

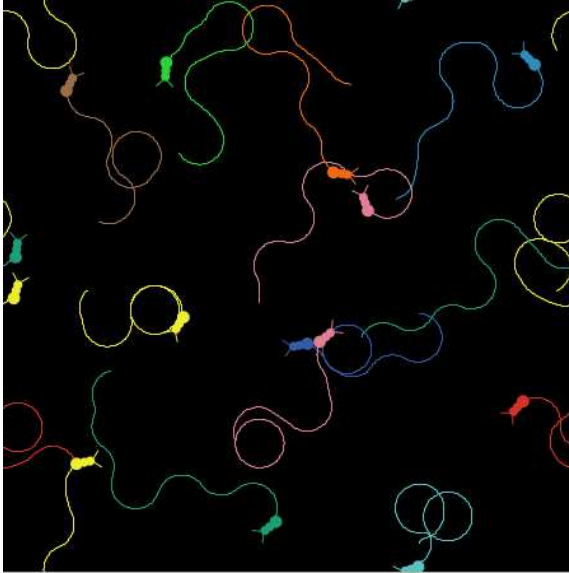
Multi-agent learning

Fictitious Play

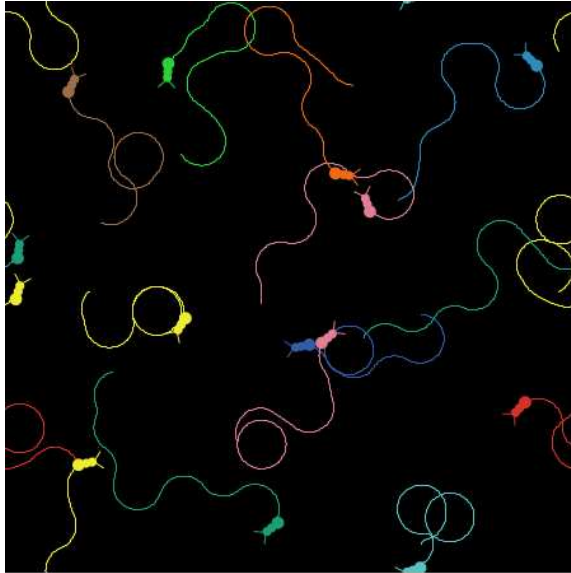
Gerard Vreeswijk, Intelligent Software Systems, Computer Science
Department, Faculty of Sciences, Utrecht University, The
Netherlands.

