Multi-agent learning

The replicator dynamic

Gerard Vreeswijk, Intelligent Software Systems, Computer Science Department, Faculty of Sciences, Utrecht University, The Netherlands.

Monday 14th June, 2021

Topics of today

Topics of today

- Symmetric games, symmetric Nash equilibria in symmetric games, a-symmetric Nash equilibria in symmetric games.

Topics of today

- Symmetric games, symmetric Nash equilibria in symmetric games, a-symmetric Nash equilibria in symmetric games.
- Evolutionary game theory: proportions, fitness, average fitness.

Topics of today

- Symmetric games, symmetric Nash equilibria in symmetric games, a-symmetric Nash equilibria in symmetric games.
- Evolutionary game theory: proportions, fitness, average fitness.
- The replicator dynamic.

Topics of today

- Symmetric games, symmetric Nash equilibria in symmetric games, a-symmetric Nash equilibria in symmetric games.
- Evolutionary game theory: proportions, fitness, average fitness.
- The replicator dynamic.
  - Fitness vector, score-matrix

Topics of today

- Symmetric games, symmetric Nash equilibria in symmetric games, a-symmetric Nash equilibria in symmetric games.
- Evolutionary game theory: proportions, fitness, average fitness.
- The replicator dynamic.
  - Fitness vector, score-matrix (= grand table if species are reply rules).

Topics of today

- Symmetric games, symmetric Nash equilibria in symmetric games, a-symmetric Nash equilibria in symmetric games.
- Evolutionary game theory: proportions, fitness, average fitness.
- The replicator dynamic.
  - Fitness vector, score-matrix (= grand table if species are reply rules).
  - The continuous replicator equation.

Topics of today

- Symmetric games, symmetric Nash equilibria in symmetric games, a-symmetric Nash equilibria in symmetric games.
- Evolutionary game theory: proportions, fitness, average fitness.
- The replicator dynamic.
  - Fitness vector, score-matrix (= grand table if species are reply rules).
  - The continuous replicator equation.
  - The discrete replicator equation.

Topics of today

- Symmetric games, symmetric Nash equilibria in symmetric games, a-symmetric Nash equilibria in symmetric games.
- Evolutionary game theory: proportions, fitness, average fitness.
- The replicator dynamic.
  - Fitness vector, score-matrix (= grand table if species are reply rules).
  - The continuous replicator equation.
  - The discrete replicator equation.
  - The discrete step equation.

Topics of today

- Symmetric games, symmetric Nash equilibria in symmetric games, a-symmetric Nash equilibria in symmetric games.
- Evolutionary game theory: proportions, fitness, average fitness.
- The replicator dynamic.
  - Fitness vector, score-matrix (= grand table if species are reply rules).
  - The continuous replicator equation.
  - The discrete replicator equation.
  - The discrete step equation.
  - Derivation of the DRE from the DSE.

Topics of today

- Symmetric games, symmetric Nash equilibria in symmetric games, a-symmetric Nash equilibria in symmetric games.
- Evolutionary game theory: proportions, fitness, average fitness.
- The replicator dynamic.
  - Fitness vector, score-matrix (= grand table if species are reply rules).
  - The continuous replicator equation.
  - The discrete replicator equation.
  - The discrete step equation.
  - Derivation of the DRE from the DSE.
  - Derivation of the CRE from the DRE.

Topics of today

- Symmetric games, symmetric Nash equilibria in symmetric games, a-symmetric Nash equilibria in symmetric games.
- Evolutionary game theory: proportions, fitness, average fitness.
- The replicator dynamic.
  - Fitness vector, score-matrix (= grand table if species are reply rules).
  - The continuous replicator equation.
  - The discrete replicator equation.
  - The discrete step equation.
  - Derivation of the DRE from the DSE.
  - Derivation of the CRE from the DRE.
  - Properties of the replicator dynamic, connection with Nash equilibria.

Symmetric games in normal form

Ways to represent the payoff matrix of a symmetric game

Symmetric game: **only your actions matter, not your role**.

Ways to represent the payoff matrix of a symmetric game

Symmetric game: **only your actions matter, not your role**. For 2-player games: it is not important whether you are row or column player:

Ways to represent the payoff matrix of a symmetric game

Symmetric game: **only your actions matter, not your role**. For 2-player games: it is not important whether you are row or column player:

$$\begin{array}{cc} & \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 1,1 & 2,3 & -4,6 \\ 3,2 & 0,0 & 8,-7 \\ 6,-4 & -7,8 & 5,5 \end{pmatrix} \end{array} = \begin{array}{cc} & \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 1,1 & 2,3 & -4,6 \\ & 0,0 & 8,-7 \\ & & 5,5 \end{pmatrix} \end{array}$$

- As a bi-matrix.
- As a partially filled bi-matrix.
- As a plain matrix.

Ways to represent the payoff matrix of a symmetric game

Symmetric game: **only your actions matter, not your role**. For 2-player games: it is not important whether you are row or column player:

$$
\begin{array}{c}
 \\ A_1 \\ A_2 \\ A_3
\end{array}
\begin{array}{c}
\begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\
\begin{pmatrix} 1,1 & 2,3 & -4,6 \\ 3,2 & 0,0 & 8,-7 \\ 6,-4 & -7,8 & 5,5 \end{pmatrix}
\end{array}
=
\begin{array}{c}
 \\ A_1 \\ A_2 \\ A_3
\end{array}
\begin{array}{c}
\begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\
\begin{pmatrix} 1,1 & 2,3 & -4,6 \\ & 0,0 & 8,-7 \\ & & 5,5 \end{pmatrix}
\end{array}
$$

$$
\sim
\begin{array}{c}
 \\ A_1 \\ A_2 \\ A_3
\end{array}
\begin{array}{c}
\begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\
\begin{pmatrix} 1 & 2 & -4 \\ 3 & 0 & 8 \\ 6 & -7 & 5 \end{pmatrix}
\end{array}.
$$

- As a bi-matrix.
- As a partially filled bi-matrix.
- As a plain matrix.

Hawk vs. Dove

Symmetric normal-form games

Example. Hawk-dove game (share V or threaten [possibly fight: $-C$]):

	H	D
H	$(V-C)/2$	V
D	0	$V/2$

Symmetric normal-form games

Example. Hawk-dove game (share V or threaten [possibly fight: $-C$]):

	H	D
H	$(V-C)/2$	V
D	0	$V/2$

	H	D
H	$-2,-2$	$2,0$
D	$0,2$	$1,1$

V=2, C=6

Symmetric normal-form games

Example. Hawk-dove game (share V or threaten [possibly fight: $-C$]):

	H	D
H	$(V-C)/2$	V
D	0	$V/2$

	H	D
H	$-2,-2$	$2,0$
D	$0,2$	$1,1$

V=2, C=6

Other instantiations: prisoner's dilemma, chicken (= hawk-dove), matching pennies, stag hunt.

Symmetric normal-form games

Example. Hawk-dove game (share V or threaten [possibly fight: $-C$]):

	H	D
H	$(V-C)/2$	V
D	0	$V/2$

	H	D
H	$-2,-2$	$2,0$
D	$0,2$	$1,1$

V=2, C=6

Other instantiations: prisoner's dilemma, chicken (= hawk-dove), matching pennies, stag hunt.

Definition. A game is symmetric when players have equal actions and payoffs:

$$u_i(a_1,\ldots,a_i,\ldots,a_j,\ldots,a_n) = u_j(a_1,\ldots,a_j,\ldots,a_i,\ldots,a_n).$$

for all i and j.

Symmetric normal-form games

Example. Hawk-dove game (share V or threaten [possibly fight: $-C$]):

	H	D
H	$(V-C)/2$	V
D	0	$V/2$

	H	D
H	$-2,-2$	$2,0$
D	$0,2$	$1,1$

V=2, C=6

Other instantiations: prisoner's dilemma, chicken (= hawk-dove), matching pennies, stag hunt.

Definition. A game is symmetric when players have equal actions and payoffs:

$$u_i(a_1,\dots,a_i,\dots,a_j,\dots,a_n) = u_j(a_1,\dots,a_j,\dots,a_i,\dots,a_n).$$

for all i and j.

So a 2-player game $G = (A, B)$ is symmetric iff $m = n$ and $B = A^T$.

Symmetric equilibrium

Symmetric equilibrium

Definition. Let p be a strategy in an n-player symmetric game. If the n-vector $(p, \ldots, p)$ is a NE, p is called a symmetric equilibrium.

Symmetric equilibrium

Definition. Let p be a strategy in an n-player symmetric game. If the n-vector $(p, \ldots, p)$ is a NE, p is called a symmetric equilibrium.

- (Theorem.) Every symmetric game has at least one symmetric equilibrium.

Symmetric equilibrium

Definition. Let p be a strategy in an n-player symmetric game. If the n-vector $(p, \ldots, p)$ is a NE, p is called a symmetric equilibrium.

- (Theorem.) Every symmetric game has at least one symmetric equilibrium.
- Symmetric equilibria can be identified with strategies

Symmetric equilibrium

Definition. Let p be a strategy in an n-player symmetric game. If the n-vector $(p, \ldots, p)$ is a NE, p is called a symmetric equilibrium.

- (Theorem.) Every symmetric game has at least one symmetric equilibrium.
- Symmetric equilibria can be identified with strategies, also with proportions.

Symmetric equilibrium

Definition. Let p be a strategy in an n-player symmetric game. If the n-vector $(p, \ldots, p)$ is a NE, p is called a symmetric equilibrium.

- (Theorem.) Every symmetric game has at least one symmetric equilibrium.
- Symmetric equilibria can be identified with strategies, also with proportions.
- Symmetric games can have a-symmetric equilibria.

Symmetric equilibrium

Definition. Let p be a strategy in an n-player symmetric game. If the n-vector $(p, \ldots, p)$ is a NE, p is called a symmetric equilibrium.

- (Theorem.) Every symmetric game has at least one symmetric equilibrium.
- Symmetric equilibria can be identified with strategies, also with proportions.
- Symmetric games can have a-symmetric equilibria.

  For example Hawk-Dove:

	H	D
H	$-2, -2$	$2, 0$
D	$0, 2$	$1, 1$

Symmetric equilibrium

Definition. Let p be a strategy in an n-player symmetric game. If the n-vector $(p, \dots, p)$ is a NE, p is called a symmetric equilibrium.

- (Theorem.) Every symmetric game has at least one symmetric equilibrium.
- Symmetric equilibria can be identified with strategies, also with proportions.
- Symmetric games can have a-symmetric equilibria.

  For example Hawk-Dove:

	H	D
H	$-2, -2$	$2, 0$
D	$0, 2$	$1, 1$

  Two pure asymmetric equilibria and one symmetric equilibrium $(1/3, 1/3)$.

Evolutionary game theory

Evolutionary game theory: the idea

Evolutionary game theory: the idea

Evolutionary game theory: the idea

- There are n, say 5, species.

Evolutionary game theory: the idea

- There are n, say 5, species. An encounter between individuals of different species yields payoffs for both.

Evolutionary game theory: the idea

- There are n, say 5, species. An encounter between individuals of different species yields payoffs for both. For row:

$$A = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{pmatrix} 6 & 7 & 0 & -1 & 0 \\ -1 & 5 & -1 & 4 & 7 \\ 9 & 0 & 8 & 9 & 6 \\ 0 & -4 & -2 & 3 & -3 \\ 3 & 0 & 6 & 0 & -1 \end{pmatrix} \end{array}.$$

Evolutionary game theory: the idea

- There are n, say 5, species. An encounter between individuals of different species yields payoffs for both. For row:

$$A = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{pmatrix} 6 & 7 & 0 & -1 & 0 \\ -1 & 5 & -1 & 4 & 7 \\ 9 & 0 & 8 & 9 & 6 \\ 0 & -4 & -2 & 3 & -3 \\ 3 & 0 & 6 & 0 & -1 \end{pmatrix} \end{array}.$$

- The population consists of a very large number of individuals, each playing a pure strategy. Individuals interact randomly.

Evolutionary game theory: the idea

- There are n, say 5, species. An encounter between individuals of different species yields payoffs for both. For row:

$$A = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{pmatrix} 6 & 7 & 0 & -1 & 0 \\ -1 & 5 & -1 & 4 & 7 \\ 9 & 0 & 8 & 9 & 6 \\ 0 & -4 & -2 & 3 & -3 \\ 3 & 0 & 6 & 0 & -1 \end{pmatrix} \end{array}.$$

- The population consists of a very large number of individuals, each playing a pure strategy. Individuals interact randomly.

- We are interested in proportions: $p = (p_1, \dots, p_5)$.

Evolutionary game theory: the idea

- There are n, say 5, species. An encounter between individuals of different species yields payoffs for both. For row:

$$A = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{pmatrix} 6 & 7 & 0 & -1 & 0 \\ -1 & 5 & -1 & 4 & 7 \\ 9 & 0 & 8 & 9 & 6 \\ 0 & -4 & -2 & 3 & -3 \\ 3 & 0 & 6 & 0 & -1 \end{pmatrix} \end{array}.$$

- The population consists of a very large number of individuals, each playing a pure strategy. Individuals interact randomly.

- We are interested in proportions: $p = (p_1, \dots, p_5)$.
- The fitness of species i is:

$$f_i = \sum_{j=1}^{5} p_j A_{ij} = (Ap)_i.$$

Evolutionary game theory: the idea

- There are n, say 5, species. An encounter between individuals of different species yields payoffs for both. For row:

$$A = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{pmatrix} 6 & 7 & 0 & -1 & 0 \\ -1 & 5 & -1 & 4 & 7 \\ 9 & 0 & 8 & 9 & 6 \\ 0 & -4 & -2 & 3 & -3 \\ 3 & 0 & 6 & 0 & -1 \end{pmatrix} \end{array}.$$

- The population consists of a very large number of individuals, each playing a pure strategy. Individuals interact randomly.

- We are interested in proportions: $p = (p_1, \dots, p_5)$.

