

# Single Variable Calculus

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# 1 Calculus A Complete Course

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## 1.1 Limits and Continuity

**Theorem 1** (The Squeeze Theorem, 4). *Suppose that  $f(x) \leq g(x) \leq h(x)$  holds for all  $x$  in some open interval containing  $c$ , except possibly at  $x = c$ . Suppose also that*

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$$

*Then  $\lim_{x \rightarrow c} g(x) = L$ .*

*Proof.* For this proof, the  $(\epsilon, \delta)$ -definition of the limit will be used.

The goal is to prove that  $\lim_{x \rightarrow c} g(x) = L$ , which is true if

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, (|x - c| < \delta \Rightarrow |g(x) - L| < \epsilon).$$

Since  $\lim_{x \rightarrow c} f(x) = L$ ,

$$\forall \epsilon > 0, \exists \delta_1 > 0 : \forall x, (|x - c| < \delta_1 \Rightarrow |f(x) - L| < \epsilon) \quad (1)$$

And since  $\lim_{x \rightarrow c} h(x) = L$ ,

$$\forall \epsilon > 0, \exists \delta_2 > 0 : \forall x, (|x - c| < \delta_2 \Rightarrow |h(x) - L| < \epsilon). \quad (2)$$

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Then we have

$$f(x) \leq g(x) \leq h(x)$$

$$f(x) - L \leq g(x) - L \leq h(x) - L$$

We can choose  $\delta = \min\{\delta_1, \delta_2\}$ , then if  $|x - c| < \delta$ , and combining (1) and (2), we have

$$-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon$$

$$-\epsilon < g(x) - L < \epsilon$$

$$|g(x) - L| < \epsilon$$

So  $\lim_{x \rightarrow c} g(x) = L$ , which completes the proof. □

**Theorem 2** (The Intermediate-Value Theorem, 9). *If  $f(x)$  is continuous on the interval  $[a, b]$  and if  $s$  is a number between  $f(a)$  and  $f(b)$ , then there exists a number  $c$  in  $[a, b]$  such that  $f(c) = s$ .*

*In particular, a continuous function defined on a closed interval takes on all values between its minimum value  $m$  and its maximum value  $M$ , so its range is also a closed interval,  $[m, M]$ .*

*Proof.* □

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### 1.1.1 Exercises 1.1

### 1.1.2 Exercises 1.2

**78. What is the domain of  $\sin \frac{1}{x}$  ? Evaluate  $\lim_{x \rightarrow 0} x \sin \frac{1}{x}$ .**

The domain of  $x \sin x$  is  $\mathbb{R}$ . The domain of  $\frac{1}{x}$  is  $(-\infty, 0) \cup (0, \infty)$ . Therefore, the domain of  $x \sin \frac{1}{x}$  is  $(-\infty, 0) \cup (0, \infty)$ .

To evaluate  $\lim_{x \rightarrow 0} x \sin \frac{1}{x}$ , we can first evaluate  $\lim_{x \rightarrow 0} \frac{1}{x}$ .

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty, \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

This means that  $\lim_{x \rightarrow 0} \sin \frac{1}{x} = \lim_{x \rightarrow \pm\infty} \sin x$ , which means that  $-1 \leq \lim_{x \rightarrow 0} \sin \frac{1}{x} \leq 1$ .

$$\lim_{x \rightarrow 0} x \sin \frac{1}{x} = (\lim_{x \rightarrow 0} x)(\lim_{x \rightarrow 0} \sin \frac{1}{x}) = 0$$

**79. Suppose  $|f(x)| \leq g(x) \forall x$ . What can you conclude about  $\lim_{x \rightarrow a} f(x)$  if  $\lim_{x \rightarrow a} g(x) = 0$  ? What if  $\lim_{x \rightarrow a} g(x) = 3$  ?**

$|f(x)| \leq g(x) \forall x \Leftrightarrow -g(x) \leq f(x) \leq g(x) \forall x$ . Since  $\lim_{x \rightarrow a} g(x) = 0$  and therefore  $\lim_{x \rightarrow a} -g(x) = 0$ , then  $\lim_{x \rightarrow a} f(x) = 0$  by the squeeze theorem.

If  $\lim_{x \rightarrow a} g(x) = 3$ , and  $-g(x) \leq f(x) \leq g(x) \forall x$ , then we can conclude that either  $-3 \leq \lim_{x \rightarrow a} f(x) \leq 3$ , or  $\lim_{x \rightarrow a} f(x)$  doesn't exist.

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### 1.1.3 Exercises 1.3

### 1.1.4 Exercises 1.4

32.

