

Analysis

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2022

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Chapter 1

Set Theory

1.1 Ordered Pairs

Definition 1.1 (Ordered Pair). The **ordered pair** (a, b) is the set whose members are $\{a\}$ and $\{a, b\}$. In symbols we have

$$(a, b) = \{\{a\}, \{a, b\}\}$$

This definition ensures that order matters. To show this, this theorem and its proof should suffice.

Theorem 1.2 (Ordered Pair Theorem). ^a

$$(a, b) = (c, d) \leftrightarrow a = c, b = d$$

^athis is a made up name by me

Proof. If $a = c$ and $b = d$, then

$$(a, b) = \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} = (c, d)$$

Conversely, suppose that $(a, b) = (c, d)$. Then by our definition we have $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. We wish to conclude that $a = c$ and $b = d$. To this end we consider two cases, depending on whether $a = b$ or $a \neq b$.

If $a = b$, then $\{a\} = \{a, b\}$, so $(a, b) = \{\{a\}\}$. Since $(a, b) = (c, d)$, we then have

$$\{\{a\}\} = \{\{c\}, \{c, d\}\}.$$

The set on the left has only one member, $\{a\}$. Thus the set on the right can have only one member, so $\{c\} = \{c, d\}$, and we can conclude that $c = d$. But then $\{\{a\}\} = \{\{c\}\}$, so $\{a\} = \{c\}$ and $a = c$. Thus $a = b = c = d$.

On the other hand, if $a \neq b$, then from the preceding argument it follows that $c \neq d$. Since $(a, b) = (c, d)$, we must have

$$\{a\} \in \{\{c\}, \{c, d\}\},$$

which means that $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$. In either case we have $c \in \{a\}$, so $a = c$. Again, since $(a, b) = (c, d)$, we must also have

$$\{a, b\} \in \{\{c\}, \{c, d\}\}.$$

Thus $\{a, b\} = \{c\}$ or $\{a, b\} = \{c, d\}$. But $\{a, b\}$ has two distinct members and $\{c\}$ has only one, so we must have $\{a, b\} = \{c, d\}$. Now $a = c$, $a \neq b$, and $b \in \{c, d\}$, which implies that $b = d$. \square

Definition 1.3 (Cartesian Product). If A and B are sets, then the **Cartesian product** (or **cross product**) of A and B , written $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. In symbols,

$$A \times B = \{(a, b) : (a \in A) \wedge (b \in B)\}.$$

1.2 Relation

Definition 1.4 (Relation). Let A and B be sets. A **relation between A and B** is any subset R of $A \times B$. We say that an element a in A is **related** by R to an element b in B if $(a, b) \in R$, and we often denote this by writing " aRb ". The first set A is referred to as the **domain** of the relation and denoted by $\text{dom } R$. If $B = A$, then we speak of a relation $R \subseteq A \times A$ being a **relation on A** .

Definition 1.5 (Equivalence Relation). A relation R on a set S is an **equivalence relation** if it has the following properties for all $x, y, z \in S$:

- **Reflexive property:** xRx
- **Symmetric property:** $xRy \leftrightarrow yRx$
- **Transitive property:** $(xRy \wedge yRz) \rightarrow xRz$

An example for a **equivalence relation** is the relation "is parallel to" when considering all lines in the plane, if we agree that a line is parallel to itself.

Definition 1.6 (Equivalence Class). Given an equivalence relation R on a set S , the **equivalence class** with respect to R of $x \in S$ is the set

$$E_x = \{y \in S : yRx\}$$

Example. Let $S = \{a : a \text{ lives in Sweden}\}$, which is the set of all people living in Sweden. Also, let a equivalence relation on this set be

$$R = \{(a, b) \in S \times S : a \text{ was born in the same year as } b\}.$$

Then

$$E_x = \{y \in S : yRx\}$$

is the set of all people living in Sweden who was born during the same year as some person x who is also living in Sweden. \diamond

Theorem 1.7. Two equivalence classes on the same set S with the same equivalence relation R must be disjoint or equal.

