Tao Analysis Solutions

Segun

21 de febrero de 2019

Pg 24 Prove that the sum of two natural numbers is again a natural number?

Solution: Let n and m be natural numbers. Using induction on n and keeping m fixed,we have the base case to be $0 + m := m$ which is a natural number as desired (by definition of addition and **Axiom 2.1**).

Now we assume inductively $n + m$ is a natural number. We have to show that $(n++) + m$ is also a natural number. But $(n++) + m = (n + m) ++$ by definition of addition and **Axiom 2.2**. However, $n + m$ is a natural number according to the inductive hypothesis and therefore $(n + m) ++$ is also a natural number by **Axiom 2.2**. $\square$

Pg 26 Prove that $n ++ = n + 1$

Solution 1: Let n be a natural number, then $n ++ = (n ++) + 0 = (n + 0) ++ = n + (0 ++) = n + 1$ according to **Lemma 2.2.2**, definition of addition, **Lemma 2.2.3** and definition of increment respectively. $\square$

Solution 2: Using induction on n, we have to show that $0 ++ = 0 + 1$ for the base case. However, $0 ++ = 1 = 0 + 1$ by definitions of increment, and addition respectively and so the base case is done.

Now suppose inductively that $n ++ = n + 1$, we have to show that $(n ++) ++ = (n ++) + 1$.

From the RHS, $(n++)+1 = (n+1)++ = (n++)++$ by definition of addition and the inductive hypothesis repectively. $\square$

Exercise 2.2.1. (Addition is associative). Prove that for any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.

Solution: Using induction on a and keeping a and c fixed. For the base case, we want to show that $(0 + b) + c = 0 + (b + c)$. However, $(0 + b) + c = b + c = 0 + (b + c)$ by the definition of addition.

Now suppose inductively that $(a+b)+c = a+(b+c)$, we have to show that $((a++)+b)+c = (a++)+(b+c)$.

From the RHS, we have that $(a++)+(b+c) = (a+(b+c))++ = ((a+b)+c)++$ by the definition of addition and the inductive hypothesis respectively. Furthermore, $((a+b)+c)++ = (((a+b)++)+c) = ((a++)+b)+c$ by definition of addition. $\square$

Exercise 2.2.2. Let a be a positive number. Then there exists exactly one natural number b such that $b ++ = a$.

Solution: First we prove the existence of a natural number b such that $b{++} = a$ for $a \in \mathbb{N}^+$ using induction on a. For the base case $a = 1$, we want to show that there exists a natural number b such that $b{++} = 1$. Clearly, $0{++} = 1$ so $b = 0$.

Now suppose inductively that for any positive number a, there exists a natural number b such that $b{++} = a$.

We have to show that there exists a natural number q such that $q ++ = a ++$. But $a ++ = (b ++) ++$ by the inductive hypothesis, so that $q{++} = (b{++}){++}$ and by **Axiom 2.4**, $q = b{++}$. So that completes the induction and the proof existence.

Now we have to prove the uniqueness of the natural number b such that $b{++} = a$ for $a \in \mathbb{N}^+$. Suppose for the sake of contradiction that there exist b, $c \in \mathbb{N}$ with $b \neq c$ such that $b{++} = a$ and $c{++} = a$. Then $b{++} = c{++}$ and by **Axiom 2.4** $b = c$ which is a contradiction. Consequently, there exists exactly one natural number b such that $b{++} = a$ for $a \in \mathbb{N}^+$. $\square$

Exercise 2.2.3. Basic properties of order for natural numbers). Let a, b, c be natural numbers. Then *Solution:*

1. (Order is reflexive) $a \geq a$.

   $a = a + 0$ by **Lemma 2.2.3** , therefore, $a \geq a$ by definition. $\square$

2. (Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$.

   By definition, $a \geq b$ implies that $a = b + d$ and $b \geq c$ implies $b = c + e$, for some d, $e \in \mathbb{N}$. Therefore, $a = b + d = (c + e) + d = c + (e + d) \geq c$ by ordering of natural numbers and associativity of addition. $\square$.

3. (Order is anti-symmetric) If $a \geq b$ and $b \geq a$, then $a = b$.

