Discrete-Time Filters

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Recommended reading:

- *Signal Processing for Communications* by Paolo Prandoni and Martin Vetterli. Free download at `http://www.sp4comm.org/getit.html`
- *Discrete-time signal processing* by Alan V. Oppenheim, Ronald W. Schafer. (library: 621.38)

Learning Objectives

- review: linear time-invariant (LTI) systems
- two filter types:
  - FIR (finite impulse response)
  - IIR (infinite impulse response)
- review: convolution–modulation theorem
- time-domain and frequency-domain
  - frequency response (magnitude and phase)
  - impulse response in the time domain
- ideal frequency response

REVIEW: linear time-invariant (LTI) systems

LTI system: $x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n]$

$$y[n] = \mathcal{H}\left\{x[n]\right\}$$

linear:

$$\mathcal{H}\left\{ax_1[n] + bx_2[n]\right\} = a\mathcal{H}\left\{x_1[n]\right\} + b\mathcal{H}\left\{x_2[n]\right\}$$

time-invariant:

$$y[n-n_0] = \mathcal{H}\left\{x[n-n_0]\right\}$$

impulse response of LTI system

- can completely characterise LTI with $x[n] = \delta[n]$;
  - define impulse response (impulse function) as

$$h[n] := \mathcal{H}\{\delta[n]\}$$

  - convolution ($*$ symbol):

$$y[n] = \mathcal{H}\{x[n]\} = x[n] * h[n]$$

- causal: O/P does not depend on future values of I/P ($h[n] = 0$ for $n < 0$)
- BIBO stable if $\sum_{k=-\infty}^{\infty} |h[n]| < \infty$
- of the 2 filter types (FIR and IIR), FIR is always stable

Moving Average Example

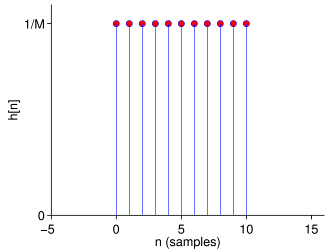
1. FIR filter

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

find impulse response ($h[n]$): let $x[n] = \delta[n]$, then

$$h[n] = \begin{cases} \frac{1}{M} & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$$

- $h[n]$ is finite $\Rightarrow$ FIR
- causal: can implement
- stable



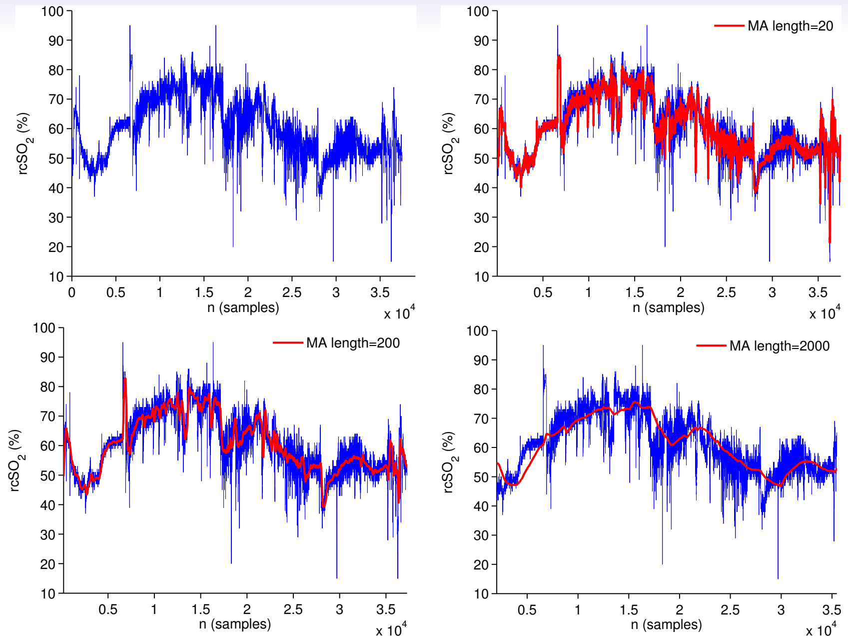
Application: NIRS for premature babies

- NIRS (near infra-red spectroscopy) measures cerebral oxygen saturation (rcSO_2)
- sensors detect changes in oxygen levels in blood
- safe for long-duration continuous recordings (even for infants)
  - $\Rightarrow$ ideal for long-term monitoring in neonatal ICUs

