

1. Given the ideal frequency response for the low-pass filter in Fig. 1(a) with cut-off  $\omega_c$ ,

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

derive the time-domain impulse response  $h[n]$ . Start with  $n = 0$  and then  $n \neq 0$ , using the identity  $\sin(x) = (e^{jx} - e^{-jx})/2j$ :

**Answer:** for  $n = 0$ :

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{1}{2\pi} [\omega]_{-\omega_c}^{\omega_c} \\ &= \frac{\omega_c}{\pi} \end{aligned}$$

for  $n \neq 0$ :

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{2\pi} \left[ \frac{e^{j\omega n}}{jn} \right]_{-\omega_c}^{\omega_c} \\ &= \frac{1}{jn2\pi} [e^{j\omega_c n} - e^{-j\omega_c n}] \\ &= \frac{1}{\pi n} \sin(\omega_c n) \\ &= \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n} \end{aligned}$$

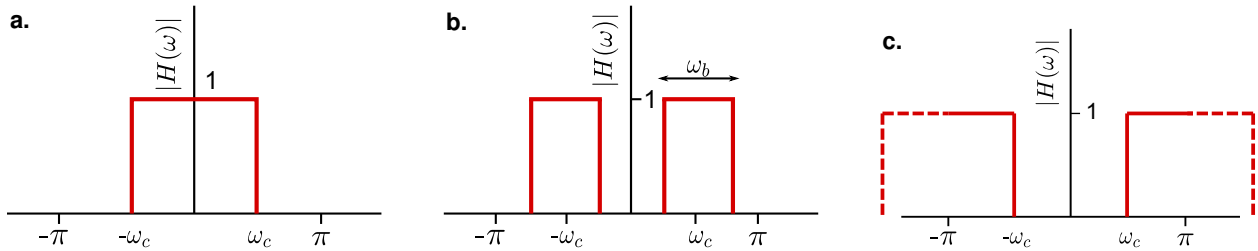


Figure 1: Ideal frequency response for (a) low-pass, (b) band-pass, and (c) high-pass filters.

2. Derive the impulse response for high-pass FIR filter with cut-off  $\omega_c$  in Fig. 1(b)

$$H_d(\omega) = \begin{cases} 1 & \omega_c - \frac{\omega_b}{2} \leq \omega \leq \omega_c + \frac{\omega_b}{2} \\ 1 & -\omega_c - \frac{\omega_b}{2} \geq \omega \geq -\omega_c + \frac{\omega_b}{2} \\ 0 & \text{otherwise} \end{cases}$$

**Answer:** Take the inverse discrete-time Fourier transform of the  $H(\omega)$ :

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{\omega_c - \omega_b/2}^{\omega_c + \omega_b/2} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{-\omega_c - \omega_b/2}^{-\omega_c + \omega_b/2} e^{j\omega n} d\omega \end{aligned}$$

Then, for  $n = 0$ :

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi} [\omega]_{\omega_c - \omega_b/2}^{\omega_c + \omega_b/2} + \frac{1}{2\pi} [\omega]_{-\omega_c - \omega_b/2}^{-\omega_c + \omega_b/2} \\ &= \frac{1}{2\pi} \left[ -\omega_c + \frac{\omega_b}{2} + \omega_c + \frac{\omega_b}{2} + \omega_c + \frac{\omega_b}{2} - \omega_c + \frac{\omega_b}{2} \right] \\ &= \frac{\omega_b}{\pi} \end{aligned}$$

And for  $n \neq 0$ :

