1

We have the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

with initial condition

$$u(x,0) = 200\sin(\pi x) \tag{2}$$

The ends of the rod is kept at 0°C which means

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

By variable separation we set

$$u(x,t) = F(x)G(t) \tag{3}$$

and

$$\begin{aligned} F(0) &= 0 \\ F(L) &= 0 \end{aligned} \tag{4}$$

which when inserted into (1)gives us

$$\begin{aligned} F\dot{G} &= c^2 F''G \\ \frac{F}{F''} &= c^2 \frac{G}{\dot{G}} \end{aligned}$$

Since F and G are independent both sides of this equation must be constant

$$\frac{F}{F''} = c^2 \frac{G}{\dot{G}} = k$$

and can be split into two ODEs

$$F - kF'' = 0 \tag{5}$$

$$c^2 G - k\dot{G} = 0 \tag{6}$$

To solve (5) we first try with positive $k = -p^2$ which gives us

$$F(x) = \frac{F'(0)}{p} \sin px$$

If we insert (4) into this we get

$$F(L) = \frac{F'(0)}{p} \sin pL = 0$$

giving us

$$p = \frac{n\pi}{L}$$

and a general solution

$$F(x) = \frac{F'(0)L}{n\pi} \sin \frac{n\pi}{L} x$$

We have from (2) that

$$F(x)G(0) = 200 \sin \pi x = \frac{G(0)F'(0)L}{n\pi} \sin \frac{n\pi}{L} x$$

which gives us

$$\begin{aligned} n &= L \\ G(0)F'(0) &= 200\pi \end{aligned}$$

Solving (6) yields

$$G(t) = G(0)e^{\frac{c^2}{k}t}$$

We know that the system has a steady state, meaning $\dot{G}(t) \to 0$ as t increases, which means we have a negative $k = -p^2$ where $p = \frac{n\pi}{L} = \pi$

$$G(t) = G(0)e^{-\frac{c^2}{\pi^2}t}$$

giving us

$$u(x,t) = F(x)G(t) = \frac{F'(0)}{\pi} \sin(\pi x) G(0) e^{-\frac{c^2}{\pi^2}t}$$

For the maximum temperature to decrease by 50°C in 20 seconds we have that

$$\begin{aligned} u(0.5, 0) &= 200 \\ u(0.5, 20) &= 150 \\ c = \pi\sqrt{ln\frac{200}{150}} &\approx 1.685 \end{aligned}$$

2

Since we have a steady state there is no change dependent on time, meaning all time derivatives are constant (or zero)

$$c^2 \frac{\partial^2 u}{\partial x^2} = k$$

We use (3) to get

$$c^2 F''(x)G(t) = k$$

We know that G(t) is constant since we are in a steady state, which means

$$\begin{aligned} u(x,\infty) &= c^2 G(\infty)F''(x) \\ F''(x) &= K \\ F(x) &= Kx^2 + \alpha x + \beta \end{aligned}$$

We have that

$$\begin{aligned} F(0) &= U_1 \\ F(L) &= U_2 \end{aligned}$$

which gives us

$$\beta = U_1$$

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