1 Models and Entailment in Propositional Logic

1.1 Page 279, Exercise 7.1

"Build the complete model table and show both entailments using model checking."

32 worlds? Ungh that is way too much work.

1.2 Page 280, Exercise 7.4

a. $False \models True$

Yes. The LHS (false) is true in no models, while the RHS (True) is true in *all* models. So the RHS is true in all the models in which the LHS is true which is what entailment requires.

b. $True \models False$

Nope.

c. $(A \land B) \models (A \Leftrightarrow B)$

Yes. Because: see the exercise description.

d. $A \Leftrightarrow B \models A \lor B$

A	B	$A \Leftrightarrow B$	$A \lor B$
0	0	1	0
0	1	0	1
1	0	0	1
1	1	1	1

Nope. See the model in which $\neg A \land \neg B$.

e. $A \Leftrightarrow B \models \neg A \lor B$

A	B	$A \Leftrightarrow B$	$\neg A \lor B$
0	0	1	1
0	1	0	1
1	0	0	0
1	1	1	1

Who would've thought it. Yes.

f. $(A \lor B) \land (\neg C \lor \neg D \lor E) \models (A \lor B \lor C) \land (B \land C \land D \Rightarrow E)$

$$\begin{aligned}
(B \land C \land D \Rightarrow E) &\equiv \neg(B \land C \land D) \lor E && \text{(implication elimination)}\\
\neg(B \land C \land D) &\equiv (\neg B \lor \neg C \lor \neg D) && \text{(De Morgan)}\\
(A \lor B \lor C) \land (B \land C \land D \Rightarrow E) &\equiv (A \lor B \lor C) \land (\neg B \lor \neg C \lor \neg D \lor E) &&\\
&\;\;\ldots &&\\
(A \lor B) \land (\neg C \lor \neg D \lor E) &\models (A \lor B \lor C) \land (\neg B \lor \neg C \lor \neg D \lor E) &&
\end{aligned}$$

Yes. Because $(A \lor B) \models (A \lor B \lor C)$ and $(\neg C \lor \neg D \lor E) \models (\neg B \lor \neg C \lor \neg D \lor E)$

g. $(A \lor B) \land (\neg C \lor \neg D \lor E) \models (A \lor B) \land (\neg D \lor E)$

Uhh... No! Because $(\neg C \lor \neg D \lor E) \not\models (\neg D \lor E)$.

In a model where we have $\neg C$, D and $\neg E$, $(\neg C \lor \neg D \lor E)$ will be true (because of $\neg C$), whle $(\neg D \lor E)$ will not be true as neither $\neg D$ or E.

h. $(A \lor B) \land \neg (A \Rightarrow B)$ **is satisfiable.**

Now well I don't know what "is satisfiable" means.

Maybe it means that it's not a contradiction?

Yeah, I like the sound of that. It's simple enough for me to handle.

Aaand yes, it is satisfiable: $\neg (A \Rightarrow B)$ is true when A is false and B is true, in this model $A \lor B$ is also true so the entire thing is true.

i. $(A \land B) \Rightarrow C \models (A \Rightarrow C) \lor (B \Rightarrow C)$

$$\begin{aligned}
(A \Rightarrow C) \lor (B \Rightarrow C) &\equiv (\neg A \lor C) \lor (\neg B \lor C) && \text{(implication elimination)}\\
(\neg A \lor C) \lor (\neg B \lor C) &\equiv \neg A \lor C \lor \neg B \lor C \\
\neg A \lor C \lor \neg B \lor C &\equiv \neg A \lor \neg B \lor C && (C \lor C \equiv C)
\end{aligned}$$

$$\begin{aligned}
((A \land B) \Rightarrow C) &\equiv \neg (A \land B) \lor C && \text{(implication elimination)}\\
\neg (A \land B) \lor C &\equiv \neg A \lor \neg B \lor C && \text{(De Morgan)}
\end{aligned}$$

So...

$$\neg A \lor \neg B \lor C \models \neg A \lor \neg B \lor C$$

Yes?