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi} \left[ \frac{e^{j\omega n}}{jn} \right]_{\omega_c - \omega_b/2}^{\omega_c + \omega_b/2} + \frac{1}{2\pi} \left[ \frac{e^{j\omega n}}{jn} \right]_{-\omega_c - \omega_b/2}^{-\omega_c + \omega_b/2} \\ &= \frac{1}{2\pi jn} \left[ e^{j(\omega_c + \omega_b/2)n} - e^{j(\omega_c - \omega_b/2)n} + e^{j(-\omega_c + \omega_b/2)n} - e^{j(-\omega_c - \omega_b/2)n} \right] \\ &= \frac{1}{2\pi jn} \left[ e^{j\omega_c n} e^{j(\omega_b/2)n} - e^{j\omega_c n} e^{-j(\omega_b/2)n} + e^{-j\omega_c n} e^{j(\omega_b/2)n} - e^{-j\omega_c n} e^{-j(\omega_b/2)n} \right] \\ &= \frac{e^{j\omega_c n}}{2\pi jn} \left[ e^{j(\omega_b/2)n} - e^{-j(\omega_b/2)n} \right] + \frac{e^{-j\omega_c n}}{2\pi jn} \left[ e^{j(\omega_b/2)n} - e^{-j(\omega_b/2)n} \right] \\ &= \frac{e^{j\omega_c n}}{\pi n} \sin((\omega_b/2)n) + \frac{e^{-j\omega_c n}}{\pi n} \sin((\omega_b/2)n) \\ &= \frac{\sin((\omega_b/2)n)}{\pi n} [e^{j\omega_c n} + e^{-j\omega_c n}] \\ &= 2 \cos(\omega_c n) \frac{\sin((\omega_b/2)n)}{\pi n} \\ &= \cos(\omega_c n) \frac{\omega_b}{\pi} \frac{\sin((\omega_b/2)n)}{(\omega_b/2)n} \end{aligned}$$

and therefore

$$h_d[n] = \begin{cases} \frac{\omega_b}{\pi} & n = 0 \\ \cos(\omega_c n) \frac{\omega_b}{\pi} \frac{\sin((\omega_b/2)n)}{(\omega_b/2)n} & n \neq 0 \end{cases}$$

3. Derive the impulse response for high-pass FIR filter with cut-off  $\omega_c$  [Fig. 1(c)] using the window method with a rectangular window of length  $M$  (assume  $M$  is odd).

**Answer:**

(a) First derive the ideal impulse response  $h_d[n]$  from the ideal frequency response:

$$H_d(\omega) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & \omega_c \leq |\omega| \leq \pi \end{cases}$$

for  $n = 0$ :

$$\begin{aligned}
h_d[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{-\omega_c} d\omega + \frac{1}{2\pi} \int_{\omega_c}^{\pi} d\omega \\
&= \frac{1}{2\pi} [\omega]_{-\pi}^{-\omega_c} + \frac{1}{2\pi} [\omega]_{\omega_c}^{\pi} \\
&= \frac{1}{2\pi} [-\omega_c + \pi] + \frac{1}{2\pi} [\pi - \omega_c] \\
&= 1 - \frac{\omega_c}{\pi}
\end{aligned}$$

for  $n \neq 0$ :

$$\begin{aligned}
h_d[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{-\omega_c} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{\omega_c}^{\pi} e^{j\omega n} d\omega \\
&= \frac{1}{2\pi} \left[ \frac{e^{j\omega n}}{jn} \right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi} \left[ \frac{e^{j\omega n}}{jn} \right]_{\omega_c}^{\pi} \\
&= \frac{1}{jn2\pi} [e^{-j\omega_c n} - e^{-j\pi n}] + \frac{1}{jn2\pi} [e^{j\pi n} - e^{j\omega_c n}] \\
&= \frac{1}{jn2\pi} [e^{j\pi n} - e^{-j\pi n}] - \frac{1}{jn2\pi} [e^{j\omega_c n} - e^{-j\omega_c n}] \\
&= \frac{\sin(\pi n)}{\pi n} - \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n} \\
&= -\frac{\sin(\omega_c n)}{\pi n}
\end{aligned}$$

and therefore

$$h_d[n] = \begin{cases} 1 - \frac{\omega_c}{\pi} & n = 0 \\ -\frac{\sin(\omega_c n)}{\pi n} & n \neq 0 \end{cases}$$

(b) Next, shift the impulse response function in time by  $M_h = (M - 1)/2$ :

$$h_d[n] \rightarrow h_d[n - M_h]$$

(c) truncate: multiple by rectangular window  $w[n]$ , where  $w[n] = 1$  for  $0 \leq n \leq M - 1$ .