Wednesday 19th May, 2021

Fictitious play: motivation

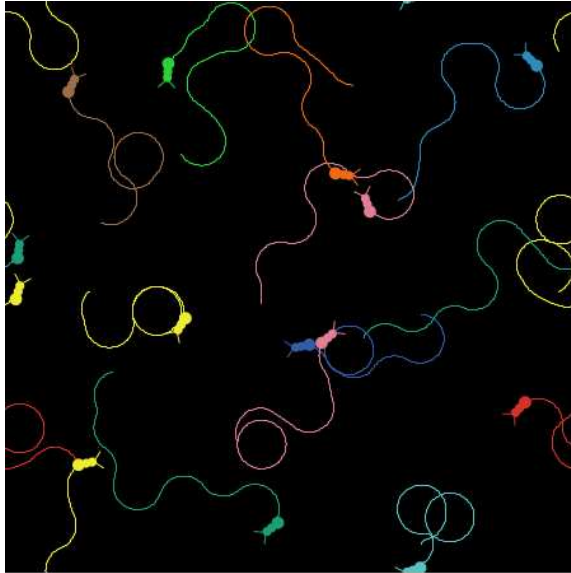


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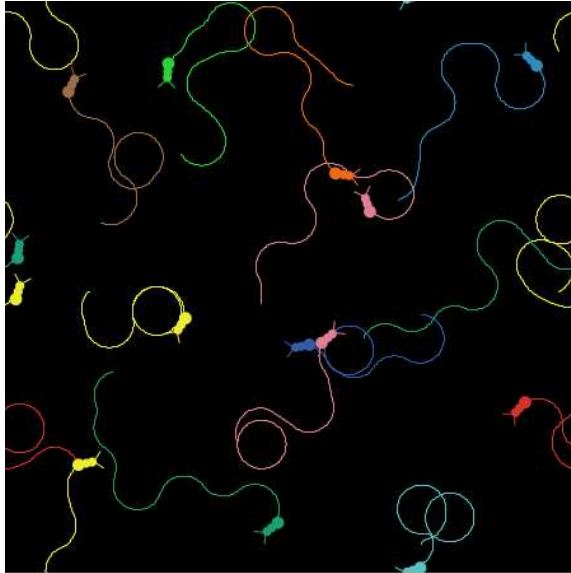
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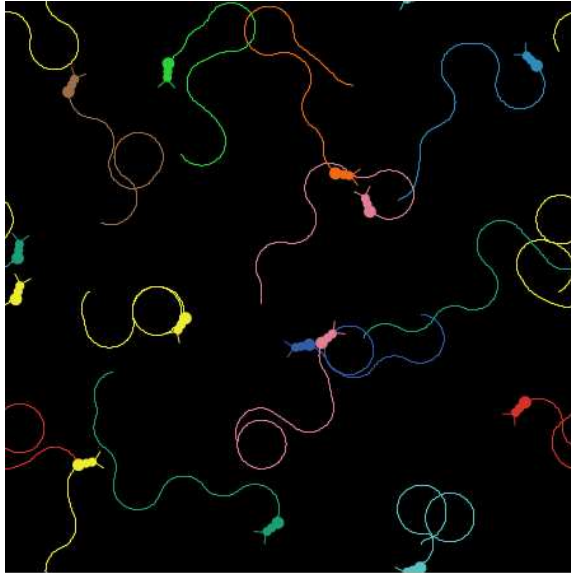
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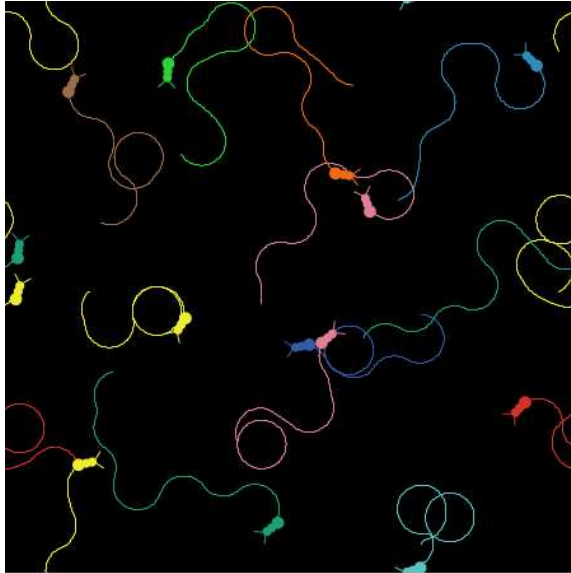
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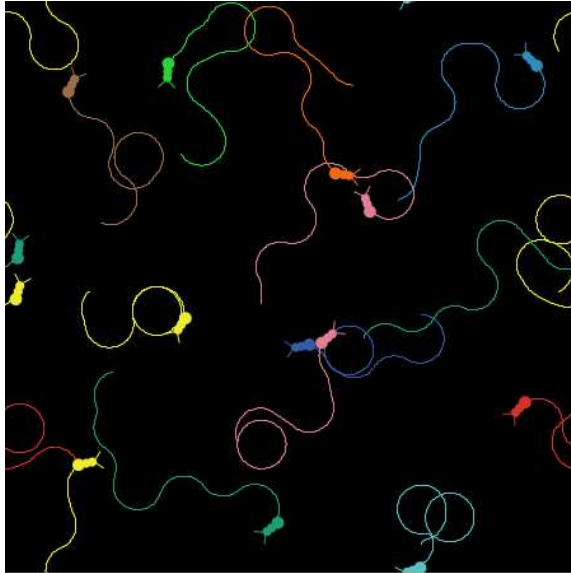
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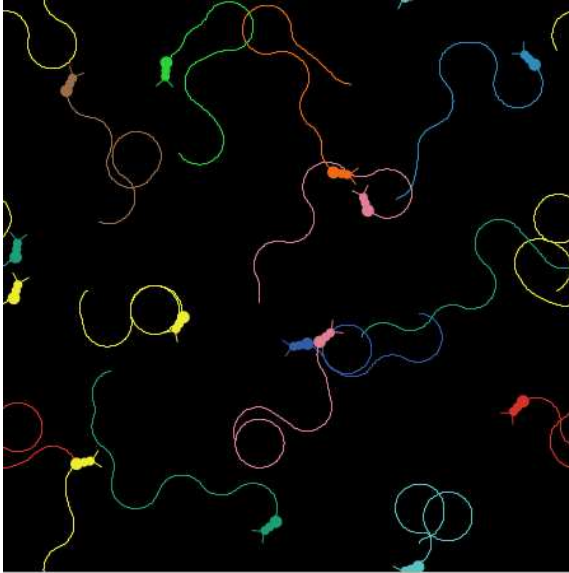
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- Brown (1951): explanation for Nash equilibrium play. In terms of current use, the name actually is a bit of a misnomer, since play actually occurs (Berger, 2005).

Plan for today

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Part I. Best reply strategy

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1. Pure fictitious play.

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Part II. Extensions and approximations of fictitious play

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3. No-regret property of smoothed fictitious play (Fudenberg *et al.*, 1995).

