Newton's Method

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The roots of a function, g(x), are simply the x values when g(x)=0 (i.e. the points at which a function crosses or touches the x-axis). The roots of second-, third- and fourth-order polynomials (i.e. a quadratic, cubic and quartic polynomials) may be found using explicit formulas. (mathematicians spent considerable time searching for such formulas for 5th and higher order polynomials until the Norwegian mathematician Neils Abel proved that no such formulas were possible).

For second-order polynomials (quadratics)

$$g(x, a, b, c) := a \cdot x^2 + b \cdot x + c$$

The formulas for the 2 roots are

$$x_{lower}(a, b, c) := \frac{-b - \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a} \qquad x_{higher}(a, b, c) := \frac{-b + \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

For example the function

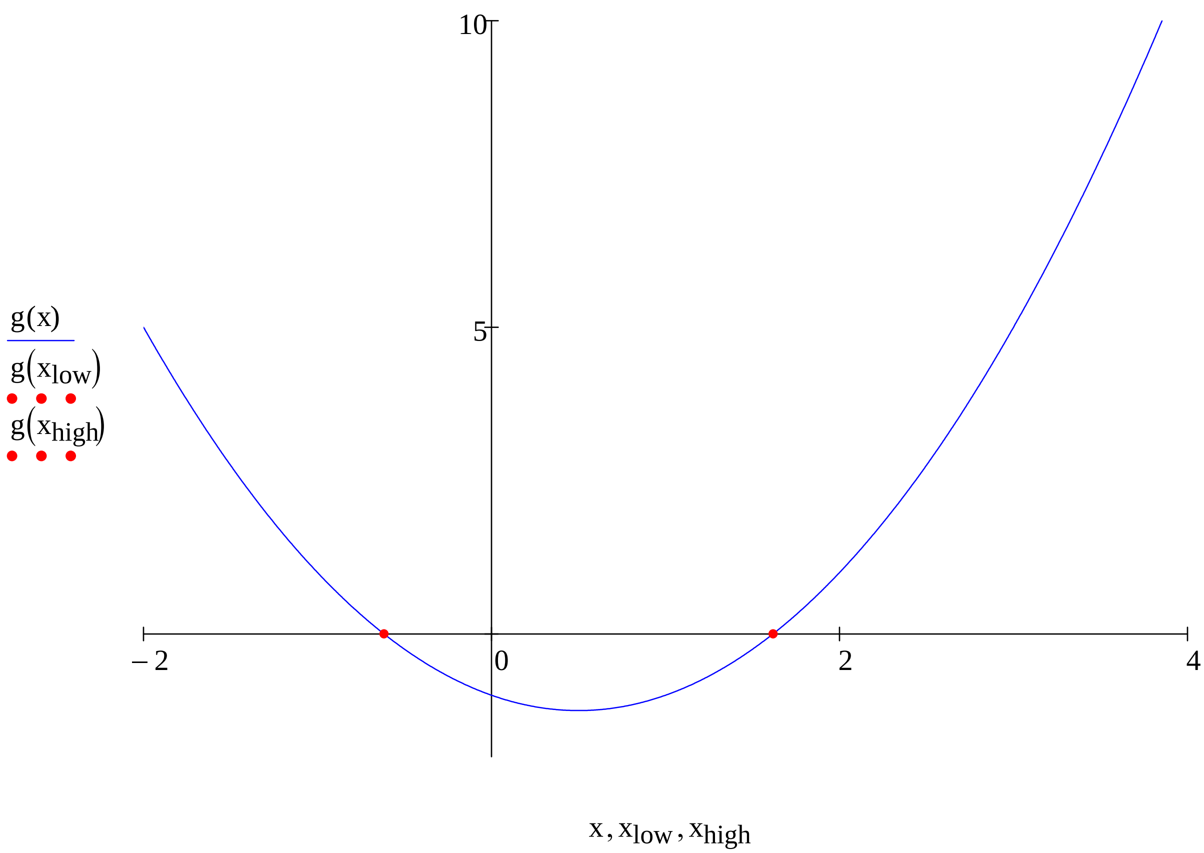
$$g(x) := x^2 - x - 1$$

Incidently the positive root of this function is the Golden Ratio.

has roots

$$x_{low} := x_{lower}(1, -1, -1) \qquad x_{high} := x_{higher}(1, -1, -1)$$

$$x_{low} = -0.618033988749895 \qquad x_{high} = 1.618033988749895$$



Newton's Method

A number of methods exist to find roots when explicit formulas are not available. The most famous of these methods is called **Newton's method** (or the **Newton-Raphson method**) developed by Issac Newton in the mid 1600s.