- current study in neonatal ICU at Cork University Maternity Hospital:
  - defining safe range of rcSO_2 for premature infants
  - monitor brain health (e.g. detect or predict serious injury, such as brain-bleed)
  - for infants born $<$32 weeks (term babies are 40 weeks)

filtering example

- data from baby 27 weeks (birth weight: 1.14 kg)
  - 48 hours of continuous rcSO_2 recording
- want to assess long-term trends of rcSO_2



2. IIR example

can rewrite as recursive equation

$$
\begin{aligned}
y_M[n] &:= \frac{1}{M}\sum_{k=0}^{M-1} x[n-k] \\
&= \frac{M-1}{M} y_{M-1}[n] + \frac{1}{M} x[n] \\
&= \lambda y_{M-1}[n-1] + (1-\lambda)x[n]
\end{aligned}
$$

with $\lambda := (M-1)/M$.

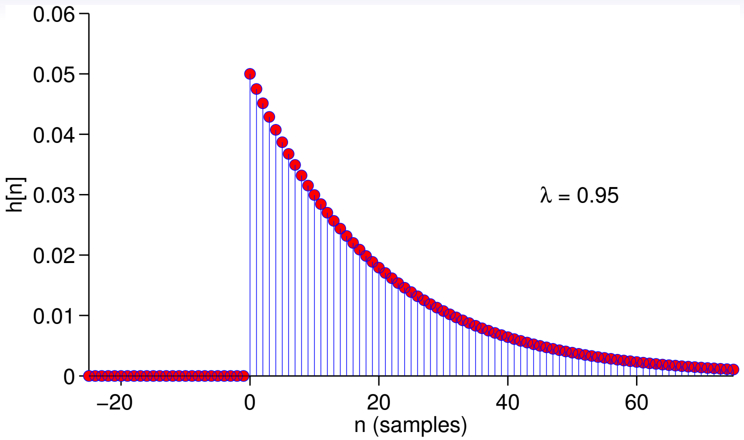
Assuming M is large and therefore $y_{M-1}[n] \approx y_M[n]$, then

$$y[n] = \lambda y[n-1] + (1-\lambda)x[n]$$

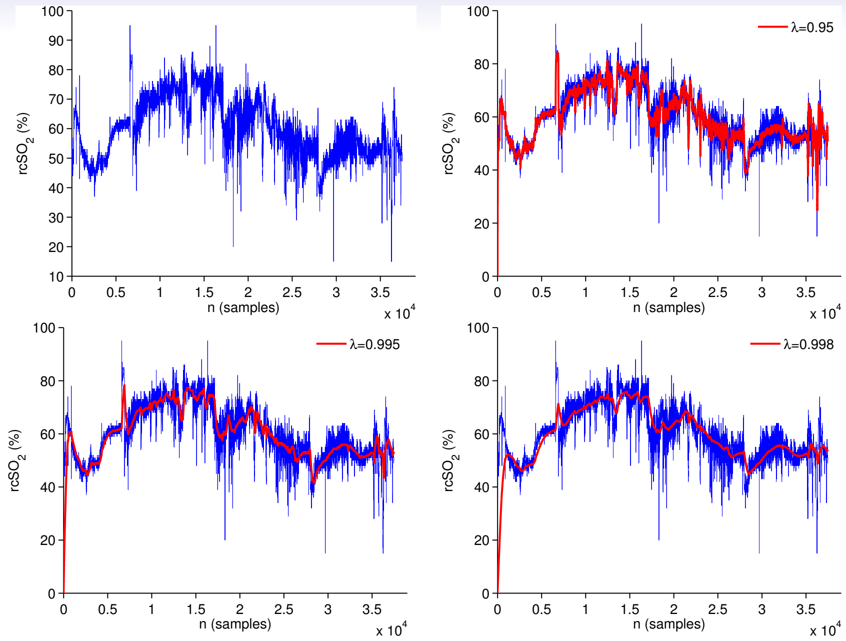
- recursive equation: $y[n] = \lambda y[n-1] + (1-\lambda)x[n]$
- is LTI? yes
- what is $h[n]$? Assume $y[n] = 0$ for $n < 0$ and let $x[n] = \delta[n]$:

