Analysis

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Contents

Chapter 1

Set Theory

1.1 Ordered Pairs

Definition 1.1 (Ordered Pair). The **ordered pair** (a, b) is the set whose members are $\{a\}$ and $\{a, b\}$. In symbols we have

$$(a, b) = \{\{a\}, \{a, b\}\}$$

This definition ensures that order matters. To show this, this theorem and its proof should suffice.

Theorem 1.2 (Ordered Pair Theorem). [a]

$$(a, b) = (c, d) \leftrightarrow a = c, b = d$$

[a] this is a made up name by me

Proof. If $a = c$ and $b = d$, then

$$(a, b) = \{\{a\}, \{a, b\} = \{\{c\}, \{c, d\}\} = (c, d)$$

Conversely, suppose that $(a, b) = (c, d)$. Then by our definition we have $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. We wish to conclude that $a = c$ and $b = d$. To this end we consider two cases, depending on whether $a = b$ or $a \neq b$.

If $a = b$, then $\{a\} = \{a, b\}$, so $(a, b) = \{\{a\}\}$. Since $(a, b) = (c, d)$, we then have

$$\{\{a\}\} = \{\{c\}, \{c, d\}\}.$$

The set on the left has only one member, $\{a\}$. Thus the set on the right can have only one member, so $\{c\} = \{c, d\}$, and we can conclude that $c = d$. But then $\{\{a\}\} = \{\{c\}\}$, so $\{a\} = \{c\}$ and $a = c$. Thus $a = b = c = d$.

On the other hand, if $a \neq b$, then from the preceding argument it follows that $c \neq d$. Since $(a, b) = (c, d)$, we must have

$$\{a\} \in \{\{c\}, \{c, d\}\},$$

which means that $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$. In either case we have $c \in \{a\}$, so $a = c$. Again, since $(a, b) = (c, d)$, we must also have

$$\{a, b\} \in \{\{c\}, \{c, d\}\}.$$

Thus $\{a, b\} = \{c\}$ or $\{a, b\} = \{c, d\}$. But $\{a, b\}$ has two distinct members and $\{c\}$ has only one, so we must have $\{a, b\} = \{c, d\}$. Now $a = c$, $a \neq b$, and $b \in \{c, d\}$, which implies that $b = d$. $\square$

Definition 1.3 (Cartesian Product). If A and B are sets, then the **Cartesian product** (or **cross product**) of A and B, written $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. In symbols,

$$A \times B = \{(a, b) : (a \in A) \wedge (b \in B)\}.$$

1.2 Relation

Definition 1.4 (Relation). Let A and B be sets. A **relation between A and B** is any subset R of $A \times B$. We say that an element a in A is **related** by R to an element b in B if $(a, b) \in \mathrm{R}$, and we often denote this by writing "$a\mathrm{R}b$". The first set A is referred to as the **domain** of the relation and denoted by dom R. If $B = A$, then we speak of a relation $\mathrm{R} \subseteq A \times A$ being a **relation on A**.

Definition 1.5 (Equivalence Relation). A relation R on a set S is an **equivalence relation** if it has the following properties for all $x, y, z \in S$:

- **Reflexive property:** $x\mathrm{R}x$
- **Symmetric property:** $x\mathrm{R}y \leftrightarrow y\mathrm{R}x$
- **Transitive property:** $(x\mathrm{R}y \wedge y\mathrm{R}z) \rightarrow x\mathrm{R}z$

An example for a **equivalence relation** is the relation "is parallel to" when considering all lines in the plane, if we agree that a line is parallel to itself.

Definition 1.6 (Equivalence Class). Given an equivalence relation R on a set S, the **equivalence class** with respect to R of $x \in S$ is the set

$$E_x = \{y \in S : y\mathrm{R}x\}$$

Example. Let $S = \{a : a \text{ lives in Sweden}\}$, which is the set of all people living in Sweden. Also, let a equivalence relation on this set be

$$R = \{(a, b) \in S \times S : a \text{ was born in the same year as } b\}.$$

Then

$$E_x = \{y \in S : y\mathrm{R}x\}$$

is the set of all people living in Sweden who was born during the same year as some person x who is also living in Sweden. ◇

Theorem 1.7. Two equivalence classes on the same set S with the same equivalence relation R must be disjoint or equal.

