Hypothesis testing 3: Two-sample tests

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Outline:

- Recap (p-value)
- Two-sample tests: paired an unpaired
- t-**test:** same variance and different variance
- Testing equality of distributions: KS-test, G-test, qqplot
- **Testing for independence**
- Confounding

Review

Definition

Statistical hypotheses:

- H : data $X_1, \ldots, X_n$ distributed according to $P \in \mathcal{C}_0$
- K : data $X_1, \ldots, X_n$ distributed according to $P \in \mathcal{C}_1$

where $\mathcal{C}_0, \mathcal{C}_1$ are COLLECTIONS OF DISTRIBUTIONS.

Remarks:

- Find statistic $T(X_1, \ldots, X_n)$ with $\approx$ same dist. under all $P \in \mathcal{C}_0$
- Test: If $T \geq t_\alpha$ then REJECT
- t_α is chosen depending on required size:

$$\max_{P \in \mathcal{C}_0} P[T \geq t_\alpha] \leq \alpha \, .$$

- Alternatively, report p-value: If $T(x_1, \ldots, x_n) = t_{obs}$

$$p = \max_{P \in \mathcal{C}_0} P[T \geq t_{obs}]$$

(aka "probability of same or more extreme data under null")

Comparison (two-sample) testing

Examples:

- A/B testing in marketing
- one algorithm vs another
- one stock vs another
- new drug vs placebo

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Two principal situations:

- Paired data:
  - each client tries both products
  - each input is evaluated using both algorithms
  - each day both stocks are evaluated
- Unpaired data:
  - each client tries only one product
  - each input is evaluated on only one product
  - each day can probe only one stock

Two-sample testing (paired data)

- Paired data:
  - each client tries both products
  - each input is evaluated using both algorithms
  - each day both stocks are evaluated
- Statistical formalism: $(X_i, Y_i) \overset{iid}{\sim} P_{X,Y}$

$$H : \mathbb{E}[X] \leq \mathbb{E}[Y] \qquad K : \mathbb{E}[X] > \mathbb{E}[Y]$$

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- For paired data can always reduce to one-sample case
- E.g. for real-valued measurements: $Z_i \triangleq X_i - Y_i$

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- Hence use z-test or t-test (or Wald test)
- Already had example before

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- What to do?
- General idea:

$$\frac{\hat{\mu}_X - \hat{\mu}_Y}{\widehat{se}} \quad \begin{matrix} < \\ > \end{matrix} \quad t_\alpha \quad \boxed{\text{ACCEPT/REJECT}}$$

(aka two-sample t-test)

Two-sample t-test: groups with same variance

- Hypothesis testing:
  - $X_i \overset{iid}{\sim} P_X$, n samples, $\mu_X = \mathbb{E}[X]$
  - $Y_i \overset{iid}{\sim} P_Y$, m samples, $\mu_Y = \mathbb{E}[Y]$
  - **Assumption:** $\boxed{\mathrm{Var}[X] = \mathrm{Var}[Y] = \sigma^2}$
  - ... same (but unknown!) variance of both samples
  - One-sided: $\boxed{H : \mu_X \leq \mu_Y \quad \text{vs} \quad K : \mu_X > \mu_Y}$
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  - One-sided: $H : \mu_X \leq \mu_Y$ vs $K :$
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$$\frac{\hat{\mu}_X - \hat{\mu}_Y}{\widehat{se}} \lessgtr t_\alpha$$ ACC

$$\boxed{\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i}$$

$$\boxed{\bar{Y}_m = \frac{1}{m}\sum_{j=1}^{m} Y_j}$$

- Let us try $T_0 = \bar{X}_n - \bar{Y}_m$ (sample means)

Need to normalize!

