Proofs for Discrete Time–Frequency Distribution Properties

Technical Report: UQCCR-2009-06-09

John M. O' Toole, Mostefa Mesbah, and B. Boashash*

The University of Queensland,
Perinatal Research Centre and UQ Centre for Clinical Research,
Royal Brisbane & Women's Hospital, Herston, QLD 4029, Australia.
phone: + 61 7 3346 6113; fax: + 61 7 3346 5594
email: j.otoole@ieee.org, m.mesbah@uq.edu.au, b.boashash@uq.edu.au

Abstract

This technical report contains proofs for a set of mathematical properties of a recently proposed discrete time–frequency distribution class.

1 Discrete Time–Frequency Distribution

We begin with some definitions. The discrete time–frequency distribution (DTFD) in [1] is defined as the time–frequency convolution of the discrete Wigner–Ville distribution (DWVD) with the discrete kernel:

$$\rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) = \left[W^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \underset{n}{\circledast} \underset{k}{\circledast} \gamma^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \right] \Bigg|_{k=0,1,\ldots,N-1}$$

where $W(n/2, k/2N)$ represents the DWVD, $\gamma^{\mathrm{C}}(n/2, k/2N)$ represents the time–frequency kernel, $\circledast$ represents circular convolution, and the DWVD is formed from the $2N$-point discrete analytic signal [2]. The DTFD over discrete frequency samples $k = 0, 1, \ldots, 2N-1$ is

$$\begin{aligned}\rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) &= W^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \underset{n}{\circledast} \underset{k}{\circledast} \gamma^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \\ &= \frac{1}{2N} \sum_{m=0}^{2N-1} \sum_{p=0}^{2N-1} K^{\mathrm{C}}(\tfrac{p}{2}, m) G^{\mathrm{C}}(\tfrac{n-p}{2}, m) \mathrm{e}^{-\mathrm{j}\pi mk/N} \qquad (1)\end{aligned}$$

*B. Boashash is also with the College of Engineering, University of Sharjah, United Arab Emirates.

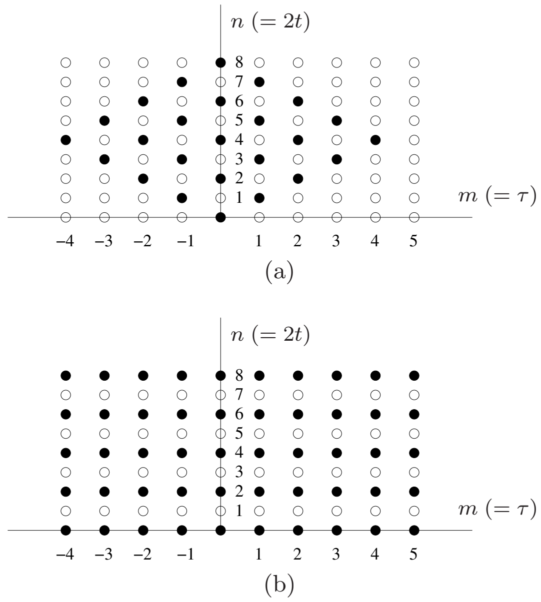

Figure 1: Discrete grids in the time–lag domain for $N = 5$. (a) Function $K^{\mathrm{C}}(n/2, m)$ and (b) kernel $G^{\mathrm{C}}(n/2, m)$. Open circles represent zero values; filled circles represent the sample points of the function.

where $K^{\mathrm{C}}(n/2, m)$ is the discrete time–lag signal function and $G^{\mathrm{C}}(n/2, m)$ is the discrete time–lag kernel [1], for $n, m = 0, 1, \dots, 2N - 1$. The time–lag kernel is zero when n is not an integer; that is, $G^{\mathrm{C}}(n + 1/2, m) = 0$.

The time–lag function $K^{\mathrm{C}}(n/2, m)$ has a nonuniform discrete grid. We write $K^{\mathrm{C}}(n/2, m)$ as a function of the analytic signal $z(n)$ in two parts. First, for $n/2$ an integer,

$$\begin{aligned} K^{\mathrm{C}}(n, 2m) &= z(n + m)\bar{z}(n - m) \\ K^{\mathrm{C}}(n, 2m + 1) &= 0 \end{aligned} \tag{2}$$

and second, for $n/2$ not an integer,

$$\begin{aligned} K^{\mathrm{C}}(n + \tfrac{1}{2}, 2m) &= 0 \\ K^{\mathrm{C}}(n + \tfrac{1}{2}, 2m + 1) &= z(n + m + 1)\bar{z}(n - m) \end{aligned} \tag{3}$$

where $\bar{z}(n)$ represents the complex conjugate of $z(n)$. Both sample grids are illustrated in Fig. 1. The time–lag kernel K^{C} is diamond shaped because $z(n) = 0$ for $N \le n \le 2N - 1$ [2, 3].