Proof. Let R be an equivalence relation on a set S , and let E_x and E_y be two equivalence classes with respect to R of $x \in S$. Suppose that they overlap, then there exists some $w \in E_x \cap E_y$. For all $x' \in E_x$ we have $x'Rx$, and because $w \in E_x$, wRx , and by symmetry, xRw . Also, $w \in E_y$ so wRy . By using transitivity, $x'Rx$ and xRw and wRy implies that $x'Ry$, which means that $x' \in E_y$ and that $E_x \subseteq E_y$.

Conversely, for all $y' \in E_y$ we have $y'Ry$, and because $w \in E_y$, wRy , and by the symmetry property, yRw . Also, $w \in E_x$ so wRx . By using the transitivity property, $y'Ry$ and yRw and wRx implies that $y'Rx$ and that $E_y \subseteq E_x$. Since $E_x \subseteq E_y$ and $E_x \supseteq E_y$, it must be that $E_y = E_x$. \square

Definition 1.8. A **partition** of a set S is a collection P of nonempty subsets of S such that

- Each $x \in S$ belongs to some subset $A \in P$.
- For all $A, B \in P$, if $A \neq B$, then $A \cap B = \emptyset$.

A member of P is called a **piece** of the partition.

Example. Two equivalence classes on the same set S with the same equivalence relation R who are not equal (and therefore disjoint) are two pieces of a partition P on the set S . \diamond

1.3 Cardinality

Definition 1.9 (Set Equivalence). Two sets S and T are called **set equivalent**, and we write $S \sim T$, if there exists a bijective function from S onto T .

This definition ensures that if two sets are set equivalent, they contain the same number of elements, since a bijective function between them will set up a one-to-one correspondence between the elements of each set.

Chapter 2

Functions

Definition 2.1 (Function between two sets). Let A and B be sets. A **function** from A to B is a nonempty relation $f \subseteq A \times B$ that satisfies the following two conditions:

1. *Existence*: $\forall a \in A, \exists b \in B \ni (a, b) \in f$
2. *Uniqueness*: $[(a, b) \in f] \wedge [(a, c) \in f] \Rightarrow (b = c)$

A is called the **domain** of f and is denoted by $\text{dom } f$. B is referred to as the **codomain** of f . We may write $f : A \rightarrow B$ to indicate that f has domain A and codomain B . The **range** of f , denoted $\text{rng } f$, is the set of

$$\text{rng } f = \{b \in B : \exists a \in A \ni (a, b) \in f\}$$

The domain of a function is either obtained from context or it is stated explicitly. Unless told otherwise, whenever a function is specified by a formula, possibly like this

$$f(x) = 3x^2 - 5,$$

then the domain of f is assumed to be the largest possible subset of \mathbb{R} for which the formula will result in a real number.

2.1 Properties of Functions

2.1.1 -jection

Definition 2.2 (Surjection). A function $f : A \rightarrow B$ is called **surjective** (or is said to map A **onto** B) if $B = \text{rng } f$. A surjective function is also referred to as a **surjection**.

Definition 2.3 (Injection). A function $f : A \rightarrow B$ is called **injective** (or **one-to-one**) if, for all a and a' in A , $f(a) = f(a')$ implies that $a = a'$. An injective function is also referred to as an **injection**.

Definition 2.4 (Bijection). A function $f : A \rightarrow B$ is called **bijective** or a **bijection** if it is both surjective and injective.

If a function is bijective, then it is particularly well behaved.

Definition 2.5 (Image and pre-image). Suppose that $f : A \rightarrow B$ and that $C \subseteq A$, then the subset $f(C) = \{f(x) : x \in C\}$ of B is called the **image** of C in B .

If we let $D \subseteq B$, then the subset $f^{-1}(D) = \{x \in A : f(x) \in D\}$ of A is called the **pre-image** of D in A , or f inverse of D .

Remark. In the second case where $D \subseteq B$ and $f^{-1}(D) = \{x \in A : f(x) \in D\}$, it must not be that $\text{rng } f$ includes all of D , because D must not be a subset of A .

Theorem 2.6. Suppose that $f : A \rightarrow B$. Let $C \subseteq A$ and let $D \subseteq B$. Then the following hold:

1. $C \subseteq f^{-1}[f(C)]$
2. $f[f^{-1}(D)] \subseteq D$

Proof. We begin with case 1.