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Shoham *et al.* (2009): *Multi-agent Systems*. Ch. 7: “Learning and Teaching”. H. Young (2004): *Strategic Learning and its Limits*, Oxford UP. D. Fudenberg and D.K. Levine (1998), *The Theory of Learning in Games*, MIT Press.

Part I:

Pure fictitious play

Repeated coordination game

Players receive a positive payoff iff they coordinate. This game possesses three Nash equilibria, viz. $(0,0)$, $(0.5,0.5)$, and $(1,1)$.

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Strict and pure Nash \Rightarrow Steady state \Rightarrow Pure Nash.

But what if all equilibria are mixed?

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Example. Matching Pennies.

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5.	H	H	(5.5, 3.0)	(4.0, 4.5)
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⋮	⋮	⋮	⋮	⋮

Frequencies of fictitious play

The interface includes a control panel on the left with buttons for 'clear-drawing' (C), 'setup' (X), 'step' (S), and 'go' (G). Below these is a 'game-type' dropdown menu set to 'matching-pennies'. A series of sliders and input fields are provided for parameters: 'nr-of-actions' (2), 'epsilon' (0.10), 'initial-cumulative' (0), 'initial-geometric' (50), 'learning-rate' (0.20), 'max-payoff' (100), 'penalty' (-15), and 'lambda' (0.100). In the center is a 2x2 game board with red and green squares. The top-left square contains the number 3, the top-right contains 1, the bottom-left contains 5, and the bottom-right contains 5. A white arrow points from the top-left square to the top-right square. To the right of the board is a text window displaying the log of fictitious play iterations, showing the matrix, action, frequencies, and expected-rewards for each step.

matrix: $\begin{bmatrix} [1 & -1] & [-1 & 1] \end{bmatrix}$ $\begin{bmatrix} [-1 & 1] & [1 & -1] \end{bmatrix}$
action: $[1 \ 1]$
frequencies: $\begin{bmatrix} [0 \ 1] & [0 \ 1] \end{bmatrix}$
expected-rewards: $\begin{bmatrix} [-1 \ 1] & [1 \ -1] \end{bmatrix}$

matrix: $\begin{bmatrix} [1 & -1] & [-1 & 1] \end{bmatrix}$ $\begin{bmatrix} [-1 & 1] & [1 & -1] \end{bmatrix}$
action: $[1 \ 0]$
frequencies: $\begin{bmatrix} [0 \ 2] & [1 \ 1] \end{bmatrix}$
expected-rewards: $\begin{bmatrix} [0 \ 0] & [1 \ -1] \end{bmatrix}$

matrix: $\begin{bmatrix} [1 & -1] & [-1 & 1] \end{bmatrix}$ $\begin{bmatrix} [-1 & 1] & [1 & -1] \end{bmatrix}$
action: $[1 \ 0]$
frequencies: $\begin{bmatrix} [0 \ 3] & [2 \ 1] \end{bmatrix}$
expected-rewards: $\begin{bmatrix} [0.3333333333333333 & -0.3333333333333333] \end{bmatrix}$

matrix: $\begin{bmatrix} [1 & -1] & [-1 & 1] \end{bmatrix}$ $\begin{bmatrix} [-1 & 1] & [1 & -1] \end{bmatrix}$
action: $[0 \ 0]$
frequencies: $\begin{bmatrix} [1 \ 3] & [3 \ 1] \end{bmatrix}$
expected-rewards: $\begin{bmatrix} [0.5 \ -0.5] & [0.5 \ -0.5] \end{bmatrix}$

matrix: $\begin{bmatrix} [1 & -1] & [-1 & 1] \end{bmatrix}$ $\begin{bmatrix} [-1 & 1] & [1 & -1] \end{bmatrix}$
action: $[0 \ 0]$
frequencies: $\begin{bmatrix} [2 \ 3] & [4 \ 1] \end{bmatrix}$
expected-rewards: $\begin{bmatrix} [0.6000000000000001 & -0.6000000000000000] \end{bmatrix}$

matrix: $\begin{bmatrix} [1 & -1] & [-1 & 1] \end{bmatrix}$ $\begin{bmatrix} [-1 & 1] & [1 & -1] \end{bmatrix}$
action: $[0 \ 0]$
frequencies: $\begin{bmatrix} [3 \ 3] & [5 \ 1] \end{bmatrix}$
expected-rewards: $\begin{bmatrix} [0.6666666666666667 & -0.6666666666666666] \end{bmatrix}$

Convergent empirical distribution of strategies

Theorem. *If the empirical distribution of strategies converges in fictitious play, then it converges to a Nash equilibrium.*

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Proof. Let i be arbitrary. If the empirical distribution converges to q , then i 's opponent model converges to q^{-i} . By definition of fictitious play, $q^i \in \text{BR}(q^{-i})$.

