15.073 Two Examples

10/2/17

Example: Line Item

An ATM in Cleveland, Ohio

Suppose that customers arrive at a single Automatic Teller Machine (ATM) under a Poisson process with mean rate one per minute. The time to complete a service is equally likely to be one minute or one-half minute. Customers are handled in the order first-come first-served; no services are interrupted because of arrivals; the machine serves each customer as soon as it has finished with all previous customers.

The owner of the ATM can make inferences about customer waiting times from data about the timings of its transactions. Suppose that, on a given day, the ATM's transaction data for the period 5:26 PM-5:29 PM looked as follows:

Time (hr/min/sec)	ATM Status
5 : 26 : 00	Machine free
5 : 26 : 30	Service starts

5 : 27 : 00	Service continues
5 : 27 : 30	Service ends; new service starts
5 : 28 : 00	Service ends; new service starts
5 : 28 : 30	Service ends; no new service
5 : 29 : 00	Machine free

Let $\mathbf{S_1}$ be the number of customer arrivals from just after 5:26:30 to 5:27:30, $\mathbf{S_2}$ the number from just after 5:27:30 to 5:28:00, and $\mathbf{S_3}$ the number from just after 5:28:00 to 5:28:30. The first arrival in the observation period (which is not counted as part of S_1) was at exactly 5:26:30.

Some questions:

(i) How many customers were served by the machine over the observation period?

(ii) What is the minimum value S_1 could take?

(iii) The values of $S_1 + S_2$ and of S_3 are exactly known given the data above. What are they?

(iv) Given the information at hand, what is the conditional probability that $S_1 = 1$?

Solution

(i) The server was free at the beginning and end of the three-minute observation period, and completed three services during the interval. Therefore, a total of *three* customers arrived over the period (and all were gone when it ended).

(ii) There were no customers in the system just before 5:26:30, when a customer arrival ended the server's free period. When that customer's service ended at 5:27:30, another service started at once, which implies that at least one new customer arrived during the initial service. Hence, $\mathbf{S_1 \geq 1}$.

(iii) Given that the server was free once the third service ended, it is clear that $S_3 = 0$. Because there were three customer arrivals in total--the first of which is not counted among the s_i's--it follows that $s_1 + s_2 = 2$.

(iv) The data demonstrate that $S_1 \geq 1$ and that $S_1 + S_2 = 2$. (The S_i's must also be integers.) Thus, if events D and E are defined by:

$$\mathbf{D: S_1 = 1 \quad \text{and} \quad E: S_1 \geq 1 \text{ and } S_1 + S_2 = 2,}$$

what we seek here is **P(D| E).**

We have the familiar formula: $\mathbf{P(D|E) = P(D \cap E)/P(E).}$

To achieve both D and E, S_1 and S_2 must both equal 1**(**think about it). Thus,

$$P(D \cap E) = P(S_1 = S_2 = 1) = P(S_1 = 1) * P(S_2 = 1)$$

= P(one customer arrival over a one-minute interval *and* one arrival over a separate half-minute interval)

Under (2-43) and independence, we have:

P(D and E) = P(one arrival in one minute and one arrival in next half minute)

$$= [\lambda e^{-\lambda}](.5\lambda)e^{-.5\lambda} = .5\lambda^2 e^{-1.5\lambda}$$

As for P(E), we have:

$$P(E) = P(E \cap D) + P(E \cap D^c) \text{ where } D^c = \text{complement of } D$$

We already know that $P(E \cap D) = .5\lambda^2 e^{-1.5\lambda}$. The event $E \cap D^c$ means that $S_1 \neq 1$, $S_1 \geq 1$, and $S_1 + S_2 = 2$. The only integers satisfying these conditions are $S_1 = 2$ and $S_2 = 0$. Thus:

$$P(E \cap D^c) = P(S_1 = 2 \cap S_2 = 0)$$

But $P(S_1 = 2 \cap S_2 = 0) = P(S_1 = 2)P(S_2 = 0) = [(\lambda^2 e^{-\lambda})/2]\, e^{-.5\lambda} = (\lambda^2 e^{-1.5\lambda})/2$.