Thus, the impulse response is:

$$\begin{aligned}
h[n] &= h_d[n - M_h] w[n] \\
&= \begin{cases} \frac{-\sin[\omega_c(n - M_h)]}{\pi(n - M_h)} & 0 \leq n \leq M - 1 \text{ and } n \neq M_h \\ 1 - \frac{\omega_c}{\pi} & n = M_h \\ 0 & n < 0 \text{ and } n > M - 1 \end{cases}
\end{aligned}$$

4. Find the phase and magnitude of the frequency response for an FIR filter with a positive symmetrical impulse response and  $M = 5$ . What is the filter Type (I, II, III, or IV) and why? Determine the group delay for this filter.

**Answer:** Filter is Type I because positive symmetrical and because  $M$  is odd.

For  $M = 5$  and with positive symmetry,

$$h[0] = h[4]; \quad h[1] = h[3]; \quad h[2]$$

Take the discrete-time Fourier transform for  $h[n]$ ,

$$\begin{aligned} H(\omega) &= \sum_{n=0}^4 h[n]e^{-j\omega n} \\ &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} \\ &= e^{-j2\omega} (h[2] + h[0]e^{j2\omega} + h[1]e^{j\omega} + h[3]e^{-j\omega} + h[4]e^{-j2\omega}) \\ &= e^{-j2\omega} (h[2] + h[0](e^{j2\omega} + e^{-j2\omega}) + h[1](e^{j\omega} + e^{-j\omega})) \\ &= e^{-j2\omega} (h[2] + 2h[0]\cos(2\omega) + 2h[1]\cos(\omega)) \end{aligned}$$

and thus

$$\begin{aligned} \angle H(\omega) &= -2\omega + \beta \\ |H(\omega)| &= |h[2] + 2h[0]\cos(2\omega) + 2h[1]\cos(\omega)| \end{aligned}$$

where  $\beta = 0$  or  $\beta = \pi$  if expression  $h[2] + 2h[0]\cos(2\omega) + 2h[1]\cos(\omega)$  is positive or negative, respectively.

Group delay:

$$-\frac{d\angle H(\omega)}{d\omega} = 2.$$

5. Derive the phase and magnitude of the frequency response for an FIR filter with an impulse response with positive symmetry and  $M$  odd (a Type I filter).

**Answer:** When  $M$  is odd and  $h[n] = h[M - 1 - n]$ , then

$$h\left[\frac{M-1}{2} - n\right] = h\left[\frac{M-1}{2} + n\right] \quad \text{for } n = 1, 2, \dots, \frac{M-1}{2}$$

Take the discrete-time Fourier transform of the impulse function  $h[n]$  to find frequency response:

$$\begin{aligned}
H(\omega) &= \sum_{n=0}^{M-1} h[n]e^{-j\omega n} \\
&= h\left[\frac{M-1}{2}\right]e^{-j\omega(M-1)/2} + \sum_{n=1}^{(M-1)/2} h\left[\frac{M-1}{2} - n\right]e^{-j\omega[(M-1)/2-n]} \\
&\quad + \sum_{n=1}^{(M-1)/2} h\left[\frac{M-1}{2} + n\right]e^{-j\omega[(M-1)/2+n]} \\
&= h\left[\frac{M-1}{2}\right]e^{-j\omega(M-1)/2} + e^{-j\omega(M-1)/2} \left\{ \sum_{n=1}^{(M-1)/2} h\left[\frac{M-1}{2} - n\right]e^{j\omega n} + \sum_{n=1}^{(M-1)/2} h\left[\frac{M-1}{2} + n\right]e^{-j\omega n} \right\} \\
&= h\left[\frac{M-1}{2}\right]e^{-j\omega(M-1)/2} + e^{-j\omega(M-1)/2} \left\{ \sum_{n=1}^{(M-1)/2} h\left[\frac{M-1}{2} - n\right](e^{j\omega n} + e^{-j\omega n}) \right\} \\
&= e^{-j\omega(M-1)/2} \left\{ h\left[\frac{M-1}{2}\right] + 2 \sum_{n=1}^{(M-1)/2} h\left[\frac{M-1}{2} - n\right] \cos(\omega n) \right\}
\end{aligned}$$

6. Derive the phase and magnitude of the frequency response for an FIR filter  $h[n]$  of length  $M$ , where  $h[n]$  has negative symmetry and  $M$  is even (a Type IV filter). Determine the group delay for this filter.