$$
\begin{aligned}
y[0] &= 1-\lambda \\
y[1] &= (1-\lambda)\lambda \\
y[2] &= (1-\lambda)\lambda^2 \\
&\vdots \\
y[n] &= (1-\lambda)\lambda^n
\end{aligned}
$$

and thus $h[n] = (1-\lambda)\lambda^n u[n]$.



- casual;
- stable for $|\lambda| < 1$
- but $h[n]$ not finite $\Rightarrow$ IIR filter



compare FIR and IIR filters

- both low-pass filters provide adequate smoothing;
- compare computational load:

$$\text{FIR:} \qquad y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

$$\text{IIR:} \qquad y[n] = \lambda y[n-1] + (1-\lambda)x[n]$$

  for $M = 2000$, then FIR requires $2,000$ additions versus 2 additions (plus 2 multiplications) for IIR
- but what about phase?

Special case for LTI: complex exponential

if $x[n] = \mathrm{e}^{j\omega_0 n}$, then $\mathrm{e}^{j\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow H(\omega_0)\mathrm{e}^{j\omega_0 n}$, where

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]\mathrm{e}^{-j\omega n} \qquad \text{(DTFT of } h[n])$$

$H(\omega)$ is the frequency response of the filter and is a complex function $\Rightarrow H(\omega) = A\mathrm{e}^{j\theta}$, and thus

$$y[n] = A_0 \mathrm{e}^{j(\omega_0 n + \theta_0)}$$

Output $y[n]$ is scaled by A_0 and shifted in phase θ_0. [Recall amplitude $A_0 = |H(\omega_0)|$ and phase $\theta_0 = \angle H(\omega_0)$.]

REVIEW: convolution–modulation theorem

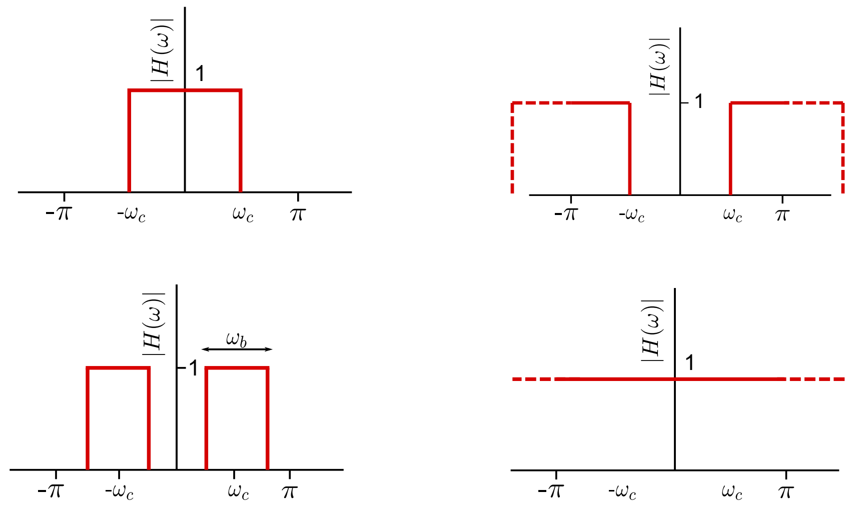
time-domain		frequency-domain
$y[n] = x[n] * h[n]$	$\leftrightarrow$	$Y(\omega) = X(\omega)H(\omega)$
$y[n] = x[n]h[n]$	$\leftrightarrow$	$Y(\omega) = X(\omega) * H(\omega)$

convolution operation is represented by $*$ and $X(\omega) = \text{DTFT}\{x[n]\}$ [and likewise for $Y(\omega)$ and $H(\omega)$].

