Hypothesis testing 2: p-value, GLRT test

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Outline:

- Recap (basics, z-test, t-test
- p-value
- GLRT test
- G-**statistic**
- **Kolmogorov-Smirnov test**

Recall: Formal setting for HT

Definition

Statistical hypotheses:

- H : data $X_1, \ldots, X_n$ distributed according to $P \in \mathcal{C}_0$
- K : data $X_1, \ldots, X_n$ distributed according to $P \in \mathcal{C}_1$

where $\mathcal{C}_0, \mathcal{C}_1$ are COLLECTIONS OF DISTRIBUTIONS.

Remarks:

- **ALWAYS (!)** make sure you can formulate in the form above
- Most common special case: X_i are i.i.d. from P
- ... So will just write things like:

$$H : \mathbb{E}[X] = 0 \qquad \text{vs.} \qquad K : \mathbb{E}[X] \neq 0 \,.$$

- Can reject H, but not "prove it"!

Design of tests

- **Before** seeing the data we announce test:

  Data $\boldsymbol{X} = (X_1, \ldots, X_n)$ *lands in set* $R_\alpha \Rightarrow$ *REJECT null.*

  with $P[\text{data} \in R_\alpha | H] \leq \alpha$ (false positive, significance)

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$$R_\alpha = \{T(\boldsymbol{X}) \geq t_\alpha\}.$$

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- We learned about two statistics with such properties: Z and T

Recap: z- and t-tests

Testing for mean

- Data $X_i \stackrel{iid}{\sim} P$, mean $\mathbb{E}[X] = \mu$
- null $H : \mu = 0$
- alt $K : \mu \neq 0$ (or $\mu > 0$)

- Good idea: compute sample mean $\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i$

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- Good idea: compute sample mean $\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i$
- z-test:
  - Applicable when $\mathrm{Var}[X|H] = \sigma_0^2$ (known, null-variance)
  - $Z = \sqrt{\frac{n}{\sigma_0^2}} (\hat{\mu} - \mu_0)$
  - Asymptotically normal: $n \gg 1$ have $Z \approx \mathcal{N}(0, 1)$ under null!
  - REJECT null if $|Z| > 1.96$ for significance $\alpha = 0.05$

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- Good idea: compute sample mean $\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i$
- t-test:
  - Applicable when variance is unknown (aka nuisance parameter)
  - $\boxed{T = \sqrt{\frac{n}{\hat{\sigma}^2}}(\hat{\mu} - \mu_0)}$
  - Asymptotically normal: $n \gg 1$ have $T \approx \mathcal{N}(0,1)$ under null!
  - REJECT null if $|T| > 1.96$ for significance $\alpha = 0.05$
  - If $X_i \sim \mathcal{N}$ then $T \sim$ Student-t dist. with $(n-1)$ d.o.f.
    Selecting $|T| > t_{\alpha/2}(n-1)$ makes test EXACT.

HT steps (again)

Hypothesis testing mindset:

1. Suppose in your experiment you will see data $\boldsymbol{X} = (X_1, \ldots, X_n)$
2. Formulate null hypothesis $H\colon\ X \sim P$ with $P \in \mathcal{C}_0$
3. Formulate alternative hypothesis $K : X \sim P$ with $P \in \mathcal{C}_1$
4. Choose statistic whose distribution under H is same for all $P \in \mathcal{C}_0$

$$T = \sqrt{n}\frac{\hat{\mu} - \mu_0}{\sqrt{\widehat{\sigma^2}}} \approx \mathcal{N}(0, 1)$$

5. Threshold test: If T "large", REJECT null H.
6. Threshold chosen s.t. $\mathbb{P}[\text{reject}|H] \leq \alpha$ for pre-specified α (typ. 0.05)
7. Only then see the data

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Q: Do we really need 5-6? Why threshold at all?

Concept of p-value

- Consider test procedure:

$$(*) \qquad T(X) \geq t_\alpha \quad \Rightarrow \quad \text{reject null } H$$

- Value of T, aka "effect size", is more important than binary decision
- **Question:** "effect size" has units, can we convert it to universal scale?

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Algorithm for computing p-value

- Got data $\boldsymbol{x} = (x_1, \ldots, x_n)$.
- Compute observed statistics $t_{obs} \triangleq T(\boldsymbol{x})$
- p-value $\triangleq P[T(\boldsymbol{X}) \geq t_{obs} | H]$

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- Mnemonic: probability of observing same or more extreme data
- For $P[\cdot|H]$ to make sense, should have "pivotality" (for general case, wait a bit)

Illustration of p-value: $\mu = \mu_0$ vs $\mu > \mu_0$

Right-tailed test

Reject when $\{T(x) > t_\alpha\}$

p-value $\triangleq \mathbb{P}[T(X) \geq t_{obs}|H]$

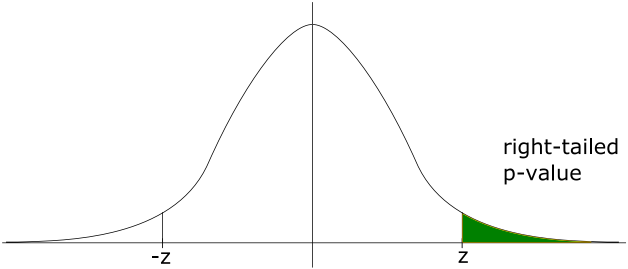

Illustration of p-value: $\mu = \mu_0$ vs $\mu \neq \mu_0$

Two-tailed test

Reject when $\{|T(x)| > t_{\frac{\alpha}{2}}\}$

p-value $\triangleq \mathbb{P}[|T(X)| \geq t_{obs}|H]$

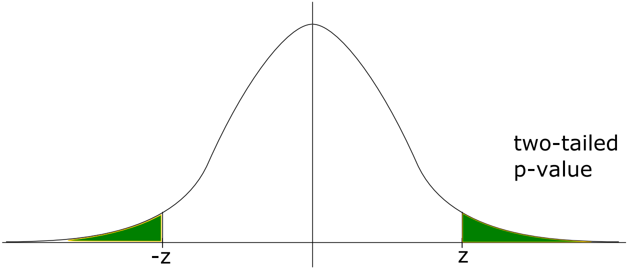

Illustration of p-value: $\mu = \mu_0$ vs $\mu < \mu_0$

Left-tailed test

Reject when $\{T(x) < t_\alpha\}$

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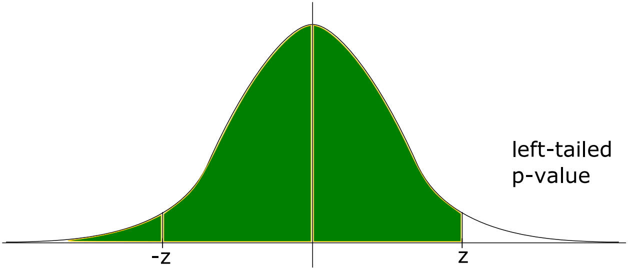

Lady tasting tea

Back story

- M. Bristol claims to be able to tell whether tea or milk was poured into cup first
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The experiment

- Design: 8 teacups are placed (4 tea first, 4 milk first)
- Data: $X =$ tasting, $Y =$ truth
  (e.g. $X = \texttt{TTTTMMMM}$, $Y = \texttt{TMTMTMTM}$)
- Test statistic: $T = \#$ of correct guesses

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Hypothesis testing formulation

- Null hypothesis $H : X, Y$ are i.i.d. uniform on $\binom{8}{4} = 70$ strings
- Distribution of T under null:

T	Prob	T	Prob
0	$1/70$	6	$16/70$
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p-value $= 1/70 \approx 0.014$

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- Then we have $t_{obs} \triangleq T(\boldsymbol{x})$ (observed value of T)

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- **Solution:** Just replace with $\max_{P \in \mathcal{C}_0}$!

Algorithm for computing p-value

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- **Problem:** What if test is not of the form "$T(x) \geq t_\alpha$"?
- **Solution:** Let R_α be a family of tests s.t.

$$P[\boldsymbol{X} \in R_\alpha] \leq \alpha \quad \forall P \in \mathcal{C}_0$$

then

$$p\text{-value} \triangleq \inf\{\alpha : \boldsymbol{x} \in R_\alpha\}$$

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- It depends on data and test.
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- ... the test for which p is computed should be specified.
- In practice hard to decipher from actual papers.
- "Reproducible research" movement is to fix this.

Interpreting p-value

- Roughly: p-value $= P[\text{data (or more extrem)}|H]$
- Value $p = 0.05$ means false REJECT in 5% of experiments
- ... often used to decide on funding, continuing drug trials etc
- Rookie mistake: think p-value is $P[H|\text{data}]$
  Mass media: "null is true w.p. $< 5\%$"

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- Fun calculation: If $p < 0.05$ is rejection threshold
  False-positive in 5 tests w.p. 22%
  False-positive in 10 tests w.p. 40%
  False-positive in 15 tests w.p. 53%
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- ... and 2500 false-positive replications, 125 triple replications, 6 quad replications
- Sensational (false?) positives get blown up by the media

Data-snooping

Another rookie blunder

- Null: $\mu = \mu_0$
- See data, observed t-statistic $t_{obs} > 0$
  (i.e. sample-mean $> \mu_0$)
- Decide to report one-sided p-value.
  I.e. write paper "On testing $\mu = \mu_0$ vs. $\mu > \mu_0$"

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- You report: "My p-value was calculated as"

$$p_{cheat} = \mathbb{P}[T(X) > t_{obs} | H]$$

  but in truth you computed

$$p_{true} = \mathbb{P}[T(X) > t_{obs} | H, T(X) > 0] \, .$$

- Under normal approximation $p_{true} \approx 2 p_{cheat}$

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$$p_{true} = \mathbb{P}[T(X) > t_{obs} | H, T(X) > 0] \, .$$

- Under normal approximation $p_{true} \approx 2 p_{cheat}$
- Example of data-snooping (beginner-level)
- Mid-level: do multiple trials, report one
  (Chicago Bears, coin tosses "$p = 2^{-14}$"?)
- Pro-level: run many tests, report one

Roadmap of tests

Tests we will learn:

- One-sample tests:
  1. for mean of population: $\mathbb{E}[X] = \mu_0$ vs $\mathbb{E}[X] \neq \mu_0$
  2. for other parameters: $\theta \in \Theta_0$ vs $\theta \notin \Theta_0$
  3. generalized likelihood-ratio test: $X \sim$ Uniform vs $X \sim$ not Uniform
  4. testing normality: $X \sim \mathcal{N}(0,1)$ vs $X \nsim \mathcal{N}(0,1)$

- Two-sample tests:
  1. Equality of means: $\mathbb{E}[X] = \mathbb{E}[Y]$ vs. $\mathbb{E}[X] \neq \mathbb{E}[Y]$
  2. Equality of distributions: $P_X = P_Y$ vs. $P_X \neq P_Y$
  3. Testing independence: $X \perp\!\!\!\perp Y$ vs $X \not\perp\!\!\!\perp Y$

Design of tests

- **Before** seeing the data we announce test:

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- How does one find such T????
- Art... (as in beautiful, cf. **exact** non-parametric tests)

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  - "consistency": Under any non-null $P \in \mathcal{C}_1$, $T(\boldsymbol{X})$ grows to ∞ with n
- How does one find such T????
- Art... (as in beautiful, cf. **exact** non-parametric tests)
- Some guidelines:
  - Use good $\hat{\theta}$
  - Shed nuisance scale parameters by Studentization
- How about cases other than $\theta \in H$ vs $\theta \in K$?

Generalized likelihood-ratio test

- How do we test for general hypotheses?
- MLE was our savior in estimation. Analog for HT?

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$$\begin{aligned} G &\triangleq -2\log\frac{P_0^*(x_1,\ldots,x_n)}{P_1^*(x_1,\ldots,x_n)} \\ P_0^*(x_1,\ldots,x_n) &= \max_{P\in\mathcal{C}_0} P(x_1,\ldots,x_n) \\ P_1^*(x_1,\ldots,x_n) &= \max_{P\in\mathcal{C}_0\cup\mathcal{C}_1} P(x_1,\ldots,x_n) \end{aligned}$$

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- Distribution of G under null? Let's find out ...

Testing for discrete distribution (goodness-of-fit)

HT problem

- X be r-valued: $[r] = \{1, \ldots, r\}$
- P_0 – a pmf on $[r]$, i.e. $P_0(1) + \cdots + P_0(r) = 1$
- Null $H : X \overset{iid}{\sim} P_0$
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Derive G-statistic

- Let $\hat{P}(\cdot) = \frac{1}{n}\#\{j : x_j = \cdot\}$ empirical dist
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- **Strong MAGIC:** as $n \to \infty$ under null

$$G \approx \chi^2(r-1)$$

regardless of P_0!

