

# Analysis

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# Chapter 1

## Set Theory

### 1.1 Ordered Pairs

**Definition 1.1** (Ordered Pair). The **ordered pair**  $(a, b)$  is the set whose members are  $\{a\}$  and  $\{a, b\}$ . In symbols we have

$$(a, b) = \{\{a\}, \{a, b\}\}$$

This definition ensures that order matters. To show this, this theorem and its proof should suffice.

**Theorem 1.2** (Ordered Pair Theorem). <sup>a</sup>

$$(a, b) = (c, d) \leftrightarrow a = c, b = d$$

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<sup>a</sup>this is a made up name by me

**Proof.** If  $a = c$  and  $b = d$ , then

$$(a, b) = \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} = (c, d)$$

Conversely, suppose that  $(a, b) = (c, d)$ . Then by our definition we have  $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ . We wish to conclude that  $a = c$  and  $b = d$ . To this end we consider two cases, depending on whether  $a = b$  or  $a \neq b$ .

If  $a = b$ , then  $\{a\} = \{a, b\}$ , so  $(a, b) = \{\{a\}\}$ . Since  $(a, b) = (c, d)$ , we then have

$$\{\{a\}\} = \{\{c\}, \{c, d\}\}.$$

The set on the left has only one member,  $\{a\}$ . Thus the set on the right can have only one member, so  $\{c\} = \{c, d\}$ , and we can conclude that  $c = d$ . But then  $\{\{a\}\} = \{\{c\}\}$ , so  $\{a\} = \{c\}$  and  $a = c$ . Thus  $a = b = c = d$ .

On the other hand, if  $a \neq b$ , then from the preceding argument it follows that  $c \neq d$ . Since  $(a, b) = (c, d)$ , we must have

$$\{a\} \in \{\{c\}, \{c, d\}\},$$

which means that  $\{a\} = \{c\}$  or  $\{a\} = \{c, d\}$ . In either case we have  $c \in \{a\}$ , so  $a = c$ . Again, since  $(a, b) = (c, d)$ , we must also have

$$\{a, b\} \in \{\{c\}, \{c, d\}\}.$$

Thus  $\{a, b\} = \{c\}$  or  $\{a, b\} = \{c, d\}$ . But  $\{a, b\}$  has two distinct members and  $\{c\}$  has only one, so we must have  $\{a, b\} = \{c, d\}$ . Now  $a = c$ ,  $a \neq b$ , and  $b \in \{c, d\}$ , which implies that  $b = d$ .  $\square$

**Definition 1.3** (Cartesian Product). If  $A$  and  $B$  are sets, then the **Cartesian product** (or **cross product**) of  $A$  and  $B$ , written  $A \times B$ , is the set of all ordered pairs  $(a, b)$  such that  $a \in A$  and  $b \in B$ . In symbols,

$$A \times B = \{(a, b) : (a \in A) \wedge (b \in B)\}.$$

## 1.2 Relation

**Definition 1.4** (Relation). Let  $A$  and  $B$  be sets. A **relation between  $A$  and  $B$**  is any subset  $R$  of  $A \times B$ . We say that an element  $a$  in  $A$  is **related** by  $R$  to an element  $b$  in  $B$  if  $(a, b) \in R$ , and we often denote this by writing " $aRb$ ". The first set  $A$  is referred to as the **domain** of the relation and denoted by  $\text{dom } R$ . If  $B = A$ , then we speak of a relation  $R \subseteq A \times A$  being a **relation on  $A$** .

**Definition 1.5** (Equivalence Relation). A relation  $R$  on a set  $S$  is an **equivalence relation** if it has the following properties for all  $x, y, z \in S$ :

- **Reflexive property:**  $xRx$
- **Symmetric property:**  $xRy \leftrightarrow yRx$
- **Transitive property:**  $(xRy \wedge yRz) \rightarrow xRz$

An example for a **equivalence relation** is the relation "is parallel to" when considering all lines in the plane, if we agree that a line is parallel to itself.

