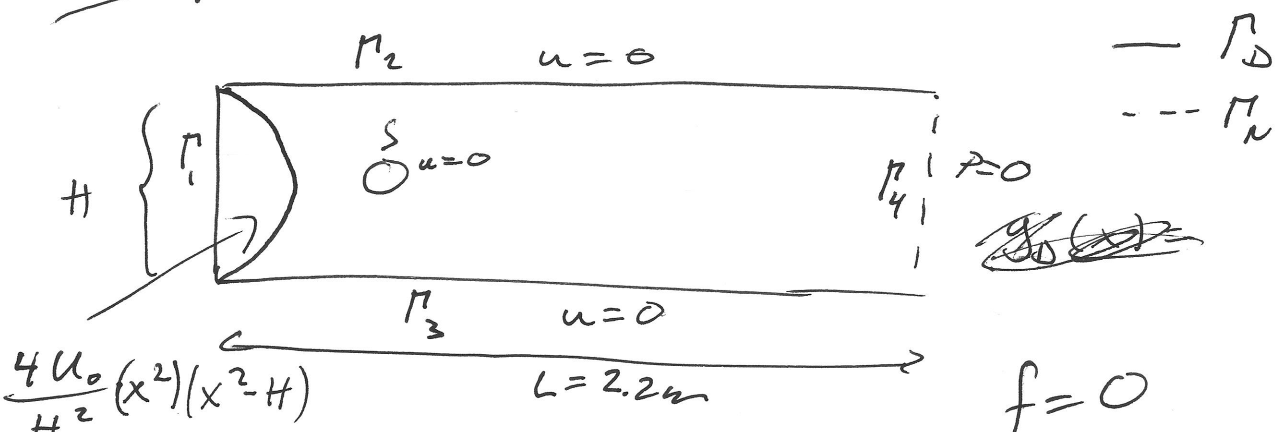
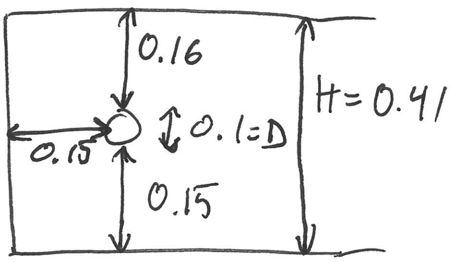
Navier-Stokes:

$$\begin{cases} \rho\left(\frac{\partial u}{\partial t} + u\cdot\nabla u\right) = -\nabla p + \mu \nabla^2 u + f & \text{in } \Omega \times (0,T] \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,T] \\ u = g_D & \text{on } \Gamma_D \times (0,T] \\ \mu \frac{\partial u}{\partial n} - pn = t_N & \text{on } \Gamma_N \times (0,T] \\ u(\cdot,0) = u_0 & \text{in } \Omega \end{cases}$$

~~We put $u(x,0) = u_0(x) =$~~



$f = 0$



<u>Inflow:</u> $U(0,y) = 4U_m\, y(H-y)/H^2$
$= g_1(y)$

mean inflow

$\bar{U} = 2U(0, H/2, t)/3 = 2U_m/3$

$Re = \bar{U}D/\nu$ $\quad \nu = \frac{\mu}{\rho}$

$= \frac{2}{3}\frac{U_m D \rho}{\mu}$

$U_m = \frac{3}{2}\frac{Re\,\mu}{D\rho}$, use $Re = 20$

$\mu = 10^{-3}\ kg\, m^{-1} s^{-1}$

$U_m = 0.3\ m/s$

$\rho = 1.0\ kg\, m^{-3}$

<u>BCs:</u>

$\Gamma_1: u = g_1(y),\ \frac{\partial p}{\partial n} = 0$

$\Gamma_2: u = 0,\ \frac{\partial p}{\partial n} = 0$

$\Gamma_3: u = 0,\ \frac{\partial p}{\partial n} = 0$

$\Gamma_4: \frac{\partial u}{\partial n} = 0,\ p = 0$

$S: u = 0,\ \frac{\partial p}{\partial n} = 0$

$F_D = \int_S \mu\, n\cdot\nabla u_t\, n_y - p n_x\, ds$

$u_t = u\cdot(n_y, -n_x)$

Discretize in time, then space

Explicit IPCS

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho}\nabla p + \nu \nabla^2 u \quad (f=0)$$

