Deliveries in an Inventory/Routing Problem Using Stochastic Dynamic Programming

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An industrial gases tanker vehicle visits n customers on a tour, with a possible $(n+1)$st customer added at the end. The amount of needed product at each customer is a known random process, typically a Wiener process. The objective is to adjust dynamically the amount of product provided on scene to each customer so as to minimize total expected costs, comprising costs of earliness, lateness, product shortfall, and returning to the depot nonempty. Earliness costs are computed by invocation of an annualized incremental cost argument. Amounts of product delivered to each customer are not known until the driver is on scene at the customer location, at which point the customer is either restocked to capacity or left with some residual empty capacity, the policy determined by stochastic dynamic programming. The methodology has applications beyond industrial gases.

Our problem is encountered within an "industrial gases" context, but is applicable to a broader range of inventory/routing problems. In this paper, to illustrate via a specific example, we discuss the problem solely within the context of industrial gases. The manufacturer/distributor of industrial gases (e.g., liquefied nitrogen, oxygen, argon) must schedule and route its large tanker vehicles to replenish product at customer sites in order to prevent customer stockout, while at the same time trying to minimize logistics costs. As is traditional in the industry, the producer/distributor has responsibility for maintaining adequate inventory levels at customer sites. Customers do not routinely call requesting replenishment nor are there regular prescheduled deliveries. For some customers there are tank-level measuring instruments on site that can be polled by the manufacturer/distributor to ascertain current tank levels.

Customer product usage is a stochastic process, where usage typically grows linearly in time, as does the variance of the amount used. Invoking the Central Limit Theorem, we have modeled the probability law of amount of product used through time t since the last replenishment as a Gaussian or normal random variable (r.v.), with mean and variance that both grow linearly with t. Such a process is called a Wiener process with drift. The amount of product used since the last replenishment, if assumed to be a complete tank refill, is equal to "tank emptiness," or available capacity for new product. Under the Wiener process model, tank emptiness is also a Gaussian r.v. with mean and variance that grow linearly in time. The complement of tank emptiness, the remaining amount of product in the tank, is equal to tank capacity C minus emptiness, also a Gaussian r.v. under the Wiener process model. Since tank capacity is finite, each of the Gaussian r.v.s discussed is in fact a truncated Gaussian r.v., with the truncations caused by tank capacity C. Our formulation is not dependent on the Gaussian assumption, and in this paper our numerical examples use other distributions.

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1. Viewing a Delivery Tour as a Sequential Decision Process

What we propose in this paper is one component of a larger inventory/routing system. Our focus is on the evaluation of costs of alternative decisions associated with one proposed tour to n (possibly $n+1$) customers in a given sequence. We assume that the tanker vehicle leaves the depot filled to capacity W and visits in sequence customers $n, n-1, n-2, \ldots,$ to Customer 1. After visiting Customer 1, the vehicle may return to the depot either empty or nonempty with some residual (nondelivered) amount of product on board. In a variation of the problem, the vehicle while "on its way back to the depot," may be detoured to a "Customer 0," to attempt to "dump" the remaining residual amount of on-board product. For those customers without tank-level monitoring equipment that can be polled from remote locations, the driver of the vehicle does not know the emptiness of any customer until he arrives on the scene of that customer, at which time the customer's emptiness is determined as a sample from its then current Gaussian (or other) distribution. For customers with tank-level monitoring equipment, the driver knows their tank level(s) up to the time of the last polling; between the time of the last polling and arrival on the scene, the Wiener or other stochastic process model is operated to obtain an updated prior/posterior probability distribution of tank level. The amount of product delivered to a customer is equal to either the customer's emptiness (i.e., the customer is "filled up") or some smaller quantity, the value of which is determined by stochastic dynamic programming. In our first formulation of a dynamic-programming solution to this problem, the driver may or may not find out precisely how much emptiness is in a particular customer's tank. The driver does not measure emptiness, and if he leaves the tank less than full, he never determines the exact emptiness. Tank levels and complementary "emptiness levels" of all customers are known as probability density functions whose functional forms and parameter values reflect all that is known (by the manufacturer/distributor) about each customer's real-time inventory level. Any customer whose inventory level is monitored continuously by technology will enjoy a zero-variance probability law for its tank level, but the numerical value of the level itself will only be known just before vehicle arrival there. So, even for those customers, advanced planning of trips may require using a positive variance. Our methodology is compatible with all these complexities of operations.

There are two uses for the methodology of this paper: (1) to estimate the "costs and benefits" of any proposed tour at any given time, thereby allowing a consistent framework for selecting best or nearly best tours to deploy over time; and (2) to operate a tour in an optimal fashion, once it is initiated. In Use 1 a planner would embed the methodology of this paper into a large tour selection and scheduling system that would be operated virtually continuously over the hours of the day and days of the week. In Use 2 a driver would use the results of this paper to allocate in an optimal manner the product delivered during execution of each tour. Use 2 is self-explanatory, while Use 1 requires a full discussion of heuristic algorithmic techniques utilized to select among a virtually limitless set of tour options. Use 1 is the subject of a separate paper (Larson et al. 2001). The cost structure and dynamic-programming approach described herein may be embedded in currently existing tour design and scheduling algorithms (virtually all of them being heuristics), and thus we feel that this effort may be of independent interest.

The novel features of the approach are: (1) modeling the product usage and emptiness as stochastic processes; (2) providing incremental costs for early deliveries as well as late deliveries; (3) allowing the amount of product delivered to be determined by the driver, with the (optimal) amount determined by the amount of product currently in the customer's tank, the amount of product currently on board the vehicle, and the requirements of the customers yet to be visited on the tour.

2. Related Literature

Inventory/routing problems have been studied by an increasing number of authors since the early 1970s. One of the earliest works is by Beltrami and Bodin (1974), who focused on modeling and simple solutions for municipal waste collection. Fisher and

Jaikumar's (1981) early work based on a generalized assignment heuristic led to a many-authored Edelman Prize paper in distribution of industrial gases for Air Products and Chemicals, Inc. (Bell et al. 1983). To our knowledge, this effort is the first inventory/routing work devoted to distribution of industrial gases, the subject of this paper. Christofides and Beasley (1984) studied the periodic vehicle-routing problem that involves designing routes for a fleet of vehicles that have a strictly periodic structure. Recent algorithmic work on the periodic vehicle-routing problem can be found in Cordeau et al. (1997). Federgruen and Zipkin (1984) addressed the problem of allocation of a scarce resource from a central depot among a number of customers experiencing probabilistic demand. The problem is to determine which delivery is to be made by the vehicles and when to order. Chien et al. (1989) investigated the problem of delivering and allocating a limited amount of commodity to customers using a fleet of vehicles at a single depot.

Our work is most closely associated with several probabilistic approaches. Assad et al. (1982) and Golden et al. (1984) describe a study of a large-scale inventory-routing problem of a company involved in the distribution of liquid propane to residential and industrial systems. Each customer has a specified resupply level and the objective is to minimize resupply costs. The main tool used was a simulation model that contains routing algorithms. Dror et al. (1985) studied a similar problem. In their model each customer has a storage tank of known capacity and a probability distribution of the daily usage, which is normally distributed. The problem is formulated as a two-stage integer program. Dror and Ball (1987) developed a procedure for reducing a long-term inventory routing problem to a short-term one. Dror and Levy (1986) introduced three node-interchange route-improvement procedures for inventory-routing problems. Dror and Trudeau (1986) and Dror et al. (1989) dealt with routing problems with probabilistic demand, as we do. Bard et al. (1998) developed a decomposition system for the problem in which a supplier is responsible for restocking customers on an intermittent basis. A customer's usage is a random variable. The paper involves several heuristics with the objective of minimizing the annual delivery cost, which includes stockout and early arrival costs. It is the paper most closely aligned with our approach. A somewhat different approach to this class of problems, focusing on heavy traffic approximations, is found in Reiman et al. (1999).

In another view of the problem, Larson (1988) studied the strategic inventory problem that focuses on the design of the distribution system rather than the actual day-to-day operations. The objective in the work is to minimize the size of the planned fleet. Webb and Larson (1995) extended this work and developed a more general version of the problem with two new heuristics. Related strategic-inventory-routing work that required simulation due to the model complexity is Larson et al. (1988) and Richetta and Larson (1997). Anily and Federgruen (1990, 1993) presented a related work called the route-sales distribution system. The objective is to minimize long-range average transportation and inventory costs for a large-scale system of retailers with deterministic demand.

Recent comprehensive surveys for analysis of heuristic algorithms for inventory-routing problems (as well as vehicle-routing problems) are found in Federgruen and Simchi-Levi (1995), Gendreau et al. (1996), and Campbell et al. (1998).

For reader convenience, a glossary of terms is included in Table 1.

3. The Incremental Cost Structure

We utilize in this work the concept of incremental costs. This allows us to evaluate the performance of a visitation to any given customer on a given day and the performance of an entire proposed tour or route for a given vehicle on a given day. We make a distinction between *fixed* costs and *variable* costs, and it is the costs above the annual minimum for service of a customer that we wish to minimize. During the year each customer j will be visited a number of times, and our efforts are directed at minimizing this number within a cost structure that penalizes both earliness and lateness of deliveries.

We find it convenient to think of the incremental costs in terms of driving our own automobiles to a service station for "filling up" the gasoline tank.

Table 1 Glossary of Terms

Variable	Definition
R_j	Ideal refill point (in % tank full) for custormer j
C_j	Units of tank capacity for customer j (assumed to be integer)
n	Number of scheduled customers to visit on a tour
F_j	Fixed cost per trip to customer j, both time- and distance-related
O_j	Cost of total stockout for customer j
Y_j	An integer random variable representing the current product level in customer j's tank
x	The amount of product currently in the vehicle
W	Vehicle capacity
$E[C_L(Y)]$	Expected cost of *L*ateness
$E[C_E(Y)]$	Expected cost of *E*arliness
$E[C_S(x, Y)]$	Expected cost of product *S*hortfall, given x units of product in the vehicle upon arrival at the customer
$E[C_R(x, Y)]$	Expected cost of *R*eturning to depot nonempty, given x units of product in the vehicle upon arrival at the customer
CE_j	Normalized fixed cost per trip $= F_j/(C_j[1-R_j])$
C_T	Total cost of a given trip or tour
$f_j(x)$	Expected differential cost of completing a tour of j customers following an optimal policy, given x gallons of product currently in the vehicle's tank

Each time we do this we incur a cost in terms of added time and expense of the trip to the service station. If our "tank replenishment policy" is to fill up the tank whenever the tank-level reading indicates exactly half a tank of gasoline remaining, then the number of such replenishment trips per year would be twice as many as necessary—assuming, of course, that we could safely implement a policy for tank replenishment precisely just before we reached "Empty." But the closer we can get to "refill on Empty," the fewer trips we will make, and the smaller the costs we will incur. For the "refill on half Empty" policy, the yearly incremental cost above the fixed minimum possible is the sum of all the travel times of the unnecessary trips plus some fraction of time spent at the service station on the unnecessary trips. The "refill on Empty" policy has zero incremental cost above the minimum possible. Of course, with multiple stops for a tanker vehicle and with huge costs associated with incurring emptiness (i.e., inventory stockout), life is not so simple. But the general idea may be useful as a first-order guide to the more complex world we describe below.

Before proceeding, we also recognize that the "cost separability" that we propose for the problem is a simplification of reality. In fact, costs may be highly cross-dependent, and our separation of them represents a modeling choice. The approach is guided by the need to create a tractable yet usable model, one that has in practice shown to be beneficial to operating firms.

3.1. The Fixed Cost per Trip

We associate with each scheduled trip to a customer a fixed dollar amount that represents in an annual sense the average added cost per trip of visiting that customer. This fixed cost per trip is composed of two parts: a fixed time-related charge and a distance-related charge.

3.1.1. Fixed Time-Related Charge per Trip. Each time a driver visits a customer's location there is a fixed time charge associated with stopping there. This fixed charge deals with everything from carefully parking the vehicle at the customer's location, to manipulating the hose and connectors, to completing paperwork, etc. It is called the fixed time charge because it is added to the variable time, which is directly proportional to the quantity of product put in the customer's tank at that visit. In industrial gas applications, one typically finds a value of 15 minutes or so for the fixed time per trip, and that is converted into dollars to obtain the cost.

To explain how the fixed time charge is allocated, suppose the ideal refill level for customer j, R_j is at the 30% mark. If we revisit the customer every time the customer is exactly at 30%, then we say the system is operating as planned. However, suppose that, on arrival, we find the customer's tank at the 65% level and we fill up his tank. Then we have incurred a fixed time charge, but we can allocate that against only half as many gallons as we would ideally like to. We have incurred 50% of the fixed time charge unnecessarily.

Another way of seeing this allocation of fixed charge is as follows: If we arrive at the 65% mark rather than the optimal 30% mark, we have, in effect, moved forward in time the entire remainder of this year's deliveries to our customer. The moving-forward time is one half of the customer's optimal fill-refill cycle time, which is the average time for the customer to use a quantity of product equivalent to 70% of the customer's tank capacity. Suppose that this cycle time is 8 days. Then, in an expected-value sense, our early delivery occurred at the four-day mark from the last complete tank fill, and the next delivery after that is now due at the $4+8=12$-day mark, not at the optimal $8+8=16$-day mark. All subsequent trips in the year are also moved forward 4 days. At the end of the year, assuming nothing else has occurred, we require, in effect, one half of an extra trip, due to the moving up of one-half cycle time. This one half of an extra trip requires us to "charge" one half of the fixed cost per visit. This argument is repeated throughout the year on subsequent trips to the customer, adding fractions of the fixed charge per visitation to our accrual account, thereby giving us a potentially major "balance" at the end of the year. By taking the fractional fixed charge now, we are in effect taking the cost of something that will occur later during the accounting year. In accounting, the issue is similar to questions of when to book revenues and costs. Here, we book costs as soon as we realize that they will ultimately be incurred during the year.

3.1.2. Fixed Distance-Related Charge per Trip. The distance-related fixed charge is to represent a reasonable accounting charge for a trip that will occur later in the year due to the earliness of the current visitation. The fixed charge is to represent the mean or average mileage-related charge per trip, when averaged over a long time period such as one year. It is not meant to represent the marginal mileage-related cost of the next trip to that customer, a charge that in practice may depend on current and projected values of customers' inventory levels.

The extra distance associated with driving to a customer would be simple to compute if all tours were one-stop tours. If one drove 200 miles to get to and return from a customer, then the extra distance traveled to get to that customer on a given tour would be 200 miles. However, the majority of tours are multistop tours, and therefore we run into the difficulty of how to assign a travel distance separately and individually to each of the customers on a tour. We believe that there is no unique way of doing this, and herein we describe a simple method that seems reasonable.

First, we take the one-way distance from the depot to each of the n respective customers and arrange these as if the customers were located on a straight line, ordered from closest to farthest customer. We assign to the fixed distance charge for each of the n customers $2/n$ of the distance between the depot and the closest customer on the line. The factor of two is required to reflect the round-trip nature of the tour. For the $n-1$ farthest customers, we add to their respective fixed distance costs $2/(n-1)$ of the distance between the closest and the second closest customer. We continue in this way until we only have remaining the distance between the $n-1$st farthest customer and the farthest customer. Twice this distance is allocated entirely to the fixed distance cost for the farthest customer. This would be a fair and equitable allocation of travel distance if all n customers were located on a straight line.

Assuming customer i is located at x_i, so that x_i is the distance between customer i and the depot, and without loss of generality $x_n < x_{n-1} < \cdots < x_1 = D_{\max}$, we can summarize this approach as follows:

$$F_n = (2/n) * x_n$$

$$F_{n-1} = F_n + (2/[n-1]) * (x_{n-1} - x_n)$$

$$\ldots$$

$$F_{n-k} = F_{n-k+1} + (2/[n-k])(x_{n-k} - x_{n-k+1})$$

$$\ldots$$

$$F_1 = F_2 + 2(x_1 - x_2).$$

Summing the quantities $F_1 + F_2 + \cdots + F_n$ reveals that the sum equals $2x_1 = 2D_{\max}$, as we require.

How does this procedure generalize to the more realistic two-dimensional case? For n customers on a tour, we first *rotate* the problem into its one-dimensional equivalent (as above) and allocate *initial* mileage costs among customers as above. We are then confronted with the task of allocating among the n customers the detour distance DET, which is defined to be the net extra distance that is required to be traveled in the (real) two-dimensional version of the problem as compared to its one-dimensional equivalent. It seems clear that as *DET* grows, the charge to each customer on the tour should grow proportionally with *DET*; any other functional form would result in differential subsidies among customers, the amount of the subsidy depending on the value of *DET*. But we must arrange the constants of proportionality so that as *DET* grows to the ultimate point of zero net savings in contrast to n one-stop tours, each customer at the same time reaches the point of zero net savings. The appropriate constants of proportionality are themselves proportional to the savings each respective customer achieves by joining the combined tour within the one-dimensional construct of the problem. For $n = 3$ customers, for instance, these savings enjoyed by the respective customers are:

$$savings1 = 2*(2/3)*x_3+2*(1/2)*(x_2-x_3);$$

$$savings2 = 2*(2/3)*x_3+2*(1/2)*(x_2-x_3)=savings1;$$

$$savings3 = 2*(2/3)*x_3.$$

Total savings is, as expected, equal to $4x_3+2(x_2-x_3) = 2(x_2+x_3.)$. So, the weights used to apportion the detour distance *DET* are, respectively,

$$w_1 = savings1/[2(x_2+x_3)];$$

$$w_2 = savings2/[2(x_2+x_3)] = w_1;$$

$$w_3 = savings3/[2(x_2+x_3)].$$

This weighting procedure generalizes in the obvious way to $n > 3$ customers and provides for us a valid method to allocate mileage costs, and ultimately to compute costs per gallon of product delivered to each customer. For instance, with n customers whose identification numbers have been assigned so that $x_n < x_{n-1} < \cdots < x_1$, we have (in the one-dimensional version of the problem) a net savings from combining all the one-stop tours into one grand n-stop tour equal to

$$savings = 2(x_2+x_3+\cdots+x_n).$$

The savings assignable to the respective customers are

$$savings(n) = 2([n-1]/n)x_n;$$

$$savings(n-1) = savings(n)+2([n-2]/[n-1])(x_{n-1}-x_n)$$

$$\ldots;$$

$$savings2 = savings3+2(1/2)(x_2-x_3)$$

$$= savings3+x_2-x_3;$$

$$savings1 = savings2.$$

The weight w_j used to apportion a fraction w_j of the detour distance *DET* to customer j is given by

$$w_j = savings(j)/savings, \quad \text{for all } j = 1, 2, \ldots, n.$$

We use this to allocate a travel cost or mileage charge to customer j equal to

$$F_j = [2x_j - savings(j)] + w_j(DET).$$

The method just described is not unique. There are probably many other reasonable ways to allocate mileage costs, but we feel that the method described is reasonable and intuitively plausible.

3.1.3. Cost of Earliness: Apportioning the Fixed Charge per Trip. We denote by F_j the sum of the fixed time per stop and the fixed mileage per stop where both are translated into dollars. We express the cost relationship in terms of a linear cost or loss function as shown in Figure 1. In Figure 1 the horizontal axis indicates the percentage of tank that is full, ranging from 0%–100%; the vertical axis reflects differential costs. The point of minimum cost on the horizontal axis is R_j, which is equal to 30% in the figure. If the vehicle arrives and the customer's tank level is 30% in this example, we assume we incur no extra fixed-charge-related cost. On the other hand, if the vehicle arrives moments after a previous vehicle has departed leaving the tank full, then we would incur a full fixed charge for that unnecessary stop. Anything in between these two extremes gives a proportional differential time-related cost associated with that visitation. This is shown in the straight-line cost relationship depicted in Figure 1.

Thus, each time a customer is visited "early," a fixed cost is incurred due to some fractional extra

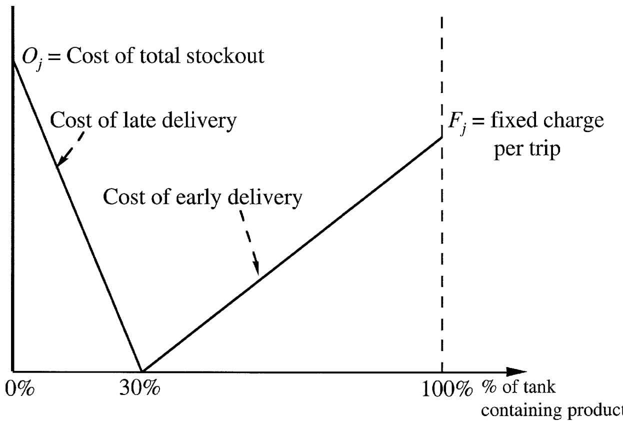

Figure 1 **Illustrative Cost Relationships**

visitation that must occur in the future. The closer the current visitation can be to the ideal refill point, the lower is the amount of total future fixed cost associated with traveling to that customer over the course of a year. In fact, if each visitation is exactly at the ideal refill point, than we are at the minimum possible, namely 0 deviation above the minimum total fixed charge for servicing that customer over the course of a year. To the extent that inaccurate or inefficient scheduling results in more of a "topping off" phenomenon upon revisitation, then we are climbing up on the right-hand side of the cost curve in Figure 1 and incurring higher than necessary annual costs associated with both driving to and remaining at that customer.

3.2. Cost of Lateness

We have shown in Figure 1 that R_j is the "ideal refill point" for customer j. What happens if we arrive "late," when the customer's inventory level is below R_j? Certainly if we arrive at the customer and the customer is at 0 product level, then the customer has suffered total stockout, and the cost to the customer and the lost-goodwill cost could be enormous. Even if we arrive at about the 20% level and the ideal refill-point level is 30%, we could suffer some loss of goodwill in the eyes of the customer. Therefore, there is some loss function or cost function associated with arriving late to a customer. In Figure 1, for illustrative purposes, we have shown a linear loss function. More likely in practice, this loss function would be near 0 just to the left of R_j and then slowly and gradually grow at an increasing rate as we get, say, to the 10% level of the tank. As shown in Figure 1, the maximum cost of lateness, incurred when total stockout occurs, is depicted as O_j.

If we incur a cost on a customer visitation due to "being late," i.e., underage, we incur the cost *immediately*. This is in contrast to the cost of "being early," i.e., "topping off," in which case we "book" the appropriate fraction of the fixed cost per visitation that will in actuality be incurred sometime later during the year.

The key observation is this: We can identify a loss or cost function for being late or early to a customer. What we want to do is minimize expected costs. Using a forecasting method which includes both the mean and the variance of hourly product usage, and given that we are dealing with normal random variables, we have a probability density function (p.d.f.) over the "sample space" of tank levels as shown in Figure 1. This p.d.f., indexed by time since the last replenishment, is sketched in Figure 2. Note that time t_k denotes the p.d.f. for the tank-level random variable at time t_k since the last replenishment. We have $t_4 > t_3 > t_2 > t_1$. The figure depicts the p.d.f. of t_1 as the rightmost position and the p.d.f. for t_4 in the leftmost. In this figure, "time moves from right to left." If the last visit to the customer depicted in Figure 2 had not resulted in a full replenishment, then the p.d.f. would start (on "Day 0") on the point at which the tank was refilled, having zero variance on that day, and then moving to the left on subsequent days with growing variance. It is this indexed set of probability laws that we use to compute the expected costs of delivery to a customer, as discussed in the next section.

3.3. Cost of Product Shortfall

There is one additional accounting or accrual cost that may arise upon visitation to a customer. That cost is associated with a visitation that leaves the customer's tank less than full. This event can happen when, due to high customer demand and/or fixed vehicle capacity, the scheduler "allocates" or rations product among the customers on a given tour, or it can happen when one or more "early" customers on a

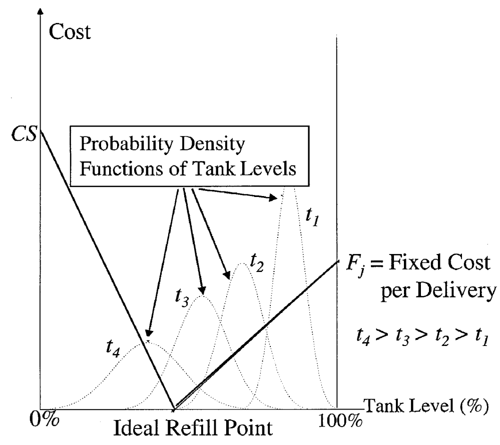

Figure 2 **Probability Density Function of Product in Tank as a Function of Time**

tour require more product than planned, leaving the amount of product for the last customer on the tour less than required to refill his tank.

To evaluate the magnitude of this cost, consider a customer whose ideal refill point is 30%, and suppose the visiting vehicle leaves the tank 90% full. Then, only 6/7 of the planned amount of product to be refilled on an ideal cycle has been placed into the tank, and the next visitation and all subsequent visitations to that customer are moved forward, on average, 1/7 or 14.3% of the customer's fill-refill idealized cycle time. If the customer's fill-refill cycle time is 10 days, for instance, then all subsequent visitations are moved forward an average of 1.43 days. Thus, the next refill would have to be 1.43 days earlier than if the tank had been completely filled. Following the same logic as in the previous subsection, in accounting terms we "charge" or accrue a cost equal to 1/7 of the fixed cost per trip, F_j.

The net effect of a customer product shortfall of X% of tank capacity is the same as an early arrival that finds the tank with an amount of product above the ideal refill point equal to X% of tank capacity. Each event—product shortfall or early arrival—moves, on average, the next and all subsequent visitations forward in time. The amount of time moved forward is equal to a fraction of the fill-refill cycle time, the fraction being the ratio X to the ideal amount of product to fill the tank upon arrival at the optimal refill point. In each case we charge that fraction of the fixed cost per trip to our accrual account, an account we are trying to minimize over a 1-year period.

The differential costing procedures we have proposed here reflect a method to account for annual schedule *compression*. By schedule compression we mean the moving forward in time of an otherwise ideal schedule, visiting customers more frequently than necessary.

3.4. Costing Examples

To illustrate the ideas discussed above, we present three examples in this section. Throughout, we focus on a customer with linear "late delivery" costs, who has an ideal refill point of 35%.

In the first example, depicted in Figure 3, a vehicle arrives at the 50% mark and fills the customer's tank; the delivery results in 50% of tank capacity being delivered, not the ideal 65%. A fraction $(65-50)/65 = 15/65 = 3/13$ of the cycle time is affected in this example in that—compared to an ideal periodic cycle—all future visitations, on average, will be moved forward by that amount. Thus, the fixed accrual charge allocated to this visitation is $(3/13)F_j$.

In the second example, depicted in Figure 4, a vehicle arrives at the 50% tank level and delivers only 40% of full-tank capacity, thereby leaving a product shortfall of 10% of tank capacity. In this case our accrual account is charged for *two* different items: The first charge is for *early arrival*, equal to $(3/13)F_j$, as discussed in the previous example; the second is

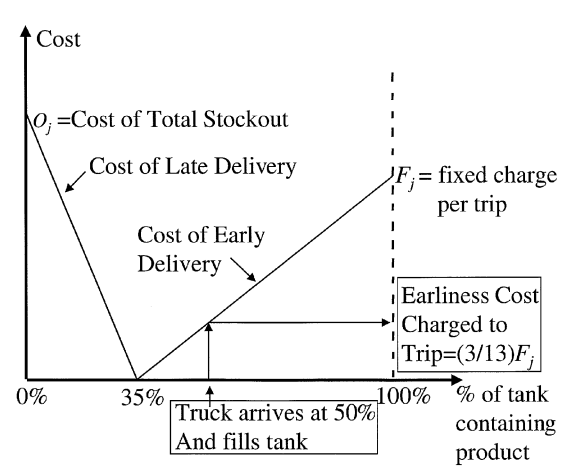

Figure 3 **Vehicle Arrives at 50% Mark and Fills Tank**

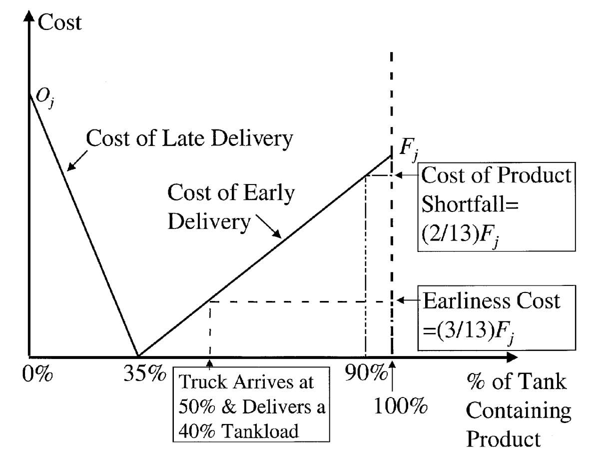

Figure 4 **Vehicle Arrives at 50% Mark and Replenishes Tank to 90%**

for a *product shortfall* equal to 10% of tank capacity or equivalent to a fraction of (10/65) of the ideal or desired delivered amount on a given cycle. The second accrual charge is equal to $(2/13)F_j$. The total fixed charge associated with this visitation is $(5/13)F_j$.

In the last example, shown in Figure 5, the vehicle arrives late at the 20% mark and places in the customer's tank an amount of product equal to 50% of tank capacity. In this case we incur an *immediate cost* due to *late arrival* and an *accrued cost* due to *product shortfall*. The cost of lateness is $(15/35)O_j$ and the accrued cost of product shortfall is $(6/13)F_j$.

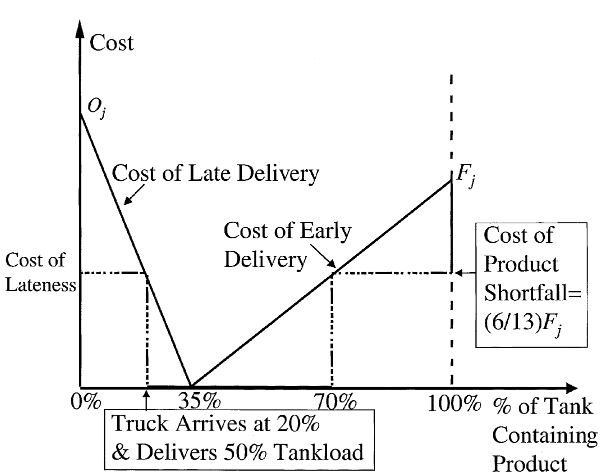

Figure 5 **Vehicle Arrives at 20% Mark and Delivers 50% of a Full Tankload**

3.5. Discussion of Costing Model

The cost structure above is illustrative of what may be done in practice. We offer the following clarifying points:

1. The cost of lateness is shown as linear in this paper only to be illustrative. In practice it may be highly nonlinear, and yet all of our methods apply. We do not explicitly assign costs to the duration of stockout because, in practice, stockout is a very rare event and, if it occurs, is rectified with an immediate emergency replenishment.
2. The "ideal refill point" R_j is included by industrial gas industry convention. In fact, this parameter could be optimized or even set equal to zero, with cost of lateness becoming nonzero prior to emptiness occurring.
3. The parameter F_j, the fixed cost of visiting a customer, is important to our work and yet represents a statistical average quantity whose computation is open to alternative approaches. We see this as an important topic for future research.

4. Computing Expected Costs of Any Particular Customer Delivery

We now develop in detail the equations for determining the costs of given deliveries. In the following, we discretize the product level in each customer's tank, letting

Y_j = an integer random variable representing the current product level in customer j's tank.

$P_{Y_j}(y_j)$ = Prob{product level in customer j's tank $= y_j$ units}, $y_j = 0, 1, 2, \ldots, C_j$;

where

C_j = tank capacity of customer j (measured in standardized "units").

In practice, the level of discretization can be conveniently specified, e.g., 50 or 100 gallons.

Imagine a vehicle carrying x units of product visiting the *last* customer on its tour. Call this last customer "Customer 1." If the amount of product in the vehicle exceeds the amount of "emptiness" in the customer's tank, then the customer will be completely refilled and the vehicle will return to the depot with a nonempty tank. If on the other hand the amount of product contained in the vehicle's tank is less than the emptiness of the customer, then the customer will not obtain a complete refill and the vehicle will return empty. There is also the (unlikely) case in which the emptiness of the customer equals precisely the amount of product on the vehicle, in which case the customer is refilled and the vehicle returns to the depot empty. The amount of emptiness for customer j is $C_j - Y_j$.

Now, consider the events that can lead to positive costs upon the arrival of the vehicle at customer 1:

1. Customer 1's tank can be below the desired refill point, R_1, in which case there is a lateness charge immediately incurred; *or*

2. Customer 1's tank can be above the desired refill point, R_1, in which case there is an accrued earliness charge incurred;

3. The vehicle's product level x can be less than the emptiness of customer 1's tank, in which case there is an accrued product shortfall cost incurred; *or*

4. The vehicle product level x can be greater than customer 1's emptiness, in which case there is a cost incurred for unnecessarily carrying excess product throughout the entire tour, returning it to the depot.

Usually two of the above four events occur, one from the first two possibilities and one from the second; only in the (lucky) case(s) of the tank level being precisely at the desired refill point and/or the vehicle product level exactly equaling the customer's emptiness level would only one (or perhaps zero) costs be incurred.

Since we are computing expected values of these random costs, there will be a positive contribution (in an expected value sense) from each of these four events. Let $E_{Y_1}[C(x)]$ or simply $E[C(x)]$ be the expectation over the random variable Y_1 of the cost of the visitation to customer 1. This cost comprises four components and may be written,

$$\begin{aligned}\mathrm{E}[\mathrm{C}(x)] = \mathrm{E}[\mathrm{C_L}(Y)] + \mathrm{E}[\mathrm{C_E}(Y)] \\ + \mathrm{E}[\mathrm{C_S}(x, Y)] + \mathrm{E}[\mathrm{C_R}(x, Y)],\end{aligned} \tag{1}$$

where:

$$\begin{aligned}
\mathrm{E}[\mathrm{C_L}(Y)] &= \text{Expected cost of } L\text{ateness;}\\
\mathrm{E}[\mathrm{C_E}(Y)] &= \text{Expected cost of } E\text{arliness;}\\
\mathrm{E}[\mathrm{C_S}(x,Y)] &= \text{Expected cost of product } S\text{hortfall,}\\
&\quad\ \text{given } x \text{ units of product in the}\\
&\quad\ \text{vehicle upon arrival at the customer;}\\
\mathrm{E}[\mathrm{C_R}(x,Y)] &= \text{Expected cost of } R\text{eturning to depot}\\
&\quad\ \text{nonempty, given } x \text{ units of product}\\
&\quad\ \text{in the vehicle upon arrival}\\
&\quad\ \text{at the customer.}
\end{aligned}$$

The expected costs of lateness and of earliness have the same functional form for any customer j, $j = 1, 2, \ldots, n$. Referring to Figure 1 and to the definitions, we can now write

$$\mathrm{E}[\mathrm{C_L}(Y)] \equiv g_{1j} = \sum_{y_j=0}^{\lfloor C_j R_j \rfloor} \big(O_j/C_j R_j\big)\big(C_j R_j - y_j\big) p_{Y_j}(y_j) \tag{2}$$

$$\begin{aligned}\mathrm{E}[\mathrm{C_E}(Y)] \equiv g_{2j} = \sum_{y_j=\lceil C_j R_j \rceil}^{C_j} \big(F_j/[C_j - C_j R_j]\big) \\ \times \big(y_j - C_j R_j\big) p_{Y_j}(y_j).\end{aligned} \tag{3}$$

In interpreting these formulas, the quantity $\lfloor x \rfloor$ denotes the largest integer not exceeding x and $\lceil x \rceil$ denotes the smallest integer not less than x.

Regarding the cost of shortfall, recall that we charge (to a fictitious accrual account) a fraction of the fixed cost per visit, F_j, the fraction being equal to the ratio of the amount of shortfall or emptiness left in the tank to the ideal refill quantity. The cost of shortfall is a function of x, the amount of product currently on board the vehicle; here the assumption is that all of that product will be placed into the customer's tank if the tank can accommodate it. We must ask what happens in a most unusual situation, namely, when a vehicle arrives late, has very little product on board, and leaves the customer with a remaining lateness condition, namely with a shortfall exceeding the normal ideal refill quantity. In this case we choose to charge more than F_j for that shortfall, the exact amount linearly prorated by the slope of the "right-hand" cost function. We can now write for the expected shortfall cost,

$$\mathrm{E}[C_S(x,Y)] \equiv g_{3j}(X) = \sum_{y_j=0}^{C_j-x} \big([C_j-x-y_j]/[C_j-C_jR_j]\big)F_j p_{Y_j}(y_j). \quad (4)$$

In examining the upper summation limit of this equation, note that no terms are added if the quantity x equals or exceeds the customer tank capacity C_j; this is clearly correct since in such a case there is no possibility of product shortfall.

Our remaining expected cost is the expected cost of returning to the depot nonempty. Here we assume a cost associated with the entire trip equal to C_T. This cost would include the cost of the driver, fuel, vehicle depreciation, and perhaps other costs as well. We charge to "overhead" as a differential cost above the minimum possible the prorated cost of carrying for the entire trip a quantity of product that is delivered to no customer. For instance, if the vehicle returns with 10% of its dispatched quantity still in the tank, then we "charge" an overhead account 10% of the trip cost, representing 10% wasted effort. We define

$$W = \text{vehicle capacity}.$$

We assume that the vehicle is filled to its capacity W upon commencement of its tour. If it arrives at customer 1 with a quantity x units of product on board, and if the customer's tank has emptiness equal to C_1-y_1, then the vehicle delivers x or C_1-y_1 units of product, whichever is smaller. The vehicle will return nonempty whenever the customer's emptiness is less than x. The amount of product remaining in the vehicle in this case would be $x-[C_1-y_1]=x+y_1-C_1$. Now we can write,

$$\mathrm{E}\big[C_R(x,Y)\big] = \sum_{y_1=\lceil C_1-x\rceil}^{C_1} \big([x+y_1-C_1]/W\big)C_T p_{Y_1}(y_1). \quad (5)$$

We have completed our depiction of the expected differential costs associated with visiting the last customer on a tour. In the next section we show how these concepts can be used in dynamic-programming formulations.

5. Dynamic-Programming Algorithms for Product Allocation

We are now in a position to analyze the expected differential costs of one preplanned trip and derive an "optimum decision rule" for the vehicle driver. The decision rule will be of the form,

> If there are x gallons left in the vehicle and the driver is currently visiting customer j on a tour of n customers, the driver will either fill customer j's tank or place $T_j(x)$ gallons in his tank, whichever is less.

The quantity $T_j(x)$ is a threshold number of gallons, a maximum possible number of gallons to be left with customer j, so that the vehicle will retain sufficient quantities of product for the remaining customers on the tour. The quantity x is a "state variable" for the vehicle, representing the net cumulative effect of previous customer off-loadings, and j is the stage variable.

5.1. Basic Dynamic-Programming Model (DPl)

We number the customers on a trip in *reverse* numerical order of visitation. Hence we visit first Customer n, then $n-1$, then $n-2, \ldots$, and finally customer 1. We define the *optimal value function*,

$f_j(x) =$ expected differential cost of completing a tour of j customers following an optimal policy, given x gallons of product currently in the vehicle's tank.

Suppose that the vehicle leaves the last customer, customer 1, with x units of product still on board. In that case we know that a prorated cost of the trip, with a proration factor equal to x/W, will be charged to "overhead." Recall that this charge is due to carrying around for the entire trip a quantity of product x, eventually returning it to the depot. So, on the trip back to the depot, the vehicle incurs a final charge equal to $(x/W)C_T$. Thus, we have the dynamic-programming boundary condition, or "terminal cost function,"

$$f_0(x) = (x/W)C_T. \quad (6)$$

Suppose the vehicle arrives at customer j with x gallons in the vehicle's tank and the driver pumps

v gallons into customer j's tank, leaving a quantity $(x-v)$ in the vehicle $(v<x)$. The expected total system differential cost for completing this tour, giving v gallons to customer j and following an optimal policy thereafter in visiting customers $j-1, j-2, \ldots, 1$, is $g_j(v)+f_{j-1}(x-v)$, where $g_j(v)$ is defined to be the sum of the *three* differential costs associated with visiting customer j:

$$\begin{aligned} g_j(v) &= \mathrm{E}_j[\mathrm{C}(x), v] \\ &= \mathrm{E}_j[\mathrm{C_L}(Y_j)]+\mathrm{E}_j[\mathrm{C_E}(Y_j)]+\mathrm{E}_j[\mathrm{C_S}(v, Y_j)], \\ &= g_{1j}+g_{2j}+g_{3j}(v). \end{aligned} \tag{7}$$

In the dynamic program, the amount of product v that the driver leaves at customer j is either the total emptiness of j's tank (in which case we leave customer j with a filled tank) or the optimal threshold value $T_j(x)$, whichever is smaller. We now write the fundamental recursion of the dynamic program:

$$\begin{aligned} f_j(x) = g_{1j}+g_{2j}+\min_{0\le T_j(x)\le x}&\Bigg\{\sum_{y_j=0}^{C_j-T_j(x)}\Big(\big[C_j-T_j(x)-y_j\big] \\ &/\big[C_j-C_jR_j\big]\Big)F_j p_{Y_j}(y_j)+\sum_{Y_j=0}^{C_j} f_{j-1} \\ &\times\big(x-\min\{C_j-y_j, T_j(x)\}\big)p_{Y_j}(y_j)\Bigg\}, \\ &\forall j=1,2,\ldots,n \end{aligned} \tag{8}$$

This dynamic program is solved starting with the last customer on the tour, customer 1, and works backwards to customer 2, customer 3, and eventually to customer n. The boundary condition utilizing the cost of returning to the depot nonempty (see Equation (6)) enters the recursion at the first iteration when evaluating the optimal value function at customer 1. Since the cost of returning nonempty with a quantity x on board the vehicle is a linearly increasing function of x, and since the cost of leaving customer 1 with an emptiness equal to v is an increasing function of v, "it always pays" to fill up customer 1. The dynamic program always finds that $T_1(x)=C_1$, the maximum possible value.

The dynamic-programming formulation has nice separability properties that allow inclusion of various important side constraints. We can easily incorporate rules such as, "Place 500 gallons in customer 3's tank, no matter what the solution says, as long as 500 gallons are present in the vehicle." Or, "Fill up customer 7's tank, if sufficient quantity of product is on the vehicle, no matter what the dynamic-programming solution says." Statements such as these simply further delimit the "search space" for an optimal value of the threshold decision variable.

Example. We consider a simple example with $n=3$ customers and a vehicle with capacity of $W=1{,}400$ gallons and tour cost $C_T=40$. As usual, customer 1 is the last customer to be visited and customer 3 is the first. The parameters for this problem are as follows:

Customer tank capacities: $C_1=1{,}000$, $C_2=1{,}200$, $C_3=800$

Ideal refill percentages: $R_1=30\%$, $R_2=30\%$, $R_3=31.25\%$

Fixed costs per trip: $F_1=28$, $F_2=32$, $F_3=22$

Costs of total stockout: $O_1=210$, $O_2=280$, $O_3=175$

The relevant probability distributions are shown in Table 2.

The dynamic-programming optimal solution to the problem is to fill the tank for each customer with as much product as possible. The resultant optimal policy is the same in this case as a naive strategy often employed in the field: "FILLALL" (fill tanks of all customers, if possible). In industry vernacular, FILLALL is sometimes also called, "FILL, FILL, DUMP," implying that the driver "dumps" into the last customer's tank whatever is left in his vehicle. The expected value of optimal cost function for this tour is 209.19.

The "normalized fixed cost per trip,"

$$CE_j = F_j/\big(C_j[1-R_j]\big),$$

represents the slope of the linear cost function associated with early delivery and product shortfall (see

Table 2 Y_j **and** P_j **Values for** $j=1,2,3$

Y_1	0	300	600	900	1000
P_1	0.2	0.3	0.1	0.2	0.2
Y_2	0	300	600	900	1200
P_2	0.3	0.1	0.2	0.2	0.2
Y_3	0	400	600	800	
P_3	0.2	0.4	0.3	0.1	

Figure 1), but with the abscissa of Figure 1 rescaled to $[0, C_j]$. One can think of CE_j as the marginal cost per unit of product shortfall. As an example, suppose $F_j = \$100$, $C_j = 4{,}000$ gallons, and $R_j = 0.30$. Then $CE_j = \$0.0357$, implying a 3.57 cent "accrual charge" for each gallon of product shortfall on any trip to customer j.

We now undertake a modest sensitivity analysis. In Table 3 we present the optimal solution (DPl SOLUTION) and its cost (DPl COST) for a range of CE_2 values ($CE_1 = 0.04 = CE_3$) from 0 to 10. We also compare the optimal solution to the solution of two other strategies:

(i) FILLALL (naively defined, not derived by dynamic programming) and
(ii) $T_j(x) = \min\{x, E[C_j - Y_j]\}$ for $j = 2, 3$; $T_1(x) = \min\{x, 1000\}$

i.e., for all customers except for the last one, the tank cannot be filled more than the expected amount of emptiness. We call this strategy "EXEM" (Expected Emptiness).

As can be seen from Table 3, for small values of CE_2 the optimal solution is to fill to the top the tanks of customers 3 and 1 but not to fill to the top the tank of customer 2 (as expected). Then there is an intermediate range of CE_2 values where FILLALL is optimal. For larger CE_2 values the optimal solution is to fill the tank of customers 2 and 1 to the top but not to fill to the top the tank of customer 3, the first one in the tour (when CE_2 becomes larger, $T_3(1{,}400)$ becomes smaller or remains the same). We can also see from the table that EXEM is never optimal and that the difference between DP1 COST and cost of the other strategies can be substantial.

Table 3 Optimal Solution and Cost of DP, FILLALL, and EXEM as a Function of CE_2

CE_2	DP1 SOLUTION (customers: 3,2,1)	DP1 COST	FILLALL COST	EXEM COST
0	FILL, $T_2(x) = \max(0, x - 700)$, FILL	196.42	199.75	201.09
0.02	FILL, $T_2(x) = \min\{\max(0, x - 100), 1200\}$, FILL	204.27	204.47	208.45
0.04	FILLALL	209.19	209.19	215.81
0.06	FILLALL	213.91	213.91	223.17
0.08	$T_3(1400) = 700$, FILL, FILL	218.61	218.63	230.53
0.1	$T_3(1400) = 500$, FILL, FILL	223.14	223.35	237.89
0.12	$T_3(1400) = 500$, FILL, FILL	227.34	228.07	245.25
0.14	$T_3(1400) = 400$, FILL, FILL	231.52	232.35	252.61
0.16	$T_3(1400) = 200$, FILL, FILL	235.43	237.62	259.97
0.2	$T_3(1400) = 200$, FILL, FILL	242.23	246.95	274.69
0.3	$T_3(1400) = 200$, FILL, FILL	259.23	270.55	311.49
10	$T_3(1400) = 200$, FILL, FILL	1908.23	2559.75	3881.09

Now we wish to find a sufficient condition such that "FILL, FILL, DUMP" is an optimal product-allocation policy. In considering the potential optimality or nonoptimality of any policy, note that the expected costs of lateness and earliness g_{1j} and g_{2j} in Equation (8) may be ignored, as their values do not depend on quantity of product delivered. Thus, we are concerned only with expected costs due to product shortfall and due to returning to the depot nonempty. Note also that FILL, FILL, DUMP minimizes the amount of product returned to the depot and thus minimizes that expected cost. To avoid technical complexities having no practical value, we assume in the following lemmas and theorems that product quantities are continuous r.v.s and that with probability one, each tank on the tour has strictly positive emptiness. (The reader may note that most of the product quantity computations carried out in the paper use discrete r.v.s. This is for illustrative computational convenience only. The continuous r.v. assumption is the one most likely to represent practical operations.)

We now can state:

Lemma 1. *Suppose for a given n-customer tour we have*

$$CE_n \geq CE_{n-1} \geq \cdots \geq CE_1$$

For such a tour, FILL, FILL, DUMP is an optimal product-allocation policy.

Proof. (contradiction) Suppose at stage j with x units of product in the vehicle, we place $v < x$ units of product into the tank, deliberately leaving an emptiness $(e_j - v) > 0$, where $e_j \equiv$ emptiness of tank j immediately before product delivery. Then, considering that $(e_j - v)$ additional units of product over the minimum possible will have to be delivered to customers $j-1, j-2, \ldots, 1$, and/or returned to the depot, we have thereby incurred an extra cost above the minimum possible cost at least equal to $(CE_j - CE_{j-1})(e_j - v) \geq 0$. Thus, it is optimal to fill customer's j's tank. □

In practice, this lemma carries several implications. First, it has no relevance for one-stop tours. For two-stop tours it suggests first visiting that customer having the higher *CE* value. Tours having three or more stops are usually designed by TSP algorithms (TSP = Traveling Salesman Problem). By considering both clockwise and counterclockwise traversals of such tours, one may find that one of the two TSP solutions satisfies the theorem and thus that FILL, FILL, DUMP is optimal. For instance one-third of all three-stop tours will have this property. In general, for an n stop TSP tour, the chance that the lemma holds for one of the two possible tour directions is $2/n!$ Moreover, in comparing the cost figures, it may be advantageous on occasion to override the TSP solution, incurring (usually small) detours above the TSP solution in order to create a FILL, FILL, DUMP tour, thereby yielding the minimum possible *total* expected cost. However, in general, for tours of three or more stops the lemma does not hold, and the value of the DP approach becomes more and more apparent as n grows.

Building from the logic above, we can also state

LEMMA 2. *Suppose for any n-customer tour we have*

$$CE_j \geq CE_i \quad \text{for all } i = j-1, j-2, \ldots, 1; \quad j = n, n-1, \ldots, 1.$$

For such a tour it is optimal to fill the tank of customer j, if possible.

PROOF. Follow the same logic as in the proof of Lemma 1, with CE_{j-1} replaced by $\max\{CE_{j-1}, CE_{j-2}, \ldots, CE_1\}$. □

Thus, at any point in the tour, if the vehicle is delivering to a customer whose *CE* value is at least as large as the *CE* value for any other subsequent customer to be visited on the tour, it is optimal to attempt to fill the tank of that customer.

For a customer having an optimal threshold of product to deliver, we need the following:

THEOREM 1. *The optimal cost function $f_j(x)$ is convex over $[0, W]$.*

PROOF. See Appendix.

Let us now interpret Lemmas 1 and 2 in view of the convexity theorem. Suppose we are at stage j with normalized cost of shortfall $CE_j \geq CE_{j-i}$ for all $i = 1, 2, \ldots, j-1$. Considering the composite construction of the expected cost function (see proof in the Appendix), averaged over all realizations of the possible piecewise linear convex functions displayed in the theorem, the maximum negative slope of $f_{j-1}(x)$ cannot exceed CE_j. Thus it is not possible to gain more savings in shortfall costs by holding back product from customer j in hopes of achieving higher savings at some future customer $j-i$, $i = 1, 2, \ldots, j-1$. Hence, we fill (if possible) the tank of any customer j having $CE_j \geq CE_{j-i}$ for all $i = 1, 2, \ldots, j-1$, thereby verifying Lemmas 1 and 2.

We have only one other type of customer to consider in an n-customer tour, and that type of customer is considered in:

LEMMA 3. *Suppose for any n-customer tour we have*

$$CE_j < CE_i \text{ for at least one } i = j-1, j-2, \ldots, 1.$$

Then for an amount of product in the vehicle equal to x, there exists an optimal maximum amount $v = v^*(x)$ of product to deliver to customer j, and that is found as follows:

1. *If $f_j(x)$ is differentiable over $[0, W]$, then attempt to find that value for x, call it $\overline{X}_j$, such that*

$$CE_j = \frac{d}{dx}\{f_{j-1}(x)\} \text{ at } x = \overline{X}_j.$$

1.1. *If on a tour at customer j we have $x < \overline{X}_j$, then place zero product in the tank of customer j, i.e., $v^*(x) = 0$ for $x < \overline{X}_j$.*

1.2. *Otherwise, place an amount of product equal to $\min\{x - \overline{X}_j$, emptiness of tank $j\}$ into customer j's tank, i.e., $v^*(x) = \min\{x - \overline{X}_j, e_j\}$ for $W \geq x \geq \overline{X}_j$.*

2. *If $f_j(x)$ is nondifferentiable over $[0, W]$, then attempt to find a minimum value of x, call it $\overline{X}_j$, satisfying*

$$f^R_{j-1}(x) < CE_j$$
$$f^L_{j-1}(x) \geq CE_j, \text{ at } x = \overline{X}_j,$$

where $f^R_{j-1}(x)$ and $f^L_{j-1}(x)$ are right and left derivatives, respectively, with respect to x.

2.1. *If on a tour at customer j we have $x < \overline{X}_j$, then place zero product in the tank of customer j, i.e., $v^*(x) = 0$ for $x < \overline{X}_j$.*

2.2. *Otherwise, place an amount of product equal to* $\min\{x - \overline{X}_j$, *emptiness of tank* $j\}$ *into customer* j*'s tank, i.e.,* $v^*(x) = \min\{x - \overline{X}_j, e_j\}$ *for* $W \geq x \geq \overline{X}_j$. *(See Figure 6.)*

Proof. Since $CE_j < CE_i$ for at least one $i = j-1, j-2, \ldots, 1$, we must have

$$\left| \frac{d}{dx}(f_{j-1}(x))\bigg|_{x=0} \right| > CE_j,$$

because there exists at least one customer in the remaining $(j-1)$ customer tour whose incremental cost of shortfall exceeds that of customer j and thus is preferred for delivery for sufficiently small x. Now if we also have

$$\left| \frac{d}{dx}(f_{j-1}(x))\bigg|_{x=W} \right| > CE_j,$$

then by convexity and continuity there exists no value for x, $0 < x < W$, satisfying the "slope conditions" of 1 or 2 above, and in that case $v^*(x) = 0$ for all x, $0 < x < W$. If, however, we have

$$\left| \frac{d}{dx}(f_{j-1}(x))\bigg|_{x=W} \right| \leq CE_j,$$

then by convexity and continuity we must have the slope condition of 1 or 2 satisfied at some point $\overline{X}_j \in [0, W]$. If at customer j we have $x < \overline{X}_j$, then the marginal cost reduction of placing product into the tank of customer j is

$$CE_j < \left| \frac{d}{dx}(f_{j-1}(x))\bigg|_{x<\overline{X}_j} \right|,$$

indicating that it is optimal to place no product into that tank and save all of the amount x for delivery to customers $j-1, j-2, \ldots, 1$. If at customer j we have $x \geq \overline{X}_j$, then the marginal cost reduction of placing product into the tank of customer j is

$$CE_j \geq \left| \frac{d}{dx}(f_{j-1}(x))\bigg|_{x\geq\overline{X}_j} \right|,$$

indicating that it is optimal to place product into that tank up to the point that the marginal savings in expected (shortfall plus overhead) cost becomes less than the expected incremental savings in cost attributable to delivering the remaining product to the remaining $j-1$ customers. □

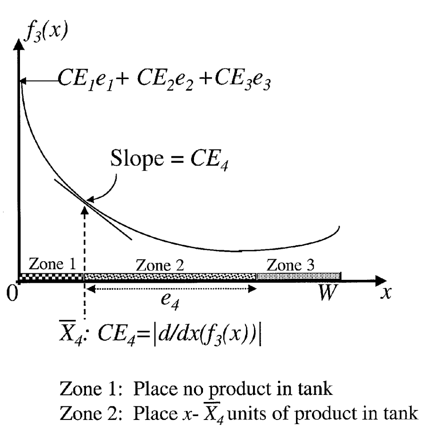

Figure 6 Three Rules for Amount of Product Delivery to Customer 4, as a Function of x

Theorem 2. *Consider all* $n!$ *orderings of replenishment visits to n given customers.*

(a) *An ordering having the least expected marginal cost is one in which the* FILL, FILL, DUMP *policy is optimal, i.e., the customers' successive CE values form a monotone nonincreasing sequence.*

(b) *An ordering having the largest expected marginal cost is one in which the customers' successive CE values form a monotone nondecreasing sequence.*

Proof. See Appendix.

The theorem raises some interesting issues. Usually the ordering of customer visitations with industrial gases (or with any number of other delivered products) is determined solely by geographical proximity analysis, often using a TSP-type algorithm. The theorem shows that other costs are also associated with alternative orderings, costs due to uncertainties in customer tank levels. As an example, we may be "lucky," and have a monotone nonincreasing sequential ordering of *CE* values correspond exactly to a TSP solution. This would mean that minimal transportation costs and minimal expected marginal (dynamic-programming) costs are achieved by the same customer ordering. But if the transportation network is bidirectional, the TSP is indifferent to clockwise or counterclockwise traversals of the tour. If the optimal ordering just cited corresponds to, say, a clockwise traversal, then a counterclockwise traversal would yield the same TSP transportation cost but the worst possible expected marginal (dynamic-programming) cost. That is because a monotone nonincreasing sequential ordering of *CE* values in one direction implies just the opposite—a monotone nondecreasing sequential ordering—in the other. More generally, additional research is needed to integrate the TSP and dynamic-programming cost constructs into one unified model.

One simple application of the theorem is as follows: Suppose that the n customers extend in one radial line from the depot and that travel is Euclidean. Then, "everything else being equal," the optimal policy is for the filled vehicle to travel to the farthest customer first, and then visit each, working back towards the depot, all the time using a greedy FILL, FILL, DUMP policy—the optimal policy in this case. If the vehicle runs out of product before filling the last of the n customer(s), then the cost of such shortfall is the minimum possible, as the cost of replenishing those close-to-the-depot customers is lower than that of replenishing distant customers.

In considering the reversal of customer orderings, such as customer o_i and o_{i+1} in this theorem, one need not actually visit o_i before o_{i+1}. The driver of the vehicle, while arriving at o_{i+1} could call ahead and learn of the tank level status of o_i. Then, since one wants to fill customer o_i if possible (i.e., to deliver e_{o_i} units of product), the driver then just "reserves" in his vehicle's tank the quantity e_{o_i} for delivery to o_i. The mathematics then work out identically to what we have shown here. More generally, this suggests that customer tank-level monitoring equipment would best be spent on customers having high *CE* values, especially if one cannot deliver to customers in the optimal sequential ordering argued for here.

5.2. DP1 Revisited: To Dump or Not to Dump?

We now modify our original formulation to allow a type of decision that is often implemented in an ad hoc manner in practice. Consider a vehicle that has just filled the tank of customer 1, the last *scheduled* customer of the current tour. Suppose the vehicle still has 200 gallons of product on board. The driver may call the scheduler and ask whether he should return to the depot with the 200 gallons or execute a *detour* to another (unscheduled) customer and "dump" the 200 gallons at that customer.

The dynamic-programming formulation discussed above does not consider this very real possibility. However, a simple modification will result in a policy that optimally addresses the "to dump or not to dump" question, as well as determines allocations to scheduled customers on the tour. The only modification required in the dynamic-programming formulation is in the terminal cost function. That function, in addition to representing the overhead cost of returning to the depot nonempty, will now include the possible cost reduction associated with executing the aforementioned detour.

To understand the logic, let's continue with our example. Suppose that the overhead cost associated with returning to the depot with 200 gallons of product is \$75. The driver could execute a detour to a "dump customer" costing an additional \$50 in direct expenses of driver time, fuel, and mileage, etc. Suppose further that the dump customer's tank can accommodate the entire 200 gallons of product and that the 200 gallons represents two days of regular product usage by that customer, a customer whose regular intervisitation time is eight days. In an average sense, we have moved the next visit to this customer 2 days backward in time, perhaps in effect deferring the "last December visit" until January of the next year. Thus, we can accrue a savings for that customer equal to one quarter (2/8) of the associated fixed charge per visit. (In this accounting scheme, the fixed charge per visit is per *scheduled* visit, and we are not counting the detour as a scheduled visit.) If the fixed charge per scheduled visit is \$100, then moving the next trip there backward by 2 days accrues a net savings of $\$100/4 = \$25$. In addition, we are now returning to the depot empty, so we are saving overhead an amount equal to \$75, yielding a net potential savings of $\$75 + \$25 - \$50 = \$50$. With a positive savings like this, we would choose to dump the product at the targeted customer, assuming that customer is the maximum savings customer. In general, we must examine an entire set of potential dump customers and select the best, assuming that at least one results in positive savings; if not, we simply return directly to the depot with all the remaining product on board.

This is the essence of the optimal dump policy:

> Dump at a chosen dump customer if the direct cost of executing the detour is less than the sum of savings accrued by deferring the next visit to that customer and the savings to overhead attributable to reducing or eliminating the overhead charge associated with returning to the depot nonempty.

Here, the savings attributed to deferral of the next trip to the "dump customer" is a simple expected value quantity, based on long-term averages.

We can now express the required new terminal cost function. We let

$$C_{trip}(u, v) = \text{direct cost of executing a trip of length } u \text{ miles, stopping a fixed time } v \text{ at the customer.}$$

Recall that the last scheduled customer in the regular tour is (arbitrarily) called customer 1, presumed to be located at node 1 of the transportation network. We assume that the depot is located at node 0 and that the proposed dump customer k is located at node k of the transportation network. Following standard transportation network notation, we let d_{ij} be the (minimum) transportation distance between nodes i and j. The fixed time charge per stop can be generalized to T_k fixed time per stop (in minutes) for customer k. Then we can write:

$$CD_k = C_{trip}(d_{1k} + d_{k0} - d_{10}, T_k).$$

The new terminal cost function may be written,

$$\begin{aligned} f_0(x) = \min\Bigg\{ (x/W)C_T, \min_{k\in\text{DUMP}} \Bigg(CD_k \\ - \sum_{y_k=0}^{C_k} \Big(\big[\min\{x, C_k - y_k\}/(C_k - C_k R_k)\big] F_k \\ - \big[\max\{0, x - C_k + y_k\} C_T/W\big] \Big) P_{Y_k}(y_k) \Bigg) \Bigg\} \end{aligned} \tag{9}$$

where in the equation we have defined

$$\text{DUMP} = \text{set of customers that are eligible for a product dump.}$$

In words, Equation (9) says the following: The minimum cost of departing from customer 1 with x units of product onboard the vehicle, following an optimal dump policy, is *either* the "old cost" of returning directly to the depot with x units of product on board *or* the net cost of detouring to the *best* dump customer, whichever is smaller. The net cost of detouring to prospective dump customer k is the sum of three quantities: (1) the direct travel- and time-related cost of the detour, CD_k; (2) the "negative cost" attributable to pushing backward in time the next scheduled delivery to customer k; (3) the cost related to the possibility of still returning to the depot nonempty, due to the chance that customer k cannot accommodate the entire on-board product quantity x. The latter two costs, one negative and one positive, are computed probabilistically, weighing each possibility by the probability that a given amount of product is in customer k's tank.

The formulation DP1 with optimal end-of-tour dumping will, for each value of on-board product x, reduce or leave unchanged the terminal cost function. Since leaving the last *scheduled* customer on the tour with x gallons of product on board is no longer as costly as it was without optimal dumping (or, at least it is no more costly), the new algorithm should be less "sensitive" to the need to distribute all on-board product to scheduled customers. Because of this behavior, we expect that the optimal state-dependent thresholds computed within the tour will be less than or equal to the threshold values found without the dump option.

5.3. A Dynamic-Programming Algorithm for Product Allocation: Measure Customer's On-Hand Product First

Dynamic-programming formulations of sequential decision problems can be created in various ways, depending on rules of operation of the process being modeled. The general rule is, the more information we use at a particular stage, the better our decisions will be. In the DP1 formulation, we took as the state variable the number of units of product currently on board the vehicle. We decided what to do when visiting a customer without knowing or at least without paying attention to the current amount of product in that customer's tank; we "pumped first, and 'asked questions' later." An alternate approach is to measure the customer's current on-hand inventory first, then make a decision on how much to pump from the vehicle into the customer's tank (or perhaps, how much to pump from the customer's tank into the vehicle!).

The decision rule is now of the form:

> If there are x gallons left in the vehicle currently visiting customer j on a tour of n customers and customer j's tank (of capacity C_j) has Y_j units of product remaining, the driver will leave customer j with $T_j(x+Y_j)$ units of product in his tank.

We note that the decision variable $T_j(x+Y_j)$ is the total quantity of product in the customer's tank at departure of the vehicle from the customer; in contrast to §4.1, it is not, in general, the amount of product *pumped* from the vehicle into the customer's tank. The quantity $T_j(x+Y_j)$ is an optimal number of units of product, so that the vehicle will retain sufficient quantities of product for the remaining customers on the tour. By allowing the optimum to depend on $x+Y_j$, the amount of product remaining on the vehicle *and* in the present customer's tank, we are managing uncertainty by capitalizing on (1) the results of previous product deliveries on the tour, each of which removed a random number of gallons from the vehicle, and (2) the amount of product in the current customer's tank, which also represents the culmination of random product utilization over the days since the last delivery to that customer. In certain situations, it may even be optimal to *remove* product from customer j's tank. The quantity $x+Y_j$ is a "state variable" for the vehicle/customer pair, representing the net cumulative effect of vehicle off-loading and product usage by customer j.

As before, we number the customers on a trip in *reverse* numerical order of visitation. We define the *optimal value function*,

$f_j(z) =$ expected differential cost of completing a tour of j customers following an optimal policy, given z units of product currently at the *delivery site*, both in the vehicle's tank and in the customer's tank.

As before, we have the usual terminal cost function,

$$f_0(x) = (x/W)C_T. \tag{10}$$

(If during the tour the driver actually pumped product from one or more customers' tanks, then x is not the quantity "carried around" during the entire trip; a minor refinement in the terminal cost function can be made if this effect needs to be captured.)

However, we have to start the tour in a special way. The vehicle leaves the depot with x units of product on board. We would like to know the expected value of the entire tour, from the moment of leaving the depot. Calling the depot "customer '$n+l$' of the tour," we can write

$$f_{n+1}(x) = E\big[C_E(Y_n)\big] + E\big[C_L(Y_n)\big] + \sum_{y_n=0}^{C_n} P_{Y_n}(y_n) f_n(x+y_n), \tag{11}$$

where we have taken the expected costs of earliness and lateness, respectively, associated with visiting customer n at stage $n+1$. Writing Equation (11) provides us with the state value transition that the vehicle experiences when going from the depot with a load of x units to the first customer, having a probabilistic amount of product Y_n in the ground, and thus yielding a value for the (summed) state variable at stage n that itself is probabilistically determined. Following the convention of Equation (11), in this version of the DP model, we will always take the expected value of customer j lateness and earliness, respectively, at stage $j+1$.

The decision variable that we optimize in this dynamic program is the state-dependent quantity $T_j(z)$, the optimal number of units of product to leave in the present customer's tank, given that we currently have z units of product on the vehicle *and* in the customer's tank.

In this dynamic program, which we call DP2, the amount of product that remains with customer j is the optimal value $T_j(x+Y_j)$. We now write, for all $j = n, n-1, \ldots, 1$, the fundamental recursion of the dynamic program, DP2,

$$f_j(z) = E\big[C_E(Y_{j-1})\big] + E\big[C_L(Y_{j-1})\big] + \min_{0 \le T_j(z) \le \min(C_j, z)} \times \left\{ \frac{C_j - T_j(z)}{C_j - C_j R_j} F_j + \sum_{y_{j-1}=0}^{C_{j-1}} P_{Y_{j-1}}(y_{j-1}) f_{j-1} \big(z - T_j(z) + y_{j-1}\big) \right\} \tag{12}$$

In using Equation (12) for $j = 1$, we assume that "Y_0" always equals 0 and that $C_E(Y_0) = C_L(Y_0) = 0$. Again, "it always pays" to fill up customer 1.

A simple change of the terminal cost function, along the lines indicated in §4.2, will transform this new dynamic-programming formulation into one that also includes optimal dumping.

In Table 4 we compare the four DP formulations: DP1 with and without dumping, and DP2 with and without dumping. The table shows that for all CE_2 values, DP2 is at least as good as DP1, and dumping is always superior to no dumping.

Conclusions

In this paper we have shown how to formulate and model the vehicle product-delivery problem in which the amount of product needed by each customer on a route is known only in a probabilistic sense. A vehicle driver traversing the route must continually address the trade-offs inherent in fulfilling the current customer's immediate product needs versus weighing the needs of other customers to be visited later on the route. We have shown how to use stochastic dynamic programming to solve the problem. Four different versions of the dynamic program have been proposed, utilizing two different state variables and allowing the possibility of "dumping" excess product on the homeward leg of the tour. Certain properties of the dynamic-programming model were derived that allow rapid and efficient computation of optimal delivery policies. We see this work as a prerequisite to additional work on selecting optimal routes, their geography, and timing, within this modeling structure.

Table 4 **Optimal Cost of DP1, DP2, With and Without Dump, as a Function of CE_2**

CE_2	Cost DP1 No Dump	Cost DP1 with Dump	Cost DP2 No Dump	Cost DP2 with Dump
0	350.9	336.8	346.0	330.8
0.1	415.7	405.5	414.5	404.6
0.2	468.5	460.5	468.5	460.5
0.3	498.4	492.4	498.4	492.4
0.4	523.0	516.6	523.0	516.6
0.5	546.6	540.2	546.6	540.2
0.6	570.2	563.8	570.2	563.8
0.8	615.5	608.7	615.5	608.7
1.0	657.4	650.3	657.4	650.3
2.0	813.5	822.7	831.5	822.7
5.0	1341.5	1332.7	1341.5	1332.7
10	2191.5	2182.7	2191.5	2182.7

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Appendix

Proofs of Two Theorems

THEOREM 1. *The optimal cost function $f_j(x)$ is convex over* $[0, W]$.

PROOF OF THEOREM 1. Define $CE^{[i]} = i$th largest in the set $\{CE_1, CE_2, \ldots, CE_n\}$, for $i = 1, 2, \ldots, n$, with ties broken arbitrarily. Similarly, customer $[j]$ is that customer having jth largest CE value. Any given tour is associated with a set of experimental values $\{e^{[1]}, e^{[2]}, \ldots, e^{[n]}\}$ for the n random variables $\{E^{[1]}, E^{[2]}, \ldots, E^{[n]}\}$ corresponding to emptinesses of the n respective customer tanks. Let $g_j(x) =$ expected cost of executing a given j customer tour with perfect information about tank emptinesses, where costs of earliness and lateness are excluded, and where there are x units of product available to deliver. The boundary condition at $j = n$ is $g_n(0) = \sum_{j=1}^{n} e^{[j]} CE^{[j]}$. If $0 \le x \le e^{[1]}$, then to minimize cost of product shortfall one follows a steepest descent, delivering all product to Customer [1], and $g_n(x) = \sum_{j=1}^{n} e^{[j]} CE^{[j]} - CE^{[1]} x$, for $0 \le x \le e^{[1]}$. Similarly, if $e^{[1]} \le x \le e^{[1]} + e^{[2]}$, again following steepest descent, Customer [1] will be filled, and the remaining product will be delivered to customer [2]. Hence, $g_n(x) = \sum_{j=1}^{n} e^{[j]} CE^{[j]} - CE^{[1]} e^{[1]} - CE^{[2]}(x - e^{[1]})$, $e^{[1]} \le x \le e^{[1]} + e^{[2]}$. This pattern continues, tracing a convex piecewise linear nonincreasing function, until either (1) $x = W$, signifying the end of the interval $[0, W]$, or (2) $x = \sum_{j=1}^{n} e^{[j]} < W$, where we have $g_n(\sum_{j=1}^{n} e^{[j]}) = 0$. In the latter case, once $W > x > \sum_{j=1}^{n} e^{[j]}$, then a quantity $(x - \sum_{j=1}^{n} e^{[j]})$ is returned to the depot, incurring an overhead cost of $(x - \sum_{j=1}^{n} e^{[j]})(C_T/W)$. In this latter case an increasing linear function over $[x - \sum_{j=1}^{n} e^{[j]}, W]$ having slope C_T/W is appended to the piecewise linear nonincreasing function created over $[0, x - \sum_{j=1}^{n} e^{[j]}]$. In either Case (1) or Case (2), we have a convex function of x over $[0, W]$, assuming perfect information. (See Figures 7 and 8.)

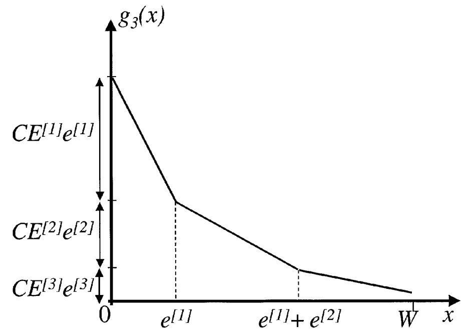

Figure 7 **Cost of Tour as a Function of *x*, Assuming Perfect Information and Total Customers' Emptiness Less than Vehicle Capacity**

A *decision rule* for allocating product to customers is any procedure that specifies the quantity to place in each customer's tank using all or a subset of information available. Now assume any given decision rule for allocating product to customers based on imperfect information but assuming the same experimental values of the random variables $\{e^{[1]}, e^{[2]}, \ldots, e^{[n]}\}$. Denote by $h_n(x|\{e^{[1]}, e^{[2]}, \ldots, e^{[n]}\})$ the (shortfall plus overhead) cost of executing the stated decision rule, given the values of the random variables $\{e^{[1]}, e^{[2]}, \ldots, e^{[n]}\}$. By arguments paralleling those above, $h_n(x|\{e^{[1]}, e^{[2]}, \ldots, e^{[n]}\})$ is a piecewise linear convex function on or above $g_n(x)$, i.e., $h_n(x|\{e^{[1]}, e^{[2]}, \ldots, e^{[n]}\}) \geq g_n(x)$, over the interval $[0, W]$.

An expected cost function for any given decision rule is found by unconditioning on the experimental values of the random variables. Let $h_n(x) =$ expected cost of executing the particular tour of n customers following the given decision rule. Clearly,

$$h_n(x) = \sum_{all\left\{e^{[1]}, e^{[2]}, \ldots, e^{[n]}\right\}} h_n\left(x|e^{[1]}, e^{[2]}, \ldots, e^{[n]}\right) \times P\left\{E^{[1]} = e^{[1]}, E^{[2]} = e^{[2]}, \ldots, E^{[n]} = e^{[n]}\right\}.$$

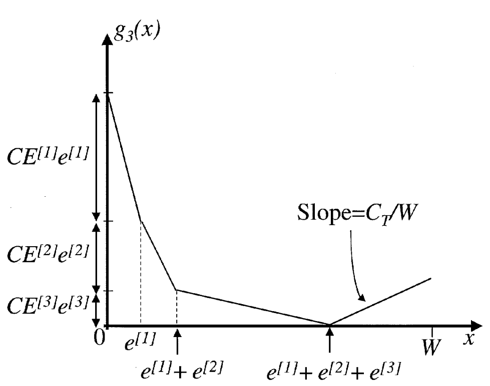

Figure 8 **Cost of Tour as a Function of *x*, Assuming Perfect Information and Total Customers' Emptiness Greater than Vehicle Capacity**

Here $h_n(x)$ is seen to be a weighted sum of convex functions over $[0, W]$. Hence, $h_n(x)$ is convex over $[0, W]$.

The decision rule found in dynamic programming is an acceptable decision rule according to our definition. Thus, the optimal cost function found by dynamic programming $f_n(x)$ is convex over $[0, W]$. □

Remark. In this paper we have discretized the probability distributions for emptinesses of the respective customers, with each respective distribution allowing only a finite number of values of the corresponding random variable. Hence, the total number constituent piecewise linear convex functions comprising $h_n(x)$ is finite, and thus $h_n(x)$ is itself piecewise linear and nondifferentiable. The convexity property carries over to continuous probability distribution functions, as long as the limits exist, and in that case $h_n(x)$ is usually differentiable.

Theorem 2. *Consider all* $n!$ *orderings of replenishment visits to* n *given customers.*

(a) An ordering having the least expected marginal cost is one in which the FILL, FILL, DUMP policy is optimal, i.e., the customers' successive *CE* values form a monotone nonincreasing sequence.

(b) An ordering having the largest expected marginal cost is one in which the customers' successive *CE* values form a monotone nondecreasing sequence.

Proof of Theorem 2. We prove Part 1; Part 2 follows from similar logic. Consider any proposed visitation ordering of the customers, $\mathbf{O} = \{o_n, o_{n-1}, \ldots, o_2, o_1\}$, where $o_i \in \{1, 2, \ldots, n\}$ and $o_i \neq o_j$ for all $i \neq j$. If the successive *CE* values associated with this ordering do not form a monotone nonincreasing sequence, there must exist at least one customer o_i for whom (i) $CE_{o_{i+1}} < CE_{o_i}$ and (ii), if $i \neq 1$, $CE_{o_i} \geq CE_{o_{i-k}} \forall k = 1, 2, \ldots, i-1$. Consider a revised ordering $\mathbf{O}'$ having customers $i+1$ and i reversed, i.e., $\mathbf{O}' = \{o_n, o_{n-1}, \ldots, o_i, o_{i+1}, \ldots, o_2, o_1\}$. Let $f_n(x)$ be the dynamic-programming optimal cost function for the sequence $\mathbf{O}$ and $f_n'(x)$ be the dynamic-programming optimal cost function for the sequence $\mathbf{O}'$. Note $f_{i-1}'(x) = f_{i-1}(x)$. Due to the separability properties of dynamic programming, in comparing $\mathbf{O}$ and $\mathbf{O}'$ we can ignore customers $\{o_n, o_{n-1}, \ldots, o_{i+2}\}$ and focus exclusively on $\{o_{i+1}, o_i, \ldots, o_2, o_1\}$ and on $f_{i+1}(x)$ and $f_{i+1}'(x)$. For customers o_i and o_{i+1}, we condition on experimental values of their respective emptinesses, i.e., $\{E_{o_{i+1}} = e_{o_{i+1}}\}$ and $\{E_{o_i} = e_{o_i}\}$. Under ordering $\mathbf{O}$, the optimal amount of product $v^*_{o_{i+1}}$ to leave with customer O_{i+1} is one of three quantities, depending on the value of $x: 0, x - \overline{X}_{o_{i+1}} < e_{o_{i+1}}$, or $e_{o_{i+1}}$, which can be written as $v^*_{o_{i+1}} = [\min(x - \overline{X}_{o_{i+1}}, e_{o_{i+1}})]^+$. The vehicle then arrives at customer o_i holding $x - v^*_{o_{i+1}}$ units of product. Since $CE_{o_i} \geq CE_{o_{i-k}} \forall k = 1, 2, \ldots, i-1$, the condition of Lemma 2 holds, and one attempts to fill up the tank of customer o_i. The vehicle then proceeds to customer o_{i-1} with $[x - v^*_{o_{i+1}} - e_{o_i}]^+$ units of product. Now consider ordering $\mathbf{O}'$: at stage $i+1$ the vehicle first visits customer o_i, deposits $\min[X, e_{o_i}]$ units of product, and proceeds to customer o_{i+1} with $[X - e_{o_i}]^+$ units of product. The amount of product delivered to o_{i+1} is the optimal amount determined by the Bellman recursion equation, with LHS equal to $f_i([x-e_{o_i}]^+)$; this quantity can be written $v^*_{o_{i+1}}(e_{o_i}) = [\min\{[x-e_{o_i}]^+ - e_{o_{i+1}}\}]^+$, where, invoking Lemma 3, $\overline{X}'_{o_i}(e_{0_i})$, is equal to that point y satisfying $|\frac{df_{i-1}(y)}{dy}|_{y=\overline{X}_{o_{i+1}}} = CE_{o_{i+1}}$ (or the equivalent if $f_{i-1}()$ is nondifferentiable). We can now write the Bellman equations for **O** and **O**$'$:

(a) $f_{i+1}(x) = (e_{o_{i+1}} - v^*_{o_{i+1}})CE_{o_{i+1}} + [e_{o_i} - (x - v^*_{o_{i+1}})]^+ CE_{o_i} + f_{i-1}([x - v^*_{o_{i+1}} - e_{o_i}]^+)$

(b) $f'_{i+1}(x) = [e_{o_{i+1}} - v^*_{o_{i+1}}(e_{o_i})]CE_{o_{i+1}} + [e_{o_i} - x]^+ CE_{o_i} + f_{i-1}([x - v^*_{o_{i+1}}(e_{o_i}) - e_{o_i}]^+)$

In comparing (a) and (b) we consider three cases:

Case 1. $x < e_{o_i}$. Here due to the magnitude of CE_{o_i}, **O**$'$ is at least as good as **O** and better if $v^*_{o_{i+1}} > 0$.

Case 2. (occurs only if $v^*_{o_{i+1}} > 0$): $e_{o_i} \le x < e_{o_i} + v^*_{o_{i+1}}$ **O**$'$ is better than **O** because **O**$'$ fills up o_{e_i} whereas **O** does not.

Case 3. $e_{o_i} + v^*_{o_{i+1}} \le x$. In this case the respective two middle terms in (a) and (b) are equal to zero, as o_{e_i} is filled up under both **O** and **O**$'$. Thus (a) and (b) may be rewritten,

(a$'$) $f_{i+1}(x) = (e_{o_{i+1}} - v^*_{o_{i+1}})CE_{o_{i+1}} + f_{i-1}([x - v^*_{o_{i+1}} - e_{o_i}]^+)$

(b$'$) $f'_{i+1}(x) = f'_i(x - e_{o_i}) = [e_{o_{i+1}} - v^*_{o_{i+1}}(e_{o_i})]CE_{o_{i+1}} + f_{i-1}([x - v^*_{o_{i+1}}(e_{o_i}) - e_{o_i}]^+)$

However, by the Bellman equation, $f'_i(x-e_{o_i}) \le [e_{o_{i+1}} - v^*_{o_{i+1}}]CE_{o_{i+1}} + f_{i-1}([x - v^*_{o_{i+1}} - e_{o_i}]^+)$, with equality certain only if the variance of E_{o_i} is zero.

Part 1 of the theorem is proved when one successively performs $O(n^2)$ pairwise interchanges of customer visitation orderings, each ordering satisfying conditions (i) and (ii) in the proof, and each improving the values of the expected marginal cost under an optimal dynamic-programming policy; eventually such pairwise interchanges lead to an ordering that is a monotone nonincreasing sequence. Reversing the pairwise interchanges used in proving Part 1 proves Part 2 of the theorem. □

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