**Definition 1.6** (Equivalence Class). Given an equivalence relation  $R$  on a set  $S$ , the **equivalence class** with respect to  $R$  of  $x \in S$  is the set

$$E_x = \{y \in S : yRx\}$$

**Example.** Let  $S = \{a : a \text{ lives in Sweden}\}$ , which is the set of all people living in Sweden. Also, let a equivalence relation on this set be

$$R = \{(a, b) \in S \times S : a \text{ was born in the same year as } b\}.$$

Then

$$E_x = \{y \in S : yRx\}$$

is the set of all people living in Sweden who was born during the same year as some person  $x$  who is also living in Sweden.  $\diamond$

**Theorem 1.7.** Two equivalence classes on the same set  $S$  with the same equivalence relation  $R$  must be disjoint or equal.

**Proof.** Let  $R$  be an equivalence relation on a set  $S$ , and let  $E_x$  and  $E_y$  be two equivalence classes with respect to  $R$  of  $x \in S$ . Suppose that they overlap, then there exists some  $w \in E_x \cap E_y$ . For all  $x' \in E_x$  we have  $x'Rx$ , and because  $w \in E_x$ ,  $wRx$ , and by symmetry,  $xRw$ . Also,  $w \in E_y$  so  $wRy$ . By using transitivity,  $x'Rx$  and  $xRw$  and  $wRy$  implies that  $x'Ry$ , which means that  $x' \in E_y$  and that  $E_x \subseteq E_y$ .

Conversely, for all  $y' \in E_y$  we have  $y'Ry$ , and because  $w \in E_y$ ,  $wRy$ , and by the symmetry property,  $yRw$ . Also,  $w \in E_x$  so  $wRx$ . By using the transitivity property,  $y'Ry$  and  $yRw$  and  $wRx$  implies that  $y'Rx$  and that  $E_y \subseteq E_x$ . Since  $E_x \subseteq E_y$  and  $E_x \supseteq E_y$ , it must be that  $E_y = E_x$ .  $\square$

**Definition 1.8.** A **partition** of a set  $S$  is a collection  $P$  of nonempty subsets of  $S$  such that

- Each  $x \in S$  belongs to some subset  $A \in P$ .
- For all  $A, B \in P$ , if  $A \neq B$ , then  $A \cap B = \emptyset$ .

A member of  $P$  is called a **piece** of the partition.

**Example.** Two equivalence classes on the same set  $S$  with the same equivalence relation  $R$  who are not equal (and therefore disjoint) are two pieces of a partition  $P$  on the set  $S$ .  $\diamond$

## 1.3 Cardinality

This subsection requires understanding of the definition of a function between two sets, and understanding of surjection and injection (and therefore bijection). This can be learned in Chapter 2.

**Definition 1.9 (Set Equivalence).** Two sets  $S$  and  $T$  are called **set equivalent**, and we write  $S \sim T$ , if there exists a bijective function from  $S$  onto  $T$ .

This definition ensures that if two sets are set equivalent, they contain the same number of elements, since a bijective function between them will set up a one-to-one correspondence between the elements of each set.

**Definition 1.10 (Finite or Infinite Set).** A set  $S$  is said to be **finite** if  $S = \emptyset$  or if there exists  $n \in \mathbb{N}$  and a bijection  $f : \{1, 2, \dots, n\} \rightarrow S$ .<sup>a</sup> If a set is not finite, it is said to be **infinite**.

<sup>a</sup>Moving forward, we will make use of the set  $I_n = \{1, 2, \dots, n\}$ .

**Definition 1.11.** The **cardinal number** of the set  $I_n = \{1, 2, \dots, n\}$  is  $n$ , and if  $S \sim I_n$ , we say that  $S$  **has  $n$  elements**. The cardinal number of  $\emptyset$  is taken to be 0. If a cardinal number is not finite, it is called **transfinite**.

**Definition 1.12.** A set  $S$  is said to be **denumerable** if there exists a bijection  $f : \mathbb{N} \rightarrow S$ . If a set is finite or denumerable, it is called **countable**. If a set is not countable, it is **uncountable**. The cardinal number of a denumerable set is denoted by  $\aleph_0$ .