Explicit Euler tentative guess

$$u^* = u^n + \Delta t\left(-u^n \cdot \nabla u^n - \frac{1}{\rho}\nabla p^n + \nu \nabla^2 u^n\right)$$

We should have

$$u^{n+1} = u^n + \Delta t\left(-u^n \cdot \nabla u^n - \frac{1}{\rho}\nabla p^{n+1} + \nu \nabla^2 u^n\right)$$

So $u^{n+1} - u^* = u^c = -\frac{\Delta t}{\rho}\nabla(p^{n+1} - p^n) = \frac{-\Delta t}{\rho}\nabla\phi, \ \phi = p^{n+1} - p^n$

Since $\nabla \cdot u^{n+1} = 0$, we have $\nabla \cdot u^c = -\nabla \cdot u^*$. Now

$$-\frac{\Delta t}{\rho}\nabla^2\phi = \nabla \cdot u^c = -\nabla \cdot u^*$$

1) $u^* = u^n + \Delta t\left(-u^n \cdot \nabla u^n - \frac{1}{\rho}\nabla p^n + \nu \nabla^2 u^n\right)$

2) $-\nabla^2\phi = -\frac{\rho}{\Delta t}\nabla \cdot u^*$

3) $u^{n+1} = u^* - \frac{\Delta t}{\rho}\nabla\phi$

4) $p^{n+1} = p^n + \phi$

BCs:

u^{n+1}, u^* have same bcs, so at fluid Dirichlet BC, $\nabla\phi \cdot n = \frac{\rho}{\Delta t}(u^{n+1} - u^*) \cdot n = 0$, homogeneous Neumann except for outflow. There, $p^{n+1} = p^n = 0$, so $\phi = p^{n+1} - p^n = 0$, homogeneous Dirichlet

Explicit IPCS

1) $\int_\Omega u^* \cdot v\,dx = \int_\Omega u^* \cdot v\,dx$

$\int_\Omega u^n \cdot v\,dx = \int_\Omega u^n \cdot v\,dx$

$\int_\Omega (u^n \cdot \nabla u^n) \cdot v\,dx = \int_\Omega (u^n \cdot \nabla u^n) \cdot v\,dx$

$\int_\Omega -\nabla p^n \cdot v\,dx = -\int_{\partial\Omega} p^n v \cdot n\,ds + \int_\Omega p^n \mathrm{div}(v)\,dx = \int_\Omega p^n \mathrm{div}(v)\,dx$

$\int_\Omega \Delta u^n \cdot v\,dx = \sum_i \int_\Omega \Delta u_i^n v_i\,dx = \sum_i \left(-\int_\Omega \nabla u_i^n \cdot \nabla v_i\,dx + \int_{\partial\Omega} \frac{\partial u_i^n}{\partial n} v_i\,ds\right)$
$= \sum_i -\int_\Omega \nabla u_i^n \cdot \nabla v_i\,dx = -\int_\Omega \nabla u^n : \nabla v\,dx$

(1) $\int_\Omega u^* \cdot v\,dx = \int_\Omega u^n \cdot v - \Delta t (u^n \cdot \nabla u^n) \cdot v + \frac{\Delta t}{\rho} p^n \mathrm{div}(v) - \Delta t \nu \nabla u^n : \nabla v\,dx$

2) $\int_\Omega -\Delta\phi\, q\,dx = \int_\Omega \nabla\phi \cdot \nabla q\,dx - \int_{\partial\Omega} \frac{\partial\phi}{\partial n} q\,ds = \int_\Omega \nabla\phi \cdot \nabla q\,dx$