We can also define the DTFD in the Doppler–frequency domain, as a function of the Doppler–frequency function $\mathcal{K}^{\mathrm{C}}$ and Doppler–frequency kernel $\mathcal{G}^{\mathrm{C}}$ as

$$\rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) = \frac{1}{4N^2} \sum_{l=0}^{2N-1} \sum_{q=0}^{2N-1} \mathcal{K}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{k-q}{2N}) \mathcal{G}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{q}{2N}) \mathrm{e}^{\,\mathrm{j}\pi l n/N}. \tag{4}$$

The Doppler–frequency kernel is a function of the analytic signal,

$$\mathcal{K}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{k}{2N}) = Z(\tfrac{k+l}{2N})\bar{Z}(\tfrac{k-l}{2N})$$

where $Z(k/2N)$ is the discrete Fourier transform (DFT) of $z(n)$.

2 Proofs for Properties

We now present proofs for a set of DTFD properties which appeared in [1].

P1) Nonnegative: to prove

$$\rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \geq 0$$

when

$$G^{\mathrm{C}}(\tfrac{n}{2}, m) = h(\tfrac{n+m}{2})\bar{h}(\tfrac{n-m}{2}) \tag{5}$$

where $h(n)$ is zero when n is not an integer.

Proof: The kernel G^{C} is only nonzero when $n/2$ is an integer and m is even,

$$G^{\mathrm{C}}(n, 2m) = h(n+m)\bar{h}(n-m) \tag{6}$$

because both $h(n/2)$ and $g^{\mathrm{C}}(n/2, m)$ are zero when $n/2$ is not an integer. The kernel form in (6) combined with the nonuniform discrete grid of the time–lag function in (2) and (3), means that the DTFD is zero at non-integer $n/2$ values. For $n/2$ integer values,

$$\begin{aligned}
\rho^{\mathrm{C}}(n, \tfrac{k}{2N}) &= \frac{1}{2N}\sum_{p=0}^{N-1}\sum_{m=0}^{N-1} K^{\mathrm{C}}(n-p, 2m)G^{\mathrm{C}}(p, 2m)\mathrm{e}^{-\mathrm{j}2\pi mk/N} \\
&= \frac{1}{2N}\sum_{p=0}^{N-1}\sum_{m=0}^{N-1} z(n-p+m)\bar{z}(n-p-m)h(p+m) \\
&\quad \cdot \bar{h}(p-m)\mathrm{e}^{-\mathrm{j}2\pi mk/N}.
\end{aligned}$$

Let $a = p+m$, $b = p-m$ and rewrite the preceding equation as

$$\begin{aligned}
&= \frac{1}{2N}\sum_{a=0}^{N-1}\sum_{b=0}^{N-1} z(2a)\bar{z}(2b)h(n-2b)\bar{h}(n-2a)\mathrm{e}^{-\mathrm{j}2\pi(a-b)k/N} \\
&\quad + \frac{1}{2N}\sum_{a=0}^{N-1}\sum_{b=0}^{N-1} z(2a+1)\bar{z}(2b+1)h(n-2b-1)\bar{h}(n-2a-1) \\
&\quad \cdot \mathrm{e}^{-\mathrm{j}2\pi(a-b)k/N} \\
&= \frac{1}{2N}\left|\sum_{a=0}^{N-1} z(2a)\bar{h}(n-2a)\mathrm{e}^{-\mathrm{j}2\pi ak/N}\right|^2 \\
&\quad + \frac{1}{2N}\left|\sum_{a=0}^{N-1} z(2a+1)\bar{h}(n-2a-1)\mathrm{e}^{-\mathrm{j}2\pi ak/N}\right|^2.
\end{aligned}$$

Hence the DTFD is nonnegative when the kernel is of the form in (5).

P2) Time marginal: to prove

$$2\sum_{k=0}^{N-1} \rho^{\mathrm{C}}(\tfrac{2n}{2}, \tfrac{k}{2N}) = |z(n)|^2 \tag{7}$$

when

$$G^{\mathrm{C}}(\tfrac{n}{2}, 0) = \delta(n). \tag{8}$$

where δ represents the Dirac function.