j. $(C \lor (\neg A \land \neg B)) \equiv ((A \Rightarrow C) \lor (B \Rightarrow C))$

$$\begin{aligned}
(C \lor (\neg A \land \neg B)) &\equiv ((C \lor \neg A) \land (C \lor \neg B)) && \text{(distributivity of } \lor \text{ over } \land)\\
((C \lor \neg A) \land (C \lor \neg B)) &\equiv ((\neg A \lor C) \land (\neg B \lor C)) && \text{(commutativity of } \lor)
\end{aligned}$$

$$(A \Rightarrow C) \lor (B \Rightarrow C) \equiv (\neg A \lor C) \lor (\neg B \lor C) \qquad \text{(implication elimination)}$$

So...

$$((C \lor \neg A) \land (C \lor \neg B)) \equiv (\neg A \lor C) \lor (\neg B \lor C)$$

So: $((C \lor \neg A) \land (C \lor \neg B)) \equiv (\neg A \lor C) \lor (\neg B \lor C)$?

Answer: No. Because if $\neg C$, B, $\neg A$ then the LHS will be false, but the RHS will be true.

A	B	C	$(\neg A \land \neg B)$	$(A \Rightarrow C)$	$(B \Rightarrow C)$	$(A \Rightarrow C) \lor (B \Rightarrow C)$	$(C \lor (\neg A \land \neg B))$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	0	1	0	1	0
0	1	1	0	1	1	1	1
1	0	0	0	0	1	1	0
1	0	1	0	1	1	1	1
1	1	0	0	0	0	0	0
1	1	1	1	1	1	1	1

k. $(A \Leftrightarrow B) \wedge (\neg A \vee B)$ **is satisfiable.**

Hey, wait, this looks a lot like e.

Oh look I already made a truth table for this. For your ease-of-viewing, I've copied it here:

A	B	$A \Leftrightarrow B$	$\neg A \vee B$
0	0	1	1
0	1	0	1
1	0	0	0
1	1	1	1

From this we see that, yes, it is satisfiable.

l. $(A \Leftrightarrow B) \Leftrightarrow C$ **has the same number of models as** $(A \Leftrightarrow B)$ **for any fixed set of proposition symbols that include** A**,** B**,** C**.**

Uhm, what? I just made a truth table. 'Cause I don't know what is being asked of me.

A	B	C	$A \Leftrightarrow B$	$(A \Leftrightarrow B) \Leftrightarrow C$
0	0	0	1	0
0	0	1	1	1
0	1	0	0	1
0	1	1	0	0
1	0	0	0	1
1	0	1	0	0
1	1	0	1	0
1	1	1	1	1

1.3 Page 281, Exercise 7.7

Consider a vocabulary with only four propositions, A, B, C, and D. How many models are there for the following sentences?

a. $B \vee C$

From page 240, "Model":

"[...] models are mathematical abstractions, each of which simply fixes the truth or falsehood of every relevant sentence."

Also:

"If a sentence α is true in model m, we say that m satisfies α or sometimes m is a model of α."

So, is the exercise asking me how many models there are in which α is true?

Because that is what I am going to answer.

There exists 2^n unique possible models for a sentence sentence involving n unique propositions.

$A \vee B$ has $2^2 = 4$ unique models, three of which are models of $A \vee B$. Since we have 16 unique models in total for our vocabulary, each of the four unique models for $A \vee B$ are repeated four times. This means that there are $\frac{3}{4} \cdot 2^4 = 12$ models of $A \vee B$.

b. $\neg A \vee \neg B \vee \neg C \vee \neg D$

$2^4 - 1 = 15$ (the only model in which it is false is the one in which $A \wedge B \wedge C \wedge D$ is true)

c. $(A \Rightarrow B) \land A \land \neg B \land C \land D$

Well okay so first off this can only be true if A and $\neg B$ and C and D, giving us one possible model. In this, however, $A \Rightarrow B$ is false as A is true while B is false.

So zero. ZERO.

1.4 Page 281, Exercise 7.10

(In this exercise, *neither* means *satisfiable but not valid.*)

"Decide whether each of the following sentences is valid, unsatisfiable, or neither. Verify your decisions using truth tables or the equivalence rules of Figure 7.11.