**Remark.** Against our intuition from finite sets, if  $E$  is the set of all even natural numbers, then  $\mathbb{N} \sim E$ , because if  $f(n) = 2n$ , then  $f : \mathbb{N} \rightarrow E$  is bijective. Therefore, both  $\mathbb{N}$  and  $E$  has the cardinal number  $\aleph_0$  even though  $E \subset \mathbb{N}$ .

**Example.**  $\mathbb{Z}$ , the set of all integers, is denumerable since  $f : \mathbb{N} \rightarrow \mathbb{Z}$  is bijective if

$$f(n) = \begin{cases} 0 & \text{if } n = 1 \\ \frac{n}{2} & \text{if } n \text{ is even} \\ \lceil -\frac{n}{2} \rceil & \text{if } n \text{ is odd} \end{cases}$$

because this leads to that

$$\begin{aligned} f(1) &\rightarrow 0 \\ f(2) &\rightarrow 1 \\ f(3) &\rightarrow (-1) \\ f(4) &\rightarrow 2 \\ f(5) &\rightarrow (-2) \\ &\vdots \end{aligned}$$

So for any  $b \in \mathbb{Z}$ , there exists a  $a \in \mathbb{N}$  such that  $f(a) = b$ , which implies that  $f$  is surjective, and there is also a one to one correspondence between the two sets so  $f$  is injective, and therefore bijective.  $\diamond$

**Notation.** For any nonempty finite set  $S$ , there exists a bijection  $f : I_n \rightarrow S$  for some  $n \in \mathbb{N}$ . Therefore, we use this function to count the members as  $f(1), f(2), f(3), \dots, f(n)$ . Letting  $f(k) = s_k$  we can write  $S = \{s_1, s_2, \dots, s_n\}$ . We can also do this for any denumerable set  $T$ , since because it is denumerable, there exists a bijection  $g : \mathbb{N} \rightarrow T$ , so we can use  $g(k) = t_k$  to write  $T = \{t_1, t_2, t_3, \dots\}$ .

**Lemma 1.13.** Every subset of a finite set is finite.

**Proof.** — NOT DONE  $\square$

**Theorem 1.14.** Let  $S$  be a countable set and let  $T \subseteq S$ . Then  $T$  is countable.

**Proof.** If  $T$  is finite, then we are done. Thus we may assume that  $T$  is infinite. This implies that  $S$  is infinite<sup>a</sup>, so  $S$  is denumerable (since it is countable and infinite). Therefore, there exists a bijection  $f : \mathbb{N} \rightarrow S$  and we can write  $S$  as a list of distinct members

$$S = \{s_1, s_2, s_3, \dots\}$$

where  $f(n) = s_n$ . Now let

$$A = \{n \in \mathbb{N} : s_n \in T\}.$$

Since  $A$  is a nonempty subset of  $\mathbb{N}$ , the *Well-Ordering Property* of  $\mathbb{N}$  implies that  $A$  has a least member, say  $a_1$ . Similarly, the set  $A \setminus \{a_1\}$  has a least



member, say  $a_2$ . In general, having chosen  $a_1, \dots, a_k$ , let  $a_{k+1}$  be the least member in  $A \setminus \{a_1, \dots, a_k\}$ . Essentially, if we select from our listing of  $S$  those terms that are in  $T$  and keep them in the same order, then  $a_n$  is the subscript of the  $n$ th term in this new list.

Now define a function  $g : \mathbb{N} \rightarrow \mathbb{N}$  by  $g(n) = a_n$ . Since  $T$  is infinite,  $g$  is defined for every  $n \in \mathbb{N}$ . Since  $a_{n+1} \notin \{a_1, \dots, a_n\}$ ,  $g$  must be injective<sup>b</sup>. Thus the composition  $f \circ g$  is also injective. Since each element of  $T$  is somewhere in the listing of  $S$ ,  $g(\mathbb{N})$  includes all the subscripts of terms in  $T$ . Thus  $f \circ g$  is a bijection from  $\mathbb{N}$  onto  $T$  and  $T$  is denumerable.  $\square$

<sup>a</sup>This implication is true by lemma 1.13

<sup>b</sup>I suppose that this is a small proof by induction that  $g$  is injective? This proof is not mine and is taken from *Analysis with an Introduction to Proof*.