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- Problem: What is $\mathrm{Var}[T_0]$?

$$\mathrm{Var}[T_0] = \mathrm{Var}[\bar{X}_n] + \mathrm{Var}[\bar{Y}_m] = \sigma^2 \left(\tfrac{1}{n} + \tfrac{1}{m}\right)$$

σ^2 unknown

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  $\mathrm{Var}[T_0] = \mathrm{Var}[\bar{X}_n] + \mathrm{Var}[\bar{Y}_m] = \sigma^2 \left(\frac{1}{n} + \frac{1}{m}\right)$ σ^2 unknown
- Unbiased estimator:

  $\widehat{\sigma^2} = \frac{1}{n+m-2} \left(\sum_{i=1}^{n} (X_i - \bar{X}_n)^2 + \sum_{j=1}^{m} (Y_j - \bar{Y}_m)^2 \right)$

  (aka pooled estimator of variance)

Two-sample t-test: groups with same variance

two-sample t-statistic (pooled variance)

$$\begin{aligned} T &= \frac{\bar{X}_n - \bar{Y}_m}{\sqrt{\frac{1}{n} + \frac{1}{m}}\sqrt{\widehat{\sigma^2}}} \\ \widehat{\sigma^2} &= \frac{1}{n+m-2}\left(\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 + \sum_{j=1}^{m}(Y_j - \bar{Y}_m)^2\right) \end{aligned}$$

and $\bar{X}_n$, $\bar{Y}_m$ are sample means.

- By LLN and CLT:

$$\begin{aligned} \frac{\bar{X}_n - \bar{Y}_m}{\sqrt{\frac{1}{n} + \frac{1}{m}}} &\to \mathcal{N}(0, \sigma^2) \\ \widehat{\sigma^2} &\to \sigma^2 \end{aligned}$$

- ... Thus $T \to \mathcal{N}(0,1)$ as $n, m \to \infty$.

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  - **Assumption:** $\mathrm{Var}[X] = \mathrm{Var}[Y] = \sigma^2$
  - One-sided: $H : \mu_X \leq \mu_Y$ vs $K : \mu_X > \mu_Y$
  - Two-sided: $H : \mu_X = \mu_Y$ vs $K : \mu_X \neq \mu_Y$
- Test statistic: $T = \frac{\bar{X}_n - \bar{Y}_m}{\sqrt{\frac{1}{n}+\frac{1}{m}}} \frac{1}{\sqrt{\widehat{\sigma^2}}}$
- Thus select thresholds from approximating $T \approx \mathcal{N}(0,1)$:

  $T > z_\alpha$ REJECT one-sided

  $|T| > z_{\frac{\alpha}{2}}$ REJECT two-sided
- If samples are $\approx \mathcal{N}$, then $\boxed{T \approx \texttt{scipy.stats.t.pdf}(\cdot, \texttt{n} + \texttt{m} - 2)}$

  (and thus replace z_α with α-quantile of ...ditto...)

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  (and thus replace z_α with α-quantile of ...ditto...)
- Remember: pooled variance assumes **HOMOSCEDASTICITY**
- Example: signals (or patients) measured on the same noisy equipment.

Two-sample t-test: general case

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- Use unbiased estimators:

  $\widehat{\sigma_X^2} = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$

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- Asymptotically normal: $T \approx \mathcal{N}(0, 1)$, so use z_α or $z_{\alpha/2}$

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- ...aka Behrens-Fisher problem

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- Welch correction: $T \approx$ Student-t with d.o.f.$=$ (hard)
- Bootstrap: Simulate dist. of T with $\tilde{X}_i \sim \mathcal{N}(0, \widehat{\sigma_X^2})$, $\tilde{Y}_i \sim \mathcal{N}(0, \widehat{\sigma_X^2})$