DOES VALID MEAN THE SAME THING AS TAUTOLOGY? YES IT DOES. I CHECKED WIKIPEDIA.

a. $Smoke \Rightarrow Smoke$

Oh come on now.

$$A \Rightarrow A \equiv \neg A \lor A \qquad \text{(implication elimination)}$$

$\neg A \lor A$ is a tautology.

So... it's valid.

b. $Smoke \Rightarrow Fire$

A	B	$A \Rightarrow B$
0	0	1
0	1	0
1	0	1
1	1	1

It is... *neither.*

c. $(Smoke \Rightarrow Fire) \Rightarrow (\neg Smoke \Rightarrow \neg Fire)$

A	B	$A \Rightarrow B$	$\neg A \Rightarrow \neg B$
0	0	1	1
0	1	0	1
1	0	1	0
1	1	1	1

Again, *neither.*

d. $Smoke \lor Fire \lor \neg Fire$

It's valid because $(Fire \lor \neg Fire)$ is valid ($True \lor A$ is valid).

e. $((Smoke \land Heat) \Rightarrow Fire) \Leftrightarrow ((Smoke \Rightarrow Fire) \lor (Heat \Rightarrow Fire))$

$$((A \land C) \Rightarrow B) \Leftrightarrow ((A \Rightarrow B) \lor (C \Rightarrow B))$$

$$\begin{aligned} ((A \land C) \Rightarrow B) &\equiv \neg (A \land C) \lor B && \text{(implication elimination)} \\ \neg (A \land C) \lor B &\equiv \neg A \lor \neg C \lor B && \text{(De Morgan)} \end{aligned}$$

$$\begin{aligned} ((A \Rightarrow B) \lor (C \Rightarrow B)) &\equiv (\neg A \lor B) \lor (\neg C \lor B) && \text{(implication elimination)} \\ (\neg A \lor B) \lor (\neg C \lor B) &\equiv \neg A \lor \neg C \lor B \end{aligned}$$

We see that both sides of the $\Leftrightarrow$ are equivalent. So it's valid.
(Doubt it? Check out Figure 7.8 on page 246)

f. *Big* $\vee$ *Dumb* $\vee$ (*Big* $\Rightarrow$ *Dumb*)

$$\begin{aligned} D \vee E \vee (D \Rightarrow E) &\equiv D \vee E \vee \neg D \vee E && \text{(implication elimination)} \\ D \vee E \vee \neg D \vee E &\equiv D \vee \neg D \vee E \\ D \vee \neg D \vee E &\equiv True \vee E \end{aligned}$$

Valid.

g. (*Big* $\wedge$ *Dumb*) $\vee$ $\neg$*Dumb*

D	E	$D \wedge E$	$(D \wedge E) \vee \neg E$
0	0	0	1
0	1	0	0
1	0	0	1
1	1	1	1

Neither.

1.5 5. At this point, I notice I didn't have to do *all* the parts of all the questions.

"Consider a logical knowledge base with 100 variables, $A_1, ..., A_{100}$. This will have $Q = 2^{100}$ possible models. For each logical sentence below, give the number of models that satisfy it. Feel free to express your answer as a fraction of Q, or to use other symbols to represent large numbers.

a. $A_1 \vee A_{73}$

Again, a sentence consisting of n unique variables has 2^n unique possible models. For a KB with z unique variables, a sentence S which includes d of these has 2^d unique possible models. That is to say, there are only 2^d unique configurations of the truth values of the variables of S. The remaining 2^{z-d} possible models are, with regards to S, not unique. If there are x unique models that satisfy S (out of the 2^d unique models), then there are $x \cdot \left(2^{z-d}\right) = \dfrac{x \cdot 2^z}{2^d}$ models that satisfy S in the entire KB.