- The fitness of species i is:

$$f_i = \sum_{j=1}^{5} p_j A_{ij} = (Ap)_i.$$

- The average fitness is

$$\bar{f} = \sum_{i=1}^{5} p_i f_i = p^T Ap.$$

The replicator equation

History of the replicator equation

History of the replicator equation

- Defined for a single species by Taylor and Jonker (1978), and named by Schuster and Sigmund (1983):

> "Several evolutionary models in distinct biological fields—population genetics, population ecology, early biochemical evolution and sociobiology—lead independently to the same class of replicator dynamics."

History of the replicator equation

- Defined for a single species by Taylor and Jonker (1978), and named by Schuster and Sigmund (1983):

> "Several evolutionary models in distinct biological fields—population genetics, population ecology, early biochemical evolution and sociobiology—lead independently to the same class of replicator dynamics."

- The replicator equation is the first game dynamics studied in connection with evolutionary game theory (as developed by Maynard Smith and Price).

History of the replicator equation

- Defined for a single species by Taylor and Jonker (1978), and named by Schuster and Sigmund (1983):

> "Several evolutionary models in distinct biological fields—population genetics, population ecology, early biochemical evolution and sociobiology—lead independently to the same class of replicator dynamics."

- The replicator equation is the first game dynamics studied in connection with evolutionary game theory (as developed by Maynard Smith and Price).

Taylor P.D., Jonker L. "Evolutionarily stable strategies and game dynamics" in: *Math. Biosci.* 1978;**40(1)**, pp. 145-156.

History of the replicator equation

- Defined for a single species by Taylor and Jonker (1978), and named by Schuster and Sigmund (1983):

> "Several evolutionary models in distinct biological fields—population genetics, population ecology, early biochemical evolution and sociobiology—lead independently to the same class of replicator dynamics."

- The replicator equation is the first game dynamics studied in connection with evolutionary game theory (as developed by Maynard Smith and Price).

Taylor P.D., Jonker L. "Evolutionarily stable strategies and game dynamics" in: *Math. Biosci.* 1978;**40(1)**, pp. 145-156.

Schuster P., Sigmund K. "Replicator dynamics" in: *J. Theor. Biol.* 1983, **100(3)**, pp. 533-538.

The replicator equation

The replicator equation

- The replicator equation models how n different specifies grow (or decline) due to mutual interaction.

The replicator equation

- The replicator equation models how n different specifies grow (or decline) due to mutual interaction.
- It is assumed that if an individual of species i interacts with an individual of species j, the expected reward for the individual of type i is a constant a_{ij}.

The replicator equation

- The replicator equation models how n different specifies grow (or decline) due to mutual interaction.
- It is assumed that if an individual of species i interacts with an individual of species j, the expected reward for the individual of type i is a constant a_{ij}.

Summarised in a relative score matrix: $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$.

The replicator equation

- The replicator equation models how n different specifies grow (or decline) due to mutual interaction.
- It is assumed that if an individual of species i interacts with an individual of species j, the expected reward for the individual of type i is a constant a_{ij}.

Summarised in a relative score matrix: $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$.

Proportions

The replicator equation

- The replicator equation models how n different specifies grow (or decline) due to mutual interaction.
- It is assumed that if an individual of species i interacts with an individual of species j, the expected reward for the individual of type i is a constant a_{ij}.

  Summarised in a relative score matrix: $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$

Proportions

- The number of individuals of species i is denoted by q_i, or $q_i(t)$.

The replicator equation

- The replicator equation models how n different specifies grow (or decline) due to mutual interaction.
- It is assumed that if an individual of species i interacts with an individual of species j, the expected reward for the individual of type i is a constant a_{ij}.

  Summarised in a relative score matrix: $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$.

Proportions

- The number of individuals of species i is denoted by q_i, or $q_i(t)$.
- $p_j =_{Def} q_j/q$ is the proportion of species i, where $q = q_1 + \cdots + q_n$.

The replicator equation

- The replicator equation models how n different specifies grow (or decline) due to mutual interaction.
- It is assumed that if an individual of species i interacts with an individual of species j, the expected reward for the individual of type i is a constant a_{ij}.

  Summarised in a relative score matrix: $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$.

Proportions

- The number of individuals of species i is denoted by q_i, or $q_i(t)$.
- $p_j =_{Def} q_j / q$ is the proportion of species i, where $q = q_1 + \cdots + q_n$.
- So $p_i \propto q_i$ and $p_1 + \cdots + p_n = 1$.

Fitness

Fitness

- The fitness of an individual is its expected reward when it encounters a random individual in the population.

Fitness

- The fitness of an individual is its expected reward when it encounters a random individual in the population.
- **Example**. Suppose

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} 0.1 \\ 0.4 \\ 0.5 \end{pmatrix}.$$

Fitness

- The fitness of an individual is its expected reward when it encounters a random individual in the population.
- **Example**. Suppose

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} 0.1 \\ 0.4 \\ 0.5 \end{pmatrix}.$$

The fitness vector, f, can now be computed as follows:

$$f = Ap = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0.1 \\ 0.4 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.4 \\ 2.3 \end{pmatrix}.$$

Fitness

- The fitness of an individual is its expected reward when it encounters a random individual in the population.
- **Example**. Suppose

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} 0.1 \\ 0.4 \\ 0.5 \end{pmatrix}.$$

  The fitness vector, f, can now be computed as follows:

$$f = Ap = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0.1 \\ 0.4 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.4 \\ 2.3 \end{pmatrix}.$$

- Average fitness:

$$\begin{aligned} \bar{f}(t) &= \sum_{j=1}^{3} p_i f_i(t) \\ &= p(Ap) = 2.29. \end{aligned}$$

Fitness

- The fitness of an individual is its expected reward when it encounters a random individual in the population.
- **Example**. Suppose

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} 0.1 \\ 0.4 \\ 0.5 \end{pmatrix}.$$

  The fitness vector, f, can now be computed as follows:

$$f = Ap = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0.1 \\ 0.4 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.4 \\ 2.3 \end{pmatrix}.$$

- Average fitness:

$$\begin{aligned} \bar{f}(t) &= \sum_{j=1}^{3} p_i f_i(t) \\ &= p(Ap) = 2.29. \end{aligned}$$

- Fitness of species 1:

$$\begin{aligned} f_1 &= \sum_{j=1}^{3} p_j a_{1j} \\ &= (Ap)_1 = 1.8. \end{aligned}$$

Fitness

- The fitness of an individual is its expected reward when it encounters a random individual in the population.
- **Example**. Suppose

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} 0.1 \\ 0.4 \\ 0.5 \end{pmatrix}.$$

The fitness vector, f, can now be computed as follows:

$$f = Ap = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0.1 \\ 0.4 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.4 \\ 2.3 \end{pmatrix}.$$

- Average fitness:

$$\begin{aligned} \bar{f}(t) &= \sum_{j=1}^{3} p_i f_i(t) \\ &= p(Ap) = 2.29. \end{aligned}$$

- Fitness of species 1:

$$\begin{aligned} f_1 &= \sum_{j=1}^{3} p_j a_{1j} \\ &= (Ap)_1 = 1.8. \end{aligned}$$

So species 1 does worse than average.

Fitness

- The fitness of an individual is its expected reward when it encounters a random individual in the population.
- **Example**. Suppose

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} 0.1 \\ 0.4 \\ 0.5 \end{pmatrix}.$$

  The fitness vector, f, can now be computed as follows:

$$f = Ap = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0.1 \\ 0.4 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.4 \\ 2.3 \end{pmatrix}.$$

- Average fitness:

$$\bar{f}(t) = \sum_{j=1}^{3} p_i f_i(t) \\ = p(Ap) = 2.29.$$

- Fitness of species 1:

$$f_1 = \sum_{j=1}^{3} p_j a_{1j} \\ = (Ap)_1 = 1.8.$$

  So species 1 does worse than average.
- Species 2 and 3 have fitness 2.4 and 2.3, respectively.

The continuous replicator equation

The continuous replicator equation

The continuous replicator equation has an extremely intuitive reading:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t)\,],$$

where $\dot{p}_i(t)$ is shorthand for the change of p_i in time:

$$\dot{p}_i(t) = p_i'(t) = dp_i(t)/dt.$$

The continuous replicator equation

The continuous replicator equation has an extremely intuitive reading:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t)\,],$$

where $\dot{p}_i(t)$ is shorthand for the change of p_i in time:

$$\dot{p}_i(t) = p_i'(t) = dp_i(t)/dt.$$

Example 1. Suppose the proportion of species 7 at time t is $p_7(t) = 0.2$

The continuous replicator equation

The continuous replicator equation has an extremely intuitive reading:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t) \,],$$

where $\dot{p}_i(t)$ is shorthand for the change of p_i in time:

$$\dot{p}_i(t) = p_i'(t) = dp_i(t)/dt.$$

Example 1. Suppose the proportion of species 7 at time t is $p_7(t) = 0.2$, the fitness of species 7 at time t is $f_7(t) = 6$

The continuous replicator equation

The continuous replicator equation has an extremely intuitive reading:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t) \,],$$

where $\dot{p}_i(t)$ is shorthand for the change of p_i in time:

$$\dot{p}_i(t) = p_i'(t) = dp_i(t)/dt.$$

Example 1. Suppose the proportion of species 7 at time t is $p_7(t) = 0.2$, the fitness of species 7 at time t is $f_7(t) = 6$, and the average fitness at time t is $\bar{f}(t) = 9$.

The continuous replicator equation

The continuous replicator equation has an extremely intuitive reading:

$$\dot{p}_i(t) = p_i(t)[\ f_i(t) - \bar{f}(t)\ ],$$

where $\dot{p}_i(t)$ is shorthand for the change of p_i in time:

$$\dot{p}_i(t) = p_i'(t) = dp_i(t)/dt.$$

Example 1. Suppose the proportion of species 7 at time t is $p_7(t) = 0.2$, the fitness of species 7 at time t is $f_7(t) = 6$, and the average fitness at time t is $\bar{f}(t) = 9$. How fast does p_7 grow at time t?

The continuous replicator equation

The continuous replicator equation has an extremely intuitive reading:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t)\,],$$

where $\dot{p}_i(t)$ is shorthand for the change of p_i in time:

$$\dot{p}_i(t) = p_i'(t) = dp_i(t)/dt.$$

Example 1. Suppose the proportion of species 7 at time t is $p_7(t) = 0.2$, the fitness of species 7 at time t is $f_7(t) = 6$, and the average fitness at time t is $\bar{f}(t) = 9$. How fast does p_7 grow at time t?

Answer. $\dot{p}_7(t)$

The continuous replicator equation

The continuous replicator equation has an extremely intuitive reading:

$$\dot{p}_i(t) = p_i(t)[\ f_i(t) - \bar{f}(t)\ ],$$

where $\dot{p}_i(t)$ is shorthand for the change of p_i in time:

$$\dot{p}_i(t) = p_i'(t) = dp_i(t)/dt.$$

Example 1. Suppose the proportion of species 7 at time t is $p_7(t) = 0.2$, the fitness of species 7 at time t is $f_7(t) = 6$, and the average fitness at time t is $\bar{f}(t) = 9$. How fast does p_7 grow at time t?

Answer. $\dot{p}_7(t) = p_7(t)[f_7(t) - \bar{f}(t)]$

The continuous replicator equation

The continuous replicator equation has an extremely intuitive reading:

$$\dot{p}_i(t) = p_i(t)[\ f_i(t) - \bar{f}(t)\ ],$$

where $\dot{p}_i(t)$ is shorthand for the change of p_i in time:

$$\dot{p}_i(t) = p_i'(t) = dp_i(t)/dt.$$

Example 1. Suppose the proportion of species 7 at time t is $p_7(t) = 0.2$, the fitness of species 7 at time t is $f_7(t) = 6$, and the average fitness at time t is $\bar{f}(t) = 9$. How fast does p_7 grow at time t?

Answer. $\dot{p}_7(t) = p_7(t)[f_7(t) - \bar{f}(t)] = 0.2(6-9)$

The continuous replicator equation

The continuous replicator equation has an extremely intuitive reading:

$$\dot{p}_i(t) = p_i(t)[\ f_i(t) - \bar{f}(t)\ ],$$

where $\dot{p}_i(t)$ is shorthand for the change of p_i in time:

$$\dot{p}_i(t) = p_i'(t) = dp_i(t)/dt.$$

Example 1. Suppose the proportion of species 7 at time t is $p_7(t) = 0.2$, the fitness of species 7 at time t is $f_7(t) = 6$, and the average fitness at time t is $\bar{f}(t) = 9$. How fast does p_7 grow at time t?

Answer. $\dot{p}_7(t) = p_7(t)[f_7(t) - \bar{f}(t)] = 0.2(6 - 9) = -0.6$. □

The continuous replicator equation

The continuous replicator equation has an extremely intuitive reading:

$$\dot{p}_i(t) = p_i(t)[\ f_i(t) - \bar{f}(t)\ ],$$

where $\dot{p}_i(t)$ is shorthand for the change of p_i in time:

$$\dot{p}_i(t) = p_i'(t) = dp_i(t)/dt.$$

Example 1. Suppose the proportion of species 7 at time t is $p_7(t) = 0.2$, the fitness of species 7 at time t is $f_7(t) = 6$, and the average fitness at time t is $\bar{f}(t) = 9$. How fast does p_7 grow at time t?

Answer. $\dot{p}_7(t) = p_7(t)[f_7(t) - \bar{f}(t)] = 0.2(6 - 9) = -0.6.$ □

Example 2. Suppose $p_5(t) = 0.2$, $f_5(t) = 6$, and $\bar{f}(t) = 4$. Same question.