Proof. Let R be an equivalence relation on a set S, and let E_x and E_y be two equivalence classes with respect to R of $x \in S$. Suppose that they overlap, then there exists some $w \in E_x \cap E_y$. For all $x' \in E_x$ we have $x' \mathrm{R} x$, and because $w \in E_x$, $w \mathrm{R} x$, and by symmetry, $x \mathrm{R} w$. Also, $w \in E_y$ so $w \mathrm{R} y$. By using transitivity, $x' \mathrm{R} x$ and $x \mathrm{R} w$ and $w \mathrm{R} y$ implies that $x' \mathrm{R} y$, which means that $x' \in E_y$ and that $E_x \subseteq E_y$.

Conversely, for all $y' \in E_y$ we have $y' \mathrm{R} y$, and because $w \in E_y$, $w \mathrm{R} y$, and by the symmetry property, $y \mathrm{R} w$. Also, $w \in E_x$ so $w \mathrm{R} x$. By using the transitivity property, $y' \mathrm{R} y$ and $y \mathrm{R} w$ and $w \mathrm{R} x$ implies that $y' \mathrm{R} x$ and that $E_y \subseteq E_x$. Since $E_x \subseteq E_y$ and $E_x \supseteq E_y$, it must be that $E_y = E_x$. □

Definition 1.8. A **partition** of a set S is a collection P of nonempty subsets of S such that

- Each $x \in S$ belongs to some subset $A \in \mathrm{P}$.
- For all $A, B \in \mathrm{P}$, if $A \neq B$, then $A \cap B = \emptyset$.

A member of P is called a **piece** of the partition.

Example. Two equivalence classes on the same set S with the same equivalence relation R who are not equal (and therefore disjoint) are two pieces of a partition P on the set S. ◇

Chapter 2

Functions

Definition 2.1 (Function between two sets)**.** Let A and B be sets. A **function** from A to B is a nonempty relation $f \subseteq A \times B$ that satisfies the following two conditions:

1. *Existance*: $\forall a \in A, \exists b \in B \ni (a, b) \in f$
2. *Uniqueness*: $([(a, b) \in f] \wedge [(a, c) \in f]) \Rightarrow (b = c)$

A is called the **domain** of f and is denoted by dom f. B is referred to as the **codomain** of f. We may write $f : A \to B$ to indicate that f has domain A and codomain B. The **range** of f, denoted rng f, is the set of

$$\text{rng } f = \{b \in B : \exists a \in A \ni (a, b) \in f\}$$

The domain of a function is either obtained from context or it is stated explicitly. Unless told otherwise, whenever a function is specified by a formula, possibly like this

$$f(x) = 3x^2 - 5,$$

then the domain of f is assumed to be the largest possible subset of $\mathbb{R}$ for which the formula will result in a real number.

2.1 Properties of Functions

2.1.1 -jection

Definition 2.2 (Surjection). A function $f : A \to B$ is called **surjective** (or is said to map A **onto** B) if $B = \operatorname{rng} f$. A surjective function is also referred to as a **surjection**.

Definition 2.3 (Injection). A function $f : A \to B$ is called **injective** (or **one-to-one**) if, for all a and a' in A, $f(a) = f(a')$ implies that $a = a'$. An injective function is also referred to as an **injection**.

Definition 2.4 (Bijection). A function $f : A \to B$ is called **bijective** or a **bijection** if it is both surjective and injective.

If a function is bijective, then it is particularly well behaved.

Definition 2.5 (Image and pre-image). Suppose that $f : A \to B$ and that $C \subseteq A$, then the subset $f(C) = \{f(x) : x \in C\}$ of B is called the **image** of C in B.

If we let $D \subseteq B$, then the subset $f^{-1}(D) = \{x \in A : f(x) \in D\}$ of A is called the **pre-image** of D in A, or f inverse of D.

Remark. In the second case where $D \subseteq B$ and $f^{-1}(D) = \{x \in A : f(x) \in D\}$, it must not be that rng f includes all of D, because D must not be a subset of A.

Theorem 2.6. Suppose that $f : A \to B$. Let $C \subseteq A$ and let $D \subseteq B$. Then the following hold:

1. $C \subseteq f^{-1}[f(C)]$
2. $f[f^{-1}(D)] \subseteq D$

Proof. We begin with case 1.

