Linear Algebra

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March 8, 2022

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1 Linear Algebra Done Right

1.1 Vector Spaces

1.1.1 Exercises 1.A

1

$$\frac{1}{a+bi} = c + di$$

$$(a+bi)(c+di) = 1$$

$$ac + adi + bci - bd = 1$$

$$(ac - bd) + (ad + bc)i = 1$$

Which leads to

$$ac - bd = 1$$

$$ad + bc = 0$$

Because 1 has no imaginary part.

$$c = \frac{1+bd}{a}$$

$$d = \frac{-bc}{a}$$

$$c = \frac{1 + b\frac{-bc}{a}}{a}$$

$$c = \frac{1}{a} - \frac{b^2c}{a^2}$$

$$c + \frac{b^2c}{a^2} = \frac{1}{a}$$

$$c\left(1 + \frac{b^2}{a^2}\right) = \frac{1}{a}$$

$$c = \frac{1}{a\left(1 + \frac{b^2}{a^2}\right)}$$

$$c = \frac{1}{a + \frac{b^2}{a}}$$

$$c = \frac{1}{\frac{a^2+b^2}{a}}$$

$$c = \frac{a}{a^2 + b^2}$$

Now that we have c, we can pick out d too.

$$d = \frac{-bc}{a}$$

$$d = \frac{-b\left(\frac{a}{a^2+b^2}\right)}{a}$$

$$d = \frac{\frac{-ab}{a^2+b^2}}{a}$$

$$d = \frac{-b}{a^2 + b^2}$$

So now we have d and c as real numbers, since a and b are real.

$$c = \frac{a}{a^2 + b^2}$$

$$d = \frac{-b}{a^2 + b^2}$$

. . . .

2

$$\frac{-1 + \sqrt{3}i}{2} =$$

$$\frac{-1}{2} + \frac{\sqrt{3}}{2}i =$$

$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} =$$

$$e^{i\frac{2\pi}{3}}$$

Which means that

$$\left(e^{i\frac{2\pi}{3}}\right)^3 =$$

$$e^{i\frac{6\pi}{3}} =$$

$$e^{i2\pi} =$$

$$1$$

Hence, the cube of $\frac{-1+\sqrt{3}i}{2}$ is equal to 1, which was to be shown.

1.1.2 Exercises 1.B

Problem 1

$$-(-v) = v$$

Add the additive inverse of $-(-v)$ to both sides

$$0 = v + (-v)$$

$$0 = 1v + (-1)v$$

$$0 = (1 + (-1))v$$

$$0v = 0$$

Which is true because

$$0v = (0+0)v = 0v + 0v$$

And if we then add the additive inverse of $0v$ to both sides and switch the sides around the equal sign we get

$$0v = 0$$

Which shows that $0v = 0$ and therefore $-(-v) = v$ is true for each $v \in V$.

So the statement is true, which was to be proved.

Problem 2 ***Suppose*** $a \in \mathbb{F}$***,*** $v \in V$***, and*** $av = 0$***. Prove that*** $a = 0$ ***or*** $v = 0$***.***

Suppose that $a \neq 0$. We can then multiply both sides by the multiplicative inverse of a.

$$v = 0\frac{1}{a}$$

$$v = 0$$

Suppose that $a = 0$. Since $0v = 0$ the value of v doesn't matter for the statement to be true.

In total, this means that $v = 0$ or $a = 0$, or both $v = 0$ and $a = 0$, which was to be proved.

Problem 3 ***Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.***

A unique x exists in V since V is closed under addition and scaling and $x = (w - v)\frac{1}{3}$.

Problem 4 ***The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?***

The empty set, $\{\}$, doesn't satisfy the requirement of an additive identity. It has no elements so it must therefore not contain an additive identity.

Problem 5 ***Show that in the definition of a vector space (1.19), the additive inverse condition can be replaced with the condition that***

$$0v = 0, \forall v \in V.$$

2 Lectures