So!
$A_1 \vee A_{73}$ is satisfied by 3 unique models.
$3 \cdot \left(2^{100-2}\right)$
Alternatively we could say that $A_1 \vee A_{73}$ is satisfied by a three fourths of all the possible models.

$$\frac{3}{4} \cdot 2^{100} = 3 \cdot 2^{98}$$

I have, by the way, no idea if this is correct.

b. $A_7 \vee (A_{19} \wedge A_{33})$

Oooh!
For $S_{b1} = A_{19} \wedge A_{33}$ there exists four unique models. S_{b1} is true in one of these (the one in which they both are true).
$S_{b2} = A_7$ is true in half of the models it is featured in.
$S_{b3} = S_{b2} \vee S_{b1}$ is true in five out of eight models (that one model where $\neg A_7 \wedge S_{b1}$ is true, and those other four in which A_7 is true)

$$\frac{5}{2^3} \cdot 2^{100} = 5 \cdot 2^{97}$$

c. $A_{11} \rightarrow A_{22}$

$$3 \cdot 2^{98}$$

2 Resolution in Propositional Logic

2.1 Convert each of the following sentences to Conjunctive Normal Form (CNF).

Right, so CNF is a sentence consisting only of $\vee$ and $\wedge$ (where distributy of $\vee$ over $\wedge$ is applied wherever possible) and where $\neg$ only appears in literals. OK!

a. $A \wedge B \wedge C$

Hah! Trick question, it's already in CNF. (A, B and C are all literals, and there's nothing anywhere as far as I can see about a clause having to contain more than one literal).

b. $A \vee B \vee C$

Again, it's already in CNF because $(A \vee B \vee C)$ is a valid clause and a CNF-sentence doesn't need to contain more than one clause.

c. $A \Rightarrow (B \vee C)$

$$A \Rightarrow (B \vee C) \equiv \neg A \vee (B \vee C) \equiv (\neg A \vee B \vee C)$$

d. $(A \vee \neg C) \Rightarrow B$

$$\begin{aligned} ((A \vee \neg C) \Rightarrow B) &\equiv \neg(A \vee \neg C) \vee B && \text{(implication elimination)} \\ \neg(A \vee \neg C) \vee B &\equiv (\neg A \wedge C) \vee B && \text{(De Morgan)} \\ (\neg A \wedge C) \vee B &\equiv ((B \vee \neg A) \wedge (B \vee C)) && \text{(distributivity of } \vee \text{ over } \wedge) \end{aligned}$$

2.2 Consider the following Knowledge Base (KB):

- $(A \vee \neg B) \Rightarrow \neg C$
- $D \wedge E \Rightarrow C$
- $A \wedge D$

Use resolution to show that $KB \models \neg E$
(page 254)

$$\begin{aligned} ((A \vee \neg B) \Rightarrow \neg C) &\equiv \neg(A \vee \neg B) \wedge \neg C && \text{(implication elimination)} \\ \neg(A \vee \neg B) \wedge \neg C &\equiv (\neg A \wedge B) \wedge \neg C && \text{(De Morgan, double-negation elimination)} \end{aligned} \tag{1}$$

$\neg(A \vee \neg B) = (\neg A \wedge B)$
So:

$$((A \vee \neg B) \Rightarrow \neg C) \wedge ((D \wedge E) \Rightarrow C) \wedge (A \wedge D)$$

You know this is really hard for me as I've never before seen a KB displayed as a bulleted list. Or is each item a new KB?

3 Representations in First-Order Logic

3.1 Page 316, Exercise 8.9

Consider a vocabulary with the following symbols:
Occupation(p, o): Predicate. Person p has occupation o.
Customer $(p1, p2)$: Predicate. Person $p1$ is a customer of person $p2$.
Boss $(p1, p2)$: Predicate. Person $p1$ is a boss of person $p2$.
Doctor, Surgeon, Lawyer, Actor: Constants denoting occupations.
Emily, Joe: Constants denoting people.
Use these symbols to write the following assertions in first-order logic:

a. Emily is either a surgeon or a lawyer.

$$Occupation(Emily,\ Surgeon) \lor Occupation(Emily,\ Lawyer)$$

b. Joe is an actor, but he also holds another job.

$$Occupation(Joe,\ Actor) \land Occupation(Joe,\ Doctor \lor Surgeon \lor Lawyer)$$

c. All surgeons are doctors

$$\forall p\,(Occupation(p,\ Surgeon) \Rightarrow Occupation(p,\ Doctor))$$

3.2 Page 319, Exercise 8.21

Arithmetic assertions can be written in first-order logic with the predicate symbol $<$, the function symbols $+$ and $\times$, and the constant symbols 0 and 1. Additional predicates can also be defined with biconditionals.

a. Represent the property "x is an even number."

