Single Variable Calculus

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Contents

1 Calculus A Complete Course

1.1 Limits and Continuity

Theorem 1 (The Squeeze Theorem, 4). *Suppose that $f(x) \leq g(x) \leq h(x)$ holds for all x in some open interval containing c, except possibly at $x = c$. Suppose also that*

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

Then $\lim_{x \to c} g(x) = L$.

Proof. For this proof, the (ϵ, δ)-definition of the limit will be used.

The goal is to prove that $\lim_{x \to c} g(x) = L$, which is true if

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, (|x - c| < \delta \Rightarrow |g(x) - L| < \epsilon).$$

Since $\lim_{x \to c} f(x) = L$,

$$\forall \epsilon > 0, \exists \delta_1 > 0 : \forall x, (|x - c| < \delta_1 \Rightarrow |f(x) - L| < \epsilon) \tag{1}$$

And since $\lim_{x \to c} h(x) = L$,

$$\forall \epsilon > 0, \exists \delta_2 > 0 : \forall x, (|x - c| < \delta_2 \Rightarrow |h(x) - L| < \epsilon). \tag{2}$$

Then we have

$$f(x) \leq g(x) \leq h(x)$$

$$f(x) - L \leq g(x) - L \leq h(x) - L$$

We can choose $\delta = \min\{\delta_1, \delta_2\}$, then if $|x - c| < \delta$, and combining (1) and (2), we have

$$-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon$$

$$-\epsilon < g(x) - L < \epsilon$$

$$|g(x) - L| < \epsilon$$

So $\lim_{x \to c} g(x) = L$, which completes the proof. □

Theorem 2 (The Intermediate-Value Theorem, 9). *If $f(x)$ is continuous on the interval $[a, b]$ and if s is a number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that $f(c) = s$.*

In particular, a continuous function defined on a closed interval takes on all values between its minimum value m and its maximum value M , so its range is also a closed interval, $[m, M]$.

Proof. □

1.1.1 Exercises 1.1

1.1.2 Exercises 1.2

78. What is the domain of $\sin \frac{1}{x}$? Evaluate $\lim_{x\to 0} x \sin \frac{1}{x}$.

The domain of $x \sin x$ is $\mathbb{R}$. The domain of $\frac{1}{x}$ is $(-\infty, 0) \cup (0, \infty)$. Therefore, the domain of $x \sin \frac{1}{x}$ is $(-\infty, 0) \cup (0, \infty)$.

To evaluate $\lim_{x\to 0} x \sin \frac{1}{x}$, we can first evaluate $\lim_{x\to 0} \frac{1}{x}$.

$\lim_{x\to 0^+} \frac{1}{x} = +\infty$, $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

This means that $\lim_{x\to 0} \sin \frac{1}{x} = \lim_{x\to \pm\infty} \sin x$, which means that $-1 \leq \lim_{x\to 0} \sin \frac{1}{x} \leq 1$.

$\lim_{x\to 0} x \sin \frac{1}{x} = (\lim_{x\to 0} x)(\lim_{x\to 0} \sin \frac{1}{x}) = 0$

79. Suppose $|f(x)| \leq g(x) \forall x$. What can you conclude about $\lim_{x\to a} f(x)$ if $\lim_{x\to a} g(x) = 0$? What if $\lim_{x\to a} g(x) = 3$?

$|f(x)| \leq g(x) \forall x \Leftrightarrow -g(x) \leq f(x) \leq g(x) \forall x$. Since $\lim_{x\to a} g(x) = 0$ and therefore $\lim_{x\to a} -g(x) = 0$, then $\lim_{x\to a} f(x) = 0$ by the squeeze theorem.

If $\lim_{x\to a} g(x) = 3$, and $-g(x) \leq f(x) \leq g(x) \forall x$, then we can conclude that either $-3 \leq \lim_{x\to a} f(x) \leq 3$, or $\lim_{x\to a} f(x)$ doesn't exist.

1.1.3 Exercises 1.3

1.1.4 Exercises 1.4

32.