In consequence, $$P(E) = P(E \cap D) + P(E \cap D^*) = \lambda^2 e^{-1.5\lambda}$$

Putting it all together, we reach a surprisingly simple answer:

$$P(D|E) = P(D \cap E)/P(E) = (.5\lambda^2 e^{-1.5\lambda}/2)/(\lambda^2 e^{-1.5\lambda}) = 1/2.$$

Note that the answer does not depend on λ. This happens because, in any Poisson process with *any* rate λ, the chance of having one event in each of two consecutive intervals—one of unit length and the other of ½ unit length—is the same as the probability of two events in the first interval and none in the second. The value of λ affects the probability of actually getting two events in 1.5 minutes, with at least one in the first minute. Once that event is known to have occurred, however, probabilities conditioned on the event no longer depend on λ.

Why would this question interest anyone? Well, the owner of the ATM might be concerned about whether people are waiting too long for service. When $S_1 = 2$, and $S_2 = 0$, the second person to arrive between 5:26:30 and 5:27:30 had to wait until the original service ended and then had to wait through the service of the first customer who arrived between 5:26:30 and 5:27:30. If $S_1 = 1$ and $S_2 = 1$, by contrast, then each arriving person went into service immediately as soon as the customer then using the ATM left. Knowing what the transaction data imply about the likelihood of these two possibilities helps in assessing the quality of customer service.

Example: Too Close for Comfort

Now, a problem about air traffic control. Suppose that two aerial routes--one Eastbound and one Northbound--cross at an altitude of 35,000 feet at junction J

(Figure 2-6). In the absence of air-traffic control, suppose that the times at which eastbound planes would arrive at the junction would reflect a Poisson process with parameter λ_E (per minute). Likewise, northbound planes would arrive under an independent Poisson process with parameter λ_N. All planes are jets that move at a speed of 600 miles per hour along their routes.

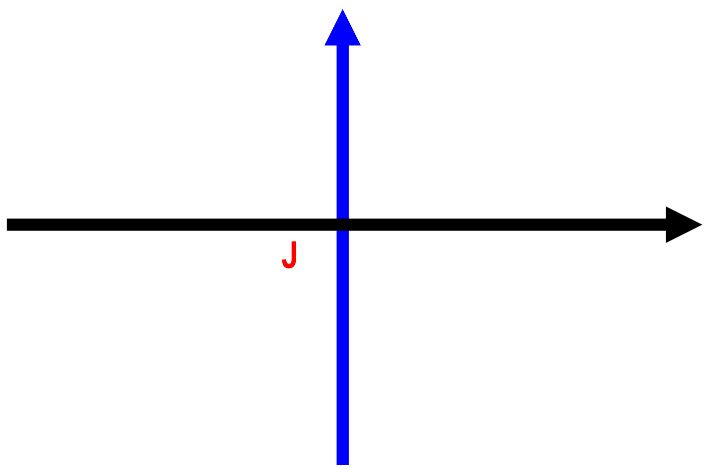

Figure 2-6: Eastbound and Northbound Air Routes Cross at J

The US Federal Aviation Administration thinks it dangerous if two planes cruising at the same altitude get within 5 miles of one another (in which case they are said to **conflict**). The idea is that, if a conflict arises, the planes are traveling so fast that they could collide if one of them deviates from its course. With the FAA standard in mind, find the probabilities of three interesting events:

(i) E: the chance that an eastbound plane that has just reached J is in conflict *at that moment* with a northbound plane

(ii) N: the chance that a northbound plane that has just reached J is at that moment in conflict with an eastbound plane

(iii)) EE: the chance that a given eastbound plane that passes through J is *at any time* in conflict with a northbound plane that passes through J.

Solution

(i) To find P(E), we note that the conflict occurs if, at the time the eastbound plane reaches J, there is a northbound plane within five miles of J. If E^c is the complement of E, then E^c requires that there be no northbound plane within five miles of the junction. It is easier to find $P(E^c)$ than P(E), so we will do so and then invoke the rule $P(E) = 1 - P(E^c)$.

We aren't told anything about planes that are not at the junction, so how can we determine whether a Northbound aircraft is within five miles of J? Well, we can exploit the clue that planes travel at 600 miles per hour (which works out to ten miles per minute, or one mile every six seconds). Suppose that a plane is north of J and within five miles of it. Then, the plane must have passed through J *within the last thirty seconds*. Similarly, if a northbound plane is still south of J but less than five miles away, it will reach J within the *next* thirty seconds.