The continuous replicator equation

The continuous replicator equation has an extremely intuitive reading:

$$\dot{p}_i(t) = p_i(t)[\ f_i(t) - \bar{f}(t)\ ],$$

where $\dot{p}_i(t)$ is shorthand for the change of p_i in time:

$$\dot{p}_i(t) = p_i'(t) = dp_i(t)/dt.$$

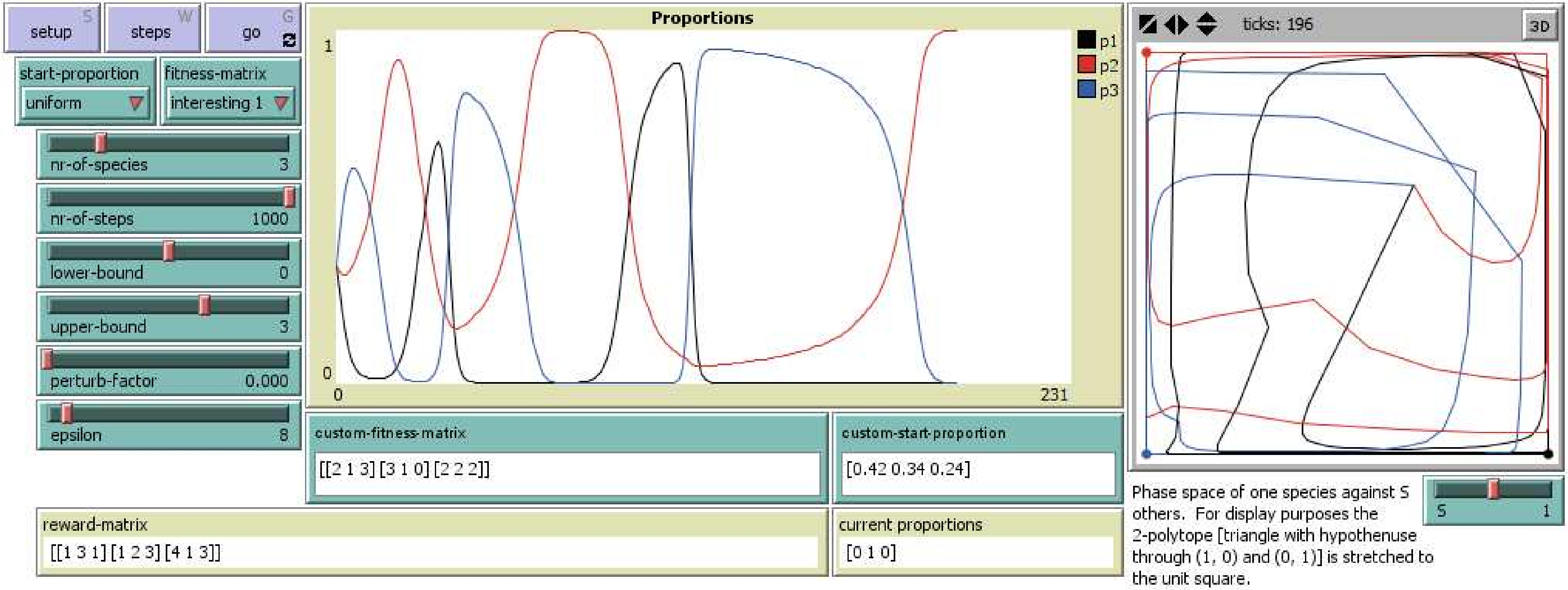
Example 1. Suppose the proportion of species 7 at time t is $p_7(t) = 0.2$, the fitness of species 7 at time t is $f_7(t) = 6$, and the average fitness at time t is $\bar{f}(t) = 9$. How fast does p_7 grow at time t?

Answer. $\dot{p}_7(t) = p_7(t)[f_7(t) - \bar{f}(t)] = 0.2(6 - 9) = -0.6$. $\square$

Example 2. Suppose $p_5(t) = 0.2$, $f_5(t) = 6$, and $\bar{f}(t) = 4$. Same question.

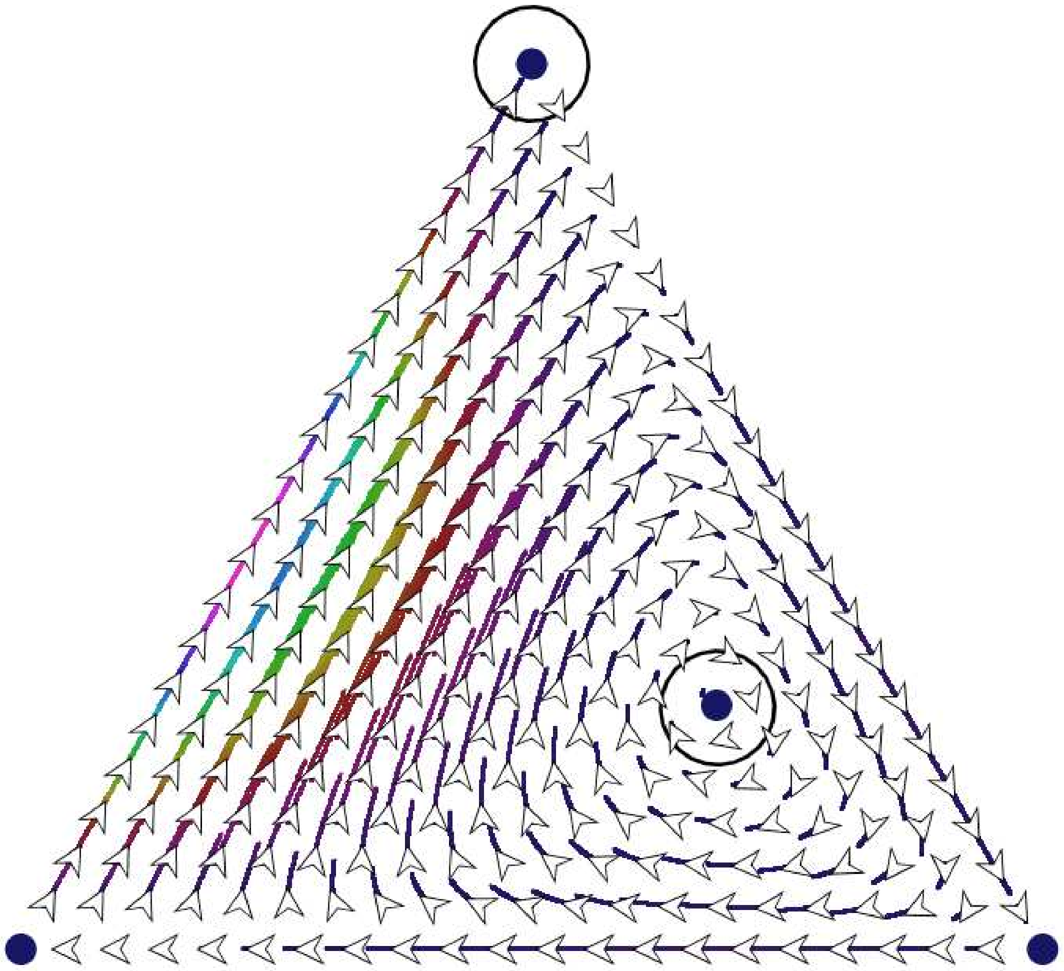
Answer. $\dot{p}_5(t) = p_5(t)[f_5(t) - \bar{f}(t)] = 0.2(6 - 4) = 0.4$. $\square$

The dynamics of the replicator equation



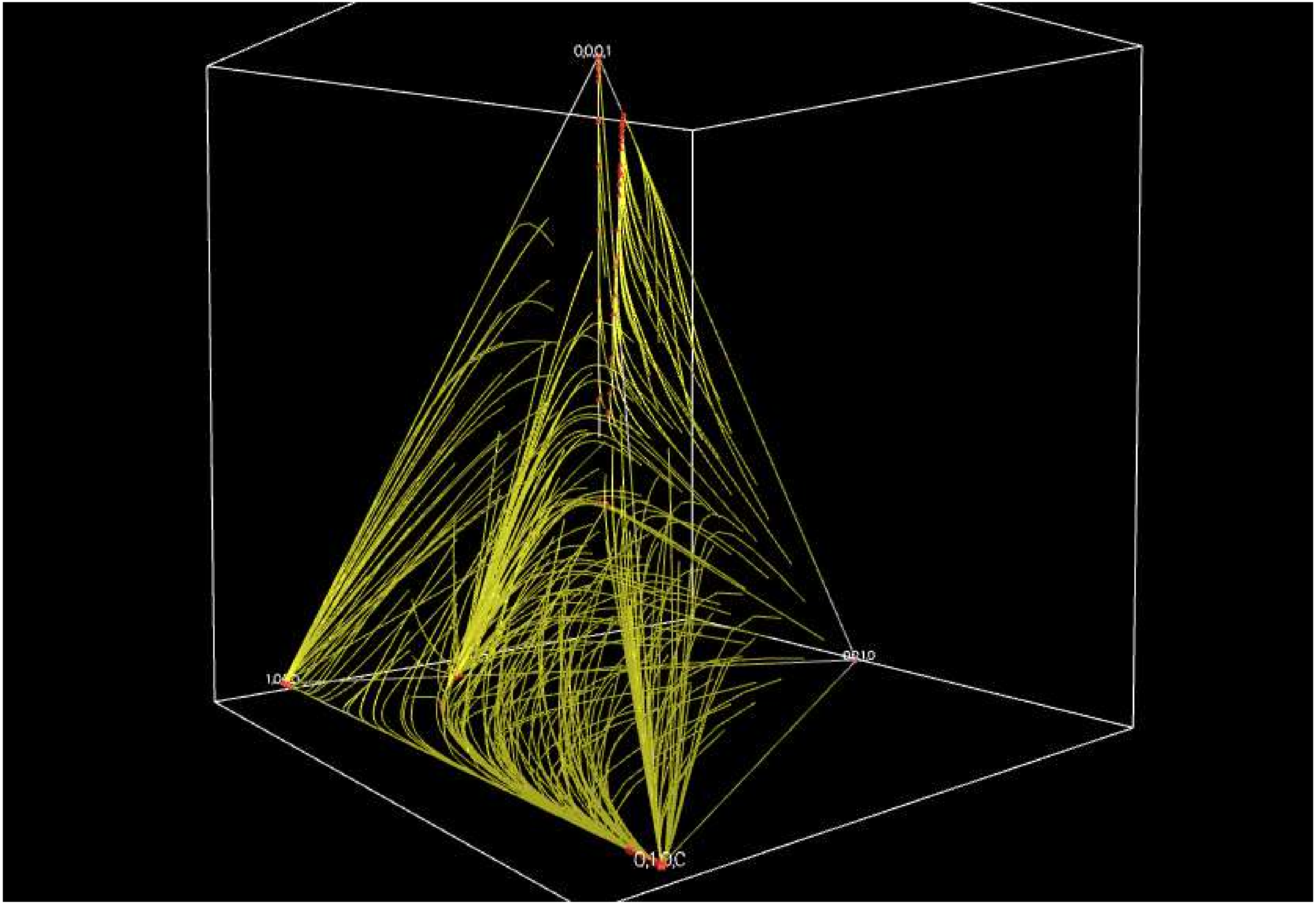
Relative score matrix $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix}$, start proportions $p = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$.

Phase space of the replicator on the previous page



Circled rest points indicate Nash equilibria of the score-matrix, interpreted as the payoff matrix of a symmetric game in normal form.

A replicator dynamic in a higher dimension



Rest point, stable point, asymptotically stable point

The continuous replicator equation:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t)\,]$$

is a system of differential equations. We have

$$p = (p_1, \ldots, p_n)$$

Rest point, stable point, asymptotically stable point

The continuous replicator equation:

$$\dot{p}_i(t) = p_i(t)[\ f_i(t) - \bar{f}(t)\ ]$$

is a system of differential equations. We have

$$p = (p_1, \ldots, p_n) \in \Delta^n$$

Rest point, stable point, asymptotically stable point

The continuous replicator equation:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t)\,]$$

is a system of differential equations. We have

$$p = (p_1, \ldots, p_n) \in \Delta^n \quad \text{and} \quad \dot{p} = (\dot{p}_1, \ldots, \dot{p}_n)$$

Rest point, stable point, asymptotically stable point

The continuous replicator equation:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t)\,]$$

is a system of differential equations. We have

$$p = (p_1, \dots, p_n) \in \Delta^n \quad \text{and} \quad \dot{p} = (\dot{p}_1, \dots, \dot{p}_n) \in \mathbb{R}^n.$$

Rest point, stable point, asymptotically stable point

The continuous replicator equation:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t)\,]$$

is a system of differential equations. We have

$$p = (p_1, \ldots, p_n) \in \Delta^n \quad \text{and} \quad \dot{p} = (\dot{p}_1, \ldots, \dot{p}_n) \in \mathbb{R}^n.$$

Definitions:

Rest point, stable point, asymptotically stable point

The continuous replicator equation:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t)\,]$$

is a system of differential equations. We have

$$p = (p_1, \dots, p_n) \in \Delta^n \quad \text{and} \quad \dot{p} = (\dot{p}_1, \dots, \dot{p}_n) \in \mathbb{R}^n.$$

Definitions:

- p is called a rest point, if $\dot{p} = 0$.

Rest point, stable point, asymptotically stable point

The continuous replicator equation:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t)\,]$$

is a system of differential equations. We have

$$p = (p_1, \ldots, p_n) \in \Delta^n \quad \text{and} \quad \dot{p} = (\dot{p}_1, \ldots, \dot{p}_n) \in \mathbb{R}^n.$$

Definitions:

- p is called a rest point, if $\dot{p} = 0$. ("If at p, then stays at p".)

Rest point, stable point, asymptotically stable point

The continuous replicator equation:

$$\dot{p}_i(t) = p_i(t)[\ f_i(t) - \bar{f}(t)\ ]$$

is a system of differential equations. We have

$$p = (p_1, \ldots, p_n) \in \Delta^n \quad \text{and} \quad \dot{p} = (\dot{p}_1, \ldots, \dot{p}_n) \in \mathbb{R}^n.$$

Definitions:

- p is called a rest point, if $\dot{p} = 0$. ("If at p, then stays at p".)
- A rest point p is called (Lyapunov) stable if, for every neighborhood U of p, there is another neighborhood U' of p such that states in U', if iterated, remain within U.

Rest point, stable point, asymptotically stable point

The continuous replicator equation:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t) \,]$$

is a system of differential equations. We have

$$p = (p_1, \dots, p_n) \in \Delta^n \quad \text{and} \quad \dot{p} = (\dot{p}_1, \dots, \dot{p}_n) \in \mathbb{R}^n.$$

Definitions:

- p is called a rest point, if $\dot{p} = 0$. ("If at p, then stays at p".)
- A rest point p is called (Lyapunov) stable if, for every neighborhood U of p, there is another neighborhood U' of p such that states in U', if iterated, remain within U. ("If close to p, then always close to p.")