**Answer:** When  $M$  is even and  $h[n] = -h[M-1-n]$ , then

$$h\left[\frac{M}{2} - n\right] = -h\left[\frac{M}{2} + n - 1\right] \quad \text{for } n = 1, 2, \dots, \frac{M}{2}$$

Discrete-time Fourier transform of the impulse function  $h[n]$ :

$$\begin{aligned}
H(\omega) &= \sum_{n=0}^{M-1} h[n]e^{-j\omega n} \\
&= \sum_{n=1}^{M/2} h\left[\frac{M}{2} - n\right]e^{-j\omega(\frac{M}{2}-n)} + \sum_{n=1}^{M/2} h\left[\frac{M}{2} + n - 1\right]e^{-j\omega(\frac{M}{2}+n-1)} \\
&= e^{-j\omega M/2} \left\{ \sum_{n=1}^{M/2} h\left[\frac{M}{2} - n\right]e^{j\omega n} - \sum_{n=1}^{M/2} h\left[\frac{M}{2} - n\right]e^{-j\omega(n-1)} \right\} \\
&= e^{-j\omega M/2} e^{j\omega/2} \left\{ \sum_{n=1}^{M/2} h\left[\frac{M}{2} - n\right](e^{j\omega n} - e^{-j\omega(n-1)})e^{-j\omega/2} \right\} \\
&= e^{-j\omega(M-1)/2} \left\{ \sum_{n=1}^{M/2} h\left[\frac{M}{2} - n\right](e^{j\omega(n-1/2)} - e^{-j\omega(n-1/2)}) \right\} \\
&= e^{-j\omega(M-1)/2} \left\{ 2j \sum_{n=1}^{M/2} h\left[\frac{M}{2} - n\right] \sin(\omega[n - \frac{1}{2}]) \right\} \\
&= je^{-j\omega(M-1)/2} \left\{ 2 \sum_{n=1}^{M/2} h\left[\frac{M}{2} - n\right] \sin(\omega[n - \frac{1}{2}]) \right\}
\end{aligned}$$

Magnitude and phase of filter:

$$|H(\omega)| = \left| 2 \sum_{n=1}^{M/2} h[\frac{M}{2} - n] \sin(\omega[n - \frac{1}{2}]) \right|$$

$$\angle H(\omega) = \begin{cases} -\omega(\frac{M-1}{2}) + \frac{\pi}{2} & \text{for } A(\omega) \geq 0 \\ -\omega(\frac{M-1}{2}) + \frac{3\pi}{2} & \text{for } A(\omega) < 0 \end{cases}$$

Group delay is:

$$-\frac{d\angle H(\omega)}{d\omega} = \frac{M-1}{2}.$$

7. Design a low-pass FIR filter  $h[n]$  using the window method with

- pass-band edge frequency:  $f_p = 4$  kHz
- transition bandwidth: 200 Hz
- sampling frequency: 16 kHz
- stopband attenuation: -60 dB.

with minimal  $M$  (length of  $h[n]$ ). Find the expression for  $h[n]$  and the value of  $h[(M-1)/2]$ .

