Problem 1.

(a) The diameter of the circle is 10 miles, thus the circumference is $10\pi \approx$ 31.4159 miles. The ERV is advantageously located on 'one side of the circle,' the two circle sides split at max distance from the incident, $10\pi/2$. By inspection, the ERV is uniformly distributed over the interval $[0, \frac{10\pi}{2} = 5\pi = 15.70795]$.

Therefore, the pdf for the distance traveled by ERV to a random incident is:

$$f_D(d) = \frac{1}{5\pi}, \qquad for\ 0 \le d \le 5\pi$$

And the cdf is:

$$F_D(d) = \frac{d}{5\pi}, \qquad for\ 0 \le d \le 5\pi$$

(b) The fraction of reported incidents will be delayed in queue is the fraction that the ERV is busy in serving one incident, which is equal to the system utilization ratio ρ.

We see that $\rho = \lambda/\mu = 1/3$, and since ***Poisson Arrivals See Time Averages*** (PASTA), the answer is 1/3.

(c) Since the incident locations are i.i.d. (identically and independently distribution), and the queue discipline has no impacts on the travel distance, no change to the answer of part (a).

(d) This is standard min and max function of random variables over a square sample space with uniform pdf. We have done the min in class, follow the same logic here:

Define $D_{close} = \min(D_1, D_2)$ as the distance traveled to the incident by the closer of the two ERV's, define $D_{far} = \max(D_1, D_2)$ as the distance traveled to the incident by the farther of the two ERV's, where D_i is the distance traveled to the incident by ERV i, $i = 1,2$.

Cdf of D_{close}:

$$F_{D_{close}}(d) = P(D_{close} \le d) = 1 - P(D_{close} > d) = 1 - P(D_1 > d \cap D_2 > d)$$

$$= 1 - P(D_1 > d)P(D_2 > d) = 1 - (1 - F_{D_1}(d)(1 - F_{D_2}(d)$$

$$= 1 - \left(1 - \frac{d}{5\pi}\right)^2 \qquad for\ 0 \le d \le 5\pi$$

Pdf of D_{close}:

$$f_{D_{close}}(d) = \frac{\alpha F_{D_{close}}(d)}{\alpha d} = -2\left(1 - \frac{d}{5\pi}\right) \times \left(-\frac{1}{5\pi}\right) = \frac{2}{5\pi}\left(1 - \frac{d}{5\pi}\right)$$
$$= \frac{2}{5\pi} - \frac{2d}{25\pi^2} \qquad for\ 0 \le d \le 5\pi$$

Cdf of D_{far}:

$$F_{D_{far}}(d) = P\left(D_{far} \le d\right) = P(D_1 \le d \cap D_2 \le d) = P(D_1 \le d)P(D_2 \le d)$$
$$= F_{D_1}(d)F_{D_2}(d) = (\frac{d}{5\pi})^2 \qquad for\ 0 \le d \le 5\pi$$

Pdf of D_{far}:

$$f_{D_{far}}(d) = \frac{\alpha F_{D_{far}}(d)}{\alpha d} = 2\left(\frac{d}{5\pi}\right) \times \left(\frac{1}{5\pi}\right) = \frac{2d}{25\pi^2} \qquad for\ 0 \le d \le 5\pi$$

Problem 2.

(a) There will be a region in the "northeast corner" of the city that corresponds to a travel path greater than the minimum distance L_1 path. Consider a CFS at location (x, y) for $y \geq x$. In this case the optimal travel path is north for 5 miles, east on the northern perimeter for x miles and then south for $(5 - y)$ miles. There will be a boundary line where all points on the boundary line will have equal travel times by this augmented route and by the L_1 route.
Travel time by augmented route:

$$T_a = \frac{5}{20} + \frac{x}{20} + \frac{5-y}{10}$$

Travel time by L_1 route:

$$T_L = \frac{y}{20} + \frac{x}{10}$$

Along this boundary line we have $T_a = T_L$, that is:

$$\frac{5}{20} + \frac{x}{20} + \frac{5-y}{10} = \frac{y}{20} + \frac{x}{10}$$

Or

$$5 + x + 2(5 - y) = y + 2x$$

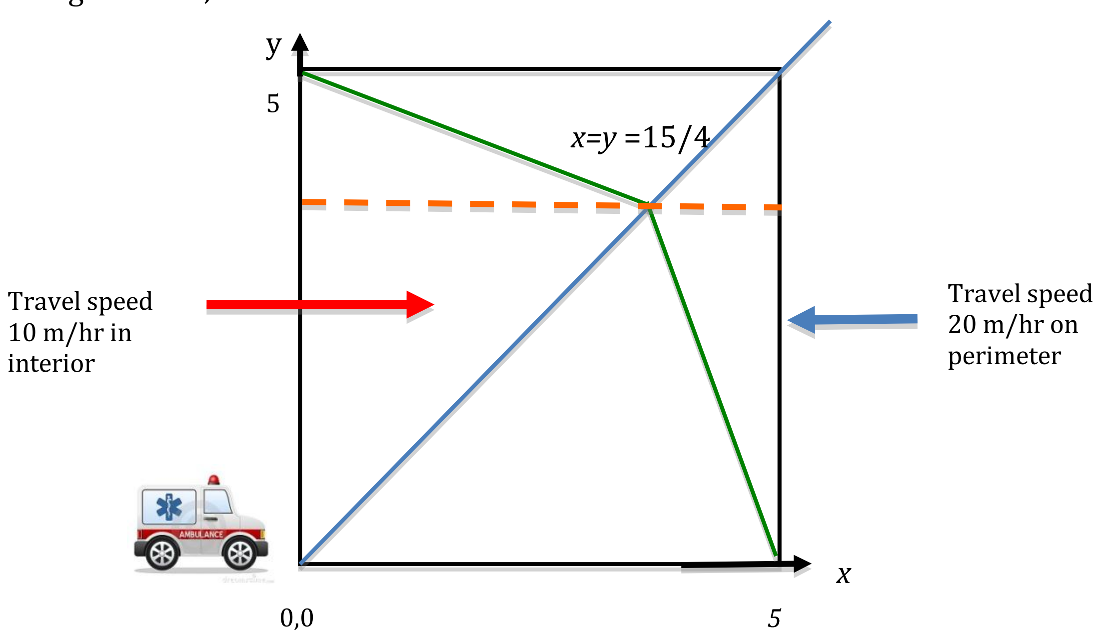
Or

$$3y = 15 - x$$

That is:

$$y = 5 - \frac{x}{3}$$

The boundary line for $y \geq x$ region is $y = 5 - x/3$, intercepted with $y = x$ at point $(15/4, 15/4)$.
By symmetric, the boundary line for $y \leq x$ region is $x = 5 - y/3$, also intercepted with $y = x$ at point $(15/4, 15/4)$.
The region within the square city in which the ambulance travels a route greater than the minimum distance path in order to get to the scene of the call for service as fast as possible is the two triangles region northeast of the green line, illustrated as follows.



(b) We can work only in zone $y \geq x$. By symmetry, the same result holds for $y < x$. We need 2 conditional probabilities, that of responding by a minimum distance path (left sub-zone) and that of the 'northern route' (upper right sub-zone). These probabilities are proportional to the areas of each respective triangle. Then need to compute $E[D_x]$ and $E[D_y]$for each instance, adjust by the respective speeds, multiply by the appropriate probabilities and sum. For each sub-zone, either $E[D_x]$ or $E[D_y]$ is available by inspection, as it has a triangular pdf anchored at zero. We hope the students recognize this and state it in their solution. The other requires a slightly messy integration. We are not asking the students to complete these details.

$E(D) = P(Left\ Sub\ Zone) \times E(D_{LSZ}) + P(Right\ Sub\ Zone) \times P(D_{RSZ})$
Where

$$P(Left\ Sub\ Zone) = \frac{Area\ of\ Left\ Sub\ Zone}{Area\ of\ y \geq x\ zone}$$

$$P(Reft\ Sub\ Zone) = \frac{Area\ of\ Right\ Sub\ Zone}{Area\ of\ y \geq x\ zone}$$

$$E(D_{LSZ}) = \frac{E(D_y|LSZ)}{20} + \frac{E(D_x|LSZ)}{10}$$

$$E(D_{RSZ}) = \frac{5}{20} + \frac{E(D_x|RSZ)}{20} + \frac{5 - E(D_y|RSZ)}{10}$$

(c) Look at the boundary line $y = 5 - x/3$, from part (a), we know it intercepts the line $y = x$ at point $(15/4,\ 15/4)$. This is the 'orange line" in the above figure. The distance and thus time spent on the perimeter boulevard is conditionally deterministic once you know whether $y \geq x$ or $y < x$. Then for each case the only probabilistic part relates to the x location that is uniform either on $[0,\ 15/4]$ or on $[15/4,\ 5]$.

Case 1: x is on $[0, 15/4]$,

$$P(Case\ 1) = \frac{\frac{15}{4}}{5} = \frac{3}{4}$$

$$T_{boulevard} = \frac{15/4}{20} = \frac{3}{16}$$

T_{inner} is $x/10$, that is uniform on $[0,\ 3/8]$.

$T = T_{boulevard} + T_{inner}$

$$f(T|Case\ 1) = \frac{8}{3}, \qquad where\ \frac{3}{16} \leq T \leq \frac{9}{16}$$

Case 2: x is on $[15/4,\ 5]$

$$P(Case\ 2) = \frac{5 - \frac{15}{4}}{5} = \frac{1}{4}$$

$T_{boulevard} = \frac{5}{20} + \frac{15/4}{20} = \frac{7}{16}$

T_{inner} is $(5 - x)/10$, that is uniform on $[0,\ 1/8]$.

$T = T_{boulevard} + T_{inner}$

$f(T|Case\ 2) = 8, \qquad where\ \frac{7}{16} \le T \le \frac{9}{16}$

Therefore, the final pdf is:

$$f(T) = P(Case\ 1) \times f(T|Case\ 1) + P(Case\ 2) \times f(T|Case\ 2)$$
$$= \begin{cases} 2, where\ \frac{3}{16} \le T \le \frac{7}{16} \\ 4, where\ \frac{7}{16} \le T \le \frac{9}{16} \end{cases}$$

The pdf plot is: