Digital Signal Processing — FIR filters

1. Given the ideal frequency response for the low-pass filter (Fig. 1, left) with cut-off ω_c,

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

derive the time-domain impulse response $h[n]$. Start with $n = 0$ and then $n \neq 0$, using the identity $\sin(x) = (e^{jx} - e^{-jx})/2j$:

for $n = 0$:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) \mathrm{d}\,\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \mathrm{d}\,\omega = \frac{1}{2\pi} \left[\omega\right]_{-\omega_c}^{\omega_c} \\ &= \frac{\omega_c}{\pi} \end{aligned}$$

for $n \neq 0$:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) \mathrm{e}^{j\omega n} \mathrm{d}\,\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \mathrm{e}^{j\omega n} \mathrm{d}\,\omega = \frac{1}{2\pi} \left[\frac{\mathrm{e}^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} \\ &= \frac{1}{jn2\pi} \left[\mathrm{e}^{j\omega_c n} - \mathrm{e}^{-j\omega_c n}\right] \\ &= \frac{1}{\pi n} \sin(\omega_c n) \\ &= \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n} \end{aligned}$$

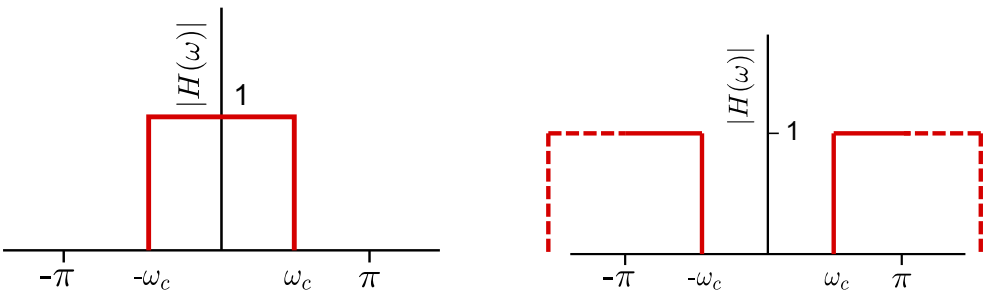

Figure 1: Ideal frequency response for low-pass (left) and high-pass (right) filters.

2. Derive the impulse response for high-pass FIR filter with cut-off ω_c (Fig. 1, right) using the window method with a rectangular window of length M (assume M is odd).

1 Appendix: identities

$$\sin(\omega) = \frac{\mathrm{e}^{j\omega} - \mathrm{e}^{-j\omega}}{2j}$$
$$\cos(\omega) = \frac{\mathrm{e}^{j\omega} + \mathrm{e}^{-j\omega}}{2}$$

The sinc function $\mathrm{sinc}(x) = \sin(x)/x$ is defined as

$$\mathrm{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Window $w(n)$	Sidelobe	$\triangle f$	Stopband Attenuation	Passband Ripple
Rectangular	-13db	$\frac{0.9}{M}$	21db	0.7416db
$w(n) = \begin{cases} 1 & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$				
Hanning	-31db	$\frac{3.1}{M}$	44db	0.0546db
$w(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$				
Hamming	-41db	$\frac{3.3}{M}$	53db	0.0194db
$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right) & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$				
Blackman	-57db	$\frac{5.5}{M}$	75db	0.0017db
$w(n) = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) + 0.08\cos\left(\frac{4\pi n}{M-1}\right) & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$				

Table 1: Window parameters