Frequency Response: Magnitude

Magnitude of frequency response $|H(\omega)|$ defines type:

- low-pass
- high-pass
- pass-band
- all-pass (delay only)



Frequency Response: Phase

Linear Phase

example: all-pass system (i.e. $|H(\omega)| = 1$) with

$$H(\omega) = \mathrm{e}^{-j\omega d}$$

as $h[n] = \delta[n-d]$, then

$$y[n] = x[n-d]$$

is a pure-delay system. (d may not be an integer $\Rightarrow$ *fractional delay*.)

More generally, filter with linear phase

$$H(\omega) = |H(\omega)|\mathrm{e}^{-j\omega d}$$

Group Delay

if $\varphi(\omega) = \angle H(\omega)$, then

$$\text{G.D.} = -\frac{\mathrm{d}\varphi(\omega)}{\mathrm{d}\omega}$$

for linear phase, G.D. is a constant.

Examples:

1. Moving average example, FIR filter response is

$$H(\omega) = \frac{1}{N}\frac{\sin(\omega N/2)}{\sin(\omega/2)}\,\mathrm{e}^{-j\frac{(N-1)}{2}\omega}$$

Therefore, the filter has linear phase

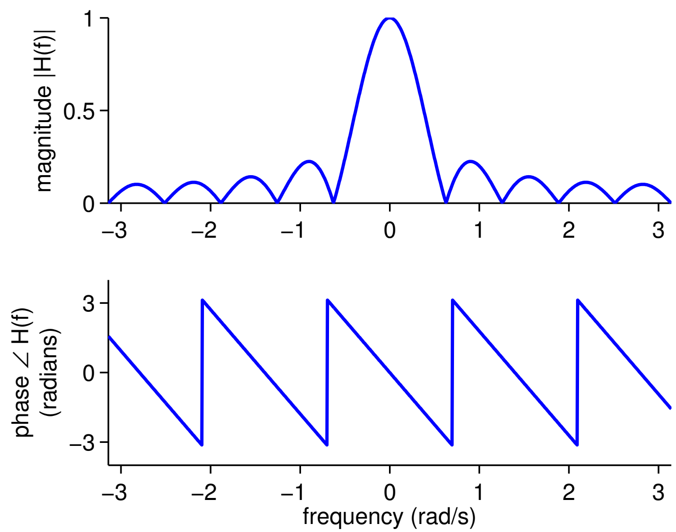
$$\varphi(\omega) = \angle H(\omega) = -\frac{(N-1)}{2}\omega$$

and

$$-\frac{\mathrm{d}\varphi(\omega)}{\mathrm{d}\omega} = \frac{N-1}{2}$$

which is a constant delay.

frequency response of moving average filter



2. for IIR filter (“leaky integrator”), nonlinear phase as

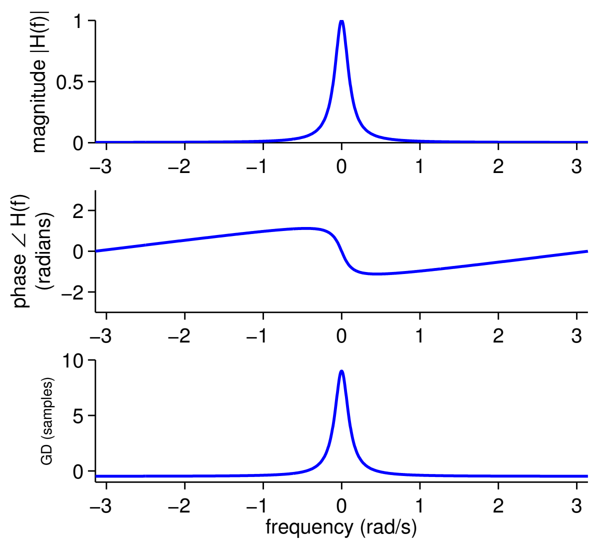
$$\angle H(\omega) = \tan^{-1}\left\{\frac{-\lambda\sin(\omega)}{1-\lambda\cos(\omega)}\right\}$$

and therefore (Prandoni and Vetterli, p. 127)