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Proof. Let i be arbitrary. If the empirical distribution converges to q , then i 's opponent model converges to q^{-i} . By definition of fictitious play, $q^i \in \text{BR}(q^{-i})$. Because of convergence, all such (mixed) best replies converge along.

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Remarks:

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Remarks:

1. The q^i may be mixed.
2. It actually suffices that the q^{-i} converge asymptotically to the actual distribution (Fudenberg & Levine, 1998).
3. If the empirical distributions converge (hence, converge to a Nash equilibrium), the actually played responses per stage need **not** be Nash equilibria of the stage game.

Empirical distributions converge to Nash \nRightarrow stage Nash

Repeated Coordination Game. Players receive payoff 1 iff they coordinate, else 0.

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1.	B	A	(1.5, 1.0)	(1.0, 1.5)
2.	A	B		

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0.			(0.5, 1.0)	(1.0, 0.5)
1.	B	A	(1.5, 1.0)	(1.0, 1.5)
2.	A	B	(1.5, 2.0)	(2.0, 1.5)

Empirical distributions converge to Nash \nRightarrow stage Nash

Repeated Coordination Game. Players receive payoff 1 iff they coordinate, else 0.

Round	<i>A</i> 's action	<i>B</i> 's action	<i>A</i> 's beliefs	<i>B</i> 's beliefs
0.			(0.5, 1.0)	(1.0, 0.5)
1.	B	A	(1.5, 1.0)	(1.0, 1.5)
2.	A	B	(1.5, 2.0)	(2.0, 1.5)
3.	B	A		

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Round	<i>A</i> 's action	<i>B</i> 's action	<i>A</i> 's beliefs	<i>B</i> 's beliefs
0.			(0.5, 1.0)	(1.0, 0.5)
1.	B	A	(1.5, 1.0)	(1.0, 1.5)
2.	A	B	(1.5, 2.0)	(2.0, 1.5)
3.	B	A	(2.5, 2.0)	(2.0, 2.5)

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Round	<i>A</i> 's action	<i>B</i> 's action	<i>A</i> 's beliefs	<i>B</i> 's beliefs
0.			(0.5, 1.0)	(1.0, 0.5)
1.	B	A	(1.5, 1.0)	(1.0, 1.5)
2.	A	B	(1.5, 2.0)	(2.0, 1.5)
3.	B	A	(2.5, 2.0)	(2.0, 2.5)
4.	A	B		

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Round	<i>A</i> 's action	<i>B</i> 's action	<i>A</i> 's beliefs	<i>B</i> 's beliefs
0.			(0.5, 1.0)	(1.0, 0.5)
1.	B	A	(1.5, 1.0)	(1.0, 1.5)
2.	A	B	(1.5, 2.0)	(2.0, 1.5)
3.	B	A	(2.5, 2.0)	(2.0, 2.5)
4.	A	B	(2.5, 3.0)	(3.0, 2.5)

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Repeated Coordination Game. Players receive payoff 1 iff they coordinate, else 0.

Round	<i>A</i> 's action	<i>B</i> 's action	<i>A</i> 's beliefs	<i>B</i> 's beliefs
0.			(0.5, 1.0)	(1.0, 0.5)
1.	B	A	(1.5, 1.0)	(1.0, 1.5)
2.	A	B	(1.5, 2.0)	(2.0, 1.5)
3.	B	A	(2.5, 2.0)	(2.0, 2.5)
4.	A	B	(2.5, 3.0)	(3.0, 2.5)
⋮	⋮	⋮	⋮	⋮

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Repeated Coordination Game. Players receive payoff 1 iff they coordinate, else 0.

Round	A 's action	B 's action	A 's beliefs	B 's beliefs
0.			(0.5, 1.0)	(1.0, 0.5)
1.	B	A	(1.5, 1.0)	(1.0, 1.5)
2.	A	B	(1.5, 2.0)	(2.0, 1.5)
3.	B	A	(2.5, 2.0)	(2.0, 2.5)
4.	A	B	(2.5, 3.0)	(3.0, 2.5)
\vdots	\vdots	\vdots	\vdots	\vdots

- This game possesses three equilibria, viz. $(0,0)$, $(0.5,0.5)$, and $(1,1)$, with expected payoffs 1, 0.5, and 1, respectively.

Empirical distributions converge to Nash \nRightarrow stage Nash

Repeated Coordination Game. Players receive payoff 1 iff they coordinate, else 0.

Round	A's action	B's action	A's beliefs	B's beliefs
0.			(0.5, 1.0)	(1.0, 0.5)
1.	B	A	(1.5, 1.0)	(1.0, 1.5)
2.	A	B	(1.5, 2.0)	(2.0, 1.5)
3.	B	A	(2.5, 2.0)	(2.0, 2.5)
4.	A	B	(2.5, 3.0)	(3.0, 2.5)
\vdots	\vdots	\vdots	\vdots	\vdots

- This game possesses three equilibria, viz. $(0,0)$, $(0.5,0.5)$, and $(1,1)$, with expected payoffs 1, 0.5, and 1, respectively.
- Empirical distribution of play converges to $(0.5,0.5)$,—with payoff 0, rather than 0.5.