Two-sample t-test: example

- Setting:
  - Hypothesis: F. Dostoevsky's sentences are longer than M. Twain's:

$$\text{null } H : \mu_{MT} < \mu_{FD}$$

  - Two novels: 1.*Tom Sawyer* vs 2.*Crime and Punishment*
  - X_i – lengths of sentences in novel 1
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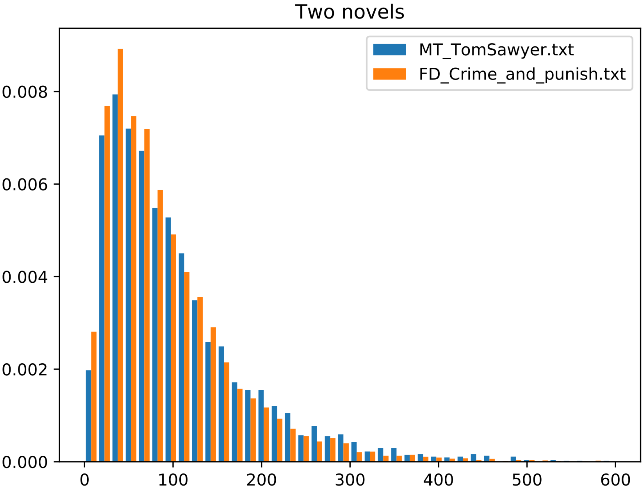
- Firs



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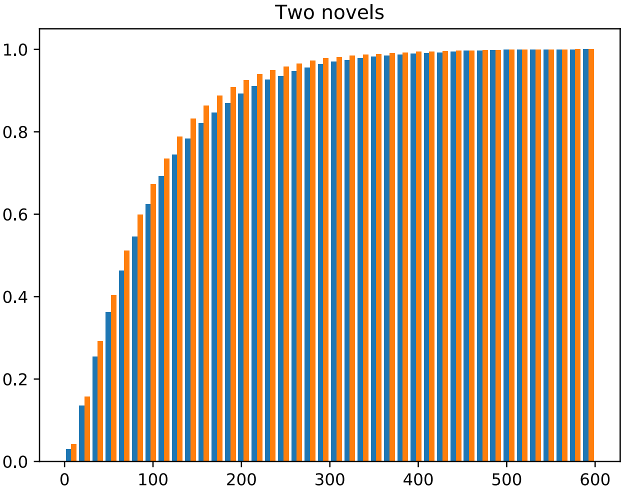

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- Ok, let's do a t-test (unequal var.)
  - Novel 1: $\hat{\mu}_1 = 105.6, \hat{\sigma}_1 = 90.1, n_1 = 3622$
  - Novel 2: $\hat{\mu}_2 = 93.8, \hat{\sigma}_2 = 79.0, n_2 = 11906$
  - T-statistic: $t = \frac{\hat{\mu}_1 - \hat{\mu}_2}{\sqrt{\hat{\sigma}_1^2/n_1 + \hat{\sigma}_2^2/n_2}} = 7.11$
  - p-value $p = 5 \cdot 10^{-13}$. Sound REJECT

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- Histograms are alike, but we got $p \ll 1$? How?

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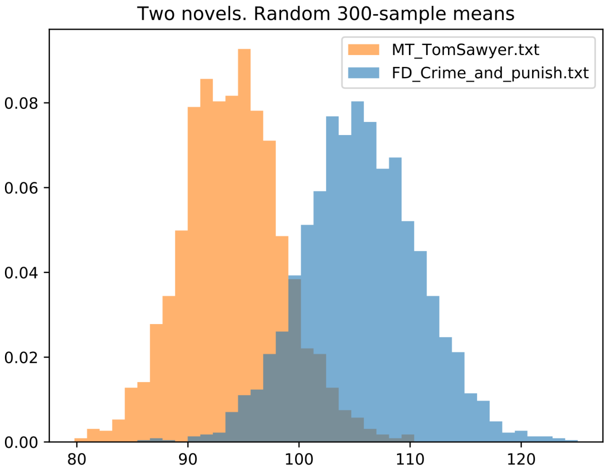

Sample-mean amplifies (and Gaussianizes) subtle differences.