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- What is $\chi^2(d)$?

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- What is $\chi^2(d)$?

$$\chi^2(d) \sim \sum_{i=1}^{d} Z_i^2 \qquad Z_i \stackrel{iid}{\sim} \mathcal{N}(0,1)$$

$$\chi^2(d) = \texttt{scipy.stats.chi2.pdf}(\cdot, \mathrm{df} = d)$$

Hacks: $\chi^2(d) \approx \mathcal{N}$ for $d \geq 500$ and $\sqrt{\chi^2} \approx \mathcal{N}$ for $d \geq 50$

- So the final test is: REJECT if $G > x_\alpha(r-1)$

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G-test

- $\hat{P}(\cdot) = \frac{1}{n}\#\{j : x_j = \cdot\}$ empirical dist
- $g_{obs} = 2n \sum_{a=1}^{r} \hat{P}(a) \log \frac{\hat{P}(a)}{P_0(a)} = 2nD(\hat{P} \| P_0)$
- p-value= $\mathbb{P}[\chi^2(r-1) > g_{obs}]$

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Remarks: Could use any other "distance" $d(\hat{P}, P_0)$ and simulate.

Social experiment

Rules of the game

- **Everyone** please think of **two** random bits
- Write them down!

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- **Everyone** please think of **two** random bits
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- Now let me collect the results

Test 1: Generated bits are uniform coin flips?

- $n_0 = \#$ of 0 bits, $n_1 = \#$ of 1 bits.
- Calculate $G = 2n_0 \log \frac{2n_0}{n} + 2n_1 \log \frac{2n_1}{n}$
- Compare to quantiles of $\chi^2(1)$:

$P[\chi^2(1) > g]$	g
0.05	3.8
0.1	2.7
0.2	1.6
0.3	1.1

Social experiment

Rules of the game

- **Everyone** please think of **two** random bits
- Write them down!
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Test 2: Generated pairs of bits are (1/4, 1/2, 1/4)?

- $m_0 = \#$ of $\{0,0\}$, $m_1 = \#$ of $\{0,1\}$, $m_2 = \#$ of $\{1,1\}$ pairs.
- Calculate $G = 2m_0 \log \frac{4m_0}{m} + 2m_1 \log \frac{2m_1}{m} + 2m_2 \log \frac{4m_2}{m}$
- Compare to quantiles of $\chi^2(2)$:

$P[\chi^2(2) > g]$	g
0.05	6.0
0.1	4.6
0.2	3.2
0.3	2.4

Testing for continuous distribution (goodness-of-fit)

- What if now null $H : X \overset{iid}{\sim} P_0$ with P_0 – continuous dist. on $\mathbb{R}$?
- For example: $P_0 = \mathcal{N}(0, 1)$?

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$$KS_n = \max_{-\infty < x < \infty} \sqrt{n} |\hat{F}_X(x) - F_0(x)|$$

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$KS_n >$ `scipy.stats.ksone.ppf`$(1-\alpha,$ `n`$)$ then REJECT

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- Non-parametric stats.: dist. of KS_n is same for all P_0 in a huge class
- Don't trust `scipy`? Can do Monte Carlo with $P_0 = \mathrm{Uniform}[0,1]$.
- For large n converges to explicit Kolmogorov distribution:

$$\mathbb{P}[KS_n \leq x] \to \frac{\sqrt{2\pi}}{x} \sum_{k=1}^{\infty} e^{-(2k-1)^2\pi^2/(8x^2)}$$

- Example: Check if Pearson's crab data is normal.