Thus, if an eastbound plane reaches J at time t, there will be a conflict at t if any northbound planes pass through J between t-0.5 (in minutes) and t + 0.5. And there will be *no* conflict if no northbound plane reaches t over the interval (t-0.5,t+0.5).

We can therefore write:

$P(E^C)$ = P(no northbound arrivals at J over (t-0.5,t+0.5)) $= e^{-\lambda_N}$

and thus that $$\mathbf{P(E) = 1 -}\ e^{-\lambda_N}$$

(ii) The reasoning is the same as for P(E), so we can write:

$$\mathbf{P(N) = 1 -}\ e^{-\lambda_E}$$

It might seem surprising that P(E) and P(N) differ, given that each conflict we are considering involves one eastbound and one northbound plane. If, however, $\lambda_N > \lambda_E$ (for example), then more northbound planes reach J per hour than do eastbound planes. Thus, if equal *numbers* of northbound and eastbound planes face conflicts, the *percentage* of conflicts is lower for northbound planes passing through J than eastbound ones. And P(E) and P(N) reflect these percentages.

c) The reader might be wondering about something: what is the difference between P(EE) and P(E)? The definitions of the events differ a bit: P(E) requires that a conflict be in progress when an eastbound plane reaches J; P(EE) requires some east/north conflict, but allows for the possibility that the conflict is already over (or has not yet begun) when the eastbound plane passes through J. Still, does this distinction really matter?

Well, yes. Suppose that, when an eastbound plane arrives at J, there is a northbound plane six miles north of J. The two planes are not then in conflict. But consider the situation twelve seconds earlier, when the northbound plane was four miles north of J and the eastbound plane two miles west of it. The Pythagorean theorem reminds us that the two planes were $\sqrt{20} = 4.5$ miles apart at that time (i.e. that they were in conflict, even though they no longer are).

Suppose that a Northbound plane is L miles north of J when an Eastbound plane reaches J. What was the shortest distance between these two planes? Recognizing that each plane travels at 10 miles per minute, their distance apart w minutes *previously* is given by: $D(w) = \sqrt{(10w)^2 + (L - 10w)^2}$. To find the value of w at which D(w) is minimized, we can observe that the value of w that minimizes D(w) is the same at that which minimizes $D^2(w) = (10w)^2 + (L - 10w)^2$. We find that minimizing value by equating to zero the derivative of

$(10w)^2 + (L - 10w)^2$ with respect to w. That derivative is 200w -20L + 200w = 400w – 20L, meaning that a zero derivative arises when w = L/20. (Because the second derivative is positive, we know we have a minimum.)

At time w = L/20, $D^2(w) = L^2/2$, from which it follows that the minimum distance between the two planes was $L/\sqrt{2}$. (We need not consider times after the Eastbound plane reached J, because its subsequent distance from the Northbound plane would always exceed L.). Setting $L/\sqrt{2} = 5$, *we conclude if the Northbound plane was less than* $5\sqrt{2}$ *= 7.07 miles North of J when the Eastbound plane reached J, the two planes are/were in conflict at some time*. If the present distance apart is between 5 and 7.07 miles, then the conflict occurred earlier but has ended. By similar reasoning, a conflict will arise in the future if a Northbound plane is between 5 and 7.07 miles south of J when an Eastbound plane passes through J.

At travel speeds of ten miles per minute, an eastbound aircraft reaching J at time t will conflict with any northbound plane that passes through J during the interval (t-.707,t+.707). It will avoid a conflict if no Northbound planes reach J during that interval of 1.414 minutes. In consequence,

$$\mathbf{P(EE) = 1 - P(EE^c) = 1 -} e^{-1.414\lambda_N}$$

This example assumed the absence of air-traffic control, and random arrivals at J under Poisson processes. In reality, aircraft arrival times at junctions would never be left to chance alone. What these calculations suggest is the frequency at which *potentially* hazardous situations would arise based on activity levels and operational randomness in the air-traffic system. They therefore indicate the magnitude of the challenge facing air-traffic controllers. The magnificence with which the controllers meet this challenge is suggested by a statistic: between 1990 and 2012, over 11 billion passengers travelled in commercial jet aircraft in the United States. The number killed in midair collisions was zero.