More than two groups, non-parametric tests

What we did not cover:

- Could have more than two groups
  - The null-hypothesis:

$$H : \mu_1 = \mu_2 = \cdots = \mu_K$$

  - Test-statistic is (sort of):

$$F = (\hat{\mu}_1 - \hat{\mu})^2 + \cdots + (\hat{\mu}_K - \hat{\mu})^2$$

  - Asymptotically $\chi^2()$-distributed under null.
  - Known as F-test
  - Such multiple-group problems have cool name: ANOVA

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- Non-parametric tests:
  - t-tests are "parametric": exactly size-α only for Gaussian distributions.
  - Exactly size-α tests w/o assumptions?
  - Yes! And they are beautiful: Wilcoxon sum-rank tests
  - Key: sort $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$. If $P_X = P_Y$ then ranks of X's and Y's are uniformly distributed on $[n+m]$.

Two-sample tests: beyond means

- Sometimes we may not be interested in means
  (e.g. data non-numerical)
- ... but still want to know if there is some effect
- Typical setting:
  - $X_i \stackrel{iid}{\sim} P_X$, n samples
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- **Two cases:** continuous and discrete data

Equality of distributions: Kolmogorov-Smirnov

- Last time: How to test $X \sim P_0$ with given (cts) P_0.
- Main observation: $\sqrt{n} \cdot \sup_t |\hat{F}_X(t) - F_0(t)|$ has known distribution (under null)

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two-sample Kolmogorov-Smirnov statistic

$$KS = \sqrt{\frac{nm}{n+m}} \cdot \sup_t |\hat{F}_X(t) - \hat{F}_Y(t)|$$

- Known distribution independent (!) of $P_X = P_Y$

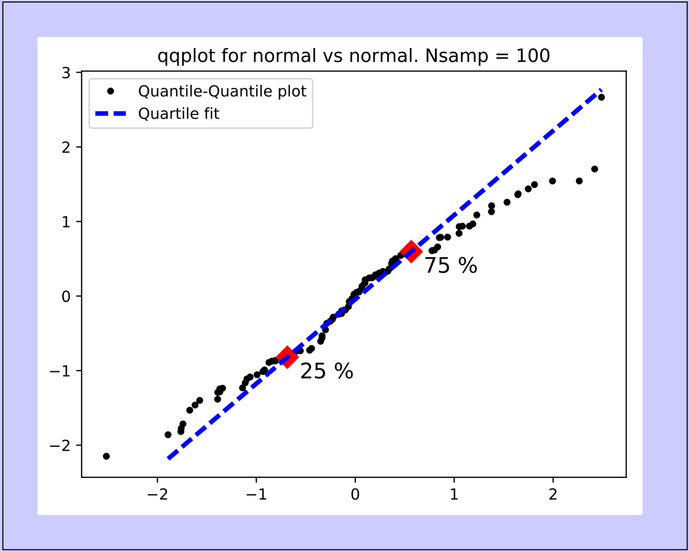
  . . . so just simulate on uniform to get p-value!
- Analytical formulae for $n, m \to \infty$. Use:

```
scipy.stats.ks_2samp(x_samp, y_samp)
```

Equality of distributions: Quantile-Quantile plots

- Setting:
  - $X_i \overset{iid}{\sim} P_X$, n samples
  - $Y_i \overset{iid}{\sim} P_Y$, m samples
  - $H : P_X = P_Y$ vs $K : P_X \neq P_Y$
- qqplot:
  - Step 1. Sort data: $X_{(1)} \leq \cdots \leq X_{(n)}$ and $Y_{(1)} \leq \cdots \leq Y_{(m)}$
  - Step 2. Suppose $m = n$ (general case is similar)
  - Step 3. Plot pairs $(X_{(i)}, Y_{(i)})$
- **MAGIC:** Under null, should get a straight line
- Example 1: $P_X = P_Y = \mathcal{N}(0, 1)$