**Theorem 1.15.** Let  $S$  be a nonempty set. The following three conditions are equivalent.

1.  $S$  is countable.
2. There exists an injection  $f : S \rightarrow \mathbb{N}$ .
3. There exists a surjection  $g : \mathbb{N} \rightarrow S$ .

**Proof.** Suppose that  $S$  is countable. Then there exists some bijection  $h : J \rightarrow S$  where  $J = I_n$  for some  $n \in \mathbb{N}$  if  $S$  is finite, or  $J = \mathbb{N}$  if  $S$  is infinite. In either case,  $h^{-1} : S \rightarrow \mathbb{N}$  is at least injective. Thus (1) implies (2).

Now suppose that there exists an injection  $f : S \rightarrow \mathbb{N}$ . Then  $f$  is a bijection from  $S$  to  $f(S)$ , so  $f^{-1}$  is a bijection from  $f(S)$  to  $S$ . Let  $g : \mathbb{N} \rightarrow S$  be defined by

$$g(n) = \begin{cases} f^{-1}(n), & \text{if } n \in f(S) \\ p, & \text{if } n \notin f(S) \end{cases}$$

where  $p \in S$ . Then  $g[f(S)] = f^{-1}[f(S)] = S$  and  $g[\mathbb{N} \setminus f(S)] = \{p\}$ , so that  $g$  is a surjection from  $\mathbb{N}$  onto  $S$ . Thus, (2) implies (3).

Finally, suppose that there exists a surjection  $g : \mathbb{N} \rightarrow S$ . Define  $h : S \rightarrow \mathbb{N}$  by

$$h(s) \text{ is the smallest } n \in \mathbb{N} \text{ such that } g(n) = s.$$

Then  $h$  is an injection from  $S$  to  $\mathbb{N}$ , and hence a bijection from  $S$  onto the subset  $h(S)$  of  $\mathbb{N}$ . Since  $\mathbb{N}$  is countable, theorem 1.14 implies that  $h(S)$  is countable. Since  $S$  and  $h(S)$  are set equivalent, because there exists a bijection between the two sets,  $S$  is also countable.  $\square$

# Chapter 2

## Functions

**Definition 2.1** (Function between two sets). Let  $A$  and  $B$  be sets. A **function** from  $A$  to  $B$  is a nonempty relation  $f \subseteq A \times B$  that satisfies the following two conditions:

1. *Existence*:  $\forall a \in A, \exists b \in B \ni (a, b) \in f$
2. *Uniqueness*:  $([(a, b) \in f] \wedge [(a, c) \in f]) \Rightarrow (b = c)$

$A$  is called the **domain** of  $f$  and is denoted by  $\text{dom } f$ .  $B$  is referred to as the **codomain** of  $f$ . We may write  $f : A \rightarrow B$  to indicate that  $f$  has domain  $A$  and codomain  $B$ . The **range** of  $f$ , denoted  $\text{rng } f$ , is the set of

$$\text{rng } f = \{b \in B : \exists a \in A \ni (a, b) \in f\}$$

The domain of a function is either obtained from context or it is stated explicitly. Unless told otherwise, whenever a function is specified by a formula, possibly like this

$$f(x) = 3x^2 - 5,$$

then the domain of  $f$  is assumed to be the largest possible subset of  $\mathbb{R}$  for which the formula will result in a real number.

## 2.1 Properties of Functions

### 2.1.1 $f$ -jection

**Definition 2.2** (Surjection). A function  $f : A \rightarrow B$  is called **surjective** (or is said to map  $A$  **onto**  $B$ ) if  $B = \text{rng } f$ . A surjective function is also referred to as a **surjection**.