Pearson crab data

BIN	COUNT
0.5385	1.0
0.5875	3.0
0.5915	5.0
0.5955	2.0
0.5995	7.0
0.6035	10.0
0.6075	13.0
0.6115	19.0
0.6155	20.0
0.6195	25.0
0.6235	40.0
0.6275	31.0
0.6315	60.0
0.6355	62.0
0.6395	54.0

BIN	COUNT
0.6435	74.0
0.6475	84.0
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0.6555	96.0
0.6595	85.0
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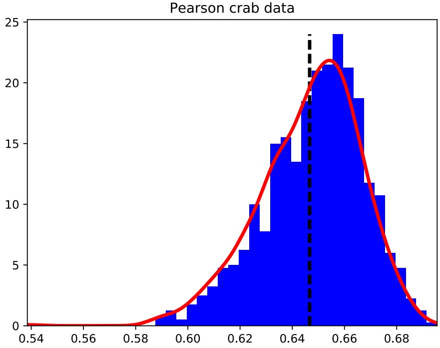

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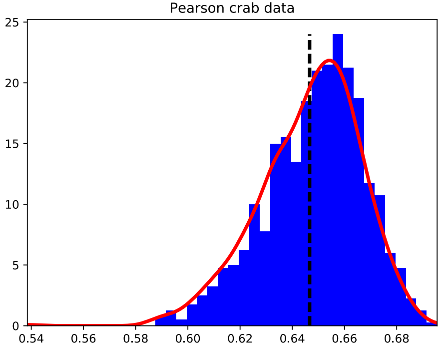

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- Use G-test with $r = 21$ (merge small-count bins).

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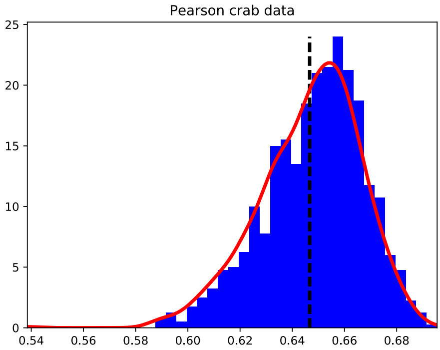

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- Got: $G \approx 64$ and $p \approx 2 \cdot 10^{-6}$

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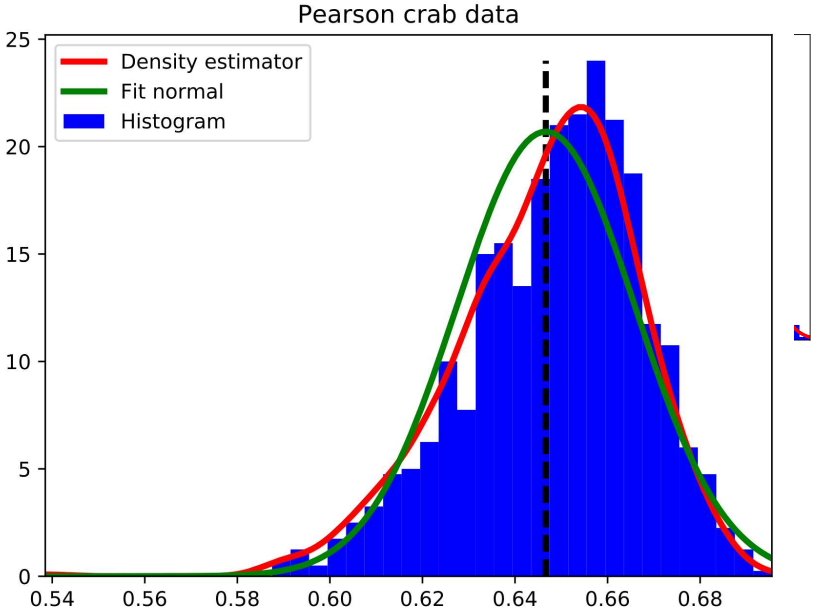

- Is
- N
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- G

Quantile-quantile plot

qqplot

- Goal: check if X_i's $\approx P_0$
- Step 1. Sort $X_{(1)} \leq \cdots \leq X_{(n)}$
- Step 2. Plot pairs $(F_0^{-1}(i/n), X_{(i)})$
- Good fit $\iff$ straightline

- Good tool for graphically inspecting goodness-of-fit

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- Good tool for graphically inspecting goodness-of-fit
- $F_0^{-1}(q)$ is q-th quantile of P_0
- Rationale:
  - empirical 10% quantile $= X_{(n/10)}$
  - it is a very stable estimate of $F_0^{-1}(0.1)$
  - ... if X_i's truly iid $\sim P_0$
- Also allows to readoff location-scale params

Quantile-quantile plot

qq