   Since $a \geq b$ and $b \geq a$, we have that $a = b + c$ and $b = a + d$ for some c, $d \in \mathbb{N}$. Thus $a = a + c + d$ which implies that $c + d = 0$ by **Lemma 2.2.2** and that further implies $c = 0$ and $d = 0$ by **Corollary 2.2.9**. Consequently, $a = b$. $\square$

4. (Addition preserves order) $a \geq b$ if and only if $a + c \geq b + c$.

   Because $a \geq b$, $a = b + d$ for some $d \in \mathbb{N}$. Hence $a + c = (b + d) + c = (b + c) + d \geq b + c$ by the cancellation law, associativity of addition and ordering of natural numbers respectively.

   Likewise, $a + c \geq b + c$ implies $a + c = (b + c) + e$, for some $e \in \mathbb{N}$. Consequently, $a + c = (b + e) + c$ by associativity of addition. By cancellation law, $a = b + e$ so $a \geq b$. $\square$

5. $a < b$ if and only if $a{++} \leq b$

   Proof. Since $a < b$, then $b = a + n$, for some $n \in \mathbb{N}$ and $b \neq a$. Consequently, $n \neq 0$ by **Lemma 2.2.2**. Then there exists a number say $m \in \mathbb{N}$ such that $m{++} = n$ by **Lemma 2.2.10**. So $b = a + n = a + m{++} = (a{++}) + m$ by **Lemma 2.2.3**. So $a{++} \leq b$.

   Likewise $a{++} \leq b$ implies $b = (a{++}) + n$ for some $n \in \mathbb{N}$. Consequently, $b = a + (n{++})$ by **Lemma 2.2.3** and by **Definition 2.2.1** (Addition of natural numbers). But we know that $n{++} \neq 0$ by **Axiom 2.3** and by implication $b \neq a$. So $a < b$. $\square$

6. $a < b$ if and only if $b = a + d$ for some positive number d.

   Proof. If $a < b$, then $b = a + d$ for some $d \in \mathbb{N}$ and $b \neq a$ by definition. We need to show that $d \neq 0$. Now suppose for the sake of contradiction that $d = 0$. Then $b = a + d$ implies $a = b$ which is a contradition so $d \neq 0$.

   Likewise, $b = a + d$, $d \neq 0$ implies $a \leq b$ by **Definition 2.2.11** (Ordering of the natural numbers). We need to show that $b \neq a$. Suppose for the sake of contradiction that $b = a$, then $b = a + d$ implies $d = 0$ by **Lemma 2.2.2** which is a contradiction. Hence $b \neq a$ and $a \leq b$ so $a < b$. $\square$

Exercise 2.2.4. Justify the three statements marked (why?) in the proof of **Proposition 2.2.13**.

1. $0 \leq b$ for all $b \in \mathbb{N}$.

   Proof. Since $b \in \mathbb{N}$, then $b = 0 + b$ by **Definition 2.2.1** (Addition of natural numbers) so $0 \leq b$ by **Definition 2.2.11** (Ordering of the natural numbers).

2. If $a > b$, then $a{++} > b$

   Proof. Because $a{++} > a$ and $a > b$. Then by **Proposition 2.2.12** (Transitivity of order), $a{++} > b$.

3. If $a = b$, the $a{++} = b + d$

   Proof. Because $a{++} > a$ implies $a{++} > b$ since $a = b$.

Exercise 2.2.5. Prove **Proposition 2.2.14**. (Strong principle of induction). Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m0$.

Proof 1: (Informal) Let $Q(n)$ be the property that $P(m)$ is true for all $m_o \leq m < n$. Then if $n \leq m_0$, then $Q(n)$ is vacuously true. If $n = m_0{+}{+}$, then $Q(m_0{+}{+})$ is true because $P(m)$ is true for all natural numbers $m_0 \leq m < m_0{+}{+}$ (since $P(m_0)$ is vacuously true) and by definition of P, $P(m_0{+}{+})$ is also true. Likewise, if $n = (m_0{+}{+}) + +$, then $Q((m_0{+}{+}) + +)$ is true becuase $P(m)$ is true for all $m_0 \leq m < (m_0{+}{+}) + +$ (since $P(m_0)$ and $P(m_0{+}{+})$ are true) and by definition of P, $P((m_0{+}{+}) + +)$ is also true. And so it goes on and on.