Suppose that $f : A \rightarrow B$, and that $C_1 \subseteq A$ and $C_2 \subseteq A$, and that $C_1 \cap C_2 = \emptyset$ and that $f(C_1) = f(C_2)$. Then $f^{-1}[f(C_1)] = C_1 \cup C_2$, which must contain more members than C_1 . Therefore, $C \subseteq f^{-1}[f(C)]$ as was to be proven.^a

For case 2, suppose that $f : A \rightarrow B$ and $D \subseteq B$. Let $D_1 = \{d \in D : \exists a \in A \ni f(a) = d\}$, and let $D_2 = \{d \in D : \forall a \in A, f(a) \neq d\}$. This implies that $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$. The definition of D_1 also means that $f[f^{-1}(D_1)] = D_1$. Also, because of the definition of D_2 , $f^{-1}(D) = f^{-1}(D_1 \cup D_2) = f^{-1}(D_1)$ since $f^{-1}(D_2) = \emptyset$.

Since $f[f^{-1}(D_1)] = D_1 = f[f^{-1}(D)]$ and $D_1 \cap D_2 = \emptyset$, it must be that $f[f^{-1}(D)] \subseteq D$ because D has equal or more members than D_1 . \square

^aif f were injective (which it isn't in the proof) then $C = f^{-1}[f(C)]$, which is shown in the proof of 2.7.

Theorem 2.7. Suppose that $f : A \rightarrow B$. Let $C \subseteq A$ and $D \subseteq B$. Then the following hold:

1. If f is injective, then $f^{-1}[f(C)] = C$.
2. If f is surjective, then $f[f^{-1}(D)] = D$.

Proof. We begin with case 1.

Suppose that $f : A \rightarrow B$, and that $C_1 \subseteq A$ and $C_2 \subseteq A$, and that $f(C_1) = f(C_2)$. Then $f^{-1}[f(C_1)] = C_1 \cup C_2$. Since f is injective, and $f(C_1) = f(C_2)$, it must be that $C_1 = C_2$, and therefore $f^{-1}[f(C_1)] = C_1$.

For case 2, suppose that $f : A \rightarrow B$ and $D \subseteq B$. Let $D_1 = \{d \in D : \exists a \in A \ni f(a) = d\}$, and let $D_2 = \{d \in D : \forall a \in A, f(a) \neq d\}$. This implies that $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$. The definition of D_1 also means that $f[f^{-1}(D_1)] = D_1$. Since f is surjective, $D_2 = \emptyset$, which means that $D = D_1$ since $D_1 \cup D_2 = D_1$, and therefore $f[f^{-1}(D_1)] = D_1$ implies that $f[f^{-1}(D)] = D$. \square

2.1.2 Composition Function

Definition 2.8 (Composition Function). Suppose that $f : A \rightarrow B$ and $g : B \rightarrow C$, then $\forall a \in A, f(a) \in B$, and since $f(a)$ is an object in B , $g(f(a)) \in C$. This is called the **composition** of f and g .

$$g \circ f = g(f(a)), \quad \forall a \in A$$

In terms of ordered pairs,

$$g \circ f = \{(a, c) \in A \times C : [\exists b \in B \ni (a, b) \in f] \wedge [(b, c) \in g]\}$$

Theorem 2.9. Let $f : A \rightarrow B$ and $g : B \rightarrow C$. Then

1. f and g are surjective $\Rightarrow g \circ f$ is surjective.
2. f and g are injective $\Rightarrow g \circ f$ is injective.
3. f and g are bijective $\Rightarrow g \circ f$ is bijective.

Proof. Case 1:

Since g is surjective, $\text{rng } g = C$, which means that $\forall c \in C, \exists b \in B \ni g(b) = c$. Now since f is surjective, $\exists a \in A \ni f(a) = b$. But then $(g \circ f)(a) = g(f(a)) = g(b) = c$, so $g \circ f$ is surjective.

Case 2:

Suppose that $b' = f(a') \in B$ and $b = f(a) \in B$, and that $g(b') = g(b) \in C$. This implies that $b' = b$ since g is injective, which means that $f(a') = f(a)$, but because f too is injective, this implies that $a' = a$. This results in that $g(f(a')) = g(f(a)) \Rightarrow a' = a$, so by definition, $g \circ f$ is injective.

