Section 2.1

Exercise 5

$\forall x\big(x \in A \leftrightarrow x \in B\big) \leftrightarrow A = B$

a) $A = B$ since A contains all elements of B and vice versa

b) $((1)) \in A \land ((1)) \in B$ but $(1) \notin A \rightarrow A \neq B$

c) $\emptyset$ is an empty set, and is not equal to a set containing an empty set.

Exercise 24

a) No. The power set of a set always include a set of an empty set: $P(\emptyset) = (\emptyset)$ meaning a power set of a set is never empty. $\forall A\big((\emptyset) \in P(A)\big) \rightarrow P(A) \neq \emptyset$

b) Yes. $P((a)) = (\emptyset, (a))$

c) No. $P((\emptyset, a)) = \big(\emptyset, (a), (\emptyset), (a, \emptyset)\big)$ The set in question is missing a $(\emptyset)$

d) Yes. $P((a, b)) = \big(\emptyset, (a), (b), (a, b)\big)$

Section 2.2

Exercise 18

c) $(A - B) \equiv \forall x \mid x \in A \land x \notin B$
$(A - B) - C \equiv \forall x \mid x \in (A - B) \land x \notin C$
$(A - B) - C \equiv \forall x \mid x \in A \land x \notin B \land x \notin C$
$(A - C) \equiv \forall x \mid x \in A \land x \notin C$
Since all elements of $(A - B) - C$ are in $(A - C)$, $(A - B) - C \subseteq (A - C)$

d) $(A - C) \equiv \forall x \mid x \in A \land x \notin C$
$(C - B) \equiv \forall x \mid x \in C \land x \notin B$
$(A - C) \cap (C - B) \equiv \forall x \mid x \in (A - C) \land x \in (C - B)$
$(A - C) \cap (C - B) \equiv \forall x \mid (x \in A \land x \notin C) \land (x \in C \land x \notin B)$
Rewriting the paranthesis we get
$(A - C) \cap (C - B) \equiv \forall x \mid x \in A \land \underbrace{(x \notin C \land x \in C)}_{x \in \emptyset} \land x \notin B$
Clearly, $(A - C) \cap (C - B) = \emptyset$

Exercise 46

The set $A \cup B$ consists of all elements in A that are also in B.

$$A \cup B = \forall x \mid x \in A \wedge x \in B$$

Another way of describing $A \cup B$ is to look at all elements in A plus all elements in B minus the duplicates, which are elements in both A and B.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

This also holds for $A \cup B \cup C$ but this time we have more duplicates. Elements in $A \cap B$, $A \cap C$ and $B \cap C$ are "counted" twice, and elements in $A \cap B \cap C$ are "counted" an extra time.

$$|A \cup B \cup C| = |A| + |B| + |C| - \underbrace{|A \cap B| - |A \cap C| - |B \cap C|}_{\text{elements covered twice by } |A|+|B|+|C|} - \underbrace{|A \cap B \cap C|}_{\text{elements covered three times by } |A|+|B|+|C|}$$

Section 2.3

Exercise 12

a) $f(n) = n - 1$ is one-to-one since the function is strictly growing. Since every Δn increase on its domain will produce a Δy increase on its codomain, all values of n maps to a unique value

b) $f(n) = n^2 + 1$ is one-to-one since $f : \mathbb{Z} \mapsto \mathbb{Z}$ and is strictly growing (even in n=0 because of $\mathbb{Z}$).

c) $f(n) = n^3$ is one to one for the same reason as in b)

d) $f(n) = \lceil n/2 \rceil$ is not one to one since every value of n maps to two values (except for n = 0).

