Single Variable Calculus

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Contents

1 Calculus A Complete Course

1.1 Limits and Continuity

Theorem 1 (The Squeeze Theorem, 4). *Suppose that $f(x) \leq g(x) \leq h(x)$ holds for all x in some open interval containing c, except possibly at $x = c$. Suppose also that*

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$$

Then $\lim_{x\to c} g(x) = L$.

Proof. For this proof, the (ϵ, δ)-definition of the limit will be used.

The goal is to prove that $\lim_{x\to c} g(x) = L$, which is true if

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, (|x - c| < \delta \Rightarrow |g(x) - L| < \epsilon).$$

Since $\lim_{x\to c} f(x) = L$,

$$\forall \epsilon > 0, \exists \delta_1 > 0 : \forall x, (|x - c| < \delta_1 \Rightarrow |f(x) - L| < \epsilon) \tag{1}$$

And since $\lim_{x\to c} h(x) = L$,

$$\forall \epsilon > 0, \exists \delta_2 > 0 : \forall x, (|x - c| < \delta_2 \Rightarrow |h(x) - L| < \epsilon). \tag{2}$$

Then we have

$$f(x) \leq g(x) \leq h(x)$$

$$f(x) - L \leq g(x) - L \leq h(x) - L$$

We can choose $\delta = \min\{\delta_1, \delta_2\}$, then if $|x - c| < \delta$, and combining (1) and (2), we have

$$-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon$$

$$-\epsilon < g(x) - L < \epsilon$$

$$|g(x) - L| < \epsilon$$

So $\lim_{x \to c} g(x) = L$, which completes the proof. $\square$

1.1.1 Exercises 1.1

1.1.2 Exercises 1.2

78. What is the domain of $\sin\frac{1}{x}$? Evaluate $\lim_{x\to 0} x\sin\frac{1}{x}$.

The domain of $x\sin x$ is $\mathbb{R}$. The domain of $\frac{1}{x}$ is $(-\infty, 0)\cup(0,\infty)$. Therefore, the domain of $x\sin\frac{1}{x}$ is $(-\infty, 0)\cup(0,\infty)$.

To evaluate $\lim_{x\to 0} x\sin\frac{1}{x}$, we can first evaluate $\lim_{x\to 0}\frac{1}{x}$.

$\lim_{x\to 0^+}\frac{1}{x} = +\infty$, $\lim_{x\to 0^-}\frac{1}{x} = -\infty$.

This means that $\lim_{x\to 0}\sin\frac{1}{x} = \lim_{x\to\pm\infty}\sin x$, which means that $-1 \leq \lim_{x\to 0}\sin\frac{1}{x} \leq 1$.

$\lim_{x\to 0} x\sin\frac{1}{x} = (\lim_{x\to 0} x)(\lim_{x\to 0}\sin\frac{1}{x}) = 0$

79. Suppose $|f(x)| \leq g(x)\forall x$. What can you conclude about $\lim_{x\to a} f(x)$ if $\lim_{x\to a} g(x) = 0$? What if $\lim_{x\to a} g(x) = 3$?

$|f(x)| \leq g(x)\forall x \Leftrightarrow -g(x) \leq f(x) \leq g(x)\forall x$. Since $\lim_{x\to a} g(x) = 0$ and therefore $\lim_{x\to a} -g(x) = 0$, then $\lim_{x\to a} f(x) = 0$ by the squeeze theorem.

If $\lim_{x\to a} g(x) = 3$, and $-g(x) \leq f(x) \leq g(x)\forall x$, then we can conclude that either $-3 \leq \lim_{x\to a} f(x) \leq 3$, or $\lim_{x\to a} f(x)$ doesn't exist.

2 Lectures