Case 3:

By the result of case 1 and 2, if f and g are bijective, then $g \circ f$ is bijective. \square

2.1.3 Inverse function

To extend the idea of pre-image from 2.5, we can define a **inverse function**.

Definition 2.10 (Inverse Function). Let $f : A \rightarrow B$ be bijective. The **inverse function** of f is the function f^{-1} given by

$$f^{-1} = \{(y, x) \in B \times A : (x, y) \in f\}$$

Remark. If $f : A \rightarrow B$ is bijective, then $f^{-1} : B \rightarrow A$ is bijective.

Definition 2.11 (Identity Function). A function defined on a set A that maps each element in A onto itself is called the **identity function** on A , and is denoted by i_a .

Remark. If $f : A \rightarrow B$ and f is bijective, then

- $f^{-1} \circ f = i_A$,
- $f \circ f^{-1} = i_B$.

Theorem 2.12. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be bijective. Then the composition $g \circ f : A \rightarrow C$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. By theorem 2.9 we know that $g \circ f$ is bijective, so there exists an inverse $(g \circ f)^{-1}$. We are asked to verify the equality of the two functions $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$, as sets of ordered pairs. To this end, suppose $(c, a) \in (g \circ f)^{-1}$. By the definition of an inverse function, this means $(a, c) \in g \circ f$. The definition of composition implies that

$$\exists b \in B \ni [(a, b) \in f] \wedge [(b, c) \in g].$$

Since f and g are bijective, this means that $(b, a) \in f^{-1}$ and $(c, b) \in g^{-1}$. That is, $f^{-1}(b) = a$ and $g^{-1}(c) = b$. But then,

$$(f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c)) = f^{-1}(b) = a \quad (2.1)$$

so that $(c, a) \in (f^{-1} \circ g^{-1})$ and $(g \circ f)^{-1} \subseteq (f^{-1} \circ g^{-1})$.

To the other end, suppose that $(c, a) \in (f^{-1} \circ g^{-1})$. The definition of

composition implies that

$$\exists b \in B \ni [(c, b) \in g^{-1}] \wedge [(b, a) \in f^{-1}].$$

This implies that $(b, c) \in g$ and that $(a, b) \in f$ and therefore $(a, c) \in g \circ f$. Since both f and g are bijective, there must exist an inverse $(g \circ f)^{-1}$ such that $(c, a) \in (g \circ f)^{-1}$. Now, since $(c, a) \in (f^{-1} \circ g^{-1})$ implies that $(c, a) \in (g \circ f)^{-1}$, and $(c, a) \in (g \circ f)^{-1}$ implies that $(c, a) \in (f^{-1} \circ g^{-1})$, it must be that $(g \circ f)^{-1} = (f^{-1} \circ g^{-1})$. \square

Chapter 3

Exercises and My Solutions

3.1 Analysis with an Introduction to Proof - Steven R. Lay

3.1.1 Sets and Functions

3.1.1.1 Exercises 3

(21) Suppose that $f : A \rightarrow B$ and let C be a subset of A .

1. Prove or give a counterexample: $f(A \setminus C) \subseteq f(A) \setminus f(C)$.
2. Prove or give a counterexample: $f(A) \setminus f(C) \subseteq f(A \setminus C)$.
3. What condition on f will ensure that $f(A \setminus C) = f(A) \setminus f(C)$? Prove your answer.
4. What condition of f will ensure that $f(A \setminus C) = B \setminus f(C)$? Prove your answer.

Proof. (1) Suppose that $f(A \setminus C) \subseteq f(A) \setminus f(C)$.

