Single Variable Calculus

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1 Analysis With an Introduction to Proof

1.1 Sets and Functions

1.1.1 Ordered Pairs

Definition 1. [Ordered Pair] The **ordered pair** (a, b) is the set whose members are $\{a\}$ and $\{a, b\}$. In symbols we have

$$(a, b) = \{\{a\}, \{a, b\}\}$$

This definition ensures that the elements have an order. To show this, this theorem and its proof should suffice.

Theorem 1.

$$(a, b) = (c, d) \Leftrightarrow a = c, b = d$$

Proof. If $a = c$ and $b = d$, then

$$(a, b) = \{\{a\}, \{a, b\} = \{\{c\}, \{c, d\}\} = (c, d)$$

Conversely, suppose that $(a, b) = (c, d)$. Then by our definition we have $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. We wish to conclude that $a = c$ and $b = d$. To this end we consider two cases, depending on whether $a = b$ or $a \neq b$.

If $a = b$, then $\{a\} = \{a, b\}$, so $(a, b) = \{\{a\}\}$. Since $(a, b) = (c, d)$, we then have

$$\{\{a\}\} = \{\{c\}, \{c, d\}\}.$$

The set on the left has only one member, $\{a\}$. Thus the set on the right can have only one member, so $\{c\} = \{c, d\}$, and we can conclude that $c = d$. But then $\{\{a\}\} = \{\{c\}\}$, so $\{a\} = \{c\}$ and $a = c$. Thus $a = b = c = d$.

On the other hand, if $a \neq b$, then from the preceding argument it follows that $c \neq d$. Since $(a, b) = (c, d)$, we must have

$$\{a\} \in \{\{c\}, \{c, d\}\},$$

which means that $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$. In either case we have $c \in \{a\}$, so $a = c$. Again, since $(a, b) = (c, d)$, we must also have

$$\{a, b\} \in \{\{c\}, \{c, d\}\}.$$

Thus $\{a, b\} = \{c\}$ or $\{a, b\} = \{c, d\}$. But $\{a, b\}$ has two distinct members and $\{c\}$ has only one, so we must have $\{a, b\} = \{c, d\}$. Now $a = c$, $a \neq b$, and $b \in \{c, d\}$, which implies that $b = d$. $\square$

1.1.2 Cartesian Product

Definition 2. [Cartesian Product] If A and B are sets, then the **Cartesian product** (or **cross product**) of A and B, written $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. In symbols,

$$A \times B = \{(a, b) : (a \in A) \wedge (b \in B)\}.$$

1.1.3 Relation

Definition 3. [Relation] Let A and B be sets. A **relation between** A **and** B is any subset R of $A \times B$. We say that an element a in A is **related** by R to an element b in B if $(a, b) \in \mathrm{R}$, and we often denote this by writing "$a\mathrm{R}b$". The first set A is referred to as the **domain** of the relation and denoted by dom R. If $B = A$, then we speak of a relation $\mathrm{R} \subseteq A \times A$ being a **relation on A**.

Definition 4. [Equivalence Relation] A relation R on a set S is an **equivalence relation** if it has the following properties for all $x, y, z \in S$:

- **Reflexive property:** $x\mathrm{R}x$
- **Symmetric property:** $x\mathrm{R}y \leftrightarrow y\mathrm{R}x$
- **Transitive property:** $(x\mathrm{R}y \wedge y\mathrm{R}z) \rightarrow x\mathrm{R}z$

An example for a **equivalence relation** is the relation "is parallel to" when considering all lines in the plane, if we agree that a line is parallel to itself.

Definition 5. [Equivalence Class] Given an equivalence relation R on a set S, the **equivalence class** with respect to R of $x \in S$ is the set

$$E_x = \{y \in S : y\mathrm{R}x\}$$

2 Thomas' Calculus Early Transcendentals

3 Calculus A Complete Course

3.1 Limits and Continuity

Theorem 2. [The Squeeze Theorem, 4] Suppose that $f(x) \leq g(x) \leq h(x)$ holds for all x in some open interval containing c, except possibly at $x = c$. Suppose also that

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

Then $\lim_{x \to c} g(x) = L$.

Proof. For this proof, the (ϵ, δ)-definition of the limit will be used.

