Digital Signal Processing — FIR filters

1. Given the ideal frequency response for the low-pass filter (Fig. 1, left) with cut-off ω_c,

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

derive the time-domain impulse response $h[n]$. Start with $n = 0$ and then $n \neq 0$, using the identity $\sin(x) = (e^{jx} - e^{-jx})/2j$:

for $n = 0$:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) \mathrm{d}\,\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \mathrm{d}\,\omega = \frac{1}{2\pi} [\omega]_{-\omega_c}^{\omega_c} \\ &= \frac{\omega_c}{\pi} \end{aligned}$$

for $n \neq 0$:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) \mathrm{e}^{j\omega n} \mathrm{d}\,\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \mathrm{e}^{j\omega n} \mathrm{d}\,\omega = \frac{1}{2\pi} \left[\frac{\mathrm{e}^{j\omega n}}{jn} \right]_{-\omega_c}^{\omega_c} \\ &= \frac{1}{jn2\pi} \left[\mathrm{e}^{j\omega_c n} - \mathrm{e}^{-j\omega_c n} \right] \\ &= \frac{1}{\pi n} \sin(\omega_c n) \\ &= \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n} \end{aligned}$$

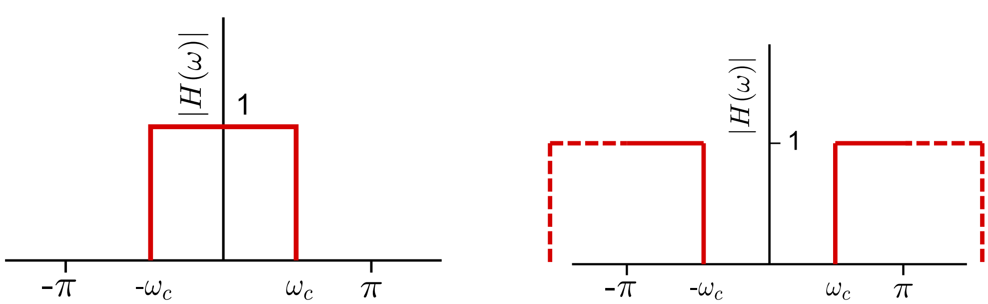

Figure 1: Ideal frequency response for low-pass (left) and high-pass (right) filters.

2. Derive the impulse response for high-pass FIR filter with cut-off ω_c (Fig. 1, right) using the window method with a rectangular window of length M (assume M is odd).

(a) First derive the ideal impulse response $h_d[n]$ from the ideal frequency response:

$$H_d(\omega) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & \omega_c \leq |\omega| \leq \pi \end{cases}$$

for $n = 0$:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) \mathrm{d}\,\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{-\omega_c} \mathrm{d}\,\omega + \frac{1}{2\pi} \int_{\omega_c}^{\pi} \mathrm{d}\,\omega \\ &= \frac{1}{2\pi} [\omega]_{-\pi}^{-\omega_c} + \frac{1}{2\pi} [\omega]_{\omega_c}^{\pi} \\ &= \frac{1}{2\pi}[-\omega_c + \pi] + \frac{1}{2\pi}[\pi - \omega_c] \\ &= 1 - \frac{\omega_c}{\pi} \end{aligned}$$

for $n \neq 0$:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) \mathrm{e}^{j\omega n} \mathrm{d}\,\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{-\omega_c} \mathrm{e}^{j\omega n} \mathrm{d}\,\omega + \frac{1}{2\pi} \int_{\omega_c}^{\pi} \mathrm{e}^{j\omega n} \mathrm{d}\,\omega \\ &= \frac{1}{2\pi} \left[\frac{\mathrm{e}^{j\omega n}}{jn} \right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi} \left[\frac{\mathrm{e}^{j\omega n}}{jn} \right]_{\omega_c}^{\pi} \\ &= \frac{1}{jn2\pi} \left[\mathrm{e}^{-j\omega_c n} - \mathrm{e}^{-j\pi n} \right] + \frac{1}{jn2\pi} \left[\mathrm{e}^{j\pi n} - \mathrm{e}^{j\omega_c n} \right] \\ &= \frac{1}{jn2\pi} \left[\mathrm{e}^{j\pi n} - \mathrm{e}^{-j\pi n} \right] - \frac{1}{jn2\pi} \left[\mathrm{e}^{j\omega_c n} - \mathrm{e}^{-j\omega_c n} \right] \\ &= \frac{\sin(\pi n)}{\pi n} - \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n} \\ &= -\frac{\sin(\omega_c n)}{\pi n} \end{aligned}$$

and therefore

$$h_d[n] = \begin{cases} 1 - \frac{\omega_c}{\pi} & n = 0 \\ -\frac{\sin(\omega_c n)}{\pi n} & n \neq 0 \end{cases}$$

(b) Next, shift the impulse response function in time by $M_h = (M-1)/2$:

$$h_d[n] \rightarrow h_d[n - M_h]$$

(c) truncate: multiple by rectangular window $w[n]$, where $w[n] = 1$ for $0 \leq n \leq M - 1$.

Thus, the impulse response is:

$$
\begin{aligned}
h[n] &= h_d\left[n - M_h\right] w[n] \\
&= \begin{cases} \frac{-\sin[\omega_c(n-M_h)]}{\pi(n-M_h)} & 0 \leq n \leq M-1 \,\text{and}\, n \neq M_h \\ 1 - \frac{\omega_c}{\pi} & n = M_h \\ 0 & n < 0 \,\text{and}\, n > M-1 \end{cases}
\end{aligned}
$$

1 Appendix: identities

$$\sin(\omega) = \frac{e^{j\omega} - e^{-j\omega}}{2j}$$
$$\cos(\omega) = \frac{e^{j\omega} + e^{-j\omega}}{2}$$

The sinc function $\mathrm{sinc}(x) = \sin(x)/x$ is defined as

$$\mathrm{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Window $w(n)$	Sidelobe	$\triangle f$	Stopband Attenuation	Passband Ripple
Rectangular	-13db	$\frac{0.9}{M}$	21db	0.7416db
$w(n) = \begin{cases} 1 & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$				
Hanning	-31db	$\frac{3.1}{M}$	44db	0.0546db
$w(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$				
Hamming	-41db	$\frac{3.3}{M}$	53db	0.0194db
$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right) & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$				
Blackman	-57db	$\frac{5.5}{M}$	75db	0.0017db
$w(n) = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) + 0.08\cos\left(\frac{4\pi n}{M-1}\right) & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$				

Table 1: Window parameters