Empirical distr. of play does not need to converge

Rock-paper-scissors. Winner receives payoff 1, else 0.

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0.				

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Round	<i>A</i> 's action	<i>B</i> 's action	<i>A</i> 's beliefs	<i>B</i> 's beliefs
0.			(0.0, 0.0, 0.5)	(0.0, 0.5, 0.0)

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Round	<i>A</i> 's action	<i>B</i> 's action	<i>A</i> 's beliefs	<i>B</i> 's beliefs
0.			$(0.0, 0.0, 0.5)$	$(0.0, 0.5, 0.0)$
1.	Rock	Scissors		

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0.			(0.0, 0.0, 0.5)	(0.0, 0.5, 0.0)
1.	Rock	Scissors	(0.0, 0.0, 1.5)	(1.0, 0.5, 0.0)

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Round	<i>A</i> 's action	<i>B</i> 's action	<i>A</i> 's beliefs	<i>B</i> 's beliefs
0.			$(0.0, 0.0, 0.5)$	$(0.0, 0.5, 0.0)$
1.	Rock	Scissors	$(0.0, 0.0, 1.5)$	$(1.0, 0.5, 0.0)$
2.	Rock	Paper		

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Round	<i>A</i> 's action	<i>B</i> 's action	<i>A</i> 's beliefs	<i>B</i> 's beliefs
0.			(0.0, 0.0, 0.5)	(0.0, 0.5, 0.0)
1.	Rock	Scissors	(0.0, 0.0, 1.5)	(1.0, 0.5, 0.0)
2.	Rock	Paper	(0.0, 1.0, 1.5)	(2.0, 0.5, 0.0)

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0.			(0.0, 0.0, 0.5)	(0.0, 0.5, 0.0)
1.	Rock	Scissors	(0.0, 0.0, 1.5)	(1.0, 0.5, 0.0)
2.	Rock	Paper	(0.0, 1.0, 1.5)	(2.0, 0.5, 0.0)
3.	Rock	Paper		

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0.			(0.0, 0.0, 0.5)	(0.0, 0.5, 0.0)
1.	Rock	Scissors	(0.0, 0.0, 1.5)	(1.0, 0.5, 0.0)
2.	Rock	Paper	(0.0, 1.0, 1.5)	(2.0, 0.5, 0.0)
3.	Rock	Paper	(0.0, 2.0, 1.5)	(3.0, 0.5, 0.0)

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0.			(0.0, 0.0, 0.5)	(0.0, 0.5, 0.0)
1.	Rock	Scissors	(0.0, 0.0, 1.5)	(1.0, 0.5, 0.0)
2.	Rock	Paper	(0.0, 1.0, 1.5)	(2.0, 0.5, 0.0)
3.	Rock	Paper	(0.0, 2.0, 1.5)	(3.0, 0.5, 0.0)
4.	Scissors	Paper		

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0.			(0.0, 0.0, 0.5)	(0.0, 0.5, 0.0)
1.	Rock	Scissors	(0.0, 0.0, 1.5)	(1.0, 0.5, 0.0)
2.	Rock	Paper	(0.0, 1.0, 1.5)	(2.0, 0.5, 0.0)
3.	Rock	Paper	(0.0, 2.0, 1.5)	(3.0, 0.5, 0.0)
4.	Scissors	Paper	(0.0, 3.0, 1.5)	(3.0, 0.5, 1.0)

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0.			(0.0, 0.0, 0.5)	(0.0, 0.5, 0.0)
1.	Rock	Scissors	(0.0, 0.0, 1.5)	(1.0, 0.5, 0.0)
2.	Rock	Paper	(0.0, 1.0, 1.5)	(2.0, 0.5, 0.0)
3.	Rock	Paper	(0.0, 2.0, 1.5)	(3.0, 0.5, 0.0)
4.	Scissors	Paper	(0.0, 3.0, 1.5)	(3.0, 0.5, 1.0)
5.	Scissors	Paper		

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0.			(0.0, 0.0, 0.5)	(0.0, 0.5, 0.0)
1.	Rock	Scissors	(0.0, 0.0, 1.5)	(1.0, 0.5, 0.0)
2.	Rock	Paper	(0.0, 1.0, 1.5)	(2.0, 0.5, 0.0)
3.	Rock	Paper	(0.0, 2.0, 1.5)	(3.0, 0.5, 0.0)
4.	Scissors	Paper	(0.0, 3.0, 1.5)	(3.0, 0.5, 1.0)
5.	Scissors	Paper	(0.0, 4.0, 1.5)	(3.0, 0.5, 2.0)

