**1)**

We have the one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

with boundary conditions:

$$\frac{\partial u(0,t)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u(L,t)}{\partial x} = 0$$

where $c \approx 340\text{m/s}$

a)

Differentiating $u(x,t) = F(x)G(t)$ with respects to $x$ and $t$ we get

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F''G$$

By inserting this into (1) we get

$$F\ddot{G} = c^2 F''G \tag{2}$$

and the boundary conditions

$$F'(0)G(t) = 0 \quad \text{and} \quad F'(L)G(t) = 0 \tag{3}$$

Simplifying (2) we get

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F}$$

We now have an equation where the left side depend on $t$ and the right side depend on $x$. Since a change in $x$ or $t$ would be independent of the other side both sides must be constant:

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k$$

This means we can split the equations into two ODEs

$$F'' - kF = 0$$

and

$$\ddot{G} - c^2 kG = 0 \tag{4}$$

We're looking for a non-trivial solution to the wave equation. We can see from (2) that if $F = 0$ or $G = 0$ all solutions become trivial. We can therefore set $F \neq 0$ and $G \neq 0$ since that is all we're interested in.

For positive $k = \mu^2$ we have the general solution

$$F = A_1 e^{\mu x} + B_1 e^{-\mu x}$$

and its derivative

$$F' = A_1' e^{\mu x} + B_1' e^{-\mu x} \tag{5}$$

Inserting the boundary conditions (3) in (5) we get

$$F'(L)G(t) = (A_1' e^{\mu L} + B_1' e^{-\mu L})G(t) = 0$$

$G(t) \neq 0$ gives

$$\mu L A_1'' + \mu L B_1'' = 0 \implies A_1'' = B_1'' = 0$$

meaning positive $k$ only gives the trivial solution $F(x)G(t) = 0$

We choose negative $k = -p^2$ which gives us the general solution

$$F(x) = A' \cos px + B' \sin px$$

and its derivative

$$F'(x) = A \sin px + B \cos px \tag{6}$$

Inserting the boundary conditions (3) into (6) gives us

$$F'(0)G(t) = \big(A \sin 0 + B \cos 0\big)G(t) = BG(t) = 0$$

Since $G(t) \neq 0$ we know that $B = 0$

To avoid trivial solutions we set $A \neq 0$

$$F'(L)G(t) = A \sin(pL)G(t) = 0$$

We now have infinitely many solutions $F(x) = F_n(x)$ where

$$F_n(x) = A \sin \frac{n\pi}{L} \quad \text{where} \quad n \in \mathbb{Z}^+$$

We can now solve (4) with $k = -p^2 = -(\frac{n\pi}{L})$

$$\ddot{G} + c^2 p^2 G = 0$$

$$\ddot{G}_n + \underbrace{\Big(\frac{cn\pi}{L}\Big)}_{\lambda_n}{}^2 G = 0$$

$$G_n(t) = G'(0) \cos \lambda_n t + G(0) \sin \lambda_n t$$

Which gives us

$$u(x,t) = F(x)G(t) = F_n(x)G_n(t) = A \sin \frac{n\pi}{L} G'(0) \cos \lambda_n t + G(0) \sin \lambda_n t$$

b)

The spectrum is the set of $\lambda_n$

$$\lambda_n = \frac{cn\pi}{L} \quad \text{where} \quad n \in \mathbb{N}$$

c)

$$f_n = \frac{cn}{2L}$$

Fundamental mode is when n = 1

$$\begin{aligned} f &= \frac{c}{2L} = \frac{340[\text{m/s}]}{2 \times 0.5[\text{m}]} \\ &= 340\,[1/\text{s}] = 340\text{Hz} \end{aligned}$$

**2)**

We have the one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} \tag{7}$$

with boundary conditions:

$$u(0,t) == 0 \qquad \text{and} \quad u(3,t) = 0$$

We set $u(x,t) = F(x)G(t)$ and see that (7) can be written as

$$F\ddot{G} = F''G$$

with boundary conditions ($G \neq 0$ since we're interested in non-trivial solutions)

$$F(0) = 0 \quad \text{and} \quad F(3) = 0$$

and separated (constant since they depend on different variables)

$$\begin{aligned} \frac{F''}{F} &= \frac{\ddot{G}}{G} = k \\ F'' - kF &= 0 \\ \ddot{G} - kG &= 0 \end{aligned}$$