Equality of distributions: Quantile-Quantile plots



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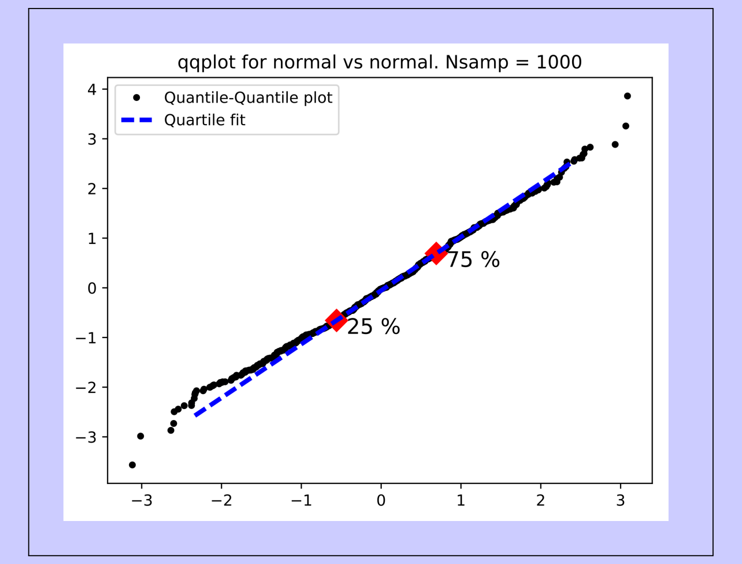

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- Example 2: $P_X = \mathcal{N}(0,1)$, $P_Y = \chi^2(\texttt{df} = 5)$.

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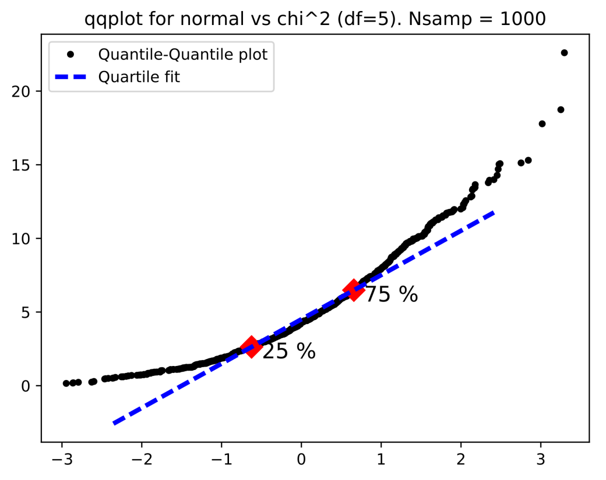

Equality of discrete distributions

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- **discrete X's and Y's.** Example:
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  - Hospital 1 sample: Cured, Cured, Died, . . . , Cured
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  - Summarize data in table:

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  - Assume $(U_i, V_i) \overset{iid}{\sim} P_{U,V}$ and have $n + m$ such samples*
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  - $H: U \perp\!\!\!\perp V$ vs $K: U \not\perp\!\!\!\perp V$
  - *subtlety: Orig. question was not symmetric in U, V

    (had samples $P_{U|V=1}$ and $P_{U|V=2}$) iid approx ok for $n, m \gg 1$

G-test for independence

- New problem
  - $(U_i, V_i) \overset{iid}{\sim} P_{U,V}$, ℓ-samples
  - U is t-valued, i.e. $U \in [t] \triangleq \{1, \ldots, t\}$
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- Recall generalized likelihood ratio test:

The G-statistic (general)

$$\begin{aligned} G &\triangleq -2 \log \frac{P_0^*(x_1, \ldots, x_n)}{P_1^*(x_1, \ldots, x_n)} \\ P_0^*(x_1, \ldots, x_n) &= \max_{P \in \mathcal{C}_0} P(x_1, \ldots, x_n) \\ P_1^*(x_1, \ldots, x_n) &= \max_{P \in \mathcal{C}_0 \cup \mathcal{C}_1} P(x_1, \ldots, x_n) \end{aligned}$$

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\begin{aligned}
G_{norm} &\triangleq 2\ell D(\hat{P}_{U,V} \| \hat{P}_U \times \hat{P}_V) \\
D(Q_1 \| Q_2) &\triangleq \sum_{(a,b)} Q_1(a,b) \log_e \frac{Q_1(a,b)}{Q_2(a,b)}
\end{aligned}
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- Empirical dist: $\hat{P}_{U,V}(a,b) = \frac{|\{i : U_i = a, V_i = b\}|}{\ell}$, $\hat{P}_U(a) = \frac{|\{i : U_i = a\}|}{\ell}$
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- **Strong MAGIC:** $G_{norm} \approx \chi^2((t-1)(s-1))$ as $\ell \to \infty$