The idea behind this method is to start with an initial guess which is close to the desired root. In the example above lets try to find the second root, x_{high} (the Golden Ratio).

We will call our initial guess x_0 and set

$$x_0 := 2$$

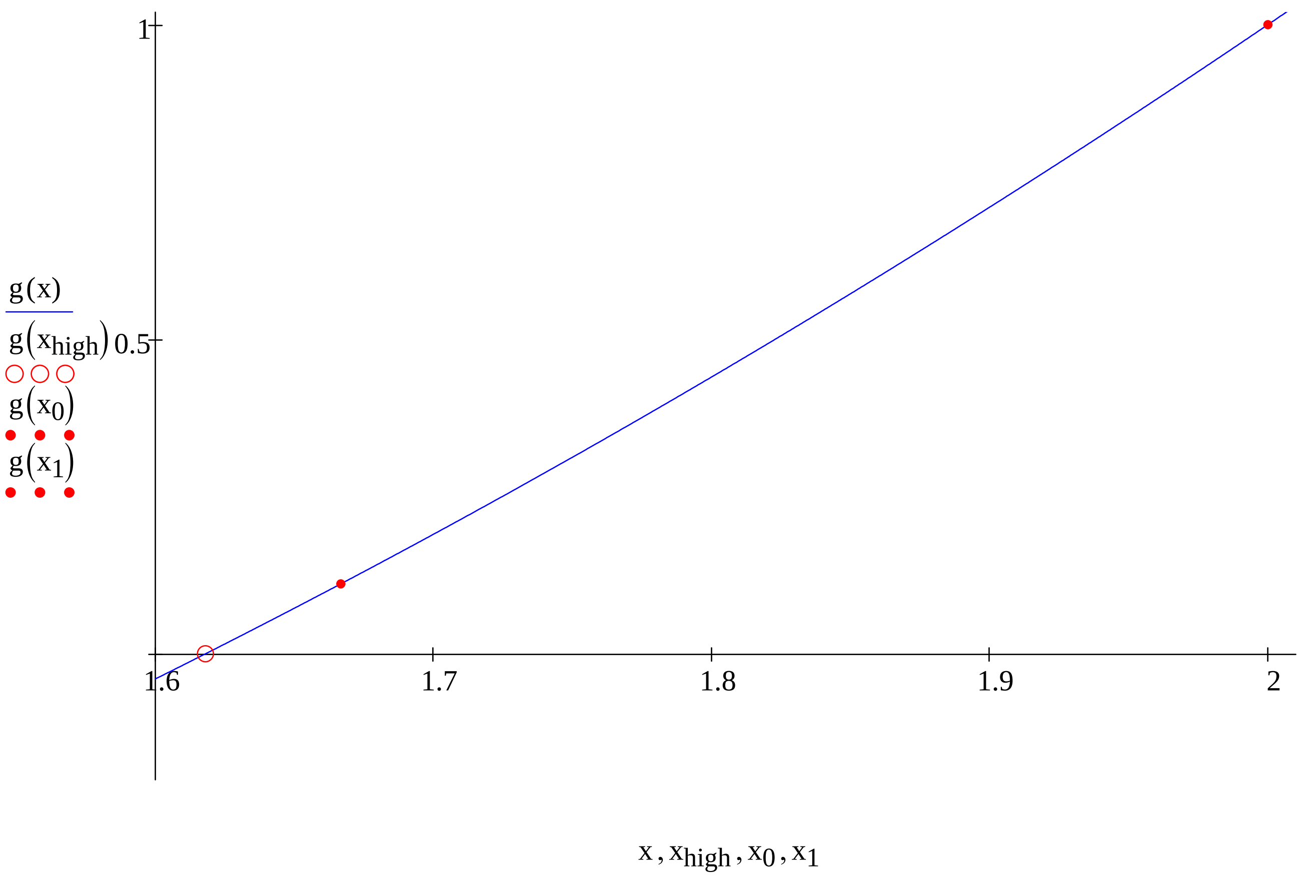
We next find the slope of g(x) at the point (x_0, g(x_0)).