Proof: Expand the DTFD in (7) using (1) but sum the DTFD over $k = 0, 1, \ldots, 2N-1$,

$$\begin{aligned}
\sum_{k=0}^{2N-1} \rho^{\mathrm{C}}(n, \tfrac{k}{2N}) &= \frac{1}{2N} \sum_{k=0}^{2N-1} \sum_{m=0}^{2N-1} \sum_{p=0}^{2N-1} K^{\mathrm{C}}(\tfrac{2n-p}{2}, m) G^{\mathrm{C}}(\tfrac{p}{2}, m) \mathrm{e}^{-\mathrm{j}\pi mk/N} \\
&= \sum_{m=0}^{2N-1} \sum_{p=0}^{2N-1} K^{\mathrm{C}}(\tfrac{2n-p}{2}, m) G^{\mathrm{C}}(\tfrac{p}{2}, m) \frac{1}{2N} \sum_{k=0}^{2N-1} \mathrm{e}^{-\mathrm{j}\pi mk/N} \\
&= \sum_{p=0}^{2N-1} K^{\mathrm{C}}(\tfrac{2n-p}{2}, 0) G^{\mathrm{C}}(\tfrac{p}{2}, 0).
\end{aligned} \tag{9}$$

as $\sum_{k=0}^{2N-1} \exp\left(-\mathrm{j}\pi mk/N\right) = 2N\delta(m)$. Apply the kernel constraint in (8) to (9), then

$$\begin{aligned}
\sum_{k=0}^{2N-1} \rho^{\mathrm{C}}(n, \tfrac{k}{2N}) &= K^{\mathrm{C}}(n, 0) \\
&= z(n)\bar{z}(n) = |z(n)|^2.
\end{aligned}$$

We can easily show, because of the periodicity of the proposed DTFD [1], that

$$\sum_{k=0}^{2N-1} \rho^{\mathrm{C}}(n, \tfrac{k}{2N}) = 2\sum_{k=0}^{N-1} \rho^{\mathrm{C}}(n, \tfrac{k}{2N})$$

and thus

$$2\sum_{k=0}^{N-1} \rho^{\mathrm{C}}(n, \tfrac{k}{2N}) = |z(n)|^2 \tag{10}$$

which concludes the proof.

P3) Frequency marginal: to prove

$$\sum_{n=0}^{2N-1} \rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) = \frac{1}{2N} \left| Z(\tfrac{k}{2N}) \right|^2$$

when

$$\mathcal{G}^{\mathrm{C}}(\tfrac{0}{N}, \tfrac{k}{2N}) = \delta(k) \tag{11}$$

where $\mathcal{G}^{\mathrm{C}}$ is the Doppler–frequency kernel.

Proof: Using the Doppler–frequency expansion in (4),

$$\begin{aligned}
\sum_{n=0}^{2N-1} \rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) &= \frac{1}{4N^2} \sum_{n=0}^{2N-1} \sum_{l=0}^{2N-1} \sum_{q=0}^{2N-1} \mathcal{K}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{k-q}{2N}) \mathcal{G}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{q}{2N}) \mathrm{e}^{\,\mathrm{j}\pi ln/N} \\
&= \frac{1}{4N^2} \sum_{l=0}^{2N-1} \sum_{q=0}^{2N-1} \mathcal{K}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{k-q}{2N}) \mathcal{G}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{q}{2N}) \sum_{n=0}^{2N-1} \mathrm{e}^{\,\mathrm{j}\pi ln/N} \\
&= \frac{1}{2N} \sum_{q=0}^{2N-1} \mathcal{K}^{\mathrm{C}}(\tfrac{0}{N}, \tfrac{k-q}{2N}) \mathcal{G}^{\mathrm{C}}(\tfrac{0}{N}, \tfrac{q}{2N})
\end{aligned} \tag{12}$$

as $\sum_{n=0}^{2N-1} \exp(\mathrm{j}\pi ln/N) = 2N\delta(l)$. Apply the kernel constraint in (11) to (12), then

$$2N \sum_{n=0}^{2N-1} \rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) = \mathcal{K}^{\mathrm{C}}(\tfrac{0}{N}, \tfrac{k}{2N}) = Z(\tfrac{k}{2N})\bar{Z}(\tfrac{k}{2N}) = \left|Z(\tfrac{k}{2N})\right|^2$$

which proves the property.

P4) Time support: to prove, for signal $z(n) = 0$ for $n < n_1$ and $n > n_2$, that

$$\rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) = 0, \qquad \text{for } n < 2n_1 \text{ and } n > 2n_2,$$

when

$$G^{\mathrm{C}}(\tfrac{n}{2}, m) = 0, \qquad \text{for } |n| > |m|. \tag{13}$$

Proof: The DTFD is the DFT of the smoothed time–lag function R^{C}, where

$$R^{\mathrm{C}}(\tfrac{n}{2}, m) = K^{\mathrm{C}}(\tfrac{n}{2}, m) \underset{n}{\circledast} G^{\mathrm{C}}(\tfrac{n}{2}, m) \tag{14}$$

as defined in (1). To satisfy time support, the smoothed time–lag function R^{C} must have the same time support as K^{C}; that is, if

$$K^{\mathrm{C}}(\tfrac{n}{2}, m) = 0, \qquad \text{for } n < 2n_1 \text{ and } n > 2n_2,$$

then the property requires that

$$R^{\mathrm{C}}(\tfrac{n}{2}, m) = 0, \qquad \text{for } n < 2n_1 \text{ and } n > 2n_2. \tag{15}$$

When the kernel has the form in (13), a *cone-shaped* kernel [4], then R^{C} satisfies (15) because the convolution of K^{C} with the kernel G^{C} in (14) does not smear nonzero energy components into the the region $n < 2n_1$ and $n > 2n_2$ for $R^{\mathrm{C}}(n/2, m)$ [4, 5].

