Chain Function Demo

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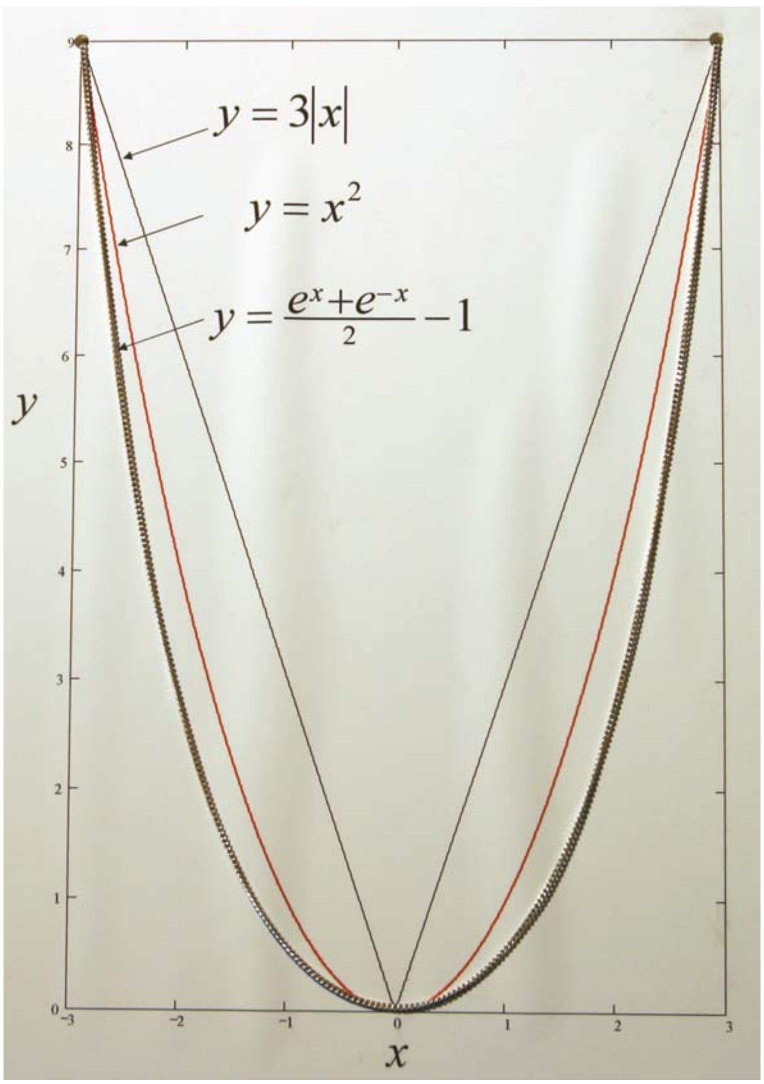
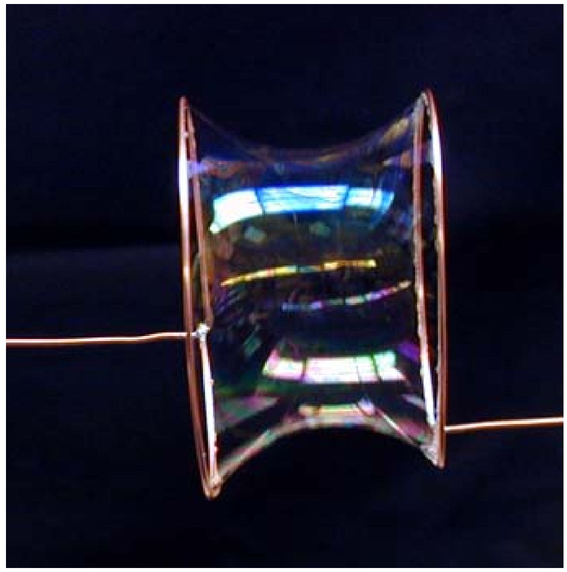
The image shows a silver chain hanging (in a uniform gravitational field) while supported at its ends. Galileo claimed that the curve formed by such a chain (or flexible wire/rope) would be a parabola but gave no proof. Others thought that it might be a quadratic. Jakob Bernoulli challenged other mathematicians to determine the form of the curve. His brother Johann Bernoulli and two other mathematicians (Leibniz and Huygens) almost simultaneously and independently proved in 1691 that the curve has a particular form which we now call a catenary (catenary comes from the Latin word for "chain"). The St. Louis Gateway Arch (shown below) was deliberately built as an inverted catenary.



The shape formed by the soap film in this image is also a catenary.