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- Resulting test:
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  - Compare to $(1-\alpha)$-quantile of $\chi^2((t-1)(s-1))$
  - Alternatively,

$$p\text{-value} = \mathbb{P}[\chi^2((t-1)(s-1)) > G_{norm}] \, .$$

  - Or `scipy.stats.chi2.sf(`G_{norm}`,df=`$(t-1)(s-1)$`)`

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p-value $= 0.28$

Alternative test

- For 2x2 case don't need to be so fancy
- Test: $X \sim \mathrm{Bino}(n, p_1), Y \sim \mathrm{Bino}(m, p_2)$ and null $H : p_1 = p_2$.
- Do the two-sided t-test:

$$T = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n} + \frac{\hat{p}_2(1-\hat{p}_2)}{m}}}$$

  with $\hat{p}_1 = X/n$ and $\hat{p}_2 = Y/m$;
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- p-value: $p = 0.262$ (Welch corrected)
- p-value: $p = 0.260 \pm 0.001$ Bootstrap 1: equal-mean Binomial X, Y
- p-value: $p = 0.261 \pm 0.005$ Bootstrap 2: equal-mean Normal X, Y

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- So why fancy G-test?
- Because it also works for r hospitals and s outcomes.

Testing quality of classifiers

Comparing quality of classifiers

Consider the following problem:

- Test set of size n is given
- Two predictors (classifiers) are tested
- The base one has 1% error.
- The new one has $e\%$ error
- Question: What e is significant (to declare new one is better)?

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- Can form a 2x2 contingency table and run a G-test
- Some sample numbers:
  - For $n = 10000$ (MNIST, CIFAR) we have
  $$e < 0.65\% \quad \text{or} \quad e > 1.45\%$$
  - For $n = 1000$ we have
  $$e < 0.13\% \quad \text{or} \quad e > 2.6\%$$

  are significant (at $p = 0.05$)

Beware of two-sample tests

- We learned how to test comparative hypotheses.
- BIG ISSUE: Confounding in observational studies
- Observational vs controlled study.
  - Observational study: groups self-selected
  - Randomized controlled study: groups assigned

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- Cartoon example: "drinking beer makes you bald"

	Bald	Not bald
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correlation $\neq$ causation

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  - Quinn et al [Nature'1999]: "Myopia and ambient lighting at night"
  - Eyeball development vs infant night sleep

Sleep condition	Fraction developing myopia
Darkness	10%
Night light	34%
Room light	55%

  - Good sample size: $n = 479$
  - Sound statistics (p-value < 0.00001)
  - Physiologically plausible: *"The duration of the daily light period has been shown to affect eye growth in chicks"*

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  - ... but: myopic parents are more likely to leave night light on

Beware of two-sample tests

What can we do about confounding?

- Use common sense:
  - E.g. want to learn about effect of third kid on women labor market
  - Cannot do R.C.T.
  - Note: families with two kids of same sex are more likely to have third (by 6%)
  - ... use this for checking if two groups have similar unemployment

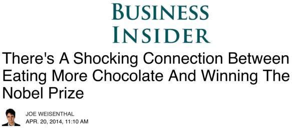
Confounding

- Big area of research (Causal Inference)
- Rough idea: conditional independence testing
- If suspect relation between X and Y is confounded by Z can test:

$$X \perp\!\!\!\perp Y | Z$$

pro-term "controlling for Z"

What can go wrong? **Predicting $\ll$ Modeling**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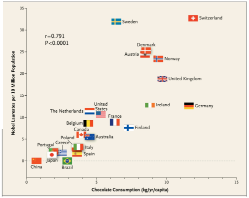

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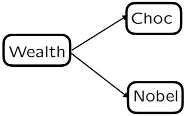

Need to infer a model/mechanism

Can we? How?