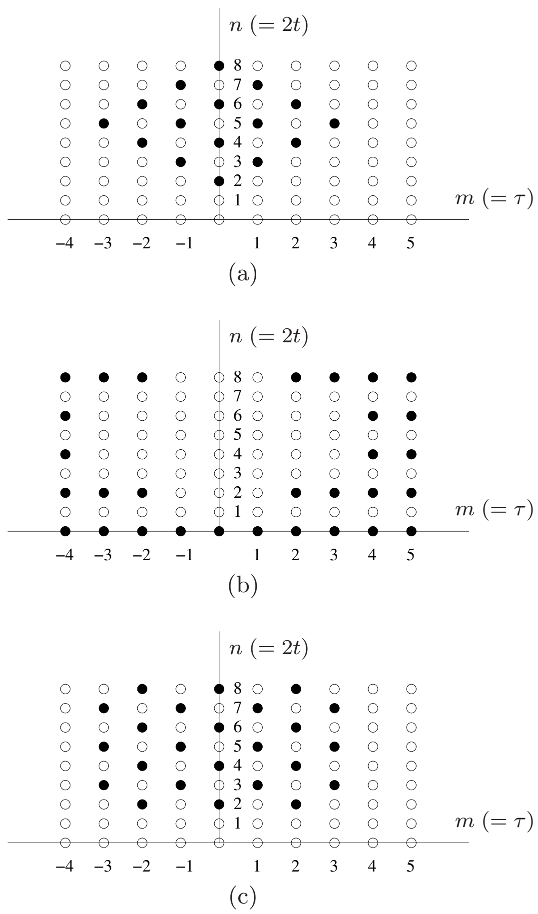

Figure 2: Time support example with $N = 5$ and $z(0) = 0$. (a) Time–lag function $K^{\mathrm{C}}(n/2, m)$, with $K^{\mathrm{C}}(0, m) = K^{\mathrm{C}}(1/2, m) = 0$, (b) time–lag kernel $G^{\mathrm{C}}(n/2, m)$, and (c) smoothed time–lag function $R^{\mathrm{C}}(n/2, m)$, where $R^{\mathrm{C}}(0, m) = R^{\mathrm{C}}(1/2, m) = 0$. Open circles represent zero values; filled circles represent the sample points of the function.

Fig. 2 shows an example of the convolution process in (14) for a signal where $z(0) = 0$ and $N = 5$. Because the kernel satisfies the constraint in (13), $R^{\mathrm{C}}(0, m) = 0$ and therefore $\rho^{\mathrm{C}}(0, k/2N) = 0$. In this example we assumed that n is positive and thus we periodically extended the kernel from $-(N-1) \leq n \leq N$ to $0 \leq n \leq 2N - 1$, hence the mirror cone-shape kernel in Fig. 1.

P5) Frequency support: to prove, for signal $Z(k/2N) = 0$ for $k < k_1$ and $k > k_2$, that

$$\rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) = 0, \qquad \text{for } k < k_1 \text{ and } k > k_2,$$

when

$$\mathcal{G}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{k}{2N}) = 0, \qquad \text{for } |k| > |l|. \tag{16}$$

Proof: The DTFD is the inverse DFT of the smoothed Doppler–frequency function $\mathcal{R}^{\mathrm{C}}$, where

$$\mathcal{R}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{k}{2N}) = \mathcal{K}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{k}{2N}) \underset{k}{\circledast} \mathcal{G}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{k}{2N}) \tag{17}$$

as defined in (4). To satisfy the property, the smoothed Doppler–frequency function $\mathcal{R}^{\mathrm{C}}$ must have the same frequency support as $\mathcal{K}^{\mathrm{C}}$; that is, if

$$\mathcal{K}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{k}{2N}) = 0, \qquad \text{for } k < k_1 \text{ and } k > k_2$$

then, the property requires that

$$\mathcal{R}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{k}{2N}) = 0, \qquad \text{for } k < k_1 \text{ and } k > k_2. \tag{18}$$

Similar to the time-support property, $\mathcal{R}^{\mathrm{C}}$ satisfies (18) when the kernel is of the form in (16) [5].