Suppose that $f : A \to B$, and that $C_1 \subseteq A$ and $C_2 \subseteq A$, and that $C_1 \cap C_2 = \emptyset$ and that $f(C_1) = f(C_2)$. Then $f^{-1}[f(C_1] = C_1 \cup C_2$, which must contain more members than C_1. Therefore, $C \subseteq f^{-1}[f(C)]$ as was to be prooven.[a]

For case 2,[b] suppose that $f : A \to B$ and $D \subseteq B$. Let $D_1 = \{d \in D : \exists a \in A \ni f(a) = d\}$, and let $D_2 = \{d \in D : \forall a \in A, f(a) \neq d\}$. This implies that $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$. The definition of D_1 also means that $f[f^{-1}(D_1)] = D_1$. Also, because of the definition of D_2, $f^{-1}(D) = f^{-1}(D_1 \cup D_2) = f^{-1}(D_1)$ since $f^{-1}(D_2) = \emptyset$.

Since $f[f^{-1}(D_1)] = D_1 = f[f^{-1}(D)]$ and $D_1 \cap D_2 = \emptyset$, it must be that $f[f^{-1}(D)] \subseteq D$ because D has equal or more members than D_1. $\square$

[a] if f were injective (which it isn't in the proof) then $C = f^{-1}[f(C)]$, which is shown in the proof of 2.7.

[b] My original proof, may contain faults

Theorem 2.7. Suppose that $f : A \to B$. Let $C \subseteq A$ and $D \subseteq B$. Then the following hold:

1. If f is injective, then $f^{-1}[f(C)] = C$.
2. If f is surjective, then $f[f^{-1}(D)] = D$.

Proof. We begin with case 1.[a]

Suppose that $f : A \to B$, and that $C_1 \subseteq A$ and $C_2 \subseteq A$, and that $f(C_1) = f(C_2)$. Then $f^{-1}[f(C_1)] = C_1 \cup C_2$. Since f is injective, and $f(C_1) = f(C_2)$, it must be that $C_1 = C_2$, and therefore $f^{-1}[f(C_1)] = C_1$.

For case 2,[b] suppose that $f : A \to B$ and $D \subseteq B$. Let $D_1 = \{d \in D : \exists a \in A \ni f(a) = d\}$, and let $D_2 = \{d \in D : \forall a \in A, f(a) \neq d\}$. This implies that $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$. The definition of D_1 also means that $f[f^{-1}(D_1)] = D_1$. Since f is surjective, $D_2 = \emptyset$, which means that $D = D_1$ since $D_1 \cup D_2 = D_1$, and therefore $f[f^{-1}(D_1)] = D_1$ implies that $f[f^{-1}(D)] = D$. $\square$

[a] My original proof, may contain faults

[b] My original proof, may contain faults

2.1.2 Composition Function

Definition 2.8 (Composition Function). Suppose that $f : A \to B$ and $g : B \to C$, then $\forall a \in A, f(a) \in B$, and since $f(a)$ is an object in B, $g(f(a)) \in C$. This is called the **composition** of f and g.

$$g \circ f = g(f(a)), \quad \forall a \in A$$

In terms of ordered pairs,

$$g \circ f = \{(a, c) \in A \times C : [\exists b \in B \ni (a, b) \in f] \wedge [(b, c) \in g]\}$$

Theorem 2.9. Let $f : A \to B$ and $g : B \to C$. Then

1. f and g are surjective $\Rightarrow g \circ f$ is surjective.
2. f and g are injective $\Rightarrow g \circ f$ is injective.
3. f and g are bijective $\Rightarrow g \circ f$ is bijective.

Proof. Case 1:

Since g is surjective, rng $g = C$, which means that $\forall c \in C, \exists b \in B \ni g(b) = c$. Now since f is surjective, $\exists a \in A \ni f(a) = b$. But then $(g \circ f)(a) = g(f(a)) = g(b) = c$, so $g \circ f$ is surjective.

Case 2:[a]

Suppose that $b' = f(a') \in B$ and $b = f(a) \in B$, and that $g(b') = g(b) \in C$. This implies that $b' = b$ since g is injective, which means that $f(a') = f(a)$, but because f too is injective, this implies that $a' = a$. This results in that $g(f(a')) = g(f(a)) \Rightarrow a' = a$, so by definition, $g \circ f$ is injective.

Case 3:

By the result of case 1 and 2, if f and g are bijective, then $g \circ f$ is bijective. □

[a] My original proof, may contain faults

2.1.3 Inverse function

To extend the idea of pre-image from 2.5, we can define a **inverse function**.

Definition 2.10 (Inverse Function)**.** Let $f : A \to B$ be bijective. The **inverse function** of f is the function f^{-1} given by

$$f^{-1} = \{(y, x) \in B \times A : (x, y) \in f\}$$

Remark. If $f : A \to B$ is bijective, then $f^{-1} : B \to A$ is bijective.