Proof: Let $n \in \mathbb{N}$ and let $Q(n)$ be the property that $P(m)$ is true for all $m_0 \leq m < n$ for $n \geq m_0$. Using induction on n, for the base case $n = 0$, we want to show that $Q(0)$ is true. However, we know that $0 \leq m_0\ \forall\ m_0 \in \mathbb{N}$. Thus, either $0 = m_0$ or $0 < m_0$ and so we split into cases. If $n = 0 < m_0$, the statement $P(m)\ \forall\ m_0 \leq m < n$ is vacuously true (since the hypothesis applies for $n \geq m_0$) and thus $Q(0)$ is true in this case. For the second case, if $n = 0 = m_0$, then the statement $P(m)\ \forall\ m_0 \leq m < n$ is also vacuously true since there is no $m' \in \mathbb{N}$ such that $0 \leq m' < 0$. Hence, $Q(0)$ is true for this case and that completes the base case of the induction.

Now suppose inductively that for some $n \geq m_0$, $Q(n)$ is true, i.e $P(m)\ \forall\ m_0 \leq m < n$ is true. We need to show that $Q(n{+}{+})$ is true.

By the definition of P in the hypothesis, $P(n)$ is also true (because $Q(n)$ is true). Since $n < n + +$, then $P(m)\ \forall\ m_0 \leq m \leq n < n + +$ is true so $P(m)\ \forall\ m_0 \leq m < n + +$ is true which in turn implies that $Q(n{+}{+})$ is true. Which closes the induction and hence we can conclude that $Q(n)\ \forall n$ is true.

However, $Q(n)$ true implies $P(m)\ \forall\ m_0 \leq m < n$ is true for all $n \geq m_0$ and by the definition of P, $P(n)$ is true for all $n \geq m_0$ which concludes the proof. $\square$

Exercise 2.2.6. Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{+}{+})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leq n$; this is known as the principle of *backwards induction.* (Hint: apply induction to the variable n.)

Proof: Let $n \in \mathbb{N}$. Using induction on n, for the base case $n = 0$, we need to show that $P(m)$ is true $\forall\ m \leq 0$. But only $0 \leq 0$ so we just need to show that $P(0)$ is true. Since $P(n)$ is true from the hypothesis, $P(0)$ is true and that completes the base case.

Suppose inductively that the principle if true for n, i.e P is such that $P(n)$ is true, and whenever $P(m{+}{+})$ is true, $P(m)$ is true $\forall m \leq n$. We have to show the principle is true for $n{+}{+}$ i.e we need to show that $P(m)$ is true $\forall\ m \leq n + +$ given that $P(n{+}{+})$ is true and given that whenever $P(m{+}{+})$ is true, $P(m)$ is true.

Since $P(n{+}{+})$ is true, then $P(n)$ is also true. So we have to show that $P(m)$ is true $m < n$. But from the inductive hypothesis, $P(m)$ is true $\forall\ m \leq n$ and that completes the inductive hypothesis. $\square$

Exercise 2.3.1. Prove Lemma 2.3.2. (Multiplication is commutative). Let n, m be natural numbers. Then $n \times m = m \times n$.

Exercise 2.3.2. Prove Lemma 2.3.3. (Positive natural numbers have no zero divisors). Let n, m be natural numbers. Then $n \times m = 0$ if and only if at least one of n, m is equal to zero. In particular, if n and m are both positive, then nm is also positive.

Exercise 2.3.3. Prove Proposition 2.3.5. (Multiplication is associative). For any natural numbers a, b, c, we have $(a \times b) \times c = a \times (b \times c)$.

Exercise 2.3.4. Prove the identity $(a + b)^2 = a^2 + 2ab + b^2$ for all natural numbers a, b.

Exercise 2.3.5. Prove Proposition 2.3.9. (Euclidean algorithm). Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$. (*Hint*: fix q and induct on n.)