P6) Instantaneous frequency: to prove,

$$\frac{1}{4\pi}\left\{\arg\left[\sum_{k=0}^{N-1}\rho^{\mathrm{C}}(\tfrac{2n}{2}, \tfrac{k}{2N})\mathrm{e}^{\mathrm{j}2\pi k/N}\right] \mod 2\pi\right\} = f(n) \tag{19}$$

when

$$G^{\mathrm{C}}(\tfrac{n}{2}, 2) = a\delta(n) \tag{20}$$

where a is a positive constant; the discrete instantaneous frequency $f(n)$ is equal to the central finite difference of the phase of $z(n)$ [6, pp. 463] as

$$f(n) = \frac{1}{2\pi}\left[\frac{\varphi(n+1)-\varphi(n-1)}{2} \mod \pi\right]. \tag{21}$$

Proof: Sum the DTFD over $k = 0, 1, \dots, 2N-1$ as follows

$$\begin{aligned}
&\sum_{k=0}^{2N-1}\rho^{\mathrm{C}}(n, \tfrac{k}{2N})\mathrm{e}^{\mathrm{j}2\pi k/N} \\
&= \frac{1}{2N}\sum_{k=0}^{2N-1}\sum_{m=0}^{2N-1}\sum_{p=0}^{2N-1} K^{\mathrm{C}}(\tfrac{2n-p}{2}, m)G^{\mathrm{C}}(\tfrac{p}{2}, m)\mathrm{e}^{-\mathrm{j}\pi mk/N}\mathrm{e}^{\mathrm{j}2\pi k/N} \\
&= \sum_{m=0}^{2N-1}\sum_{p=0}^{2N-1} K^{\mathrm{C}}(\tfrac{2n-p}{2}, m)G^{\mathrm{C}}(\tfrac{p}{2}, m)\frac{1}{2N}\sum_{k=0}^{2N-1}\mathrm{e}^{-\mathrm{j}\pi k(m-2)/N} \\
&= \sum_{p=0}^{2N-1} K^{\mathrm{C}}(\tfrac{2n-p}{2}, 2)G^{\mathrm{C}}(\tfrac{p}{2}, 2)
\end{aligned}$$

as $\sum_{k=0}^{2N-1} \exp\left[-\mathrm{j}\pi k(m-2)/N\right] = 2N\delta(m-2)$. Because of the constraint on the kernel in (20),

$$\begin{aligned} K^{\mathrm{C}}(\tfrac{2n-p}{2}, 2)G^{\mathrm{C}}(\tfrac{p}{2}, 2) &= aK^{\mathrm{C}}(n, 2) \\ &= az(n+1)\bar{z}(n-1) \\ &= aA(n+1)A(n-1)\mathrm{e}^{\mathrm{j}[\varphi(n+1)-\varphi(n-1)]} \end{aligned}$$

using the polar notation $z(n) = A(n)\exp\left[\mathrm{j}\varphi(n)\right]$. Thus,

$$\arg\left[\sum_{k=0}^{2N-1} \rho^{\mathrm{C}}(n, \tfrac{k}{2N})\mathrm{e}^{\mathrm{j}2\pi k/N}\right] = \varphi(n+1) - \varphi(n-1) \tag{22}$$

and because

$$\sum_{k=0}^{2N-1} \rho^{\mathrm{C}}(n, \tfrac{k}{2N})\mathrm{e}^{\mathrm{j}2\pi k/N} = 2\sum_{k=0}^{N-1} \rho^{\mathrm{C}}(n, \tfrac{k}{2N})\mathrm{e}^{\mathrm{j}2\pi k/N},$$

then

$$\begin{aligned} &\frac{1}{4\pi}\left\{\arg\left[\sum_{k=0}^{N-1} \rho^{\mathrm{C}}(\tfrac{2n}{2}, \tfrac{k}{2N})\mathrm{e}^{\mathrm{j}2\pi k/N}\right] \mod 2\pi\right\} \\ &\qquad = \frac{1}{4\pi}\left\{\varphi(n+1) - \varphi(n-1) \mod 2\pi\right\} \\ &\qquad = f(n). \end{aligned}$$

thus proving the property in (19).

P7) Group delay: to prove

$$-\frac{N}{2\pi}\left\{\arg\left[\sum_{n=0}^{2N-1} \rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N})\mathrm{e}^{-\mathrm{j}\pi n/N}\right] \mod -2\pi\right\} = \tau(\tfrac{k}{2N}). \tag{23}$$

when

$$\mathcal{G}^{\mathrm{C}}(\tfrac{1}{N}, \tfrac{k}{2N}) = a\delta(k). \tag{24}$$

where a is a positive constant. The discrete group delay function $\tau(k/2N)$ is defined as

$$\tau(\tfrac{k}{2N}) = -\frac{N}{2\pi}\left[\frac{\theta(k+1) - \theta(k-1)}{2} \mod -\pi\right].$$