Rest point, stable point, asymptotically stable point

The continuous replicator equation:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t)\,]$$

is a system of differential equations. We have

$$p = (p_1, \ldots, p_n) \in \Delta^n \quad \text{and} \quad \dot{p} = (\dot{p}_1, \ldots, \dot{p}_n) \in \mathbb{R}^n.$$

Definitions:

- p is called a rest point, if $\dot{p} = 0$. ("If at p, then stays at p".)
- A rest point p is called (Lyapunov) stable if, for every neighborhood U of p, there is another neighborhood U' of p such that states in U', if iterated, remain within U. ("If close to p, then always close to p.")
- A rest point p is called asymptotically stable if p has a neighborhood U such that all proportion vectors in U, if iterated, converge to p.

Rest point, stable point, asymptotically stable point

The continuous replicator equation:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t) \,]$$

is a system of differential equations. We have

$$p = (p_1, \dots, p_n) \in \Delta^n \quad \text{and} \quad \dot{p} = (\dot{p}_1, \dots, \dot{p}_n) \in \mathbb{R}^n.$$

Definitions:

- p is called a rest point, if $\dot{p} = 0$. ("If at p, then stays at p".)
- A rest point p is called (Lyapunov) stable if, for every neighborhood U of p, there is another neighborhood U' of p such that states in U', if iterated, remain within U. ("If close to p, then always close to p.")
- A rest point p is called asymptotically stable if p has a neighborhood U such that all proportion vectors in U, if iterated, converge to p. ("if close to p, then convergence to p.").

Relation with Nash equilibria

If for all i: $f_i \leq \bar{f}$, then it must be that for all i: $f_i = \bar{f}$

Relation with Nash equilibria

If for all i: $f_i \leq \bar{f}$, then it must be that for all i: $f_i = \bar{f}$ (check!), which means we have a rest point. Such a rest point is called saturated.

Relation with Nash equilibria

If for all i: $f_i \leq \bar{f}$, then it must be that for all i: $f_i = \bar{f}$ (check!), which means we have a rest point. Such a rest point is called saturated.

Further, state p is a NE

$$\begin{aligned} &\Leftrightarrow \quad \forall q : q(Ap) \leq p(Ap) \\ &\Leftrightarrow \quad \forall q : qf \leq pf \\ &\Leftrightarrow \quad \forall q : q_1 f_1 + \cdots + q_n f_n \leq \bar{f}. \end{aligned}$$

Relation with Nash equilibria

If for all i: $f_i \leq \bar{f}$, then it must be that for all i: $f_i = \bar{f}$ (check!), which means we have a rest point. Such a rest point is called saturated.

Further, state p is a NE

$$\begin{aligned} &\Leftrightarrow \quad \forall q : q(Ap) \leq p(Ap) \\ &\Leftrightarrow \quad \forall q : qf \leq pf \\ &\Leftrightarrow \quad \forall q : q_1 f_1 + \cdots + q_n f_n \leq \bar{f}. \end{aligned}$$

- Nash equilibrium $\Leftrightarrow$ saturated rest point.

Relation with Nash equilibria

If for all i: $f_i \leq \bar{f}$, then it must be that for all i: $f_i = \bar{f}$ (check!), which means we have a rest point. Such a rest point is called saturated.

Further, state p is a NE

$$\begin{aligned} &\Leftrightarrow \quad \forall q : q(Ap) \leq p(Ap) \\ &\Leftrightarrow \quad \forall q : qf \leq pf \\ &\Leftrightarrow \quad \forall q : q_1 f_1 + \cdots + q_n f_n \leq \bar{f}. \end{aligned}$$

- Nash equilibrium $\Leftrightarrow$ saturated rest point.

  Proof.

Relation with Nash equilibria

If for all i: $f_i \leq \bar{f}$, then it must be that for all i: $f_i = \bar{f}$ (check!), which means we have a rest point. Such a rest point is called saturated.

Further, state p is a NE

$$\begin{aligned}
&\Leftrightarrow \quad \forall q : q(Ap) \leq p(Ap) \\
&\Leftrightarrow \quad \forall q : qf \leq pf \\
&\Leftrightarrow \quad \forall q : q_1 f_1 + \cdots + q_n f_n \leq \bar{f}.
\end{aligned}$$

- Nash equilibrium $\Leftrightarrow$ saturated rest point.

  Proof. $\Rightarrow$: take $q = e^i$.

Relation with Nash equilibria

If for all i: $f_i \leq \bar{f}$, then it must be that for all i: $f_i = \bar{f}$ (check!), which means we have a rest point. Such a rest point is called saturated.

Further, state p is a NE

$$
\begin{aligned}
&\Leftrightarrow \quad \forall q : q(Ap) \leq p(Ap) \\
&\Leftrightarrow \quad \forall q : qf \leq pf \\
&\Leftrightarrow \quad \forall q : q_1 f_1 + \cdots + q_n f_n \leq \bar{f}.
\end{aligned}
$$

- Nash equilibrium $\Leftrightarrow$ saturated rest point.

  Proof. $\Rightarrow$: take $q = e^i$. $\Leftarrow$: if for all i: $f_i \leq f$, then no convex combination of those f_i can supersede $\bar{f}$. $\square$

Relation with Nash equilibria

If for all i: $f_i \leq \bar{f}$, then it must be that for all i: $f_i = \bar{f}$ (check!), which means we have a rest point. Such a rest point is called saturated.

Further, state p is a NE

$$\begin{aligned} &\Leftrightarrow \quad \forall q : q(Ap) \leq p(Ap) \\ &\Leftrightarrow \quad \forall q : qf \leq pf \\ &\Leftrightarrow \quad \forall q : q_1 f_1 + \cdots + q_n f_n \leq \bar{f}. \end{aligned}$$

- Nash equilibrium $\Leftrightarrow$ saturated rest point.

  Proof. $\Rightarrow$: take $q = e^i$. $\Leftarrow$: if for all i: $f_i \leq f$, then no convex combination of those f_i can supersede $\bar{f}$. $\square$

- Nash equilibrium $\Rightarrow$ rest point.

Relation with Nash equilibria

If for all i: $f_i \leq \bar{f}$, then it must be that for all i: $f_i = \bar{f}$ (check!), which means we have a rest point. Such a rest point is called saturated.

Further, state p is a NE

$$\begin{aligned} &\Leftrightarrow \quad \forall q : q(Ap) \leq p(Ap) \\ &\Leftrightarrow \quad \forall q : qf \leq pf \\ &\Leftrightarrow \quad \forall q : q_1 f_1 + \cdots + q_n f_n \leq \bar{f}. \end{aligned}$$

- Nash equilibrium $\Leftrightarrow$ saturated rest point.

  Proof. $\Rightarrow$: take $q = e^i$. $\Leftarrow$: if for all i: $f_i \leq f$, then no convex combination of those f_i can supersede $\bar{f}$. $\square$

- Nash equilibrium $\Rightarrow$ rest point. (Now trivial.)

Relation with Nash equilibria

If for all i: $f_i \leq \bar{f}$, then it must be that for all i: $f_i = \bar{f}$ (check!), which means we have a rest point. Such a rest point is called saturated.

Further, state p is a NE

$$\begin{aligned} &\Leftrightarrow \quad \forall q : q(Ap) \leq p(Ap) \\ &\Leftrightarrow \quad \forall q : qf \leq pf \\ &\Leftrightarrow \quad \forall q : q_1 f_1 + \cdots + q_n f_n \leq \bar{f}. \end{aligned}$$

- Nash equilibrium $\Leftrightarrow$ saturated rest point.

  Proof. $\Rightarrow$: take $q = e^i$. $\Leftarrow$: if for all i: $f_i \leq f$, then no convex combination of those f_i can supersede $\bar{f}$. $\square$

- Nash equilibrium $\Rightarrow$ rest point. (Now trivial.)
- Fully mixed rest point $\Rightarrow$ Nash equilibrium. (Because fully mixed implies saturated.)

Relation with Nash equilibria

If for all i: $f_i \leq \bar{f}$, then it must be that for all i: $f_i = \bar{f}$ (check!), which means we have a rest point. Such a rest point is called saturated.

Further, state p is a NE

$$\begin{aligned} &\Leftrightarrow \quad \forall q : q(Ap) \leq p(Ap) \\ &\Leftrightarrow \quad \forall q : qf \leq pf \\ &\Leftrightarrow \quad \forall q : q_1 f_1 + \cdots + q_n f_n \leq \bar{f}. \end{aligned}$$

- Nash equilibrium $\Leftrightarrow$ saturated rest point.

  Proof. $\Rightarrow$: take $q = e^i$. $\Leftarrow$: if for all i: $f_i \leq f$, then no convex combination of those f_i can supersede $\bar{f}$. $\square$

- Nash equilibrium $\Rightarrow$ rest point. (Now trivial.)
- Fully mixed rest point $\Rightarrow$ Nash equilibrium. (Because fully mixed implies saturated.)
- Strict Nash equilibrium $\Rightarrow$ asymptotically stable.

Relation with Nash equilibria

If for all i: $f_i \leq \bar{f}$, then it must be that for all i: $f_i = \bar{f}$ (check!), which means we have a rest point. Such a rest point is called saturated.

Further, state p is a NE

$$
\begin{aligned}
&\Leftrightarrow \quad \forall q : q(Ap) \leq p(Ap) \\
&\Leftrightarrow \quad \forall q : qf \leq pf \\
&\Leftrightarrow \quad \forall q : q_1 f_1 + \cdots + q_n f_n \leq \bar{f}.
\end{aligned}
$$

- Nash equilibrium $\Leftrightarrow$ saturated rest point.

  Proof. $\Rightarrow$: take $q = e^i$. $\Leftarrow$: if for all i: $f_i \leq f$, then no convex combination of those f_i can supersede $\bar{f}$. $\square$

- Nash equilibrium $\Rightarrow$ rest point. (Now trivial.)
- Fully mixed rest point $\Rightarrow$ Nash equilibrium. (Because fully mixed implies saturated.)
- Strict Nash equilibrium $\Rightarrow$ asymptotically stable.
- Limit point in the interior of Δ^n $\Rightarrow$ Nash equilibrium.

Relation with Nash equilibria

If for all i: $f_i \leq \bar{f}$, then it must be that for all i: $f_i = \bar{f}$ (check!), which means we have a rest point. Such a rest point is called saturated.

Further, state p is a NE

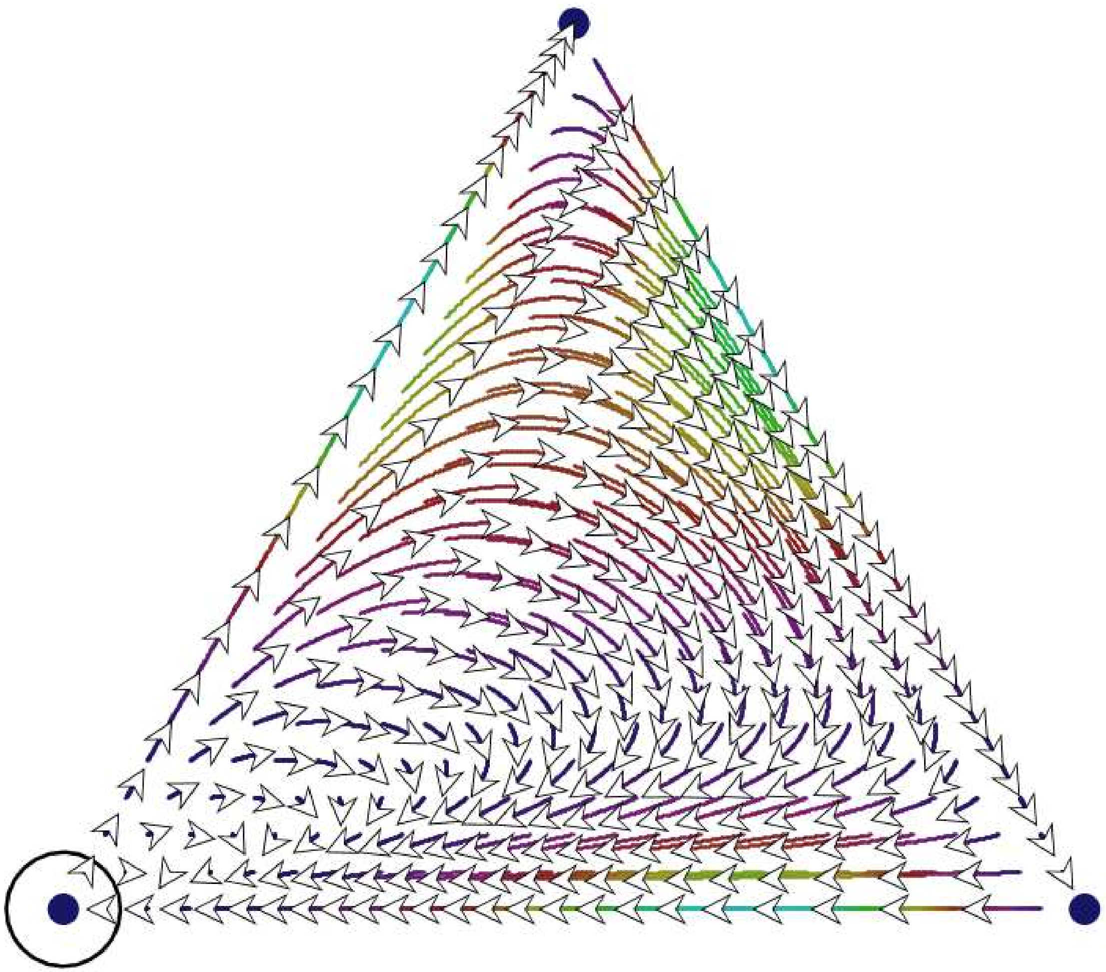
$$\begin{aligned} &\Leftrightarrow \quad \forall q : q(Ap) \leq p(Ap) \\ &\Leftrightarrow \quad \forall q : qf \leq pf \\ &\Leftrightarrow \quad \forall q : q_1 f_1 + \cdots + q_n f_n \leq \bar{f}. \end{aligned}$$

- Nash equilibrium $\Leftrightarrow$ saturated rest point.

  Proof. $\Rightarrow$: take $q = e^i$. $\Leftarrow$: if for all i: $f_i \leq f$, then no convex combination of those f_i can supersede $\bar{f}$. $\square$

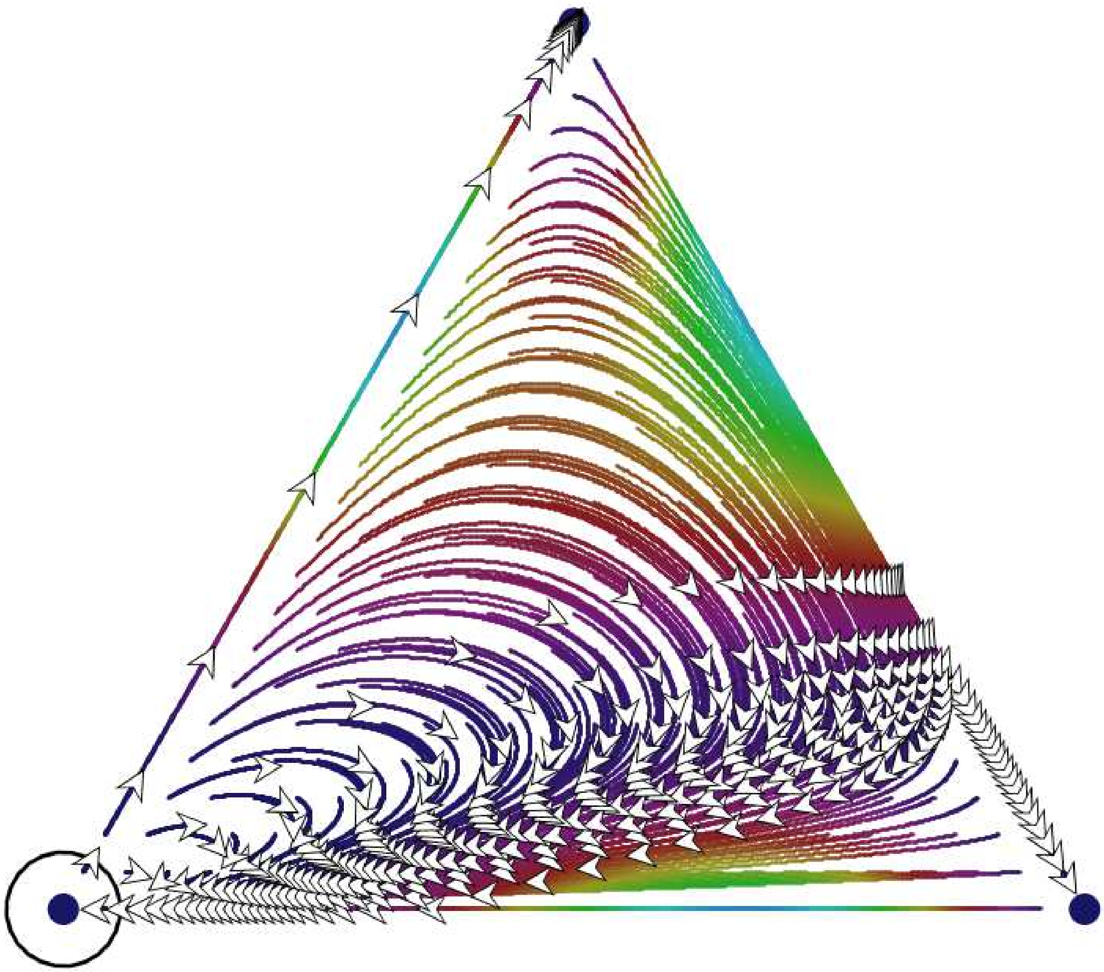
- Nash equilibrium $\Rightarrow$ rest point. (Now trivial.)
- Fully mixed rest point $\Rightarrow$ Nash equilibrium. (Because fully mixed implies saturated.)
- Strict Nash equilibrium $\Rightarrow$ asymptotically stable.
- Limit point in the interior of Δ^n $\Rightarrow$ Nash equilibrium.
- Asymptotically stable in the interior of Δ^n $\Rightarrow$ isolated trembling-hand perfect Nash equilibrium.

Not all Nash equilibria are Lyapunov stable



$(1, 0, 0)$ is Nash but not Lyapunov stable. (The picture is merely suggestive, since it only contains a few traces of the dynamics.)

Not all Nash equilibria are Lyapunov stable



$(1, 0, 0)$ is Nash but not Lyapunov stable. (The picture is merely suggestive, since it only contains a few traces of the dynamics.)

The discrete replicator equation

The discrete step equation

The discrete step equation

- The discrete step equation is given by

$$q_i(t+1) =_{Def} q_i(t)[1 + \beta + f_i(t)],$$

where 1 is the reproduction factor, β is the birth and death rate, and $f_i(t)$ indicates the percentage that is added / subtracted due to fitness.

The discrete step equation

- The discrete step equation is given by

$$q_i(t+1) =_{Def} q_i(t)[1+\beta+f_i(t)],$$

  where 1 is the reproduction factor, β is the birth and death rate, and $f_i(t)$ indicates the percentage that is added / subtracted due to fitness.

- To prevent negative proportions and sudden extermination, it is required that fitness, hence scores, must be defined such that $1+\beta+f_i(t) > 0$, for all t and i.

The discrete step equation

- The discrete step equation is given by

$$q_i(t+1) =_{Def} q_i(t)[1 + \beta + f_i(t)],$$

  where 1 is the reproduction factor, β is the birth and death rate, and $f_i(t)$ indicates the percentage that is added / subtracted due to fitness.

- To prevent negative proportions and sudden extermination, it is required that fitness, hence scores, must be defined such that $1 + \beta + f_i(t) > 0$, for all t and i. This requirement is often left implicit in the literature.

The discrete step equation

- The discrete step equation is given by

$$q_i(t+1) =_{Def} q_i(t)[1+\beta+f_i(t)],$$

  where 1 is the reproduction factor, β is the birth and death rate, and $f_i(t)$ indicates the percentage that is added / subtracted due to fitness.

- To prevent negative proportions and sudden extermination, it is required that fitness, hence scores, must be defined such that $1+\beta+f_i(t) > 0$, for all t and i. This requirement is often left implicit in the literature.

- The absolute growth of species i is

$$\begin{aligned} \Delta q_i(t) &= q_i(t+1) - q_i(t) \\ &= q_i(t)[1+\beta+f_i(t)] - q_i(t) = q_i(t)[\beta + f_i(t)]. \end{aligned}$$

The discrete replicator equation

The discrete replicator equation

- The discrete replicator equation:

$$p_i(t+1) = p_i(t)\frac{1+\beta+f_i(t)}{1+\beta+\bar{f}(t)}$$

The discrete replicator equation

- The discrete replicator equation:

$$p_i(t+1) = p_i(t)\frac{1+\beta+f_i(t)}{1+\beta+\bar{f}(t)}$$

- The idea of the discrete replicator equation: between two generations, an individual of type i produces

$$1+\beta+f_i(t)$$

  offspring. Here, β is the intrinsic birth and death ratio, and f_i is the fitness dependent birth ratio.

The discrete replicator equation

- The discrete replicator equation:

$$p_i(t+1) = p_i(t)\frac{1+\beta+f_i(t)}{1+\beta+\bar{f}(t)}$$

- The idea of the discrete replicator equation: between two generations, an individual of type i produces

$$1+\beta+f_i(t)$$

offspring. Here, β is the intrinsic birth and death ratio, and f_i is the fitness dependent birth ratio.

- The DRE follows from the discrete step equation:

$$q_i(t+1) =_{Def} q_i(t)[1+\beta+f_i(t)].$$

Derivation of the DRE from the DSE

$$
\begin{aligned}
p_i(t+1) &= \frac{q_i(t+1)}{\sum_{j=1}^n q_j(t+1)} = \frac{q_i(t)[1+\beta+f_i(t)]}{\sum_{j=1}^n q_j(t)[1+\beta+f_j(t)]} \\
&= \frac{\frac{1}{q(t)} q_i(t)[1+\beta+f_i(t)]}{\frac{1}{q(t)} \sum_{j=1}^n q_j(t)[1+\beta+f_j(t)]} \\
&= \frac{p_i(t)[1+\beta+f_i(t)]}{\sum_{j=1}^n p_j(t)[1+\beta+f_j(t)]} \\
&= \frac{p_i(t)[1+\beta+f_i(t)]}{\sum_{j=1}^n p_j(t) + \beta \sum_{j=1}^n p_j(t) + \sum_{j=1}^n p_j(t) f_j(t)]} \\
&= \frac{p_i(t)[1+\beta+f_i(t)]}{1+\beta+\bar{f}(t)} \\
&= p_i(t) \frac{1+\beta+f_i(t)}{1+\beta+\bar{f}(t)}.
\end{aligned}
$$

Properties of the DRE

Properties of the DRE

Claim. If a species is present, is was present and will remain present forever. Same for absent.

Proof. Just look at the discrete replicator equation:

$$p_i(t+1) = p_i(t)\frac{1+\beta+f_i(t)}{1+\beta+\bar{f}(t)}$$

and recall that $1+\beta+f_i(t) > 0$ for all t and i, hence $1+\beta+\bar{f}(t) > 0$ for all t. So all the p_i are always multiplied by a positive number.

Properties of the DRE

Claim. If a species is present, is was present and will remain present forever. Same for absent.

Proof. Just look at the discrete replicator equation:

$$p_i(t+1) = p_i(t)\frac{1+\beta+f_i(t)}{1+\beta+\bar{f}(t)}$$

and recall that $1+\beta+f_i(t) > 0$ for all t and i, hence $1+\beta+\bar{f}(t) > 0$ for all t. So all the p_i are always multiplied by a positive number.

- If p_i was 0 it remains 0.

Properties of the DRE

Claim. If a species is present, is was present and will remain present forever. Same for absent.

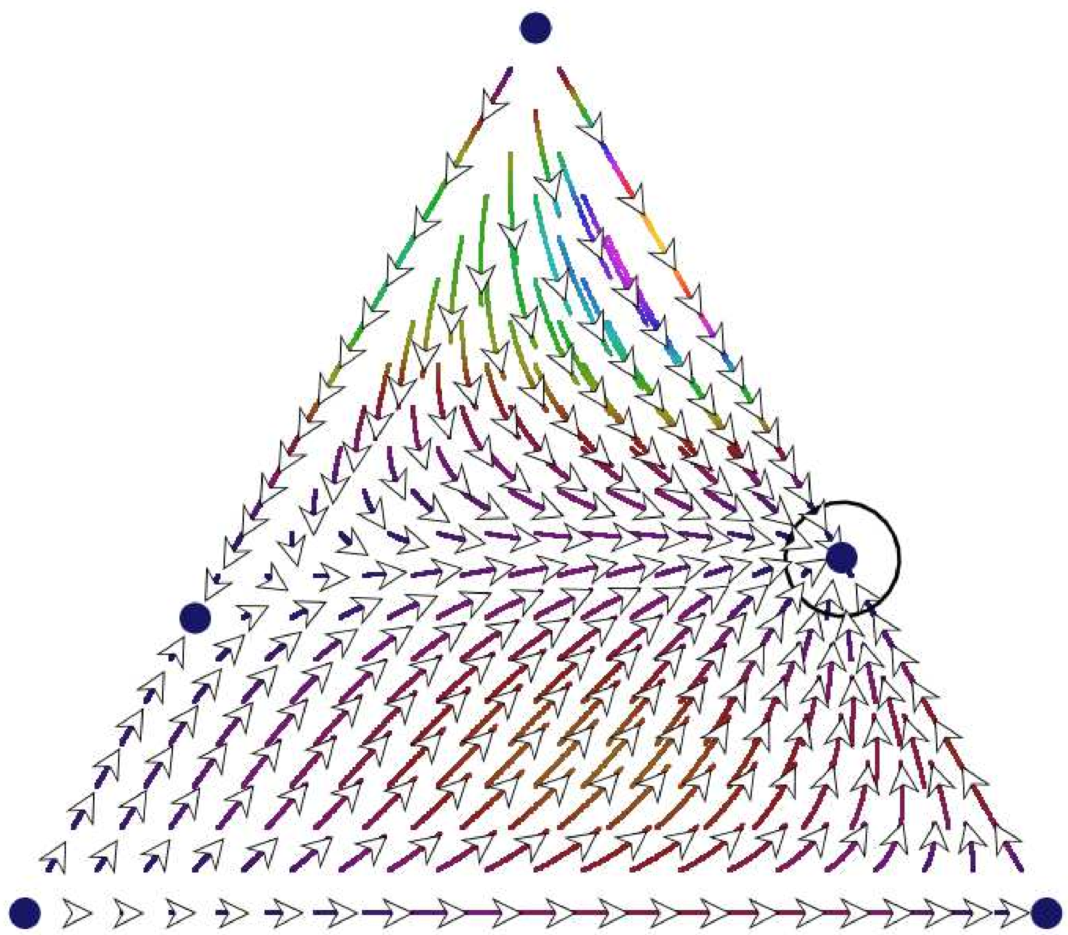
Proof. Just look at the discrete replicator equation:

$$p_i(t+1) = p_i(t)\frac{1+\beta+f_i(t)}{1+\beta+\bar{f}(t)}$$

and recall that $1+\beta+f_i(t) > 0$ for all t and i, hence $1+\beta+\bar{f}(t) > 0$ for all t. So all the p_i are always multiplied by a positive number.

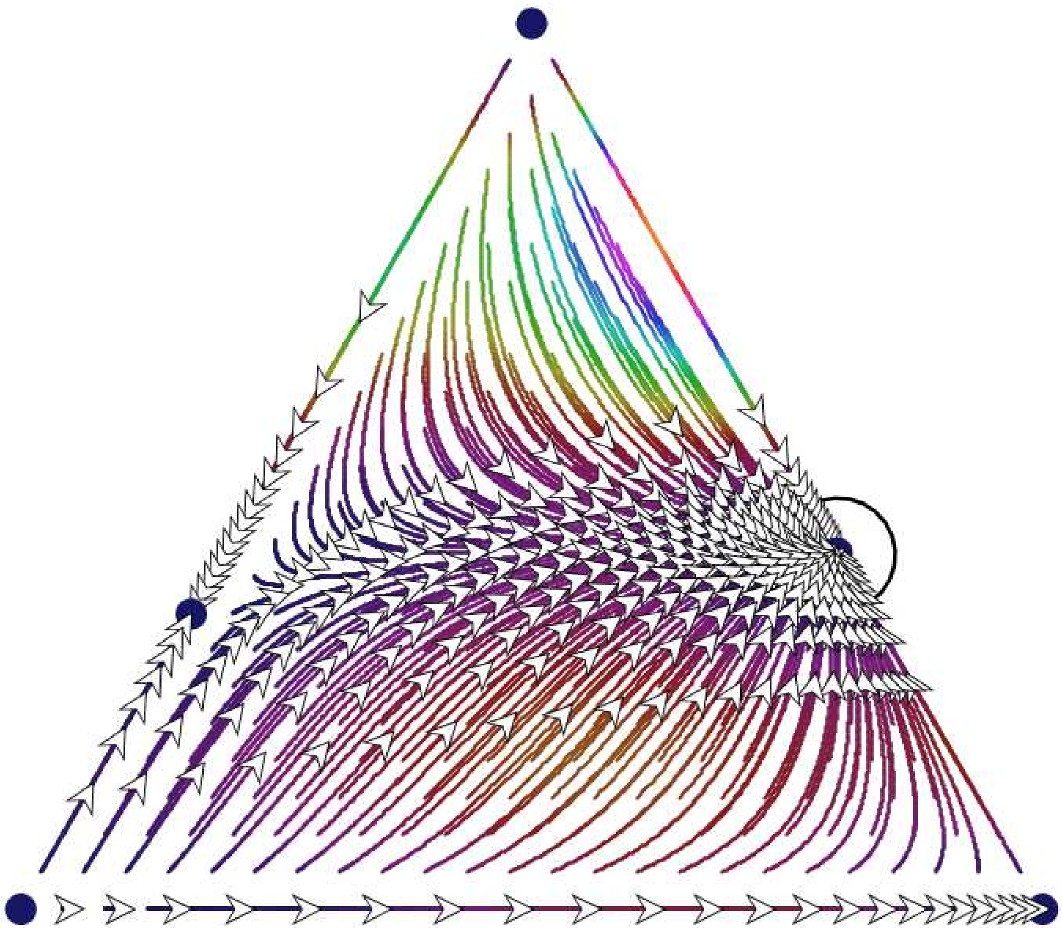
- If p_i was 0 it remains 0.
- If p_i was positive it remains positive.

If a species is absent, it will remain absent forever



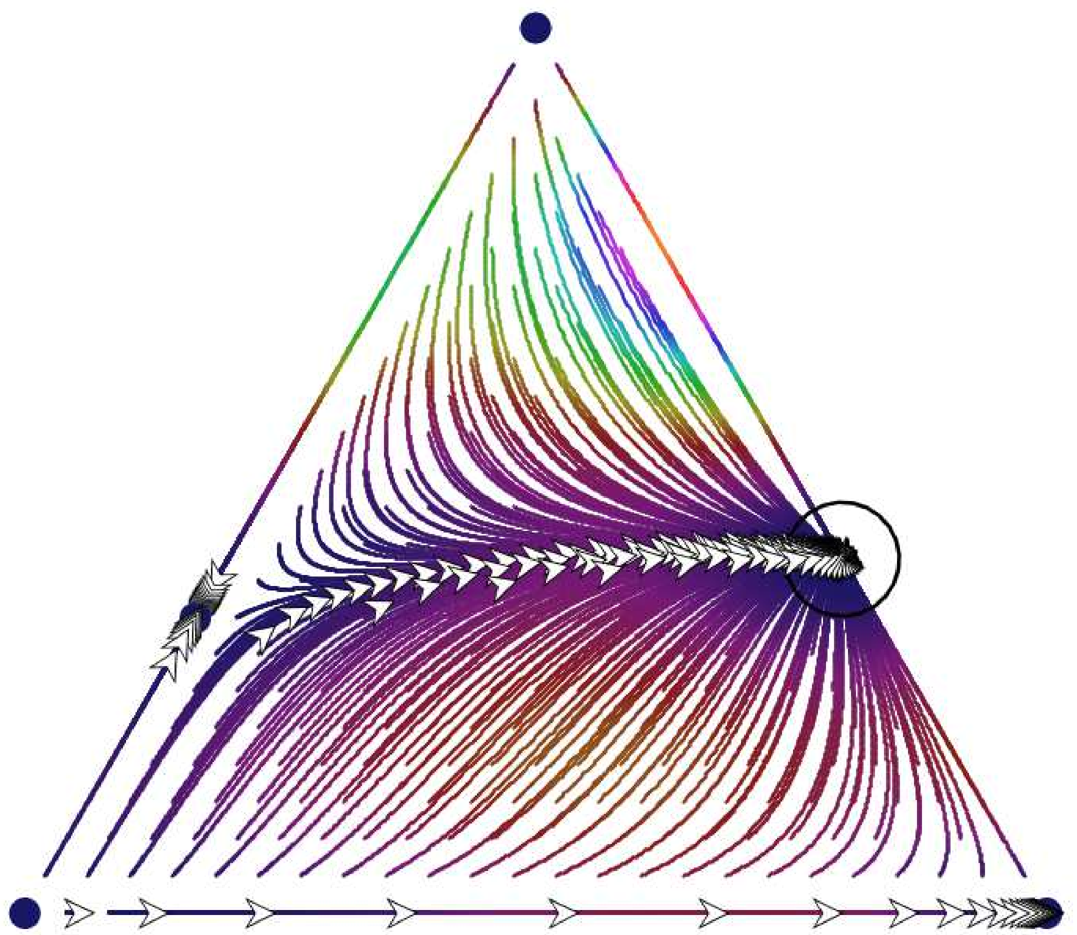
Phase space of a replicator. Notice that corners, edges, and the interior map into themselves. This is always the case.

If a species is absent, it will remain absent forever



Phase space of a replicator. Notice that corners, edges, and the interior map into themselves. This is always the case.

If a species is absent, it will remain absent forever



Phase space of a replicator. Notice that corners, edges, and the interior map into themselves. This is always the case.

Properties of the DRE

Properties of the DRE

Claim. Species i grows if and only if $p_i(t) > 0$ and it has above average fitness.

Properties of the DRE

Claim. Species i grows if and only if $p_i(t) > 0$ and it has above average fitness.

Proof.

$$
\begin{aligned}
& p_i(t+1) > p_i(t) \\
\Leftrightarrow \quad & p_i(t)\frac{1+\beta+f_i(t)}{1+\beta+\bar{f}(t)} > p_i(t) \\
\Leftrightarrow \quad & 1+\beta+f_i(t) > 1+\beta+\bar{f}(t), \qquad p_i(t) > 0 \\
\Leftrightarrow \quad & f_i(t) > \bar{f}(t), \qquad p_i(t) > 0
\end{aligned}
$$

Properties of the DRE

Claim. Species i grows if and only if $p_i(t) > 0$ and it has above average fitness.

Proof.

$$
\begin{aligned}
& p_i(t+1) > p_i(t) \\
\Leftrightarrow \quad & p_i(t)\frac{1+\beta+f_i(t)}{1+\beta+\bar{f}(t)} > p_i(t) \\
\Leftrightarrow \quad & 1+\beta+f_i(t) > 1+\beta+\bar{f}(t), \qquad p_i(t) > 0 \\
\Leftrightarrow \quad & f_i(t) > \bar{f}(t), \qquad p_i(t) > 0
\end{aligned}
$$

Question. What if β is large?

Properties of the DRE

Claim. Species i grows if and only if $p_i(t) > 0$ and it has above average fitness.

Proof.

$$
\begin{aligned}
& p_i(t+1) > p_i(t) \\
\Leftrightarrow \quad & p_i(t)\frac{1+\beta+f_i(t)}{1+\beta+\bar{f}(t)} > p_i(t) \\
\Leftrightarrow \quad & 1+\beta+f_i(t) > 1+\beta+\bar{f}(t), \qquad p_i(t) > 0 \\
\Leftrightarrow \quad & f_i(t) > \bar{f}(t), \qquad p_i(t) > 0
\end{aligned}
$$

Question. What if β is large? **Answer**. If β is large then the differences in growth among species is smaller, and the dynamics is slower ("bluer").

The continuous replicator equation

Derivation of the CRE from the DRE

Discrete step equation:

$$q_i(t+1) = q_i(t)[1 + \beta + f_i(t)]$$

Derivation of the CRE from the DRE

Discrete step equation:

$$q_i(t+1) = q_i(t)[1 + \beta + f_i(t)]$$

- Idea: reduce time steps $t = 1$ to smaller time steps $t = \delta$, where $0 \leq \delta \leq 1$.

Derivation of the CRE from the DRE

Discrete step equation:

$$q_i(t+1) = q_i(t)[1 + \beta + f_i(t)]$$

- Idea: reduce time steps $t = 1$ to smaller time steps $t = \delta$, where $0 \leq \delta \leq 1$.
- The idea is the following:

  Per small step $t = \delta$ the largest part $1 - \delta$ of species i remains unchanged, while a smaller part δ of species i does change:

$$q_i(t+\delta) = (1-\delta)\ \underbrace{q_i(t)}_{\text{remains}}\ + \delta\, \underbrace{q_i(t)(1 + \beta + f_i(t))}_{\text{changes}}.$$

Derivation of the CRE from the DRE

Discrete step equation:

$$q_i(t+1) = q_i(t)[1+\beta+f_i(t)]$$

- Idea: reduce time steps $t = 1$ to smaller time steps $t = \delta$, where $0 \leq \delta \leq 1$.
- The idea is the following:

  Per small step $t = \delta$ the largest part $1 - \delta$ of species i remains unchanged, while a smaller part δ of species i does change:

$$q_i(t+\delta) = (1-\delta)\ \underbrace{q_i(t)}_{\text{remains}}\ + \delta \underbrace{q_i(t)(1+\beta+f_i(t))}_{\text{changes}}.$$

- What if $\delta = 0$?

Derivation of the CRE from the DRE

Discrete step equation:

$$q_i(t+1) = q_i(t)[1+\beta+f_i(t)]$$

- Idea: reduce time steps $t = 1$ to smaller time steps $t = \delta$, where $0 \leq \delta \leq 1$.
- The idea is the following:

  Per small step $t = \delta$ the largest part $1-\delta$ of species i remains unchanged, while a smaller part δ of species i does change:

$$q_i(t+\delta) = (1-\delta)\ \underbrace{q_i(t)}_{\text{remains}}\ + \delta \underbrace{q_i(t)(1+\beta+f_i(t))}_{\text{changes}}.$$

- What if $\delta = 0$? What if $\delta = 1$?

Derivation of the CRE from the DRE

We have:

$$\frac{q_i(t+\delta) - q_i(t)}{\delta} = \frac{(1-\delta)q_i(t) + \delta q_i(t)(1+\beta+f_i(t)) - q_i(t)}{\delta}$$
$$= \dots$$
$$= q_i(\beta + f_i(t)).$$

So:

$$\dot{q}_i = \frac{dq_i(t)}{dt}$$
$$= \lim_{\delta \to 0} \frac{q_i(t+\delta) - q_i(t)}{\delta}$$
$$= \lim_{\delta \to 0} q_i(\beta + f_i(t))$$
$$= q_i(\beta + f_i(t)).$$

Derivation of the CRE from the DRE

$$
\begin{aligned}
p_i(t+\delta) &= \frac{q_i(t+\delta)}{\sum_{j=1}^n q_j(t+\delta)} \\
&= \frac{(1-\delta)q_i(t) + \delta q_i(t)(1+\beta+f_i(t))}{\sum_{j=1}^n \left[(1-\delta)q_j(t) + \delta q_j(t)(1+\beta+f_j(t))\right]} && (/q(t)) \\
&= \frac{(1-\delta)p_i(t) + \delta p_i(t)(1+\beta+f_i(t))}{\sum_{j=1}^n \left[(1-\delta)p_j(t) + \delta p_j(t)(1+\beta+f_j(t))\right]} && \text{(yields proportions)} \\
&= \frac{p_i(t)[1+\delta(\beta+f_i(t))]}{\sum_{j=1}^n p_j(t)[1+\delta(\beta+f_j(t))]} && (\sum p_i(t) = 1) \\
&= p_i(t)\frac{1+\delta\left((\beta+f_i(t)\right)}{1+\delta\left(\beta+\bar{f}(t)\right)}.
\end{aligned}
$$

Derivation of the CRE from the DRE

$$\begin{aligned}
p_i(t+\delta) &= \frac{q_i(t+\delta)}{\sum_{j=1}^n q_j(t+\delta)} \\
&= \frac{(1-\delta)q_i(t) + \delta q_i(t)(1+\beta+f_i(t))}{\sum_{j=1}^n \left[(1-\delta)q_j(t) + \delta q_j(t)(1+\beta+f_j(t))\right]} && (/q(t)) \\
&= \frac{(1-\delta)p_i(t) + \delta p_i(t)(1+\beta+f_i(t))}{\sum_{j=1}^n \left[(1-\delta)p_j(t) + \delta p_j(t)(1+\beta+f_j(t))\right]} && \text{(yields proportions)} \\
&= \frac{p_i(t)[1+\delta(\beta+f_i(t))]}{\sum_{j=1}^n p_j(t)[1+\delta(\beta+f_j(t))]} && (\textstyle\sum p_i(t) = 1) \\
&= p_i(t)\frac{1+\delta\left((\beta+f_i(t)\right)}{1+\delta\left(\beta+\bar{f}(t)\right)}.
\end{aligned}$$

- What if $\delta = 0$?

Derivation of the CRE from the DRE

$$
\begin{aligned}
p_i(t+\delta) &= \frac{q_i(t+\delta)}{\sum_{j=1}^n q_j(t+\delta)} \\
&= \frac{(1-\delta)q_i(t) + \delta q_i(t)(1+\beta+f_i(t))}{\sum_{j=1}^n \left[(1-\delta)q_j(t) + \delta q_j(t)(1+\beta+f_j(t))\right]} && (/q(t)) \\
&= \frac{(1-\delta)p_i(t) + \delta p_i(t)(1+\beta+f_i(t))}{\sum_{j=1}^n \left[(1-\delta)p_j(t) + \delta p_j(t)(1+\beta+f_j(t))\right]} && \text{(yields proportions)} \\
&= \frac{p_i(t)[1+\delta(\beta+f_i(t))]}{\sum_{j=1}^n p_j(t)[1+\delta(\beta+f_j(t))]} && (\textstyle\sum p_i(t) = 1) \\
&= p_i(t)\frac{1+\delta\left((\beta+f_i(t)\right)}{1+\delta\left(\beta+\bar{f}(t)\right)}.
\end{aligned}
$$

- What if $\delta = 0$? What if $\delta = 1$?

Derivation of the CRE from the DRE

Now $$\frac{p_i(t+\delta)-p_i(t)}{\delta} = \frac{p_i(t)\frac{1+\delta((\beta+f_i(t))}{1+\delta\left(\beta+\bar{f}(t)\right)} - p_i(t)}{\delta}$$

$$= p_i(t)\frac{\frac{1+\delta((\beta+f_i(t))}{1+\delta\left(\beta+\bar{f}(t)\right)} - 1}{\delta} \quad \text{multiply w. } 1+\delta\left(\beta+\bar{f}(t)\right)$$

$$= p_i(t)\frac{1+\delta\left((\beta+f_i(t)\right) - \left(1+\delta\left(\beta+\bar{f}(t)\right)\right)}{\delta\left(1+\delta\left(\beta+\bar{f}(t)\right)\right)}$$

$$= p_i(t)\frac{f_i(t)-\bar{f}(t)}{1+\delta(\beta+\bar{f}(t))}.$$

So

$$\dot{p}_i = \lim_{\delta\to 0}\frac{p_i(t+\delta)-p_i(t)}{\delta}$$

$$= \lim_{\delta\to 0} p_i(t)\frac{f_i(t)-\bar{f}(t)}{1+\delta(\beta+\bar{f}(t))} = p_i(t)\frac{f_i(t)-\bar{f}(t)}{1+0\cdot C} = p_i(t)[\, f_i(t)-\bar{f}(t)\,].$$

Calculating stationary points of the replicator

Finding stationary points of the replicator: example

Consider the replicator with

$$A = \begin{pmatrix} 6 & 1 & 6 \\ 4 & 10 & 1 \\ 8 & 5 & 1 \end{pmatrix} \text{ and } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Finding stationary points of the replicator: example

Consider the replicator with

$$A = \begin{pmatrix} 6 & 1 & 6 \\ 4 & 10 & 1 \\ 8 & 5 & 1 \end{pmatrix} \text{ and } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Stationary points (fixed points, rest points):

$$\begin{cases} p \in \Delta_2 \\ x((Ap)_x - p(Ap)) = 0 \\ y((Ap)_y - p(Ap)) = 0 \\ z((Ap)_z - p(Ap)) = 0. \end{cases}$$

Finding stationary points of the replicator: example

Consider the replicator with

$$A = \begin{pmatrix} 6 & 1 & 6 \\ 4 & 10 & 1 \\ 8 & 5 & 1 \end{pmatrix} \text{ and } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This is equivalent with

$$\begin{cases} (x, y, z) \in \Delta_2 \\ \\ \\ \end{cases}$$

Stationary points (fixed points, rest points):

$$\begin{cases} p \in \Delta_2 \\ x((Ap)_x - p(Ap)) = 0 \\ y((Ap)_y - p(Ap)) = 0 \\ z((Ap)_z - p(Ap)) = 0. \end{cases}$$

Finding stationary points of the replicator: example

Consider the replicator with

$$A = \begin{pmatrix} 6 & 1 & 6 \\ 4 & 10 & 1 \\ 8 & 5 & 1 \end{pmatrix} \text{ and } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Stationary points (fixed points, rest points):

$$\begin{cases} p \in \Delta_2 \\ x((Ap)_x - p(Ap)) = 0 \\ y((Ap)_y - p(Ap)) = 0 \\ z((Ap)_z - p(Ap)) = 0. \end{cases}$$

This is equivalent with

$$\begin{cases} (x, y, z) \in \Delta_2, \text{ i.e., } x, y, z \in [0, 1] \text{ and } x + y + z = 1 \\ \\ \\ \end{cases}$$

Finding stationary points of the replicator: example

Consider the replicator with

$$A = \begin{pmatrix} 6 & 1 & 6 \\ 4 & 10 & 1 \\ 8 & 5 & 1 \end{pmatrix} \text{ and } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Stationary points (fixed points, rest points):

$$\begin{cases} p \in \Delta_2 \\ x((Ap)_x - p(Ap)) = 0 \\ y((Ap)_y - p(Ap)) = 0 \\ z((Ap)_z - p(Ap)) = 0. \end{cases}$$

This is equivalent with

$$\begin{cases} (x,y,z) \in \Delta_2, \text{ i.e., } x,y,z \in [0,1] \text{ and } x+y+z=1 \\ x = 0 \\ \\ \end{cases}$$

Finding stationary points of the replicator: example

Consider the replicator with

$$A = \begin{pmatrix} 6 & 1 & 6 \\ 4 & 10 & 1 \\ 8 & 5 & 1 \end{pmatrix} \text{ and } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Stationary points (fixed points, rest points):

$$\begin{cases} p \in \Delta_2 \\ x((Ap)_x - p(Ap)) = 0 \\ y((Ap)_y - p(Ap)) = 0 \\ z((Ap)_z - p(Ap)) = 0. \end{cases}$$

This is equivalent with

$$\begin{cases} (x, y, z) \in \Delta_2, \text{ i.e., } x, y, z \in [0, 1] \text{ and } x + y + z = 1 \\ x = 0 \text{ or } 6x - 6x^2 + y - 5xy - 10y^2 + 6z - 14xz - 6yz - z^2 = 0 \\ \\ \end{cases}$$

Finding stationary points of the replicator: example

Consider the replicator with

$$A = \begin{pmatrix} 6 & 1 & 6 \\ 4 & 10 & 1 \\ 8 & 5 & 1 \end{pmatrix} \text{ and } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Stationary points (fixed points, rest points):

$$\begin{cases} p \in \Delta_2 \\ x((Ap)_x - p(Ap)) = 0 \\ y((Ap)_y - p(Ap)) = 0 \\ z((Ap)_z - p(Ap)) = 0. \end{cases}$$

This is equivalent with

$$\begin{cases} (x,y,z) \in \Delta_2, \text{ i.e., } x,y,z \in [0,1] \text{ and } x+y+z=1 \\ x = 0 \text{ or } 6x - 6x^2 + y - 5xy - 10y^2 + 6z - 14xz - 6yz - z^2 = 0 \\ y = 0 \\ \end{cases}$$

Finding stationary points of the replicator: example

Consider the replicator with

$$A = \begin{pmatrix} 6 & 1 & 6 \\ 4 & 10 & 1 \\ 8 & 5 & 1 \end{pmatrix} \text{ and } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Stationary points (fixed points, rest points):

$$\begin{cases} p \in \Delta_2 \\ x((Ap)_x - p(Ap)) = 0 \\ y((Ap)_y - p(Ap)) = 0 \\ z((Ap)_z - p(Ap)) = 0. \end{cases}$$

This is equivalent with

$$\begin{cases} (x, y, z) \in \Delta_2, \text{ i.e., } x, y, z \in [0, 1] \text{ and } x + y + z = 1 \\ x = 0 \text{ or } 6x - 6x^2 + y - 5xy - 10y^2 + 6z - 14xz - 6yz - z^2 = 0 \\ y = 0 \text{ or } 4x - 6x^2 + 10y - 5xy - 10y^2 + z - 14xz - 6yz - z^2 = 0 \\ \end{cases}$$

Finding stationary points of the replicator: example

Consider the replicator with

$$A = \begin{pmatrix} 6 & 1 & 6 \\ 4 & 10 & 1 \\ 8 & 5 & 1 \end{pmatrix} \text{ and } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Stationary points (fixed points, rest points):

$$\begin{cases} p \in \Delta_2 \\ x((Ap)_x - p(Ap)) = 0 \\ y((Ap)_y - p(Ap)) = 0 \\ z((Ap)_z - p(Ap)) = 0. \end{cases}$$

This is equivalent with

$$\begin{cases} (x, y, z) \in \Delta_2, \text{ i.e., } x, y, z \in [0, 1] \text{ and } x + y + z = 1 \\ x = 0 \text{ or } 6x - 6x^2 + y - 5xy - 10y^2 + 6z - 14xz - 6yz - z^2 = 0 \\ y = 0 \text{ or } 4x - 6x^2 + 10y - 5xy - 10y^2 + z - 14xz - 6yz - z^2 = 0 \\ z = 0 \end{cases}$$

Finding stationary points of the replicator: example

Consider the replicator with

$$A = \begin{pmatrix} 6 & 1 & 6 \\ 4 & 10 & 1 \\ 8 & 5 & 1 \end{pmatrix} \text{ and } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Stationary points (fixed points, rest points):

$$\begin{cases} p \in \Delta_2 \\ x((Ap)_x - p(Ap)) = 0 \\ y((Ap)_y - p(Ap)) = 0 \\ z((Ap)_z - p(Ap)) = 0. \end{cases}$$

This is equivalent with

$$\begin{cases} (x, y, z) \in \Delta_2, \text{ i.e., } x, y, z \in [0, 1] \text{ and } x + y + z = 1 \\ x = 0 \text{ or } 6x - 6x^2 + y - 5xy - 10y^2 + 6z - 14xz - 6yz - z^2 = 0 \\ y = 0 \text{ or } 4x - 6x^2 + 10y - 5xy - 10y^2 + z - 14xz - 6yz - z^2 = 0 \\ z = 0 \text{ or } 8x - 6x^2 + 5y - 5xy - 10y^2 + z - 14xz - 6yz - z^2 = 0 \end{cases}$$

Finding stationary points of the replicator: example

Consider the replicator with

$$A = \begin{pmatrix} 6 & 1 & 6 \\ 4 & 10 & 1 \\ 8 & 5 & 1 \end{pmatrix} \text{ and } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Stationary points (fixed points, rest points):

$$\begin{cases} p \in \Delta_2 \\ x((Ap)_x - p(Ap)) = 0 \\ y((Ap)_y - p(Ap)) = 0 \\ z((Ap)_z - p(Ap)) = 0. \end{cases}$$

This is equivalent with

$$\begin{cases} (x, y, z) \in \Delta_2, \text{ i.e., } x, y, z \in [0, 1] \text{ and } x + y + z = 1 \\ x = 0 \text{ or } 6x - 6x^2 + y - 5xy - 10y^2 + 6z - 14xz - 6yz - z^2 = 0 \\ y = 0 \text{ or } 4x - 6x^2 + 10y - 5xy - 10y^2 + z - 14xz - 6yz - z^2 = 0 \\ z = 0 \text{ or } 8x - 6x^2 + 5y - 5xy - 10y^2 + z - 14xz - 6yz - z^2 = 0 \end{cases}$$

Solve with Maple / Mathematica / SciPy / ...

Finding stationary points of the replicator: example

Consider the replicator with

$$A = \begin{pmatrix} 6 & 1 & 6 \\ 4 & 10 & 1 \\ 8 & 5 & 1 \end{pmatrix} \text{ and } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Stationary points (fixed points, rest points):

$$\begin{cases} p \in \Delta_2 \\ x((Ap)_x - p(Ap)) = 0 \\ y((Ap)_y - p(Ap)) = 0 \\ z((Ap)_z - p(Ap)) = 0. \end{cases}$$

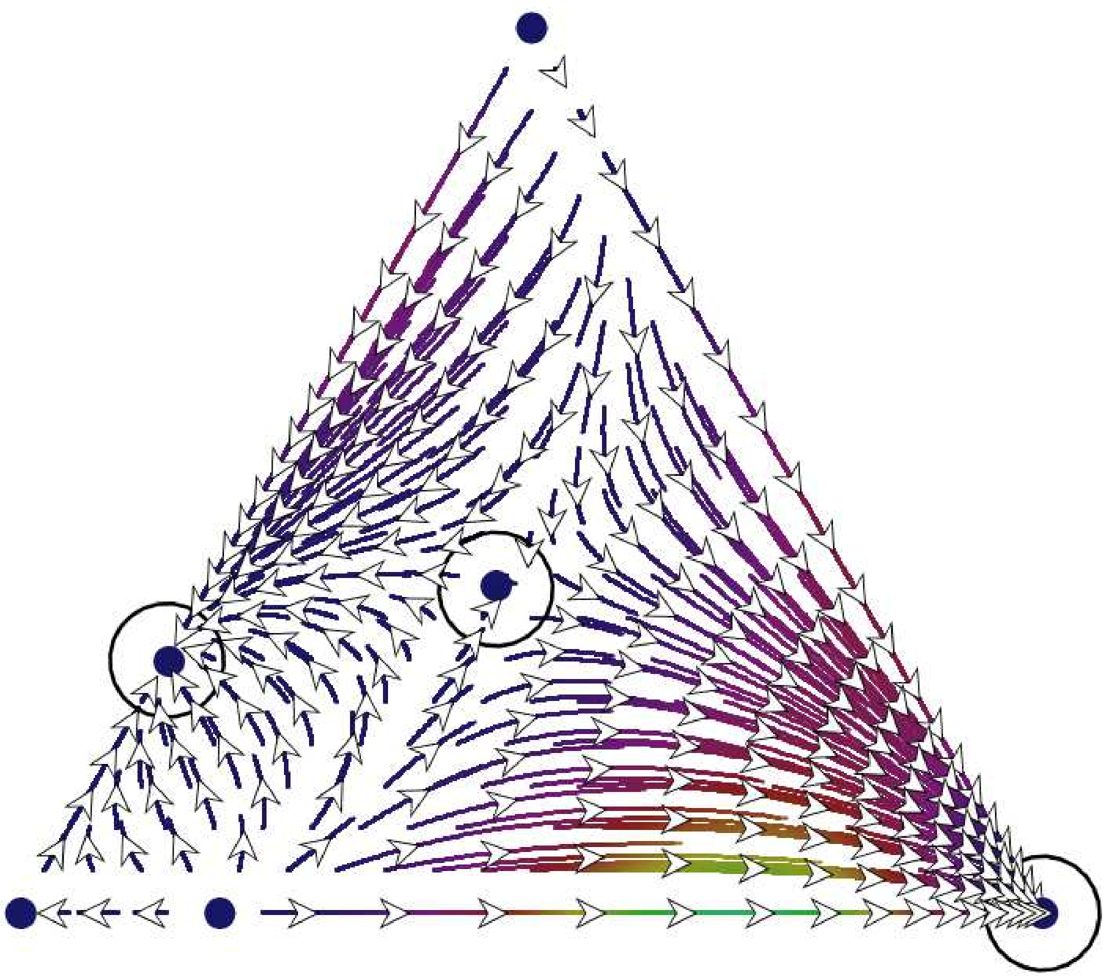
This is equivalent with

$$\begin{cases} (x, y, z) \in \Delta_2, \text{ i.e., } x, y, z \in [0, 1] \text{ and } x + y + z = 1 \\ x = 0 \text{ or } 6x - 6x^2 + y - 5xy - 10y^2 + 6z - 14xz - 6yz - z^2 = 0 \\ y = 0 \text{ or } 4x - 6x^2 + 10y - 5xy - 10y^2 + z - 14xz - 6yz - z^2 = 0 \\ z = 0 \text{ or } 8x - 6x^2 + 5y - 5xy - 10y^2 + z - 14xz - 6yz - z^2 = 0 \end{cases}$$

Solve with Maple / Mathematica / SciPy / ... (Nash equilibria are blue):

$$\left\{ (1,0,0), (0,1,0), (0,0,1), \left(\frac{25}{71}, \frac{20}{71}, \frac{26}{71}\right), \left(\frac{5}{7}, 0, \frac{2}{7}\right), \left(\frac{9}{11}, \frac{2}{11}, 0\right) \right\}.$$

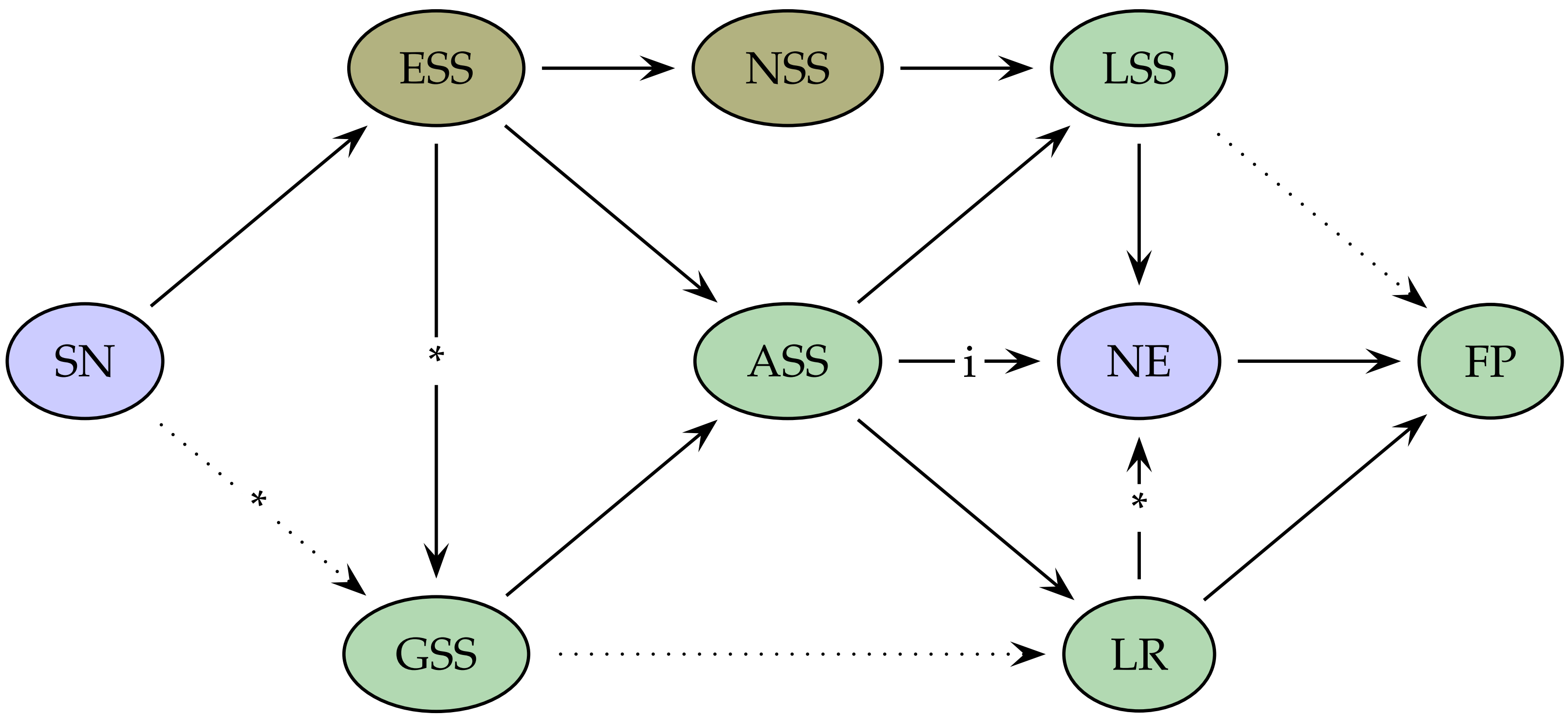
Finding stationary points of the replicator: example



Phase space of the replicator as discussed. Circled rest points indicate Nash equilibria in the corresponding symmetric game.

Summary

Implications



SN = strict Nash, ESS - evol'y stable strategy, GSS = glob'y stable state, ASS = asymp'y stable state, NSS = neutrally stable strategy, LR = limit of replicator, LSS = Lyapunov stable state, FP = fixed point, * = only if fully mixed, i = isolated NE. Dotted: indirect implication.
Blue: game theory; olive: evolutionary game theory; green: the replicator dynamic.

Justifications of the implications

- SN $\Rightarrow$ ESS: cf. slides evolutionary games and, e.g., Th 7.7.12 of Sh&LB.
- ESS $\Rightarrow$ NSS: cf. slides evolutionary games and, e.g., Game Theory Evolving (2nd ed.) by H. Gintis.
- ESS $\Rightarrow$ NE: cf. slides evolutionary games and, e.g., Sh&LB Th 7.7.11.
- ESS $\Rightarrow_*$ GSS: cf., e.g., Th. 12.7 Gintis.
- ESS $\Rightarrow$ ASS: cf., e.g., Th. 7.7.13 Sh&LB, Th. 12.7 Gintis, Sec. 3.5 (begin) of Evol. Game Theory by J.G. Weibull.
- NSS $\Rightarrow$ LSS: cf. Sec. 3.5 Weibull.
- GSS $\Rightarrow$ ASS: by definition of the two concepts.
- ASS $\Rightarrow$ LSS: by definition of the two concepts.
- ASS $\Rightarrow$ LR: by definition of the two concepts.
- ASS $\Rightarrow_i$ NE: Th 7.7.8 Sh&LB, Th. 12.6 Gintis.
- LSS $\Rightarrow$ NE: Th 7.7.6 Sh&LB, 7.2.1(c) Hofbauer & Sigmund.
- LR $\Rightarrow_*$ NE: Th. 7.2.1(b) H&S.
- NE $\Rightarrow$ FP: Th. 7.2.1(a) H&S, Th 7.7.5 Sh&LB, Th. 12.6 Gintis.
- LR $\Rightarrow$ FP: Ch. 6 Weibull.

Sources referred to

Weibull, J. W. (1997). Evolutionary game theory. MIT press.

Hofbauer, J., & Sigmund, K. (1998). Evolutionary games and population dynamics. Cambridge university press.

Gintis, H. (2001). Game theory evolving: A problem-centered introduction to modeling strategic interaction (2nd ed.). Princeton: Princeton University Press.

Shoham, Y., & Leyton-Brown, K. (2008). Multiagent systems: Algorithmic, game-theoretic, and logical foundations. Cambridge University Press.

Vreeswijk, G.A.W. (2011). Evolutionary game theory. (Slides)

Some problems

Examples

Examples

Problem (Discrete replicator).

Examples

Problem (Discrete replicator).
Suppose the proportion of species 7 at time t is $p_7(t) = 0.5$

Examples

Problem (Discrete replicator).
Suppose the proportion of species 7 at time t is $p_7(t) = 0.5$, the fitness of species 2 at time t is $f_7(t) = 5$

Examples

Problem (Discrete replicator). Suppose the proportion of species 7 at time t is $p_7(t) = 0.5$, the fitness of species 2 at time t is $f_7(t) = 5$, the average fitness at time t is $\bar{f}(t) = 2$

Examples

Problem (Discrete replicator). Suppose the proportion of species 7 at time t is $p_7(t) = 0.5$, the fitness of species 2 at time t is $f_7(t) = 5$, the average fitness at time t is $\bar{f}(t) = 2$, and the death and birth rate is $\beta = 0$.

Examples

Problem (Discrete replicator). Suppose the proportion of species 7 at time t is $p_7(t) = 0.5$, the fitness of species 2 at time t is $f_7(t) = 5$, the average fitness at time t is $\bar{f}(t) = 2$, and the death and birth rate is $\beta = 0$. Compute $p_7(t+1)$.

Examples

Problem (Discrete replicator). Suppose the proportion of species 7 at time t is $p_7(t) = 0.5$, the fitness of species 2 at time t is $f_7(t) = 5$, the average fitness at time t is $\bar{f}(t) = 2$, and the death and birth rate is $\beta = 0$. Compute $p_7(t+1)$.

Solution.

Examples

Problem (Discrete replicator). Suppose the proportion of species 7 at time t is $p_7(t) = 0.5$, the fitness of species 2 at time t is $f_7(t) = 5$, the average fitness at time t is $\bar{f}(t) = 2$, and the death and birth rate is $\beta = 0$. Compute $p_7(t+1)$.

Solution.

$$\begin{aligned} p_7(t+1) &= p_7(t)\frac{1+\beta+f_7(t)}{1+\beta+\bar{f}(t)} \\ &= 0.5\frac{1+0+5}{1+0+2} \\ &= 1. \end{aligned}$$

Examples

Problem (Discrete replicator). Suppose the proportion of species 7 at time t is $p_7(t) = 0.5$, the fitness of species 2 at time t is $f_7(t) = 5$, the average fitness at time t is $\bar{f}(t) = 2$, and the death and birth rate is $\beta = 0$. Compute $p_7(t+1)$.

Solution.

$$\begin{aligned} p_7(t+1) &= p_7(t)\frac{1+\beta+f_7(t)}{1+\beta+\bar{f}(t)} \\ &= 0.5\frac{1+0+5}{1+0+2} \\ &= 1. \end{aligned}$$

Problem (Continuous replicator).

Examples

Problem (Discrete replicator). Suppose the proportion of species 7 at time t is $p_7(t) = 0.5$, the fitness of species 2 at time t is $f_7(t) = 5$, the average fitness at time t is $\bar{f}(t) = 2$, and the death and birth rate is $\beta = 0$. Compute $p_7(t+1)$.

Solution.

$$\begin{aligned} p_7(t+1) &= p_7(t)\frac{1+\beta+f_7(t)}{1+\beta+\bar{f}(t)} \\ &= 0.5\frac{1+0+5}{1+0+2} \\ &= 1. \end{aligned}$$

Problem (Continuous replicator). Same question, approximate $p_7(t+0.01)$.

Examples

Problem (Discrete replicator). Suppose the proportion of species 7 at time t is $p_7(t) = 0.5$, the fitness of species 2 at time t is $f_7(t) = 5$, the average fitness at time t is $\bar{f}(t) = 2$, and the death and birth rate is $\beta = 0$. Compute $p_7(t+1)$.

Solution.

$$\begin{aligned} p_7(t+1) &= p_7(t)\frac{1+\beta+f_7(t)}{1+\beta+\bar{f}(t)} \\ &= 0.5\frac{1+0+5}{1+0+2} \\ &= 1. \end{aligned}$$

Problem (Continuous replicator). Same question, approximate $p_7(t+0.01)$.

Solution.

Examples

Problem (Discrete replicator). Suppose the proportion of species 7 at time t is $p_7(t) = 0.5$, the fitness of species 2 at time t is $f_7(t) = 5$, the average fitness at time t is $\bar{f}(t) = 2$, and the death and birth rate is $\beta = 0$. Compute $p_7(t+1)$.

Solution.

$$\begin{aligned} p_7(t+1) &= p_7(t)\frac{1+\beta+f_7(t)}{1+\beta+\bar{f}(t)} \\ &= 0.5\frac{1+0+5}{1+0+2} \\ &= 1. \end{aligned}$$

Problem (Continuous replicator). Same question, approximate $p_7(t+0.01)$.

Solution.

$$\begin{aligned} \dot{p}_7(t) &= p_7(t)[f_7(t) - \bar{f}(t)] \\ &= 0.5(5-2) = 1.5. \end{aligned}$$

Examples

Problem (Discrete replicator). Suppose the proportion of species 7 at time t is $p_7(t) = 0.5$, the fitness of species 2 at time t is $f_7(t) = 5$, the average fitness at time t is $\bar{f}(t) = 2$, and the death and birth rate is $\beta = 0$. Compute $p_7(t+1)$.

Solution.

$$
\begin{aligned}
p_7(t+1) &= p_7(t)\frac{1+\beta+f_7(t)}{1+\beta+\bar{f}(t)} \\
&= 0.5\frac{1+0+5}{1+0+2} \\
&= 1.
\end{aligned}
$$

Problem (Continuous replicator). Same question, approximate $p_7(t+0.01)$.

Solution.

$$
\begin{aligned}
\dot{p}_7(t) &= p_7(t)[f_7(t) - \bar{f}(t)] \\
&= 0.5(5-2) = 1.5.
\end{aligned}
$$

So

$$
\begin{aligned}
p_7(t+\Delta t) &\approx p_7(t) + \frac{d}{dt}p_7(t) \cdot \Delta t \\
&= 0.5 + 1.5 \cdot 0.01 \\
&= 0.5015.
\end{aligned}
$$