$Even(x) \Leftrightarrow (x \bmod 2 = 0)$

b. Represent the property "x is prime."

$Prime(x) \Leftrightarrow \neg\exists\,(y \in \mathbb{Z})\ ((x \div y) \in \mathbb{Z}) \land 2 < y < x$

uh, basically what I'm trying to say is: if x is prime then there is no $y \in \mathbb{Z}$, $2 < y < x$, for which $x \div y =$ some integer

c. Goldbach's conjecture is the conjecture (unproven as of yet) that every even number is equal to the sum of two primes. Represent this conjecture as a logical sentence.

$Even(x) \Leftrightarrow \exists\ y,\ z\ Prime(y) \land Prime(z) \land (y + z = x)$

3.3 Page 319, Exercise 8.23

Write in first-order logic the assertion that every key and at least one of every pair of socks will eventually be lost forever, using only the following vocabulary: $Key\,(x)$, x is a key; $Sock\,(x)$, x is a sock; $Pair\,(x, y)$, x and y are a pair; Now, the current time; $Before\,(t_1, t_2)$, time t_1 comes before t_2; $\mathrm{Lost}(x, t)$, object x is lost at time t.

$\forall x\ Key\,(x) \Rightarrow \exists t\ Lost\,(x, t)$

Wait. $\forall x\ \exists y\ Pair(x, y)$?

Because if not then ahh fuck it I'm stumped.

3.4 Page 319, Exercise 8.24

Translate into first-order logic the sentence: "Everyone's DNA is unique and is derived from their parents' DNA." You must specify the precise intended meaning of your vocabulary terms. (*Hint:* Do not use the predicate $Unique(x)$, since uniqueness is not really a property of an object itself!)
$\forall x\ DNA\,(x) \implies \neg\exists y\ DNA\,(y) \wedge Equal\,(x,y) \wedge Derived\,(DNA\,(Parent\,(x)))$

4 Resolution in First-Order Logic

4.1 Find the unifier (θ) - if possible - for each pair of atomic sentences. Here, Owner, Horse and Rides are predicates while FastestHorse is a function that maps a person to the name of heir fastest horse:

a. Horse(x) ... Horse(Rocky) Answer: $\theta = \{x/Rocky\}$

b. Owner(Leo,Rocky) ... Owner(x,y)

Answer: $\theta = \{x/Leo, y/Rocky\}$

c. Owner(Leo,x) ... Owner(y, Rocky)

Answer: $\theta = \{y/Leo, x/Rocky\}$

d. Owner(Leo,x) ... Rides(Leo,Rocky)

Answer: $\theta = \{x/Rocky\}$

4.2 Use resolution to prove Green(Linn) given the information below. You must first convert each sentence into CNF. Feel free to show only the applications of the resolution rule that lead to the desired conclusion. For each application of the resolution rule, show the unification bindings, θ.

- $Hybrid\,(Prius)$
- Drives($Linn$, $Prius$)
- $\forall x : Green\,(x) \Leftrightarrow Bikes\,(x) \vee [\exists y : Drives\,(x,y) \wedge Hybrid\,(y)]$

$Hybrid\,(Prius)$ ... $Hybrid\,(y)$
$\theta = \{y/Prius\}$
Drives($Linn$, $Prius$) ... $Drives\,(x, Prius)$
$\theta = \{x/Linn,\ y/Prius\}$
$Green\,(x)$ ... $Drives\,(Linn, Prius) \wedge Hybrid\,(Prius)$
$\theta = \{x/Linn,\ y/Prius\}$
Did I mention that I have no idea what I am doing?