Exercise 38

$$f(x) = ax + b$$
$$g(x) = cx + d$$
$$f \circ g = f(g(x)) = a(cx + d) + b = acx + ad + b$$
$$g \circ f = g(f(x)) = c(ax + b) + d = acx + cb + d$$
$$f \circ g = g \circ f \iff ad + b = cb + d$$

Exercise 42

$f : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = x^2$ has an inverse f^{-1} given by $f^{-1}(x) = \pm\sqrt{x}$

a) $f^{-1}(\{1\}) = \pm 1$

b) $f^{-1}(\{x \mid 0 < x < 1\}) = \pm\{x \mid 0 < x < \sqrt{1}\} = \{x \mid -1 < x < 1 \underbrace{\wedge x \neq 0}_{\text{input not defined for } x = 0} \}$

c) $f^{-1}(\{x \mid x > 4\}) = \pm\{x \mid x > \sqrt{4}\} = \{x \mid -2 > x \wedge x > 2\}$

Section 2.4

Exercise 12

$a_n = -3a_{n-1} + 4a_{n-2}$

a)

$$a_n = 0 \implies -3a_{n-1} + 4a_{n-2} = -3(0) + 4(0) = 0 = a_n$$

Which means $a_n = 0$ is a trivial solution to the recurrence relation

b)

$$a_n = 1 \implies -3a_{n-1} + 4a_{n-2} = -3 + 4 = 1 = a_n$$

$a_n = 1$ is a solution to the recurrence relation

c)

$$\begin{aligned} a_n = (-4)^n \implies -3a_{n-1} + 4a_{n-2} &= -3((-4)^{n-1}) + 4((-4)^{n-2}) \\ &= -3((-4)(-4)^{n-2}) + 4((-4)^{n-2}) \end{aligned}$$

We pull $(-4)^{n-2}$ from the parentheses

$$\begin{aligned} &= (-4)^{n-2}(-3)((-4)) + (-4)^{n-2}4(1)) \\ &= (-4)^{n-2}((-3)(-4) + 4) \\ &= (-4)^{n-2}(18) \neq a_n \end{aligned}$$

$a_n = (-4)^n$ is not a solution to the recurrence relation

d)

$$\begin{aligned} a_n = 2(-4)^n + 3 \implies -3a_{n-1} + 4a_{n-2} &= -3\big(2(-4)^{n-1} + 3\big) + 4\big(2(-4)^{n-2} + 3\big) \\ &= -6(-4)^{n-1} - 9 + 8(-4)^{n-2} + 12 \\ &= -6(-4)(-4)^{n-2} + 8(-4)^{n-2} + 3 \\ &= 24(-4)^{n-2} + 8(-4)^{n-2} + 3 \\ &= 32(-4)^{n-2} + 3 \\ &= 2(-4)^2(-4)^{n-2} + 3 \\ &= 2(-4)^n + 3 = a_n \end{aligned}$$

$a_n = 2(-4)^n + 3$ is a solution to the recurrence relation

Exercise 33

a)

$$\begin{aligned}\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j) &= \sum_{i=1}^{2}\big(3i+\sum_{j=1}^{3}j\big)\\ &= \sum_{i=1}^{2}(3i+6)\\ &= 12+\sum_{i=1}^{2}3i\\ &= 12+9=21\end{aligned}$$

b)

$$\begin{aligned}\sum_{i=0}^{2}\sum_{j=0}^{3}(2i+3j) &= \sum_{i=0}^{2}\big(8i+\sum_{j=0}^{3}3j\big)\\ &= \sum_{i=0}^{2}\big(8i+21\big)\\ &= 63+\sum_{i=0}^{2}8i\\ &= 63+24=87\end{aligned}$$

c)

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=0}^{2}i &= \sum_{i=1}^{3}\big(3i+\sum_{j=0}^{2}0\big)\\ &= \sum_{i=1}^{3}3i\\ &= 3+6+9=18\end{aligned}$$

d)

$$\begin{aligned}\sum_{i=0}^{2}\sum_{j=1}^{3}ij &= \sum_{i=0}^{2}\big(i\sum_{j=1}^{3}j\big)\\ &= \sum_{i=0}^{2}6i\\ &= 0+6+12=18\end{aligned}$$

Section 2.5

Exercise 16

Take $A \subset B$ where B is countable. If $A \subset B$ then an inclusion mapping $I : A \mapsto B$ can be defined. $|A| \leq |B|$ if there exists a function f that is one-to-one and $f : A \mapsto B$. Since $|B|$ is countable we know that $|B| \leq \aleph_0$. $|A| \leq |B| \wedge |B| \leq \aleph_0 \implies |A| \leq \aleph_0$ which means A is countable.