$\int_\Omega -\nabla \cdot u^* q\,dx = \int_\Omega -\nabla \cdot u^* q\,dx$

(2) $\int_\Omega \nabla\phi \cdot \nabla q\,dx = \int_\Omega -\frac{\rho}{\Delta t} \nabla \cdot u^* q\,dx$

3) $\int_\Omega u^{n+1} \cdot v\,dx = \int_\Omega u^{n+1} \cdot v\,dx$

$\int_\Omega u^* \cdot v\,dx = \int_\Omega u^* \cdot v\,dx$

$\int_\Omega -\nabla\phi \cdot v\,dx = \int_\Omega -\nabla\phi \cdot v\,dx$

(3) $\int_\Omega u^{n+1} \cdot v\,dx = \int u^* \cdot v - \frac{\Delta t}{\rho} \nabla\phi \cdot v\,dx$

4) $p^{n+1} = p^n + \phi$

Semi-implicit IPCS

$$\frac{\partial u}{\partial t} + u\cdot\nabla u = \frac{1}{\rho}(-\nabla p + \mu\Delta u) \text{ in } \Omega$$

$$\cancel{\nabla\cdot u = 0 \text{ in } \Omega \quad u = g_D \text{ in } \Gamma_D, \ \partial u}$$

Comp u^* semi-implicit wrt. u,

$$\frac{u^* - u^n}{\Delta t} + u^n\cdot\nabla u^* = -\frac{1}{\rho}\nabla p^n + \nu\Delta u^*$$

$$u^* - \Delta t\left(-u^n\cdot\nabla u^* - \frac{1}{\rho}\nabla p^n + \nu\Delta u^*\right) = u^n \quad (3)$$

Want

$$u^{n+1} - \Delta t\left(-u^n\cdot\nabla u^{n+1} - \frac{1}{\rho}\nabla p^{n+1} + \nu\Delta u^{n+1}\right) = u^n \quad (4)$$

$u^{n+1} = u^* + u^c$. Let $S(u) = \Delta t(-u^n\cdot\nabla u + \nu\Delta u)$, linear

$(3) \Leftrightarrow (3)'$ $u^* - S(u^*) = u^n - \Delta t\frac{1}{\rho}\nabla p^n$

$(4) \Leftrightarrow (4)'$ $u^{n+1} - S(u^{n+1}) = u^n - \Delta t\frac{1}{\rho}\nabla p^{n+1}$

$$p^{n+1} - p^n = \phi$$

$$u^c - S(u^c) = -\Delta t/\rho\,\nabla(p^{n+1} - p^n) = -\frac{\Delta t}{\rho}\nabla\phi$$

$$\nabla\cdot u^{n+1} = 0 \Rightarrow \nabla\cdot u^c = -\nabla\cdot u^*$$

$S(u^c)$ is first order in time, ignore and still first order approximation.

$$u^c = -\frac{\Delta t}{\rho}\nabla\phi, \quad \nabla\cdot u^c = -\nabla\cdot u^*$$

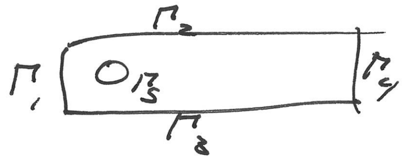
$$\Rightarrow -\frac{\Delta t}{\rho}\Delta\phi = -\nabla\cdot u^*$$

$u^*|_{\Gamma_D} = u^{n+1}|_{\Gamma_D}$ and $u^c = -\frac{\Delta t}{\rho}\nabla\phi \Rightarrow \frac{\partial\phi}{\partial n}\cdot n = 0$ on Γ_D

$(p^{n+1} - p^n)|_{\Gamma_{Dp}} = \phi|_{\Gamma_{Dp}} = 0$: $-\frac{\Delta t}{\rho}\Delta\phi = -\nabla\cdot u^*$