**Definition 2.3** (Injection). A function  $f : A \rightarrow B$  is called **injective** (or **one-to-one**) if, for all  $a$  and  $a'$  in  $A$ ,  $f(a) = f(a')$  implies that  $a = a'$ . An injective function is also referred to as an **injection**.

**Definition 2.4** (Bijection). A function  $f : A \rightarrow B$  is called **bijective** or a **bijection** if it is both surjective and injective.

If a function is bijective, then it is particularly well behaved.

**Definition 2.5** (Image and pre-image). Suppose that  $f : A \rightarrow B$  and that  $C \subseteq A$ , then the subset  $f(C) = \{f(x) : x \in C\}$  of  $B$  is called the **image** of  $C$  in  $B$ .

If we let  $D \subseteq B$ , then the subset  $f^{-1}(D) = \{x \in A : f(x) \in D\}$  of  $A$  is called the **pre-image** of  $D$  in  $A$ , or  $f$  inverse of  $D$ .

**Remark.** In the second case where  $D \subseteq B$  and  $f^{-1}(D) = \{x \in A : f(x) \in D\}$ , it must not be that  $\text{rng } f$  includes all of  $D$ , because  $D$  must not be a subset of  $A$ .

**Theorem 2.6.** Suppose that  $f : A \rightarrow B$ . Let  $C \subseteq A$  and let  $D \subseteq B$ . Then the following hold:

1.  $C \subseteq f^{-1}[f(C)]$
2.  $f[f^{-1}(D)] \subseteq D$

**Proof.** We begin with case 1.

Suppose that  $f : A \rightarrow B$ , and that  $C_1 \subseteq A$  and  $C_2 \subseteq A$ , and that  $C_1 \cap C_2 = \emptyset$  and that  $f(C_1) = f(C_2)$ . Then  $f^{-1}[f(C_1)] = C_1 \cup C_2$ , which must contain more members than  $C_1$ . Therefore,  $C \subseteq f^{-1}[f(C)]$  as was to be proven.<sup>a</sup>

For case 2, suppose that  $f : A \rightarrow B$  and  $D \subseteq B$ . Let  $D_1 = \{d \in D : \exists a \in A \ni f(a) = d\}$ , and let  $D_2 = \{d \in D : \forall a \in A, f(a) \neq d\}$ . This implies that  $D = D_1 \cup D_2$  and  $D_1 \cap D_2 = \emptyset$ . The definition of  $D_1$  also means that  $f[f^{-1}(D_1)] = D_1$ . Also, because of the definition of  $D_2$ ,  $f^{-1}(D) = f^{-1}(D_1 \cup D_2) = f^{-1}(D_1)$  since  $f^{-1}(D_2) = \emptyset$ .

Since  $f[f^{-1}(D_1)] = D_1 = f[f^{-1}(D)]$  and  $D_1 \cap D_2 = \emptyset$ , it must be that  $f[f^{-1}(D)] \subseteq D$  because  $D$  has equal or more members than  $D_1$ .  $\square$

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<sup>a</sup>if  $f$  were injective (which it isn't in the proof) then  $C = f^{-1}[f(C)]$ , which is shown in the proof of 2.7.

**Theorem 2.7.** Suppose that  $f : A \rightarrow B$ . Let  $C \subseteq A$  and  $D \subseteq B$ . Then the following hold:

1. If  $f$  is injective, then  $f^{-1}[f(C)] = C$ .
2. If  $f$  is surjective, then  $f[f^{-1}(D)] = D$ .

**Proof.** We begin with case 1.

Suppose that  $f : A \rightarrow B$ , and that  $C_1 \subseteq A$  and  $C_2 \subseteq A$ , and that  $f(C_1) = f(C_2)$ . Then  $f^{-1}[f(C_1)] = C_1 \cup C_2$ . Since  $f$  is injective, and  $f(C_1) = f(C_2)$ , it must be that  $C_1 = C_2$ , and therefore  $f^{-1}[f(C_1)] = C_1$ .