We can get this by differentiation

$$g'(x) := \frac{d}{dx} g(x)$$

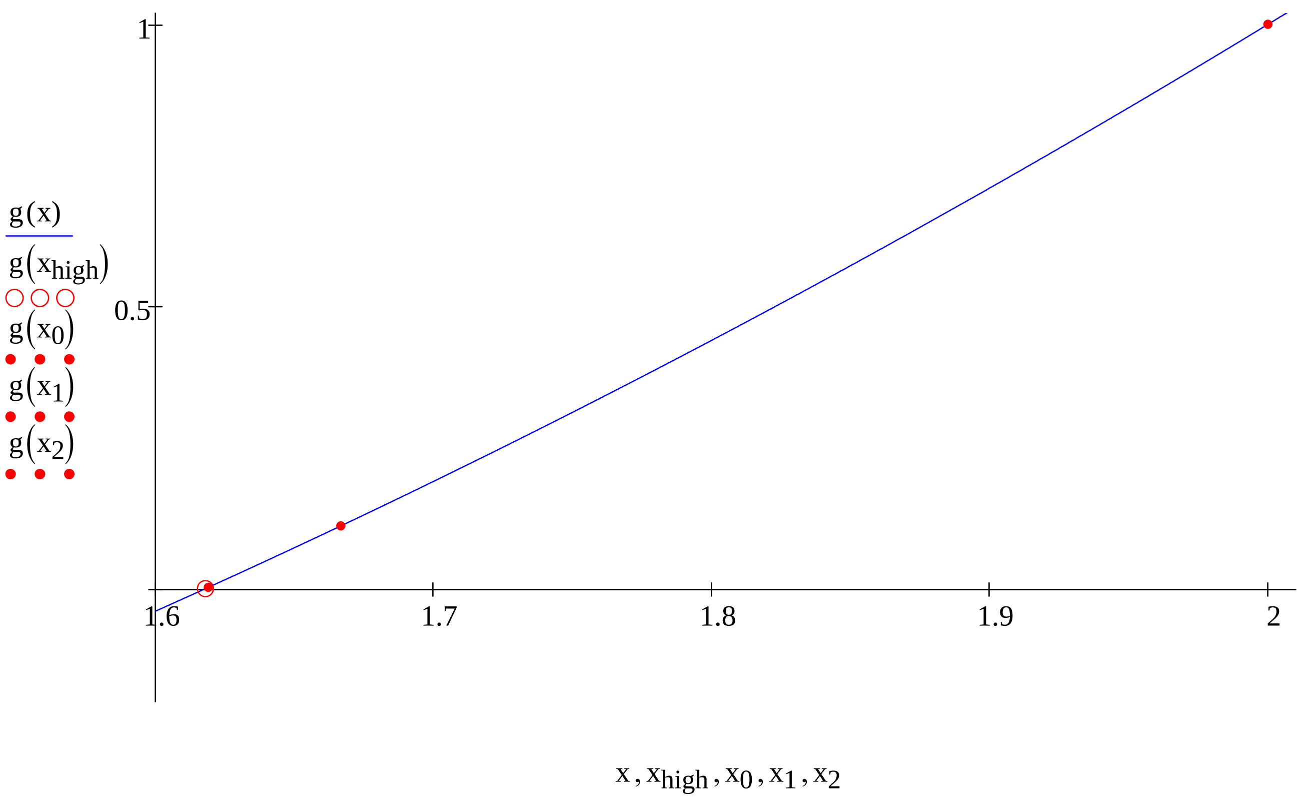
Newton's method specifies the next "guess"as

$$x_1 := x_0 - \frac{g(x_0)}{g'(x_0)} \qquad x_1 = 1.666666666666667$$



Newton's method specifies the next "guess"as

$$x_2 := x_1 - \frac{g(x_1)}{g'(x_1)} \qquad x_2 = 1.619047619047619 \qquad \text{very close}$$



It is clear from this example that Newton's method converges very rapidly to the root.