$\frac{\partial\phi}{\partial n} = 0$ on $\partial\Omega\setminus\Gamma_4$, $\phi = 0$ on Γ_4

SF-IPCS:



1) Solve for u^*

$$u^* + \Delta t\, u^n \cdot \nabla u^* - \Delta t\, \nu \Delta u^* = u^n - \Delta t \frac{1}{\rho} \nabla p^n$$

~~2~~ w/ $u^* = 0$ on $\Gamma_{1,2,3,5}$, $\frac{\partial u^*}{\partial n} = 0$ on Γ_4

2) Solve for ϕ

$$-\Delta \phi = -\frac{\rho}{\Delta t} \operatorname{div}(u^*)$$

w/ $\phi = 0$ on Γ_4, $\frac{\partial \phi}{\partial n} = 0$ on $\Gamma_{1,2,3,5}$

3) Update $p^{n+1} = p^n + \phi$

4) Solve for u^{n+1}

$$u^{n+1} = u^* + u^c = u^n - \frac{\Delta t}{\rho} \nabla \phi$$

SI-IPCS

1) $\int_\Omega u^* \cdot v \, dx = \int_\Omega u^* \cdot v \, dx$

$\int_\Omega (u^n \cdot \nabla u^*) \cdot v \, dx = \int_\Omega (u^n \cdot \nabla u^*) \cdot v \, dx$

$\int_\Omega -\Delta u^* \cdot v \, dx = \sum_i \int_\Omega -\Delta u_i^* \, v_i \, dx = \sum_i \left(\int_\Omega \nabla u_i^* \cdot \nabla v_i \, dx - \int_{\partial\Omega} \frac{\partial u_i^*}{\partial n} v_i \, ds \right)$

$= \sum_i \int_\Omega \nabla u_i^* \cdot \nabla v_i \, dx = \int_\Omega \nabla u^* : \nabla v \, dx$

$\int_\Omega u^n \cdot v \, dx = \int_\Omega u^n \cdot v \, dx$

$-\int_\Omega \nabla p^n \cdot v \, dx = -\int_{\partial\Omega} p^n v \cdot n \, ds + \int_\Omega p^n \nabla \cdot v \, dx = \int_\Omega p^n \operatorname{div}(v) \, dx$

(1) $\int_\Omega u^* \cdot v + \Delta t (u^n \cdot \nabla u^*) \cdot v + \Delta t \, \nu \nabla u^* : \nabla v \, dx = \int_\Omega u^n \cdot v + \frac{\Delta t}{\rho} p^n \operatorname{div}(v) \, dx$

2) $\int_\Omega -\Delta \phi \, q \, dx = \int_\Omega \nabla \phi \cdot \nabla q \, dx - \int_{\partial\Omega} \frac{\partial \phi}{\partial n} q \, ds = \int_\Omega \nabla \phi \cdot \nabla q \, dx$

$\int_\Omega -\frac{\rho}{\Delta t} \operatorname{div}(u^*) \, q \, dx = \int_\Omega -\frac{\rho}{\Delta t} \operatorname{div}(u^*) \, q \, dx$

(2) $\int_\Omega \nabla \phi \cdot \nabla q \, dx = \int_\Omega -\frac{\rho}{\Delta t} \operatorname{div}(u^*) \, q \, dx$

4) $\int_\Omega u^{n+1} \cdot v \, dx = \int_\Omega u^{n+1} \cdot v \, dx$

$\int_\Omega u^* \cdot v \, dx = \int_\Omega u^* \cdot v \, dx$

$\int_\Omega -\frac{\Delta t}{\rho} \nabla \phi \cdot v \, dx = \int_\Omega -\frac{\Delta t}{\rho} \nabla \phi \cdot v \, dx$

(4) $\int_\Omega u^{n+1} \cdot v \, dx = \int_\Omega u^* \cdot v - \frac{\Delta t}{\rho} \nabla \phi \cdot v \, dx$