For case 2, suppose that  $f : A \rightarrow B$  and  $D \subseteq B$ . Let  $D_1 = \{d \in D : \exists a \in A \ni f(a) = d\}$ , and let  $D_2 = \{d \in D : \forall a \in A, f(a) \neq d\}$ . This implies that  $D = D_1 \cup D_2$  and  $D_1 \cap D_2 = \emptyset$ . The definition of  $D_1$  also means that  $f[f^{-1}(D_1)] = D_1$ . Since  $f$  is surjective,  $D_2 = \emptyset$ , which means that  $D = D_1$  since  $D_1 \cup D_2 = D_1$ , and therefore  $f[f^{-1}(D_1)] = D_1$  implies that  $f[f^{-1}(D)] = D$ .  $\square$

### 2.1.2 Composition Function

**Definition 2.8** (Composition Function). Suppose that  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , then  $\forall a \in A, f(a) \in B$ , and since  $f(a)$  is an object in  $B$ ,  $g(f(a)) \in C$ . This is called the **composition** of  $f$  and  $g$ .

$$g \circ f = g(f(a)), \quad \forall a \in A$$

In terms of ordered pairs,

$$g \circ f = \{(a, c) \in A \times C : [\exists b \in B \ni (a, b) \in f] \wedge [(b, c) \in g]\}$$

**Theorem 2.9.** Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . Then

1.  $f$  and  $g$  are surjective  $\Rightarrow g \circ f$  is surjective.
2.  $f$  and  $g$  are injective  $\Rightarrow g \circ f$  is injective.
3.  $f$  and  $g$  are bijective  $\Rightarrow g \circ f$  is bijective.

**Proof.** Case 1:

Since  $g$  is surjective,  $\text{rng } g = C$ , which means that  $\forall c \in C, \exists b \in B \ni g(b) = c$ . Now since  $f$  is surjective,  $\exists a \in A \ni f(a) = b$ . But then  $(g \circ f)(a) = g(f(a)) = g(b) = c$ , so  $g \circ f$  is surjective.

Case 2:

Suppose that  $b' = f(a') \in B$  and  $b = f(a) \in B$ , and that  $g(b') = g(b) \in C$ . This implies that  $b' = b$  since  $g$  is injective, which means that  $f(a') = f(a)$ , but because  $f$  too is injective, this implies that  $a' = a$ . This results in that  $g(f(a')) = g(f(a)) \Rightarrow a' = a$ , so by definition,  $g \circ f$  is injective.

Case 3:

By the result of case 1 and 2, if  $f$  and  $g$  are bijective, then  $g \circ f$  is bijective.  $\square$

### 2.1.3 Inverse function

To extend the idea of pre-image from 2.5, we can define a **inverse function**.

**Definition 2.10** (Inverse Function). Suppose that  $f : A \rightarrow B$ . The **inverse function** of  $f$  is the function  $f^{-1}$  given by

$$f^{-1} = \{(y, x) \in B \times A : (x, y) \in f\}$$

**Remark.** If  $f : A \rightarrow B$  is bijective, then  $f^{-1} : B \rightarrow A$  is bijective.

**Definition 2.11** (Identity Function). A function defined on a set  $A$  that maps each element in  $A$  onto itself is called the **identity function** on  $A$ , and is denoted by  $i_A$ .

**Remark.** If  $f : A \rightarrow B$  and  $f$  is bijective, then

- $f^{-1} \circ f = i_A$ ,
- $f \circ f^{-1} = i_B$ .

**Theorem 2.12.** Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be bijective. Then the composition  $g \circ f : A \rightarrow C$  is bijective and  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ .

**Proof.** By theorem 2.9 we know that  $g \circ f$  is bijective, so there exists an inverse  $(g \circ f)^{-1}$ . We are asked to verify the equality of the two functions  $(g \circ f)^{-1}$  and  $f^{-1} \circ g^{-1}$ , as sets of ordered pairs. To this end, suppose  $(c, a) \in (g \circ f)^{-1}$ . By the definition of an inverse function, this means  $(a, c) \in g \circ f$ . The definition of composition implies that

$$\exists b \in B \ni [(a, b) \in f] \wedge [(b, c) \in g].$$

Since  $f$  and  $g$  are bijective, this means that  $(b, a) \in f^{-1}$  and  $(c, b) \in g^{-1}$ . That is,  $f^{-1}(b) = a$  and  $g^{-1}(c) = b$ . But then,

$$(f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c)) = f^{-1}(b) = a \quad (2.1)$$

so that  $(c, a) \in (f^{-1} \circ g^{-1})$  and  $(g \circ f)^{-1} \subseteq (f^{-1} \circ g^{-1})$ .

To the other end, suppose that  $(c, a) \in (f^{-1} \circ g^{-1})$ . The definition of

composition implies that

$$\exists b \in B \ni [(c, b) \in g^{-1}] \wedge [(b, a) \in f^{-1}].$$

This implies that  $(b, c) \in g$  and that  $(a, b) \in f$  and therefore  $(a, c) \in g \circ f$ . Since both  $f$  and  $g$  are bijective, there must exist an inverse  $(g \circ f)^{-1}$  such that  $(c, a) \in (g \circ f)^{-1}$ . Now, since  $(c, a) \in (f^{-1} \circ g^{-1})$  implies that  $(c, a) \in (g \circ f)^{-1}$ , and  $(c, a) \in (g \circ f)^{-1}$  implies that  $(c, a) \in (f^{-1} \circ g^{-1})$ , it must be that  $(g \circ f)^{-1} = (f^{-1} \circ g^{-1})$ .  $\square$



## Chapter 3

# Exercises and My Solutions

### 3.1 Analysis with an Introduction to Proof - Steven R. Lay

### 3.1.1 Sets and Functions

#### 3.1.1.1 Exercises 3

(21) Suppose that  $f : A \rightarrow B$  and let  $C$  be a subset of  $A$ .

1. Prove or give a counterexample:  $f(A \setminus C) \subseteq f(A) \setminus f(C)$ .
2. Prove or give a counterexample:  $f(A) \setminus f(C) \subseteq f(A \setminus C)$ .
3. What condition on  $f$  will ensure that  $f(A \setminus C) = f(A) \setminus f(C)$ ? Prove your answer.
4. What condition of  $f$  will ensure that  $f(A \setminus C) = B \setminus f(C)$ ? Prove your answer.

**Proof.** (1) Suppose that  $f(A \setminus C) \subseteq f(A) \setminus f(C)$ .

Let  $x \in A \setminus C$ ,  $x' \in C$  and  $f(x) = f(x')$ . Then,  $f(x) \in f(A \setminus C)$ , and therefore  $f(x') \in f(A \setminus C)$ . But since  $f(x') \in f(C)$  and therefore  $f(x) \in f(C)$ , neither  $f(x)$  or  $f(x')$  is in  $f(A) \setminus f(C)$ . This contradicts our original statement because there exists a member in  $f(A \setminus C)$  which is not in  $f(A) \setminus f(C)$ , so  $f(A \setminus C) \not\subseteq f(A) \setminus f(C)$ .  $\square$

**Proof.** (2) For any  $y \in f(A) \setminus f(C)$ , there exists an  $x \in A$  such that  $f(x) = y$ . If  $x \in C$ , then  $f(x) \in f(C)$  which means that  $f(x) \neq y$ , so by contradiction it must be that  $x \notin C$ . This implies that  $x \in A \setminus C$ , and therefore that  $f(x) \in f(A \setminus C)$  and  $y \in f(A \setminus C)$ . Since  $y \in f(A) \setminus f(C)$  implies that  $y \in f(A \setminus C)$ , the statement  $f(A) \setminus f(C) \subseteq f(A \setminus C)$  must be true.  $\square$

**Proof.** (3) Proof 2 have already shown that  $f(A) \setminus f(C) \subseteq f(A \setminus C)$ , so to prove that  $f(A) \setminus f(C) = f(A \setminus C)$  I must only prove the reverse of the first statement.