Lets do a few more iterations and zoom in on the root.

$$x_3 := x_2 - \frac{g(x_2)}{g'(x_2)} \qquad x_3 = 1.618034447821682$$

$$x_4 := x_3 - \frac{g(x_3)}{g'(x_3)} \qquad x_4 = 1.618033988749989 \qquad \text{very close}$$

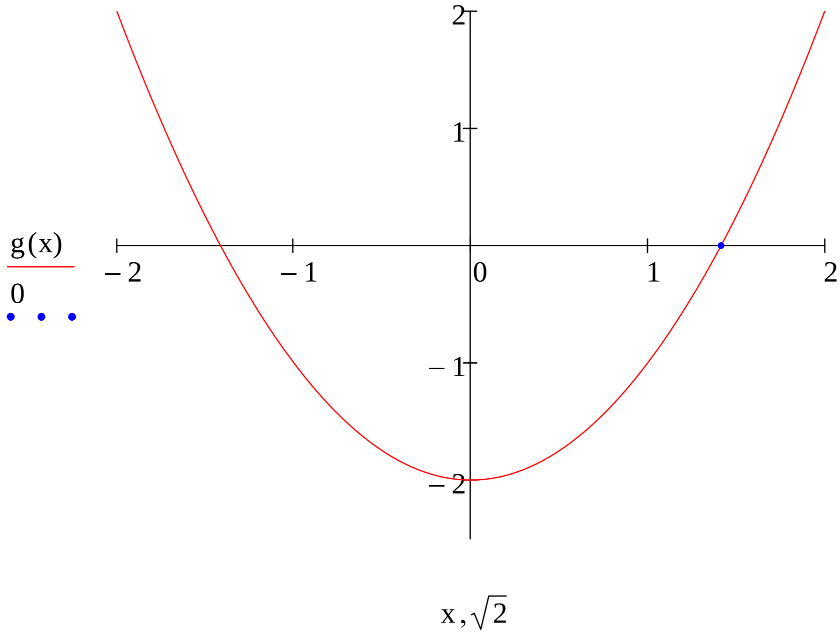
$$x_5 := x_4 - \frac{g(x_4)}{g'(x_4)} \qquad x_5 = 1.618033988749895$$

$$x_6 := x_5 - \frac{g(x_5)}{g'(x_5)} \qquad x_6 = 1.618033988749895 \qquad \text{no change}$$

Find the Square Root of 2

If $x = \sqrt{2}$ then $x^2 = 2$ and the root of

$g(x) := x^2 - 2$ is $\sqrt{2}$



Using Newton's method we determine the derivative

and then start with a guess

$$g'(x) := \frac{d}{dx} g(x)$$

$x_0 := 2$

$x_1 := x_0 - \frac{g(x_0)}{g'(x_0)}$ $x_1 = 1.5$ $x_0 - x_1 = 0.5$

$x_2 := x_1 - \frac{g(x_1)}{g'(x_1)}$ $x_2 = 1.416666666666667$ $x_1 - x_2 = 0.083$

$x_3 := x_2 - \frac{g(x_2)}{g'(x_2)}$ $x_3 = 1.41421568627451$ $x_2 - x_3 = 2.451 \times 10^{-3}$

$x_4 := x_3 - \frac{g(x_3)}{g'(x_3)}$ $x_4 = 1.41421356237469$ $x_3 - x_4 = 2.124 \times 10^{-6}$