The goal is to prove that $\lim_{x \to c} g(x) = L$, which is true if

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, (|x - c| < \delta \Rightarrow |g(x) - L| < \epsilon).$$

Since $\lim_{x \to c} f(x) = L$,

$$\forall \epsilon > 0, \exists \delta_1 > 0 : \forall x, (|x - c| < \delta_1 \Rightarrow |f(x) - L| < \epsilon) \tag{1}$$

And since $\lim_{x \to c} h(x) = L$,

$$\forall \epsilon > 0, \exists \delta_2 > 0 : \forall x, (|x - c| < \delta_2 \Rightarrow |h(x) - L| < \epsilon). \tag{2}$$

Then we have

$$f(x) \leq g(x) \leq h(x)$$

$$f(x) - L \leq g(x) - L \leq h(x) - L$$

We can choose $\delta = \min\{\delta_1, \delta_2\}$, then if $|x - c| < \delta$, and combining (1) and (2), we have

$$-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon$$

$$-\epsilon < g(x) - L < \epsilon$$

$$|g(x) - L| < \epsilon$$

So $\lim_{x \to c} g(x) = L$, which completes the proof. □

Theorem 3. [The Intermediate-Value Theorem, 9] If $f(x)$ is continuous on the interval $[a, b]$ and if s is a number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that $f(c) = s$.

In particular, a continuous function defined on a closed interval takes on all values between its minimum value m and its maximum value M , so its range is also a closed interval, $[m, M]$.

Proof. □

3.1.1 Exercises 1.1

3.1.2 Exercises 1.2

78. What is the domain of $\sin\frac{1}{x}$ **? Evaluate** $\lim_{x\to 0} x\sin\frac{1}{x}$**.**

The domain of $x\sin x$ is $\mathbb{R}$. The domain of $\frac{1}{x}$ is $(-\infty,0)\cup(0,\infty)$. Therefore, the domain of $x\sin\frac{1}{x}$ is $(-\infty,0)\cup(0,\infty)$.

To evaluate $\lim_{x\to 0} x\sin\frac{1}{x}$, we can first evaluate $\lim_{x\to 0}\frac{1}{x}$.

$\lim_{x\to 0^+}\frac{1}{x} = +\infty$, $\lim_{x\to 0^-}\frac{1}{x} = -\infty$.

This means that $\lim_{x\to 0}\sin\frac{1}{x} = \lim_{x\to\pm\infty}\sin x$, which means that $-1 \leq \lim_{x\to 0}\sin\frac{1}{x} \leq 1$.

$\lim_{x\to 0} x\sin\frac{1}{x} = (\lim_{x\to 0} x)(\lim_{x\to 0}\sin\frac{1}{x}) = 0$

79. Suppose $|f(x)| \leq g(x)\forall x$**. What can you conclude about** $\lim_{x\to a} f(x)$ **if** $\lim_{x\to a} g(x) = 0$ **? What if** $\lim_{x\to a} g(x) = 3$ **?**

$|f(x)| \leq g(x)\forall x \Leftrightarrow -g(x) \leq f(x) \leq g(x)\forall x$. Since $\lim_{x\to a} g(x) = 0$ and therefore $\lim_{x\to a} -g(x) = 0$, then $\lim_{x\to a} f(x) = 0$ by the squeeze theorem.

If $\lim_{x\to a} g(x) = 3$, and $-g(x) \leq f(x) \leq g(x)\forall x$, then we can conclude that either $-3 \leq \lim_{x\to a} f(x) \leq 3$, or $\lim_{x\to a} f(x)$ doesn't exist.

3.1.3 Exercises 1.3

3.1.4 Exercises 1.4

32.

Let $g(x) = f(x) - x$. Since $0 \leq f(x) \leq 1$ for $0 \leq x \leq 1$, then $0 \leq g(0)$. By the same argument, $g(1) \leq 0$. Because $g(x)$ is continuous in the interval $[0,1]$, there must be some value $c \in [0,1]$ such that $g(c)=0$, by the Intermediate-Value Theorem. If $g(c)=0$, then $f(c)=c$, which was to be shown.

33.

Since $f(x)$ is even, it is symmetric around the y-axis. The symmetric equivilance of $\lim_{x\to0^+}$ around the y-axis is $\lim_{x\to0^-}$. Since $f(x)$ is right-continuous, it means that $\lim_{x\to0+} f(x) = f(0)$ and because of the symmetry, $\lim_{x\to0-} f(x) = f(0)$. Because $\lim_{x\to0+} f(x) = \lim_{x\to0-} f(x) = f(0)$, f is continuous at $x=0$.