Let  $g(x) = f(x) - x$ . Since  $0 \leq f(x) \leq 1$  for  $0 \leq x \leq 1$ , then  $0 \leq g(0)$ . By the same argument,  $g(1) \leq 0$ . Because  $g(x)$  is continuous in the interval  $[0, 1]$ , there must be some value  $c \in [0, 1]$  such that  $g(c) = 0$ , by the Intermediate-Value Theorem. If  $g(c) = 0$ , then  $f(c) = c$ , which was to be shown.

33.

Since  $f(x)$  is even, it is symmetric around the y-axis. The symmetric equivariance of  $\lim_{x \rightarrow 0^+}$  around the y-axis is  $\lim_{x \rightarrow 0^-}$ . Since  $f(x)$  is right-continuous, it means that  $\lim_{x \rightarrow 0^+} f(x) = f(0)$  and because of the symmetry,  $\lim_{x \rightarrow 0^-} f(x) = f(0)$ . Because  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0)$ ,  $f$  is continuous at  $x = 0$ .

34.

$\lim_{x \rightarrow 0^+} f(x) = f(0)$  because  $f$  is right continuous. Since  $f$  is odd, it is symmetric around the origin, and therefore  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0) = 0$ . Since  $f$  is both right and left continuous at  $x = 0$ , it is continuous at  $x = 0$ .

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### 1.1.5 Exercises 1.5

31.

$$(\lim_{x \rightarrow a} f(x) = L) \Leftrightarrow (\forall \epsilon > 0, \exists \delta_1 > 0 : 0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \epsilon)$$

and

$$(\lim_{x \rightarrow a} f(x) = M) \Leftrightarrow (\forall \epsilon > 0, \exists \delta_2 > 0 : 0 < |x - a| < \delta_2 \Rightarrow |f(x) - M| < \epsilon)$$

We assume that  $L \neq M$ . If we choose  $\delta = \min\{\delta_1, \delta_2\}$ , then  $0 < |x - a| < \delta \Rightarrow |f(x) - L| + |f(x) - M| < \epsilon + \epsilon = 2\epsilon$ .

By the triangle inequality,

$$|f(x) - L| + |f(x) - M| \geq |(L - f(x)) + (f(x) - M)| = |L - M|.$$

Since  $L \neq M$ , we can let  $\epsilon = |L - M|/4$  because  $|L - M|$  is positive.

This means that  $|L - M| \leq 2\epsilon = |L - M|/2 \Rightarrow 2 \leq 1$ , which is obviously false. Therefore  $L = M$  and the limit is unique, which was to be shown.

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32.

Since  $\lim_{x \rightarrow a} g(x) = M$ ,

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, 0 < |x - a| < \delta \Rightarrow |g(x) - M| = |g(x) + (-M)| \leq |g(x)| + |(-M)| < \epsilon \Leftrightarrow |g(x)| < \epsilon - |M|$$

Since we're supposed to show that there exists some  $\delta > 0$  which implies that  $|g(x)| < 1 + |M|$ , we can just choose  $\epsilon = 1 + 2|M|$  so that  $\epsilon - |M| = 1 + |M|$ . This results in that

$$0 < |x - a| < \delta \Rightarrow |g(x)| < 1 + |M|$$

Which was to be shown.

33.

$$\lim_{x \rightarrow a} f(x) \Leftrightarrow (\forall \epsilon > 0, \exists \delta_1 > 0 : \forall x, (0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \epsilon))$$

And

$$\lim_{x \rightarrow a} g(x) \Leftrightarrow (\forall \epsilon > 0, \exists \delta_2 > 0 : \forall x, (0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \epsilon))$$

Lets assume that  $\lim_{x \rightarrow a} f(x)g(x) \neq LM$ .

Let  $\delta = \min\{\delta_1, \delta_2\}$ . This would result in that

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, (0 < |x - a| < \delta \Rightarrow |f(x) - L| + |g(x) - M| < 2\epsilon) \quad (3)$$

$$|f(x) - L| + |g(x) - M| \geq |g(x)||f(x) - L| + |L||g(x) - M| = |g(x)(f(x) - L)| + |L(g(x) - M)| \geq |g(x)(f(x) - L) + L(g(x) - M)| = |f(x)g(x) -$$



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$$|Lg(x) + Lg(x) - LM| = |(f(x)g(x)) - (LM)|$$

This together with (3) means that

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, (0 < |x - a| < \delta \Rightarrow |(f(x)g(x)) - (LM)| < 2\epsilon)$$

Which shows that  $\lim_{x \rightarrow a} f(x)g(x) = LM$ .

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## 2 Lectures