**Answer:** Ideal impulse response is given as (see Question 1):

$$h_d[n] = \begin{cases} \frac{\sin(\omega_c n)}{\pi n} & n \neq 0 \\ \frac{\omega_c}{\pi} & n = 0 \end{cases}$$

Next, we need to select a window. From Table 1, we see that only the Blackman window has stopband attenuation less than -60 dB. With a transition width  $\Delta\omega \leq 200$  Hz, we find  $M$  using the relation in Table 1:

$$\begin{aligned} M_{\text{blackman}} &= \frac{12\pi}{\Delta\omega} \\ &= \left\lceil \frac{12\pi}{(200/16000)2\pi} \right\rceil_{\text{odd}} \\ &= 479 \end{aligned}$$

(decreasing  $\Delta\omega$  will increase  $M_{\text{blackman}}$  so we set  $\Delta\omega = 200$  Hz).

If we use the Kaiser window, then

$$\begin{aligned} M_{\text{kaiser}} &= \left\lceil \frac{A-8}{2.285\Delta\omega} \right\rceil_{\text{odd}} \\ &= \left\lceil 289.7525 \right\rceil_{\text{odd}} = 291 \end{aligned}$$

where  $A = -20 \log_{10}(\delta) = 60$  (see Appendix 1.1). Thus, with stopband attenuation of -60 dB,  $M_{\text{kaiser}} < M_{\text{blackman}}$  therefore we select the Kaiser window of length 291.

Next, calculate  $\beta$  for the Kaiser window (again see Appendix 1.1):

$$\beta = 0.1102(A - 8.7) = 5.633$$

Finally, shift the impulse response function  $h_d[n]$  and multiply by Kaiser window  $w[n]$ :

$$h[n] = h_d[n - (M-1)/2]w[n]$$

which equates to

$$h[n] = \begin{cases} \frac{\omega_c}{\pi} \frac{I_0[\beta]}{I_0[\beta]} = \frac{\omega_c}{\pi} & n = M_h \\ \frac{\omega_c}{\pi} \frac{\sin(\omega_c(n-M_h))}{\omega_c(n-M_h)} \frac{I_0[\beta(1-[(n-M_h)/M_h]^2)^{1/2}]}{I_0(\beta)} & n \neq M_h \end{cases}$$

for  $\beta = 5.633$ ,  $\omega_c = 2\pi(4000/16000) = \pi/4$ , and  $M_h = (M-1)/2 = 145$ .

For  $n = (M-1)/2 = M_h$ , then

$$h[\frac{M-1}{2}] = \frac{\omega_c}{\pi} = \frac{\pi}{4\pi} = 0.25$$

# 1 Appendix: identities

$$\sin(\omega) = \frac{e^{j\omega} - e^{-j\omega}}{2j}$$

$$\cos(\omega) = \frac{e^{j\omega} + e^{-j\omega}}{2}$$

window type	side-lobe amplitude (dB)	approximate width of main lobe ( $\approx \Delta\omega$ )	approximate stopband $20 \log_{10}(\delta)$ (dB)	approximate passband $20 \log_{10}(1 + \delta)$ (dB)
rectangular	-13	$4\pi/(M + 1)$	-21	0.7416
Hanning	-31	$8\pi/M$	-44	0.0546
Hamming	-41	$8\pi/M$	-53	0.0194
Blackman	-57	$12\pi/M$	-74	0.0017

Table 1: Window parameters for length  $M$  window with relation to filter transition width  $\Delta\omega$ .

## 1.1 Kaiser Window

Kaiser window with parameter  $\beta$  and length  $M$  is defined as:

$$w[n] = \begin{cases} \frac{I_0[\beta(1 - [(n - M_h)/M_h]^2)^{1/2}]}{I_0(\beta)} & 0 \leq n \leq M - 1 \\ 0 & \text{otherwise} \end{cases}$$

where  $M_h = (M - 1)/2$ ,  $I_0$  is zero-order modified Bessel function of the first kind.

For filter transition  $\Delta\omega$  with stopband attenuation  $A = -20 \log_{10} \delta$ ,

$$\beta = \begin{cases} 0.1102(A - 8.7) & A > 50 \\ 0.5842(A - 21)^{0.4} + 0.07886(A - 21) & 21 \leq A \leq 50 \\ 0 & A < 21 \end{cases}$$

and

$$M = \frac{A - 8}{2.285\Delta\omega}$$