Let $g(x) = f(x) - x$. Since $0 \leq f(x) \leq 1$ for $0 \leq x \leq 1$, then $0 \leq g(0)$. By the same argument, $g(1) \leq 0$. Because $g(x)$ is continuous in the interval

$[0, 1]$, there must be some value $c \in [0, 1]$ such that $g(c) = 0$, by the Intermediate-Value Theorem. If $g(c) = 0$, then $f(c) = c$, which was to be shown.

33.

Since $f(x)$ is even, it is symmetric around the y-axis. The symmetric equivilance of $\lim_{x \to 0^+}$ around the y-axis is $\lim_{x \to 0^-}$. Since $f(x)$ is right-continuous, it means that $\lim_{x \to 0+} f(x) = f(0)$ and because of the symmetry, $\lim_{x \to 0-} f(x) = f(0)$. Because $\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = f(0)$, f is continuous at $x = 0$.

34.

$\lim_{x \to 0^+} f(x) = f(0)$ because f is right continuous. Since f is odd, it is symmetric around the origin, and therefore $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$. Since f is both right and left continuous at $x = 0$, it is continuous at $x = 0$.

1.1.5 Exercises 1.5

31.

$(\lim_{x\to a} f(x) = L) \Leftrightarrow (\forall\epsilon > 0, \exists\delta_1 > 0 : 0 < |x-a| < \delta_1 \Rightarrow |f(x)-L| < \epsilon)$

and

$(\lim_{x\to a} f(x) = M) \Leftrightarrow (\forall\epsilon > 0, \exists\delta_2 > 0 : 0 < |x-a| < \delta_2 \Rightarrow |f(x)-M| < \epsilon)$

We assume that $L \neq M$. If we choose $\delta = \min\{\delta_1, \delta_2\}$, then $0 < |x-a| < \delta \Rightarrow |f(x)-L| + |f(x)-M| < \epsilon + \epsilon = 2\epsilon$.

By the triangle inequality,

$|f(x)-L| + |f(x)-M| \geq |(L-f(x)) + (f(x)-M)| = |L-M|$.

Since $L \neq M$, we can let $\epsilon = |L-M|/4$ because $|L-M|$ is positive.

This means that $|L-M| \leq 2\epsilon = |L-M|/2 \Rightarrow 2 \leq 1$, which is obviously false. Therefore $L = M$ and the limit is unique, which was to be shown.

32.

Since $\lim_{x\to a} g(x) = M$, the following must be true

$$|g(x)| = |(g(x) - M) + M| \leq |g(x) - M| + |M| < \epsilon + |M|$$

If we choose $\epsilon = 1$ then

$$|g(x)| < 1 + |M|$$

Which was to be shown.

33.

$\lim_{x\to a} f(x) \Leftrightarrow (\forall\epsilon > 0, \exists\delta_1 > 0 : \forall x, (0 < |x-a| < \delta_1 \Rightarrow |f(x)-L| < \epsilon))$

And

$\lim_{x\to a} g(x) \Leftrightarrow (\forall\epsilon > 0, \exists\delta_2 > 0 : \forall x, (0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \epsilon))$

Lets assume that $\lim_{x\to a} f(x)g(x) \neq LM$.

Let $\delta = \min\{\delta_1, \delta_2\}$. This would result in that

$$\forall\epsilon > 0, \exists\delta > 0 : \forall x, (0 < |x - a| < \delta \Rightarrow |f(x) - L| + |g(x) - M| < 2\epsilon) \tag{3}$$

$|f(x)-L|+|g(x)-M| \geq |g(x)||f(x)-L|+|L||g(x)-M| = |g(x)(f(x)-L)| + |L(g(x) - M)| \geq |g(x)(f(x) - L) + L(g(x) - M)| = |f(x)g(x) - Lg(x) + Lg(x) - LM| = |(f(x)g(x)) - (LM)|$

This together with (3) means that

$\forall \epsilon > 0, \exists \delta > 0 : \forall x, (0 < |x - a| < \delta \Rightarrow |(f(x)g(x)) - (LM)| < 2\epsilon)$

Which shows that lim $\lim_{x \to a} f(x)g(x) = LM$.

2 Lectures