Let  $f$  be injective<sup>a</sup>. For any  $y \in f(A \setminus C)$ , there exists one and only one  $x \in A \setminus C$  such that  $f(x) = y$ . Since  $x \in A \setminus C$ ,  $x \in A$  and  $f(x) \in f(A)$ . Also, since  $x \in A \setminus C$ ,  $x \notin C$  and  $f(x) \notin f(C)$ . This implies that  $f(x) \in f(A) \setminus f(C)$  and thus  $y \in f(A) \setminus f(C)$ . Since  $y \in f(A \setminus C)$  implies  $y \in f(A) \setminus f(C)$ , and  $y \in f(A) \setminus f(C)$  implies  $y \in f(A \setminus C)$  from proof 2, it must be that  $f(A \setminus C) = f(A) \setminus f(C)$ .  $\square$

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<sup>a</sup>this is the necessary condition such that  $f(A \setminus C) = f(A) \setminus f(C)$ .

**Proof.** (4) Proof 3 in combination with that  $f$  is surjective<sup>a</sup> means that  $f(A \setminus C) = f(A) \setminus f(C) = B \setminus f(C)$  since  $B = \text{rng } f = f(A)$ .  $\square$

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<sup>a</sup>Proof 3 needed the condition that  $f$  was injective, and since proof 4 needs  $f$  to be surjective and is based on proof 3,  $f$  is now bijective.

(32) Suppose that  $f : A \rightarrow B$  is any function. Then a function  $g : B \rightarrow A$  is called a

- **left inverse** for  $f$  if  $g(f(x)) = x$  for all  $x \in A$ ,
- **right inverse** for  $f$  if  $f(g(y)) = y$  for all  $y \in B$ .

1. Prove that  $f$  has a left inverse iff  $f$  is injective.
2. Prove that  $f$  has a right inverse iff  $f$  is surjective.

**Proof.** (1) Suppose that  $f$  is injective. Let  $g = \{(b, a) \in B \times A : (a, b) \in f\} \cup \{(b, a) \in B \times A : b \notin f(A)\}^a$ . By definition, each  $a \in A$  corresponds to one and only one  $b \in B$  such that  $f(a) = b$ , and because of the definition of  $g$ , for each  $b \in B$  such that  $f(a) = b$ ,  $g(b) = a$ , which implies that  $g(f(a)) = a$  for all  $a \in A$ .

Conversely, suppose that  $f(x) \in B$  and  $f(x') \in B$ , and that  $f(x) = f(x')$ . If  $g(f(a)) = a$  for all  $a \in A$ ,  $g(f(x)) = g(f(x'))$  implies that  $x = x'$ . Therefore,  $f$  is injective.  $\square$

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<sup>a</sup>I added the part  $\cup \{(b, a) \in B \times A : b \notin f(A)\}$  to  $g$  to show that  $f$  must not be surjective.

**Proof.** (2) Suppose that  $f$  has a right inverse and therefore  $f(g(y)) = y$  for all  $y \in B$ . This implies that  $f$  is surjective, since for all  $y \in B$  there exists some  $x \in A$ , which may be  $g(y)$ , such that  $f(x) = y$ .  $\square$

**(33)** Let  $S$  be a nonempty set and let  $F$  be the set of all functions that map  $S$  into  $S$ . Suppose that for every  $f$  and  $g$  in  $F$  we have

$$(f \circ g)(x) = (g \circ f)(x), \forall x \in S$$

Prove that  $S$  has only one element.

**Proof.** If  $S$  contains more than one element, then there exists some functions  $f$  and  $g$  in  $F$  that are neither surjective nor injective. Suppose that  $x, x' \in S$  and that  $x \neq x'$ , and that  $f(x) = x'$  and  $f(x') = x'$ , and that  $g(x) = x$  and  $g(x') = x$ . Then  $f(g(x)) = f(x) = x'$ , and  $g(f(x)) = g(x') = x$ , which contradicts the statement that  $(f \circ g)(x) = (g \circ f)(x), \forall x \in S$ , so  $S$  must contain less than two elements. Since  $S$  is nonempty, it must therefore contain one element.  $\square$