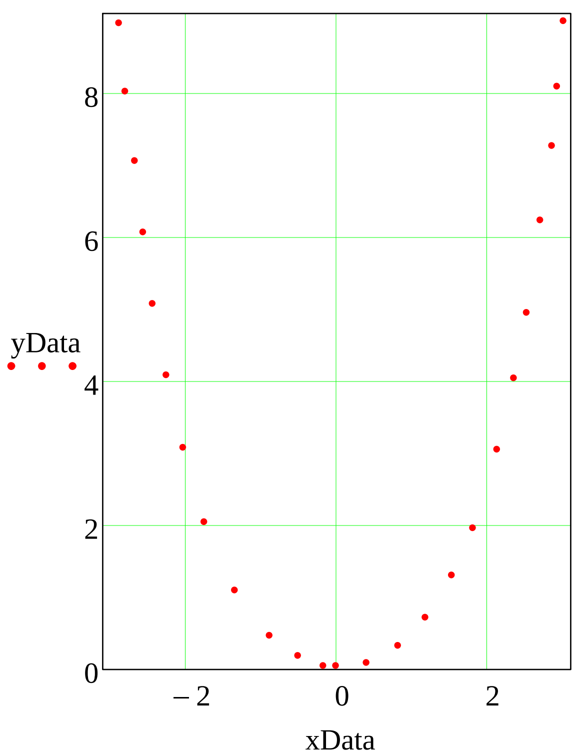


The data points were obtained by using Logger Pro to measure a number of (x,y) points along the chain in the image (previous page).

$$\mathrm{Data} := \begin{pmatrix} -2.888 & 8.972 \\ -2.805 & 8.023 \\ -2.679 & 7.06 \\ -2.567 & 6.07 \\ -2.442 & 5.079 \\ -2.26 & 4.088 \\ -2.037 & 3.084 \\ -1.758 & 2.051 \\ -1.353 & 1.102 \\ -0.893 & 0.474 \\ -0.516 & 0.195 \\ -0.181 & 0.056 \\ -0.014 & 0.056 \\ 0.391 & 0.098 \\ 0.809 & 0.335 \\ 1.172 & 0.726 \\ 1.521 & 1.312 \\ 1.8 & 1.967 \\ 2.121 & 3.056 \\ 2.344 & 4.047 \\ 2.512 & 4.953 \\ 2.693 & 6.237 \\ 2.847 & 7.27 \\ 2.916 & 8.093 \\ 3 & 9 \end{pmatrix}$$

$\mathrm{xData} := \mathrm{Data}^{\langle 0 \rangle}$

$\mathrm{yData} := \mathrm{Data}^{\langle 1 \rangle}$



$n := \mathrm{length}(\mathrm{xData})$

$n = 25$

The equation of a simple catenary is $$f(x) := \frac{e^{x} + e^{-x}}{2}$$

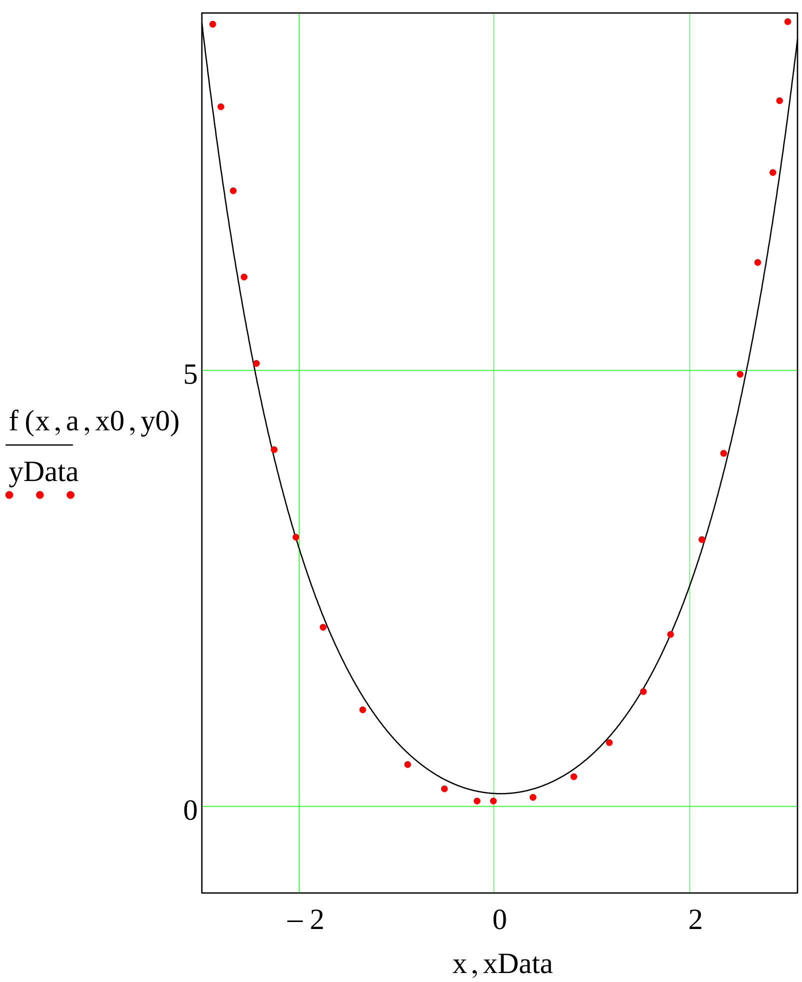
A scaling parameter, a, is usually added $$f(x, a) := a \cdot \frac{e^{\frac{x}{a}} + e^{\frac{-x}{a}}}{2} \quad = \quad a \cdot \cosh\left(\frac{x}{a}\right)$$