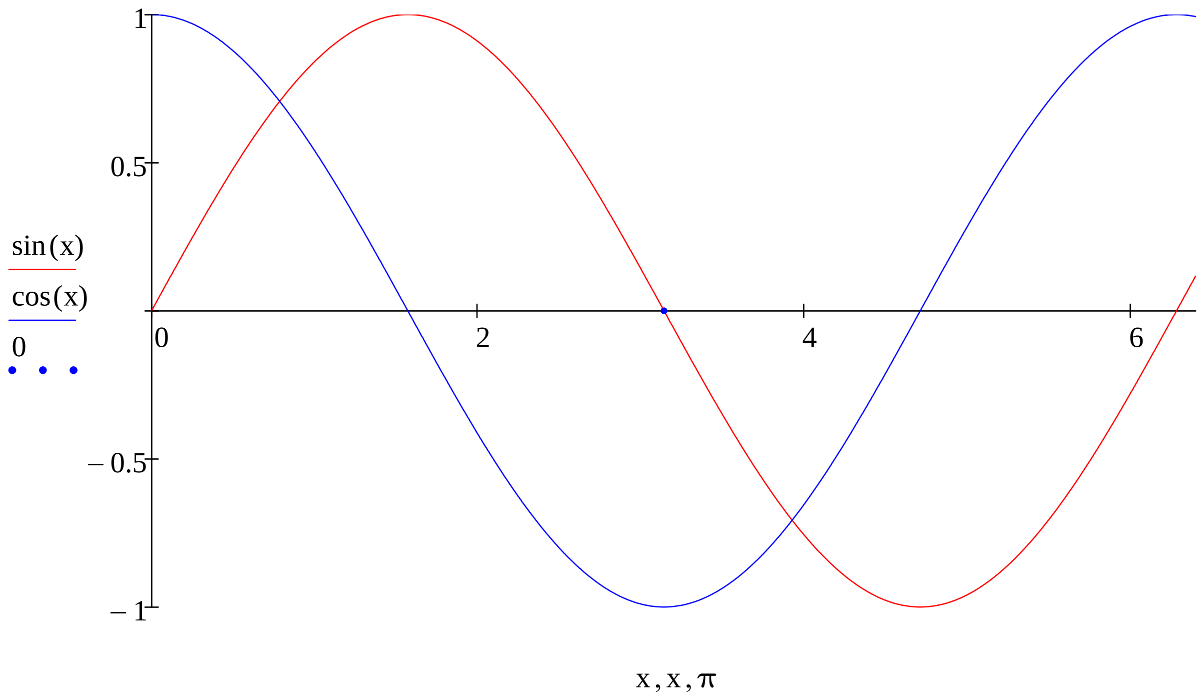
$x_5 := x_4 - \frac{g(x_4)}{g'(x_4)}$ $x_5 = 1.414213562373095$ $x_4 - x_5 = 1.595 \times 10^{-12}$

$x_6 := x_5 - \frac{g(x_5)}{g'(x_5)}$ $x_6 = 1.414213562373095$ $x_5 - x_6 = 0$ no change

$\sqrt{2} = 1.414213562373095$

Find Pi

$g(x) := \sin(x)$ $\qquad\qquad$ $\frac{d}{dx}\sin(x) \rightarrow \cos(x)$



The root of f(x) near 3 is π .

$x_0 := 3$

$x_1 := x_0 - \frac{\sin(x_0)}{\cos(x_0)}$ $\qquad$ $x_1 = 3.142546543074278$ $\qquad$ $x_0 - x_1 = -0.143$

$x_2 := x_1 - \frac{\sin(x_1)}{\cos(x_1)}$ $\qquad$ $x_2 = 3.141592653300477$ $\qquad$ $x_1 - x_2 = 9.539 \times 10^{-4}$

$x_3 := x_2 - \frac{\sin(x_2)}{\cos(x_2)}$ $\qquad$ $x_3 = 3.141592653589793$ $\qquad$ $x_2 - x_3 = -2.893 \times 10^{-10}$

$x_4 := x_3 - \frac{\sin(x_3)}{\cos(x_3)}$ $\qquad$ $x_4 = 3.141592653589793$ $\qquad$ $x_3 - x_4 = 0$

$x_5 := x_4 - \frac{\sin(x_4)}{\cos(x_4)}$ $\qquad$ $x_5 = 3.141592653589793$ $\qquad$ $x_4 - x_5 = 0$