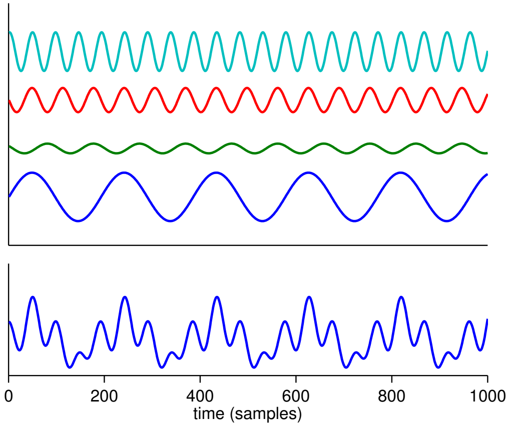
$$\mathrm{G.D.} = \frac{\lambda\cos(\omega) - \lambda^2}{1+\lambda^2 - 2\lambda\cos(\omega)}$$

thus delay is a function of frequency (ω).

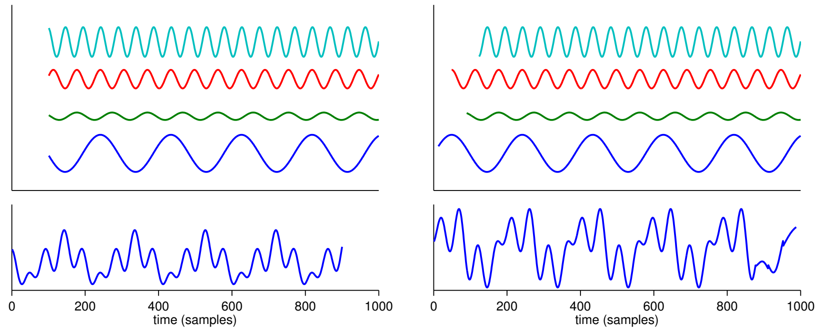
frequency response of IIR filter



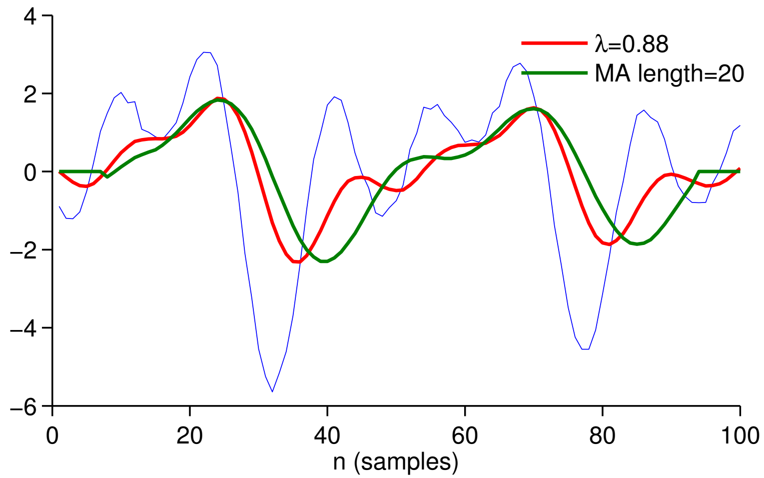
Example of difference between nonlinear and linear phase



same delay for all components (linear phase, LEFT) vs. different delay (nonlinear phase, RIGHT):



example for FIR (MA) and IIR (leaky integrator) filters:

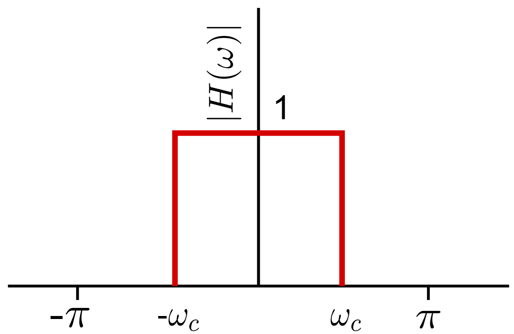


Ideal Low-Pass Filter

Frequency Response

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

Magnitude response:

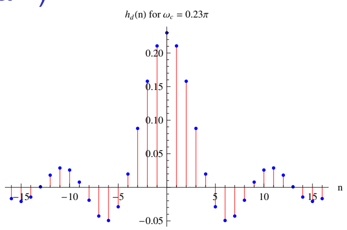


Phase response:
$\angle H(\omega) = 0$

Impulse Response (time-domain)

Inverse DTFT

$$
\begin{aligned}
h(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\
&= \begin{cases} \frac{\omega_c}{\pi} & n = 0 \\ \frac{\omega_c}{\pi}\frac{\sin\omega_c n}{\omega_c n} & n \neq 0 \end{cases}
\end{aligned}
$$



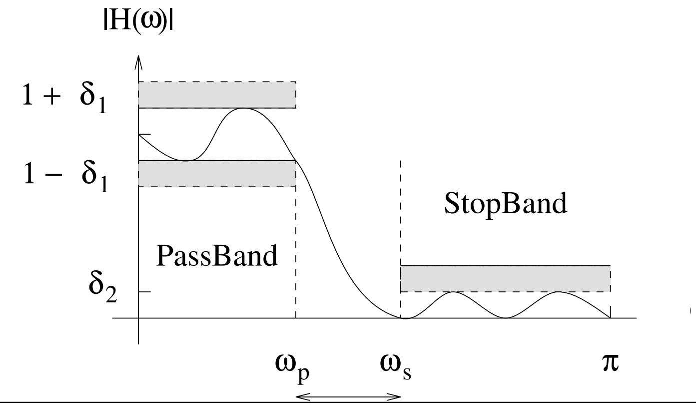
Problems:

- ideal filter is non-causal
- not realisable for stored data ($n = -\infty$)
- unstable

Non-Ideal Filters

Compromise:

- let $|H(\omega)| \approx 0$ in stopband
- let $|H(\omega)| \approx 1$ in passband
- smooth transition from passband to stopband



IIR Filter

Causal IIR filter:

$$y[n] = \sum_{k=0}^{\infty} h[n]x[n-k]$$

Problem: how to implement when $k = 0, 1, \ldots, \infty$?

Answer: a **causal** LTI system can be described using a constant coefficient difference equation:

$$y[n] = -\sum_{k=1}^{N} a[k]y[n-k] + \sum_{k=0}^{M} b[k]x[n-k]$$

IIR Frequency Response

$$H(\omega) = \frac{\sum_{k=0}^{M} b[k]e^{-j\omega k}}{1 + \sum_{k=1}^{N} a[k]e^{-j\omega k}}$$

- IIR filter design:
  - **determine** $a(k)$ **and** $b(k)$
  - to satisfy limits set for δ_1, δ_2 and $\omega_s - \omega_p$
- Accuracy depends on $a[k], b[k]$ and N, M.

FIR vs IIR

FIR filters:

- can have linear phase response;
- are aways stable;
- have less roundoff noise and coefficient quantization errors than IIR filters.

IIR filters:

- a higher order FIR filter is required to get the same sharp cutoff as lower order IIR: $(N+M)_{IIR} << (M)_{FIR}$
- can transform analog filter design to discrete IIR filter.

which one?

- IIR when sharp cutoff and high throughput required.
- FIR when little or no-phase distortion is required.

Summary

- moving average example for FIR and IIR filters:
  - IIR is more computational efficient, but FIR has linear phase.
- convolution and modulation:
  - modulation (multiplication) in time-domain is equivalent to convolution in frequency-domain.

$$x[n]w[n] \leftrightarrow X(\omega) * W(\omega)$$

- frequency response $H(\omega)$ has both magnitude and phase:
  - magnitude scales frequencies according to “shape” (e.g. low-pass, high-pass)
  - phase provides information on time-delays
  - linear phase: constant time-delay
  - nonlinear phase: group delay is a function of frequency (different oscillations have different delays)