When a catenary is measured there will usually be an offset in both x and y. We can therefore include two extra parameters, x_0 and y_0, to represent this offset.

$$f(x, a, x0, y0) := a \cdot \frac{e^{\frac{x+x0}{a}} + e^{\frac{-(x+x0)}{a}}}{2} - y0$$

$a :=$ $x0 :=$ $y0 :=$

$a = 1.04$ $x0 = -0.06$ $y0 = 0.9$



Try fitting the catenary to the data by adjusting the three parameters.

Parameter Estimation

We now consider the problem of estimating the three parameters, a, x_0 and y_0.

We might take the usual approach and use a nonlinear minimization technique to estimate the values of the three parameters which minimize the sum of squared difference between the measured yData and the predicted y values. Such an objective function would look like

$$SS(a, x0, y0) := \sum_{i=0}^{n-1} \left(f\left(xData_i, a, x0, y0\right) - yData_i\right)^2$$

The problem with this approach is that we are effectively assuming that all the measurement error is in the yData. This is not a valid assumption as the xData probably has about the same error as the yData. We are therefore faced with the problem of fitting a nonlinear equation to a data set having equal measurement error in both axes.

If the measurement errors are assumed to be Gaussian distributed and independent (i.e. white) then we want to find the parameters which minimize the sum of squared perpendicular distances between the measured data points and the function. This is not nearly as straight forward and indeed is rarely undertaken.

The distance between the function, f(x), and a measured data point (x_i, y_i) is

$$\text{Distance}\left(x, x_i, y_i, a, x0, y0\right) := \sqrt{\left(x - x_i\right)^2 + \left(f(x, a, x0, y0) - y_i\right)^2}$$

The minimum distance occurs when the line from the data point to the function is normal (perpendicular) to the function. This minimum distance (or equivalently the minimum squared distance) may be found by finding the root of the differentiated distance squared.

We now define the perpendicular sum of squares objective function

$$SS(a, x0, y0) := \left|\begin{array}{l} ss \leftarrow 0 \\ \text{for } \ i \in 0..n-1 \\ \quad \left|\begin{array}{l} xmin \leftarrow xData_i - 0.5 \\ xmax \leftarrow xData_i + 0.5 \\ x_f \leftarrow \text{root}\left[\frac{d}{dx}\left[\left(x - xData_i\right)^2 + \left(f(x, a, x0, y0) - yData_i\right)^2\right], x, xmin, xmax\right] \\ y_f \leftarrow f\left(x_f, a, x0, y0\right) \\ ss \leftarrow ss + \left(x_f - xData_i\right)^2 + \left(y_f - yData_i\right)^2 \end{array}\right. \\ \text{return } \ ss \end{array}\right.$$

We provide some initial parameter estimates

$a := 1$ $\qquad$ $x0 := 0$ $\qquad$ $y0 := 1$

Given $\quad a > 0 \quad a < 2$

$$\begin{pmatrix} a_{est} \\ x0_{est} \\ y0_{est} \end{pmatrix} := \text{Minimize}(SS, a, x0, y0)$$

$a_{est} = 0.991$ $\quad x0_{est} = -0.048$ $\quad y0_{est} = 0.969$

$$SS\left(a_{est}, x0_{est}, y0_{est}\right) = 9.751 \times 10^{-3}$$