Proof: First, expand part of left hand side expression in (23) as follows:

$$\begin{aligned}
&\sum_{n=0}^{2N-1} \rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \mathrm{e}^{-\mathrm{j}\pi n/N} \\
&\quad = \frac{1}{4N^2} \sum_{n=0}^{2N-1} \sum_{l=0}^{2N-1} \sum_{q=0}^{2N-1} \mathcal{K}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{k-q}{2N}) \mathcal{G}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{q}{2N}) \mathrm{e}^{\mathrm{j}\pi(l-1)n/N} \\
&\quad = \frac{1}{4N^2} \sum_{l=0}^{2N-1} \sum_{q=0}^{2N-1} \mathcal{K}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{k-q}{2N}) \mathcal{G}^{\mathrm{C}}(\tfrac{l}{N}, \tfrac{q}{2N}) \sum_{n=0}^{2N-1} \mathrm{e}^{\mathrm{j}\pi(l-1)n/N} \\
&\quad = \frac{1}{2N} \sum_{q=0}^{2N-1} \mathcal{K}^{\mathrm{C}}(\tfrac{1}{N}, \tfrac{k-q}{2N}) \mathcal{G}^{\mathrm{C}}(\tfrac{1}{N}, \tfrac{q}{2N})
\end{aligned}$$

as $\sum_{n=0}^{2N-1} \exp\left[\mathrm{j}\pi(l-1)n/N\right] = 2N\delta(l-1)$. Substituting the kernel constraint (24) into the previous expression, then

$$\begin{aligned}
\sum_{n=0}^{2N-1} \rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \mathrm{e}^{-\mathrm{j}\pi n/N} &= a\mathcal{K}^{\mathrm{C}}(\tfrac{1}{N}, \tfrac{k}{2N}) \\
&= aZ(\tfrac{k+1}{2N})\bar{Z}(\tfrac{k-1}{2N}) \\
&= ab(k+1)b(k-1)\mathrm{e}^{\mathrm{j}[\theta(k+1)-\theta(k-1)]}
\end{aligned}$$

using the polar notation $Z(k/2N) = b(k)\exp\left[\mathrm{j}\theta(k)\right]$. Combing the previous relation with the rest of the expression in (22),

$$\begin{aligned}
-\frac{N}{2\pi} &\left\{ \arg\left[\sum_{n=0}^{2N-1} \rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \mathrm{e}^{-\mathrm{j}\pi n/N}\right] \mod -2\pi \right\} \\
&= -\frac{N}{2\pi} \left\{ \left[\theta(k+1) - \theta(k-1)\right] \mod -2\pi \right\} \\
&= \tau(\tfrac{k}{2N})
\end{aligned}$$

thus proving the property.

P8) Moyal's Formula: to prove

$$4N \sum_{n=0}^{2N-1} \sum_{k=0}^{N-1} \rho_x^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \bar{\rho}_y^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) = \left| \sum_{n=0}^{N-1} x(n)\bar{y}(n) \right|^2 \tag{25}$$

when

$$g^{\mathrm{C}}(\tfrac{l}{N}, m)\bar{g}^{\mathrm{C}}(\tfrac{l}{N}, m) = 1. \tag{26}$$

Proof: Rewrite the DTFD inner product in (25) in terms of the smoothed ambiguity functions $S(l/N, m)$, where

$$S(\tfrac{l}{N}, m) = A(\tfrac{l}{N}, m)g^{\mathrm{C}}(\tfrac{l}{N}, m)$$

and the discrete ambiguity function $A(l/N, m)$ is defined as

$$A(\tfrac{l}{N}, m) = \frac{1}{2N} \sum_{n=0}^{2N-1} K_x^{\mathrm{C}}(\tfrac{n}{2}, m) \mathrm{e}^{-\mathrm{j}\pi l n/N}.$$

Summing the DTFD products over $k = 0, 1, \ldots, 2N-1$,

$$\begin{aligned}
&\sum_{n=0}^{2N-1} \sum_{k=0}^{2N-1} \rho_x^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \bar{\rho}_y^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \\
&\quad = \frac{1}{4N^2} \sum_{n=0}^{2N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2N-1} \sum_{m=0}^{2N-1} S_x(\tfrac{l}{N}, m) \mathrm{e}^{-\mathrm{j}\pi(mk - ln)/N} \\
&\qquad \cdot \sum_{l'=0}^{2N-1} \sum_{m'=0}^{2N-1} \bar{S}_y(\tfrac{l'}{N}, m') \mathrm{e}^{\,\mathrm{j}\pi(m'k - l'n)/N} \\
&\quad = \frac{1}{4N^2} \sum_{l=0}^{2N-1} \sum_{m=0}^{2N-1} \sum_{l'=0}^{2N-1} \sum_{m'=0}^{2N-1} S_x(\tfrac{l}{N}, m) \bar{S}_y(\tfrac{l'}{N}, m') \\
&\qquad \cdot \sum_{n=0}^{2N-1} \mathrm{e}^{\,\mathrm{j}\pi n(l-l')/N} \sum_{k=0}^{2N-1} \mathrm{e}^{-\mathrm{j}\pi k(m-m')/N} \\
&\quad = \sum_{l=0}^{2N-1} \sum_{m=0}^{2N-1} S_x(\tfrac{l}{N}, m) \bar{S}_y(\tfrac{l}{N}, m)
\end{aligned} \tag{27}$$