Let $x \in A \setminus C$, $x' \in C$ and $f(x) = f(x')$. Then, $f(x) \in f(A \setminus C)$, and therefore $f(x') \in f(A \setminus C)$. But since $f(x') \in f(C)$ and therefore $f(x) \in f(C)$, neither $f(x)$ or $f(x')$ is in $f(A) \setminus f(C)$. This contradicts our original statement because there exists a member in $f(A \setminus C)$ which is not in $f(A) \setminus f(C)$, so $f(A \setminus C) \not\subseteq f(A) \setminus f(C)$. \square

Proof. (2) For any $y \in f(A) \setminus f(C)$, there exists an $x \in A$ such that $f(x) = y$. If $x \in C$, then $f(x) \in f(C)$ which means that $f(x) \neq y$, so by contradiction it must be that $x \notin C$. This implies that $x \in A \setminus C$, and therefore that $f(x) \in f(A \setminus C)$ and $y \in f(A \setminus C)$. Since $y \in f(A) \setminus f(C)$ implies that $y \in f(A \setminus C)$, the statement $f(A) \setminus f(C) \subseteq f(A \setminus C)$ must be true. \square

Proof. (3) Proof 2 have already shown that $f(A) \setminus f(C) \subseteq f(A \setminus C)$, so to prove that $f(A) \setminus f(C) = f(A \setminus C)$ I must only prove the reverse of the first statement.

Let f be injective^a. For any $y \in f(A \setminus C)$, there exists one and only one $x \in A \setminus C$ such that $f(x) = y$. Since $x \in A \setminus C$, $x \in A$ and $f(x) \in f(A)$. Also, since $x \in A \setminus C$, $x \notin C$ and $f(x) \notin f(C)$. This implies that $f(x) \in f(A) \setminus f(C)$ and thus $y \in f(A) \setminus f(C)$. Since $y \in f(A \setminus C)$ implies $y \in f(A) \setminus f(C)$, and $y \in f(A) \setminus f(C)$ implies $y \in f(A \setminus C)$ from proof 2, it must be that $f(A \setminus C) = f(A) \setminus f(C)$. \square

^athis is the necessary condition such that $f(A \setminus C) = f(A) \setminus f(C)$.

Proof. (4) Proof 3 in combination with that f is surjective^a means that $f(A \setminus C) = f(A) \setminus f(C) = B \setminus f(C)$ since $B = \text{rng } f = f(A)$. \square

^aProof 3 needed the condition that f was injective, and since proof 4 needs f to be surjective and is based on proof 3, f is now bijective.

(32) Suppose that $f : A \rightarrow B$ is any function. Then a function $g : B \rightarrow A$ is called a

- **left inverse** for f if $g(f(x)) = x$ for all $x \in A$,
- **right inverse** for f if $f(g(y)) = y$ for all $y \in B$.

1. Prove that f has a left inverse iff f is injective.
2. Prove that f has a right inverse iff f is surjective.

Proof. (1) Suppose that f is injective. Let $g = \{(b, a) \in B \times A : (a, b) \in f\} \cup \{(b, a) \in B \times A : b \notin f(A)\}^a$. By definition, each $a \in A$ corresponds to one and only one $b \in B$ such that $f(a) = b$, and because of the definition of g , for each $b \in B$ such that $f(a) = b$, $g(b) = a$, which implies that $g(f(a)) = a$ for all $a \in A$.

Conversely, suppose that $f(x) \in B$ and $f(x') \in B$, and that $f(x) = f(x')$. If $g(f(a)) = a$ for all $a \in A$, $g(f(x)) = g(f(x'))$ implies that $x = x'$. Therefore, f is injective. \square

^aI added the part $\cup \{(b, a) \in B \times A : b \notin f(A)\}$ to g to show that f must not be surjective.

Proof. (2) Suppose that f has a right inverse and therefore $f(g(y)) = y$ for all $y \in B$. This implies that f is surjective, since for all $y \in B$ there exists some $x \in A$, which may be $g(y)$, such that $f(x) = y$. \square

(33) Let S be a nonempty set and let F be the set of all functions that map S into S . Suppose that for every f and g in F we have

$$(f \circ g)(x) = (g \circ f)(x), \forall x \in S$$

Prove that S has only one element.

Proof. If S contains more than one element, then there exists some functions f and g in F that are neither surjective nor injective. Suppose that $x, x' \in S$ and that $x \neq x'$, and that $f(x) = x'$ and $f(x') = x'$, and that $g(x) = x$ and $g(x') = x$. Then $f(g(x)) = f(x) = x'$, and $g(f(x)) = g(x') = x$, which contradicts the statement that $(f \circ g)(x) = (g \circ f)(x), \forall x \in S$, so S must contain less than two elements. Since S is nonempty, it must therefore contain one element. \square