We can now solve the two ODEs. Setting positive $k = \mu^2$ gives us trivial solutions, so we set negative $k = -p^2$

$$F(x) = F'(0) \sin px$$

set boundary conditions

$$
\begin{aligned}
F(3) &= F'(0)\sin(3p) = 0 \\
p &= \frac{n\pi}{3}
\end{aligned}
$$

Using $p = \frac{n\pi}{3}$ we can solve for $G(t)$

$$G(t) = G(0)\cos\frac{n\pi}{3}t + G'(0)\sin\frac{n\pi}{3}t$$

which gives us

$$u(x,t) = \underbrace{F'(0)\sin\frac{n\pi}{3}x}_{F(x)}\underbrace{(G(0)\cos\frac{n\pi}{3}t + G'(0)\sin\frac{n\pi}{3}t)}_{G(t)}$$

a)

We have the initial conditions

$$
\begin{aligned}
F(x)G(0) &= x(3-x) \quad \text{and} \quad G'(0) = 0 \\
u(x,t) &= x(3-x)\cos\frac{n\pi}{3}t
\end{aligned}
$$

b)

We have the initial conditions

$$
\begin{aligned}
G(0) &= 0 \quad \text{and} \quad F(x)G'(0) = \sin(\pi x) - \frac{1}{2}\sin(2\pi x) \\
u(x,t) &= \underbrace{F'(0)\sin\frac{n\pi}{3}x}_{F(x)}(G(0)\cos\frac{n\pi}{3}t \\
&= \underbrace{F'(0)\sin\frac{n\pi}{3}xG'(0)}_{F(x)G'(0)}\sin\frac{n\pi}{3}t) \\
&= (\sin(\pi x) - \frac{1}{2}\sin(2\pi x))\sin\frac{n\pi}{3}t)
\end{aligned}
$$

c)

We have the initial conditions

$$F(x)G(0) = x(3-x) \quad \text{and} \quad F(x)G'(0) = \sin(\pi x) - \frac{1}{2}\sin(2\pi x)$$

$$\begin{aligned} u(x,t) &= \underbrace{F'(0)\sin\frac{n\pi}{3}x}_{F(x)}\underbrace{\left(G(0)\cos\frac{n\pi}{3}t + G'(0)\sin\frac{n\pi}{3}t\right)}_{G(t)} \\ &= \underbrace{F'(0)\sin\frac{n\pi}{3}xG(0)}_{F(x)G(0)}\cos\frac{n\pi}{3}t + \underbrace{F'(0)\sin\frac{n\pi}{3}xG'(0)}_{F(x)G'(0)}\sin\frac{n\pi}{3}t \\ &= x(3-x)\cos\frac{n\pi}{3}t + \big(\sin(\pi x) - \frac{1}{2}\sin(2\pi x)\big)\sin\frac{n\pi}{3}t \end{aligned}$$

**3)**

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{8}$$

with boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(2,t) = 0$$

We separate the variables by defining functions $F(x)$ and $G(t)$

$$u(x,t) = F(x)G(t)$$

We have that

$$\dot{G}F = \frac{\partial u}{\partial t} \quad \text{and} \quad GF'' = \frac{\partial^2 u}{\partial x^2} \tag{9}$$

We insert (9) into (8) and get

$$\begin{aligned} \dot{G}F &= F''G \\ \frac{\dot{G}}{G} &= \frac{F''}{F} \end{aligned}$$

Since these are separated they must be constant

$$\frac{\dot{G}}{G} = \frac{F''}{F} = k$$

Splitting these into two ODEs gives

$$\dot{G} - kG = 0$$

and

$$F'' - kF = 0$$

Like in 1a) we're not interested in trivial solutions where $F(x)G(t) = 0$ so we set $F \neq 0$ and $G \neq 0$

Solving for G(t) gives

$$G(t) = Ae^{kt}$$

and for F(x) gives