34.

$\lim_{x\to0^+} f(x) = f(0)$ because f is right continuous. Since f is odd, it is symmetric around the origin, and therefore $\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = f(0) = 0$. Since f is both right and left continuous at $x=0$, it is continuous at $x=0$.

3.1.5 Exercises 1.5

31.

$(\lim_{x\to a} f(x) = L) \Leftrightarrow (\forall\epsilon > 0, \exists\delta_1 > 0 : 0 < |x-a| < \delta_1 \Rightarrow |f(x)-L| < \epsilon)$

and

$(\lim_{x\to a} f(x) = M) \Leftrightarrow (\forall\epsilon > 0, \exists\delta_2 > 0 : 0 < |x-a| < \delta_2 \Rightarrow |f(x) - M| < \epsilon)$

We assume that $L \neq M$. If we choose $\delta = \min\{\delta_1, \delta_2\}$, then $0 < |x-a| < \delta \Rightarrow |f(x) - L| + |f(x) - M| < \epsilon + \epsilon = 2\epsilon$.

By the triangle inequality,

$|f(x) - L| + |f(x) - M| \geq |(L - f(x)) + (f(x) - M)| = |L - M|$.

Since $L \neq M$, we can let $\epsilon = |L - M|/4$ because $|L - M|$ is positive.

This means that $|L-M| \leq 2\epsilon = |L-M|/2 \Rightarrow 2 \leq 1$, which is obviously false. Therefore $L = M$ and the limit is unique, which was to be shown.

32.

Since $\lim_{x \to a} g(x) = M$, the following must be true

$$|g(x)| = |(g(x) - M) + M| \leq |g(x) - M| + |M| < \epsilon + |M|$$

If we choose $\epsilon = 1$ then

$$|g(x)| < 1 + |M|$$

Which was to be shown.

33.

$\lim_{x \to a} f(x) \Leftrightarrow (\forall \epsilon > 0, \exists \delta_1 > 0 : \forall x, (0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \epsilon))$

And

$\lim_{x \to a} g(x) \Leftrightarrow (\forall \epsilon > 0, \exists \delta_2 > 0 : \forall x, (0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \epsilon))$

Lets assume that $\lim_{x \to a} f(x)g(x) \neq LM$.

Let $\delta = \min\{\delta_1, \delta_2\}$. This would result in that

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, (0 < |x - a| < \delta \Rightarrow |f(x) - L| + |g(x) - M| < 2\epsilon) \tag{3}$$

$|f(x)-L|+|g(x)-M| \geq |g(x)||f(x)-L|+|L||g(x)-M| = |g(x)(f(x)-L)|+|L(g(x)-M)| \geq |g(x)(f(x)-L)+L(g(x)-M)| = |f(x)g(x)-Lg(x)+Lg(x)-LM| = |(f(x)g(x))-(LM)|$

This together with (3) means that

$\forall \epsilon > 0, \exists \delta > 0 : \forall x, (0 < |x-a| < \delta \Rightarrow |(f(x)g(x))-(LM)| < 2\epsilon)$

Which shows that lim $\lim_{x \to a} f(x)g(x) = LM$.

3.2 Tangent Lines and Their Slopes

3.2.1 Exercises 2.1

32.

$$P(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n$$

$$P^{(n)}(a) = n!a_n$$

This means that $P'(a) = a_1$.

$$l(x) = m(x-a) + b$$

$$l'(a) = m$$

$$P(x) - l(x) = (x-a)^2 Q(x)$$

$$P(x) = l(x) + (x-a)^2 Q(x)$$

$$P(x) = b + m(x-a) + (x-a)^2 Q(x)$$

$$P'(x) = m + 2(x-a)Q(x) + (x-a)^2 Q'(x)$$

$$P'(a) = m$$

Since $P'(a) = a_1 = m = l'(a)$, P and l has the same tangent in $x = a$.

3.2.2 Exercises 2.2

52.

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$$

$$\frac{d}{dx}x^{-n} = \lim_{h\to 0} \frac{(x+h)^{-n} - x^{-n}}{h} =$$

$$= \lim_{h\to 0} \left(\frac{1}{h(x+h)^{-n}} + \frac{1}{hx^{-n}} \right) =$$

4 Lectures