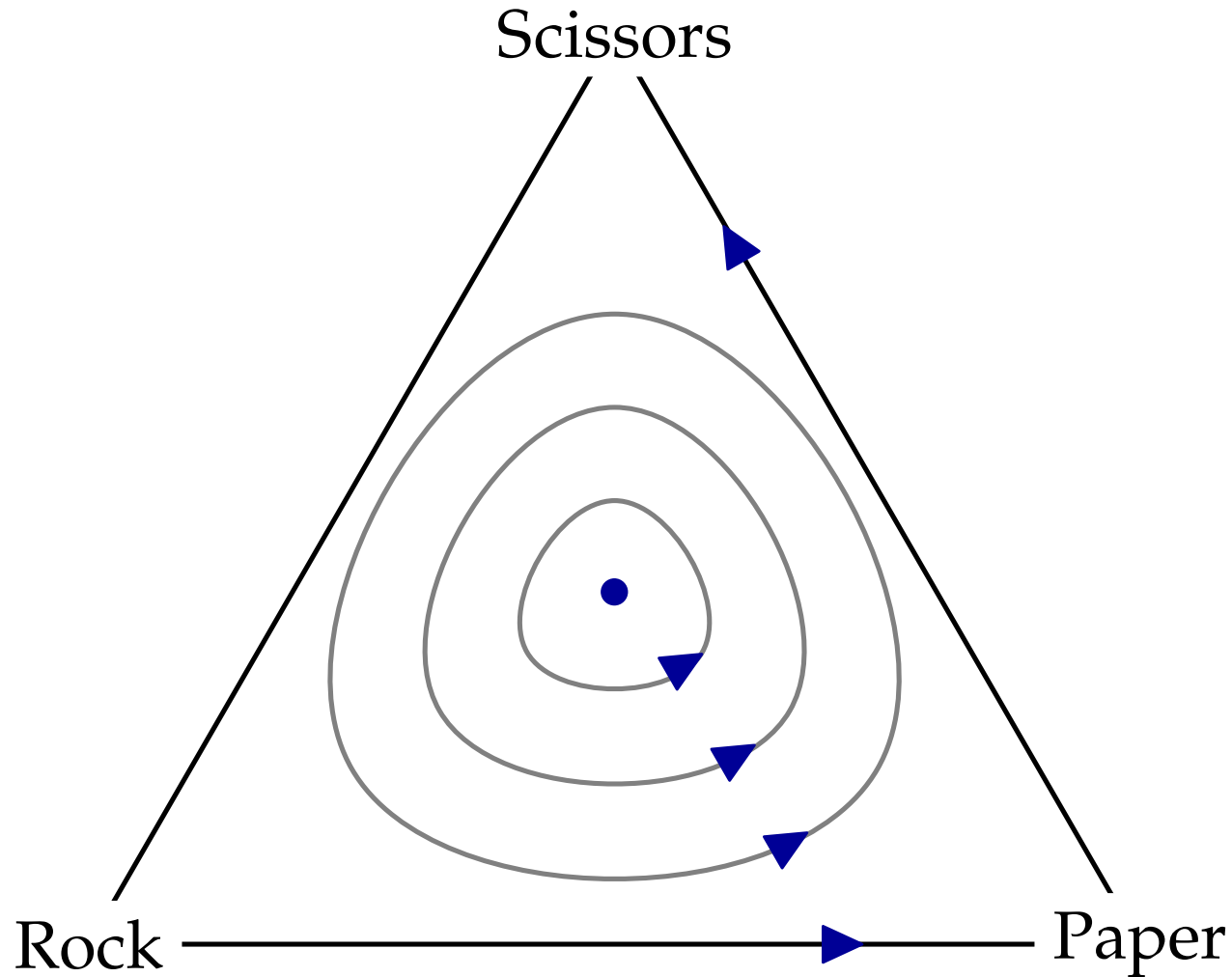
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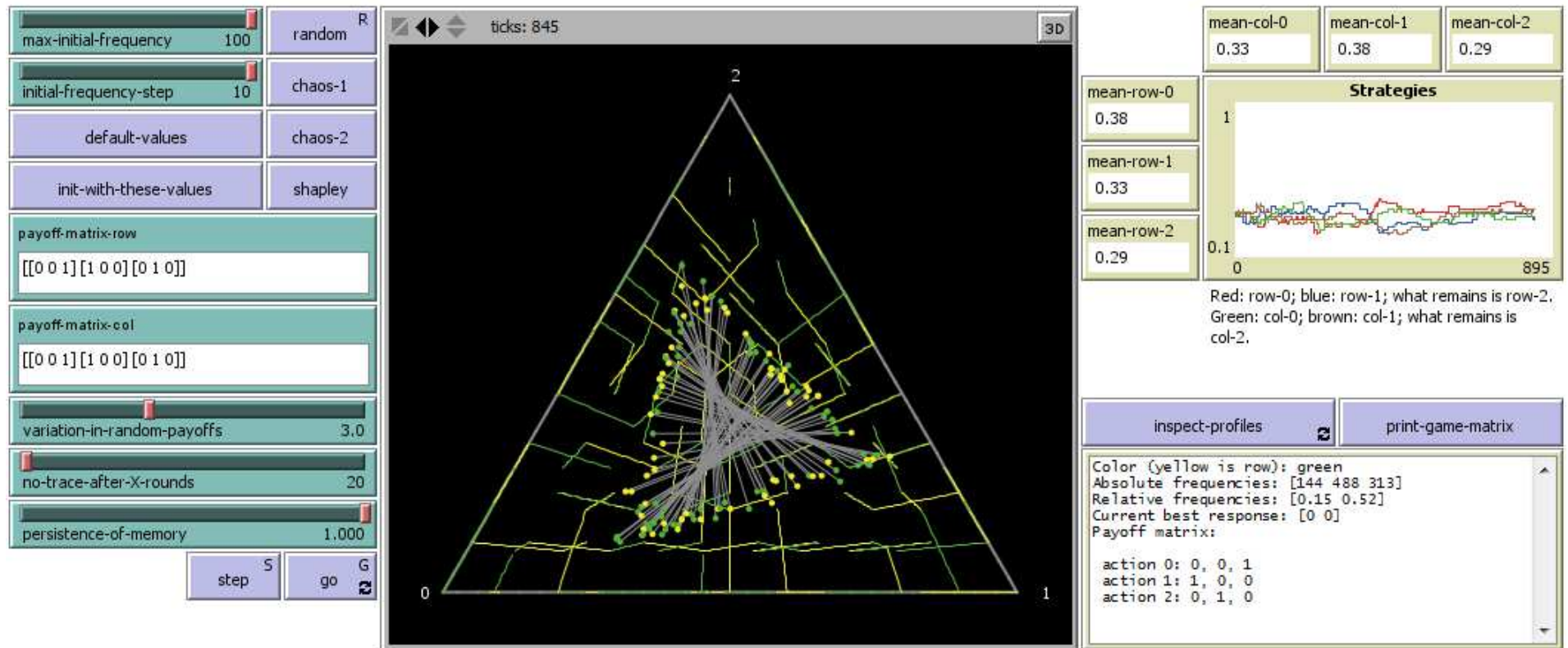
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0.			(0.0, 0.0, 0.5)	(0.0, 0.5, 0.0)
1.	Rock	Scissors	(0.0, 0.0, 1.5)	(1.0, 0.5, 0.0)
2.	Rock	Paper	(0.0, 1.0, 1.5)	(2.0, 0.5, 0.0)
3.	Rock	Paper	(0.0, 2.0, 1.5)	(3.0, 0.5, 0.0)
4.	Scissors	Paper	(0.0, 3.0, 1.5)	(3.0, 0.5, 1.0)
5.	Scissors	Paper	(0.0, 4.0, 1.5)	(3.0, 0.5, 2.0)
⋮	⋮	⋮	⋮	⋮

