$$\underline{q}(s) = 0 \le s \le 2\pi$$
$$\underline{q}(0) = \underline{q}(2\pi)$$

$$\underline{Q}(s,t) \tag{1}$$

Parameterisation invariant..

The template $\underline{q}_A$ is the curve at $t = 0$, and the target $\underline{q}_B$ is the curve at $t = 1$. The scheme finds a velocity $\underline{u}(s)$ to minimize distance between the two curves. A BFGS algorithm is used to find the minimal functional $S[u]$,

$$S = \frac{1}{2}\int_{t=0}^{1}\int_{s=0}^{2\pi}\left[|\underline{u}(s,t)|^2\left|\frac{\partial \underline{q}}{\partial s}\right|^2 + \alpha^2\frac{\left|\frac{\partial \underline{u}}{\partial s}\right|^2}{\left|\frac{\partial \underline{q}}{\partial s}\right|}\right] ds\, dt + \frac{1}{\sigma^2}\int_{s=0}^{2\pi}|q(s,1) - q_B(s)|^2\, ds. \tag{2}$$

Subject to

$$\underline{q}_A(s) = \underline{q}(s,0) \tag{3a}$$
$$\underline{q}_B(s) = \underline{q}(s,1) \tag{3b}$$
$$\frac{\partial \underline{q}}{\partial t}(s,t) = \underline{u}(s,t) \tag{3c}$$

where α is dependent on the mesh size ????, and σ controls the importance of error term $|q(s,1) - q_B(s)|^2$.

Given a velocity $\underline{u}(s,t)$, the scheme first finds the vector $\underline{q}$ at each time step from $t = 0$ to $t = 1$. Eq. 3c is discretized using a forward finite difference

$$\frac{\underline{q}(s)^{n+1} - \underline{q}(s)^n}{\Delta t} = \underline{u}(s). \tag{4}$$

Using a test function $\underline{r}(s)$, Eq. 5 is solved for $\underline{q}^{n+1}$ at each time step.

$$\int \underline{r}(s)\cdot\underline{q}^{n+1}(s)\, ds = \int \underline{r}(s)\cdot\underline{q}^{n}(s)\, ds + \Delta t\int \underline{r}(s)\cdot\underline{v}(s)\, ds \tag{5}$$

The variational derivative $\frac{\delta S}{\delta \underline{u}}$ is need to improve the performance of the optimiser.

For a single timestep, the variational derivative can be found with the equation

$$\langle \underline{v}, \frac{\delta S}{\delta \underline{u}}\rangle = \langle \underline{v}, \left|\frac{\partial \underline{q}}{\partial s}\right|\rangle + \alpha^2\langle\frac{\partial \underline{v}}{\partial s}, \frac{\frac{\partial \underline{v}}{\partial s}}{\left|\frac{\partial \underline{q}}{\partial s}\right|}\rangle - \langle \underline{v}, \hat{\underline{q}}\rangle, \tag{6}$$

where $\underline{v}$ is a test function, and $\hat{\underline{q}}$ is ???

For each time step, $\underline{\hat{q}}$ is found using backwards finite difference method where $\underline{\hat{q}}(s,1)$ is found by

$$\langle \underline{\hat{p}}, \underline{\hat{q}}(s,1)\rangle = -\frac{1}{\sigma^2}\langle \underline{\hat{p}}, \underline{q}(s,1) - q_B\rangle \tag{7}$$

where $\underline{\hat{p}}$ is a test function.

$$\langle \underline{\hat{p}}, \frac{\partial \underline{\hat{q}}}{\partial t}\rangle = \langle \left(\frac{1}{2}\frac{|\underline{u}|^2}{\left|\frac{\partial \underline{q}}{\partial s}\right|^2} + \frac{\alpha^2}{2}\frac{\left|\frac{\partial \underline{u}}{\partial s}\right|^2}{\left|\frac{\partial \underline{q}}{\partial s}\right|^2} \right) \frac{\partial \underline{\hat{p}}}{\partial s}, \frac{\partial \underline{\hat{q}}}{\partial s}\rangle \tag{8}$$

Substituting $\frac{\partial \hat{q}}{\partial t}$ with a backwards finite difference and rearranging Eq. 8,

$$\int \underline{\hat{p}} \cdot \underline{q}^n \, ds = \int \underline{\hat{p}} \cdot \underline{q}^{n+1} \, ds + \left(\frac{1}{2}\frac{|\underline{u}|^2}{\left|\frac{\partial \underline{q}}{\partial s}\right|^2} + \frac{\alpha^2}{2}\frac{\left|\frac{\partial \underline{u}}{\partial s}\right|^2}{\left|\frac{\partial \underline{q}}{\partial s}\right|^2} \right) \int \frac{\partial \underline{\hat{p}}}{\partial s} \cdot \frac{\partial \underline{\hat{q}}^{n+1}}{\partial s} \, ds \tag{9}$$

which is solved for q^n at each timestep.