as

$$\frac{1}{2N} \sum_{n=0}^{2N-1} \mathrm{e}^{\mathrm{j}\pi n(l-l')/N} = \delta(l - l')$$

$$\frac{1}{2N} \sum_{k=0}^{2N-1} \mathrm{e}^{-\mathrm{j}\pi k(m-m')/N} = \delta(m - m').$$

Because of (26), $S_x(\tfrac{l}{N}, m)\bar{S}_y(\tfrac{l}{N}, m) = A_x(\tfrac{l}{N}, m)\bar{A}_y(\tfrac{l}{N}, m)$ and therefore

$$\sum_{n=0}^{2N-1} \sum_{k=0}^{2N-1} \rho_x^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \bar{\rho}_y^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) = \sum_{l=0}^{2N-1} \sum_{m=0}^{2N-1} A_x(\tfrac{l}{N}, m) \bar{A}_y(\tfrac{l}{N}, m)$$

Rewriting this expression in terms of the time–lag function K^{C} as

$$\begin{aligned}
&\sum_{l=0}^{2N-1} \sum_{m=0}^{2N-1} A_x(\tfrac{l}{N}, m) \bar{A}_y(\tfrac{l}{N}, m) \\
&\quad = \frac{1}{4N^2} \sum_{l=0}^{2N-1} \sum_{m=0}^{2N-1} \sum_{n=0}^{2N-1} K_x^{\mathrm{C}}(\tfrac{n}{2}, m) \mathrm{e}^{-\mathrm{j}\pi l n/N} \sum_{n'=0}^{2N-1} \bar{K}_y^{\mathrm{C}}(\tfrac{n}{2}, m) \mathrm{e}^{\,\mathrm{j}\pi l n'/N}
\end{aligned}$$

$$= \frac{1}{4N^2} \sum_{m=0}^{2N-1} \sum_{n=0}^{2N-1} K_x^{\mathrm{C}}(\tfrac{n}{2}, m) \sum_{n'=0}^{2N-1} \bar{K}_y^{\mathrm{C}}(\tfrac{n}{2}, m) \sum_{l=0}^{2N-1} \mathrm{e}^{-\mathrm{j}\pi l(n-n')/N}$$

and because $\sum_{l=0}^{2N-1} \exp[-\mathrm{j}\pi l(n-n')/N] = 2N\delta(n-n')$,

$$\begin{aligned}
&= \frac{1}{2N} \sum_{m=0}^{2N-1} \sum_{n=0}^{2N-1} K_x^{\mathrm{C}}(\tfrac{n}{2}, m) \bar{K}_y^{\mathrm{C}}(\tfrac{n}{2}, m) \\
&= \frac{1}{2N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x(n+m)\bar{x}(n-m)\bar{y}(n+m)y(n-m) \\
&\quad + \frac{1}{2N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x(n+m+1)\bar{x}(n-m)\bar{y}(n+m+1)y(n-m).
\end{aligned}$$

By substituting $a = n - m$ in the preceding equation we now have

$$\begin{aligned}
&= \frac{1}{2N} \sum_{a=0}^{2N-1} \sum_{m=0}^{N-1} x(a+2m)\bar{x}(a)\bar{y}(a+2m)y(a) \\
&\quad + \frac{1}{2N} \sum_{a=0}^{2N-1} \sum_{m=0}^{N-1} x(a+2m+1)\bar{x}(a)\bar{y}(a+2m+1)y(a) \\
&= \frac{1}{2N} \sum_{a=0}^{2N-1} \bar{x}(a)y(a) \left[\sum_{m=0}^{N-1} x(a+2m)\bar{y}(a+2m) \right. \\
&\quad \left. + \sum_{m=0}^{N-1} x(a+2m+1)\bar{y}(a+2m+1) \right] \\
&= \frac{1}{2N} \sum_{a=0}^{2N-1} \bar{x}(a)y(a) \sum_{m=0}^{2N-1} x(a+m)\bar{y}(a+m) \\
&= \frac{1}{2N} \left| \sum_{a=0}^{2N-1} x(a)\bar{y}(a) \right|^2 .
\end{aligned}$$

Thus,

$$2N \sum_{n=0}^{2N-1} \sum_{k=0}^{2N-1} \rho_x^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \bar{\rho}_y^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) = \left| \sum_{n=0}^{N-1} x(n)\bar{y}(n) \right|^2$$

Summing over half the frequency extent $k = 0, 1, \ldots, N-1$ is proportional to summing over the full frequency extent $k = 0, 1, \ldots, 2N-1$,

$$\sum_{n=0}^{2N-1} \sum_{k=0}^{2N-1} \rho_x^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \bar{\rho}_y^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) = 2 \sum_{n=0}^{2N-1} \sum_{k=0}^{N-1} \rho_x^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \bar{\rho}_y^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N})$$

therefore

$$4N \sum_{n=0}^{2N-1} \sum_{k=0}^{N-1} \rho_x^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \bar{\rho}_y^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) = \left| \sum_{n=0}^{N-1} x(n)\bar{y}(n) \right|^2$$

thus proving the relation in (25).

P9) Signal recovery: to prove,

$$2 \sum_{k=0}^{N-1} \rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \mathrm{e}^{\mathrm{j}\pi kn/N} = z(n)\bar{z}(0)$$

when

$$G^{\mathrm{C}}(\tfrac{n}{2}, m) = \delta(n). \tag{28}$$

Proof: Expand as follows:

$$\begin{aligned}
&\sum_{k=0}^{2N-1} \rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \mathrm{e}^{\mathrm{j}\pi kn/N} \\
&\quad = \frac{1}{2N} \sum_{k=0}^{2N-1} \sum_{m=0}^{2N-1} \sum_{p=0}^{2N-1} K^{\mathrm{C}}(\tfrac{n-p}{2}, m) G^{\mathrm{C}}(\tfrac{p}{2}, m) \mathrm{e}^{-\mathrm{j}\pi mk/N} \mathrm{e}^{\mathrm{j}\pi kn/N} \\
&\quad = \frac{1}{2N} \sum_{m=0}^{2N-1} \sum_{p=0}^{2N-1} K^{\mathrm{C}}(\tfrac{n-p}{2}, m) G^{\mathrm{C}}(\tfrac{p}{2}, m) \sum_{k=0}^{2N-1} \mathrm{e}^{-\mathrm{j}\pi(m-n)k/N} \\
&\quad = \sum_{p=0}^{2N-1} K^{\mathrm{C}}(\tfrac{n-p}{2}, n) G^{\mathrm{C}}(\tfrac{p}{2}, n)
\end{aligned}$$

as $\sum_{k=0}^{2N-1} \exp\left[-\mathrm{j}\pi(m-n)k/N\right] = 2N\delta(m-n)$. Using the kernel constraint in (28), and the definition of time–lag signal function K^{C} in (2) and (3),

$$\begin{aligned}
\sum_{k=0}^{2N-1} \rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \mathrm{e}^{\mathrm{j}\pi kn/N} &= K^{\mathrm{C}}(\tfrac{n}{2}, n) \\
&= z(n)\bar{z}(0)
\end{aligned}$$

and as

$$\sum_{k=0}^{2N-1} \rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \mathrm{e}^{\mathrm{j}\pi kn/N} = 2 \sum_{k=0}^{N-1} \rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \mathrm{e}^{\mathrm{j}\pi kn/N}$$

then

$$2 \sum_{k=0}^{N-1} \rho^{\mathrm{C}}(\tfrac{n}{2}, \tfrac{k}{2N}) \mathrm{e}^{\mathrm{j}\pi kn/N} = z(n)\bar{z}(0)$$

which concludes the proof.

References

[1] J. M. O' Toole, M. Mesbah, and B. Boashash, "Improved discrete definition of quadratic time–frequency distributions," *IEEE Trans. Signal Processing*, vol. submitted for publication, 2009.

[2] ——, "A new discrete analytic signal for reducing aliasing in the discrete Wigner–Ville distribution," *IEEE Trans. Signal Processing*, vol. 56, no. 11, pp. 5427–5434, Nov. 2008.

[3] M. A. Poletti, "The development of a discrete transform for the Wigner distribution and ambiguity function," *J. Acoust. Soc. Am.*, vol. 84, no. 1, pp. 238–252, July 1988.

[4] Y. Zhao, L. E. Atlas, and R. J. Marks II, "The use of cone-shaped kernels for generalized time–frequency distributions of nonstationary signals," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 38, no. 7, pp. 1084–1091, 1990.

[5] T. Claasen and W. Mecklenbräuker, "The Wigner distribution—a tool for time-frequency signal analysis. Part III: Relations with other time-frequency signal transformations," *Philips J. Res*, vol. 35, pp. 372–389, 1980.

[6] B. Boashash, "Time–frequency signal analysis," in *Advances in Spectrum Estimation*, S. Haykin, Ed. Englewood Cliffs, NJ: Prentice-Hall, 1991, ch. 9, pp. 418–517.