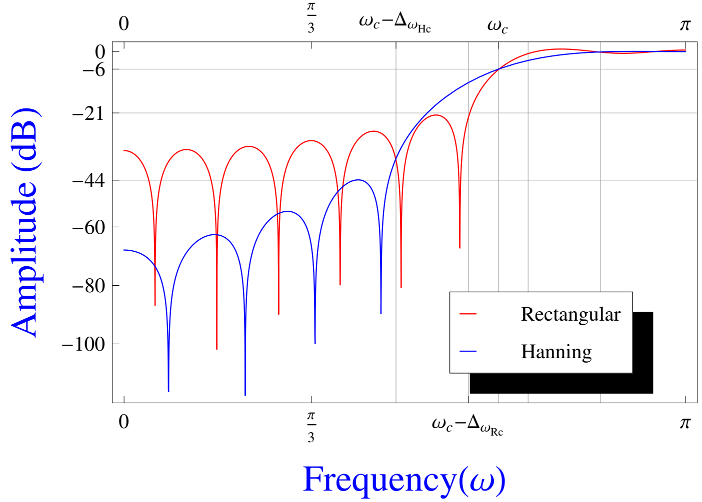
Rectangular vs Hanning:

- Approximate width of the main lobe in the magnitude spectrum of the Hanning window is $4\pi/(M-1)$.
  - Width of the main lobe of the rectangular window spectrum is $2\pi/M$.
- Hanning window results in a wider transition region.
  - But the height of the sidelobes in the Hanning window spectrum is less than that for the rectangular window.
- $\Rightarrow$ Hanning window has less overshoots and undershoots in the passband and less leakages in the stopband.

Same Transition Bandwidth



High-pass filter design (with $\omega_c = 2\pi/3$ and $M = 17$)



Windows

- Several window functions have been proposed which lead to filters with varying transition band widths and stopband attenuation for a fixed filter length.
- Can relate the transition band width ($\triangle f$) to the filter length. For example the Rectangular Window $\triangle f = 0.9/M$.
- Can determine approximately the minimum stopband attenuation that can be achieved with a particular window. For example with the Hanning Window it is 44 dB.
- Can determine approximately the passband ripple. For example with the Hamming Window it is 0.0194 dB.

Windows II

Window $w[n]$	Sidelobe	$\triangle f$	Stopband Attenuation	Passband Ripple
Rectangular	-13 dB	$\frac{0.9}{M}$	21 dB	0.7416 dB
$w[n] = \begin{cases} 1 & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$				
Hanning	-31 dB	$\frac{3.1}{M}$	44 dB	0.0546 dB
$w[n] = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$				
Hamming	-41 dB	$\frac{3.3}{M}$	53 dB	0.0194 dB
$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right) & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$				
Blackman	-57 dB	$\frac{5.5}{M}$	75 dB	0.0017 dB
$w[n] = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) + 0.08\cos\left(\frac{4\pi n}{M-1}\right) & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$				

Passband Ripple and Stopband Attenuation

recall that:

- passband ripple (in dBs): $A_p = 20\log_{10}(1+\delta_1)$
- stopband attenuation (in dBs): $A_s = -20\log_{10}\delta_2$

Rectangular

$$\begin{aligned} 20\log(1+\delta_p) &= 0.7416 \text{ dB} \\ \delta_p &= 0.08913 \\ -20\log(\delta_s) &= 21 \text{ dB} \\ \delta_s &= 0.089125 \end{aligned}$$

Blackman

$$\begin{aligned} 20\log(1+\delta_p) &= 0.0017 \text{ dB} \\ \delta_p &= 0.0002 \\ -20\log(\delta_s) &= 75 \text{ dB} \\ \delta_s &= 0.0001778 \end{aligned}$$

Summary

1. Specify the "ideal" or desired frequency response $H_d(\omega)$
2. Obtain the impulse response $h_d[n]$ by evaluating the inverse Fourier transform
3. Select a window function with the appropriate passband or stopband properties
4. Determine number (M) of coefficients necessary to get appropriate transition band width.
5. "shift and truncate": $h[n] = h_d\left[n - \frac{M-1}{2}\right] w[n]$

Summary (cont.)

Also, the window is important:

- main lobe of window $\Rightarrow$ transition bandwidth
  - increase in M (window length) results in a decrease in the transition band
- side lobe of window $\Rightarrow$ passband ripple and stopband attenuation
  - different window types have different levels of passband ripple