Repeated Shapley game: phase diagram

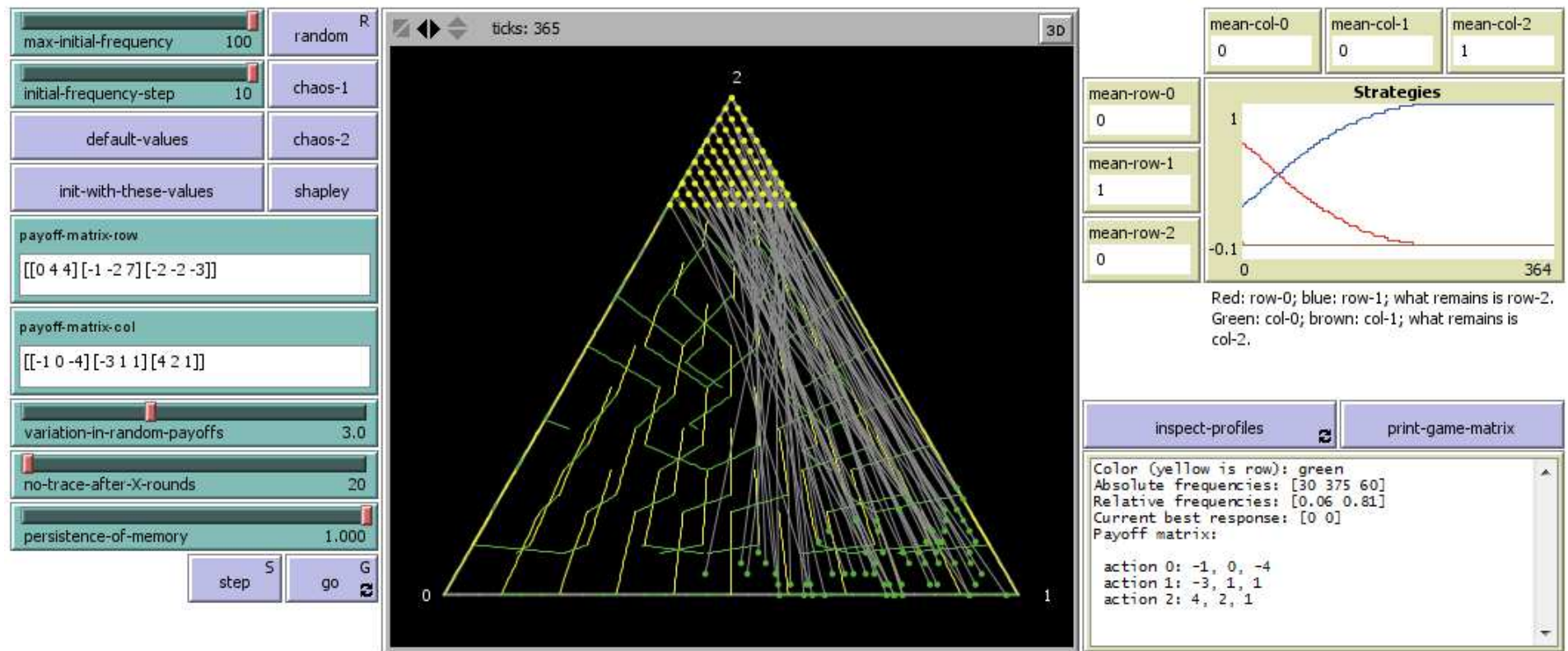


FP on Shapley's game; strategy profiles in a simplex



There are many player couples. Each couple is connected by a gray line. Yellow is row; green is column. Player location is determined by the mixed strategy it projects on its opponent (i.e., normalised action count of its opponent). Each player starts with a biased action count. For example, with $[100, 0, 0]$ (lower left) or $[0, 100, 0]$ (lower right) or $[33, 33, 33]$ (center). Initial action counts of player pairs are unrelated.

FP on a 3x3 game; strategy profiles in a simplex



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Part II:

Extensions and approximations of fictitious play

Proposed extensions to fictitious play

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1. Build forecasts, not on *complete history*, but on

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 - **Recent data**

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Proposed extensions to fictitious play

1. Build forecasts, not on *complete history*, but on
 - **Recent data**, say on m most recent rounds.
 - **Discounted data**, say with discount factor γ .

Proposed extensions to fictitious play

1. Build forecasts, not on *complete history*, but on
 - **Recent data**, say on m most recent rounds.
 - **Discounted data**, say with discount factor γ .
 - **Perturbed data**

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1. Build forecasts, not on *complete history*, but on
 - **Recent data**, say on m most recent rounds.
 - **Discounted data**, say with discount factor γ .
 - **Perturbed data**, say with error ϵ on individual observations.
 - **Random samples** of historical data

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2. Give not necessarily *best responses*, but
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 - **Perturbed throughout**, with small random shocks.
 - Randomly, and **proportional to expected payoff**.

Jordan's framework for FP

Framework for predictive learning (e.g. fictitious play)

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$$f_i : H \rightarrow \Delta(X_{-i}).$$

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- Forecasting and response functions are **deterministic**.
- Reinforcement and regret do not fit.

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- Reinforcement and regret do not fit. (They are not involved with prediction.)

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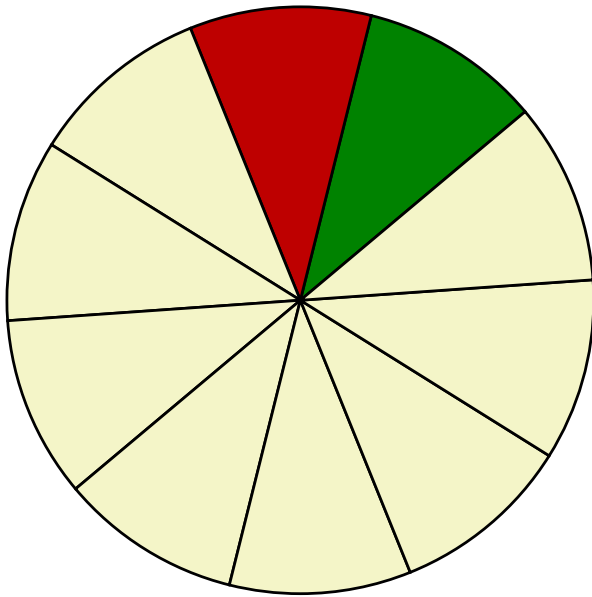
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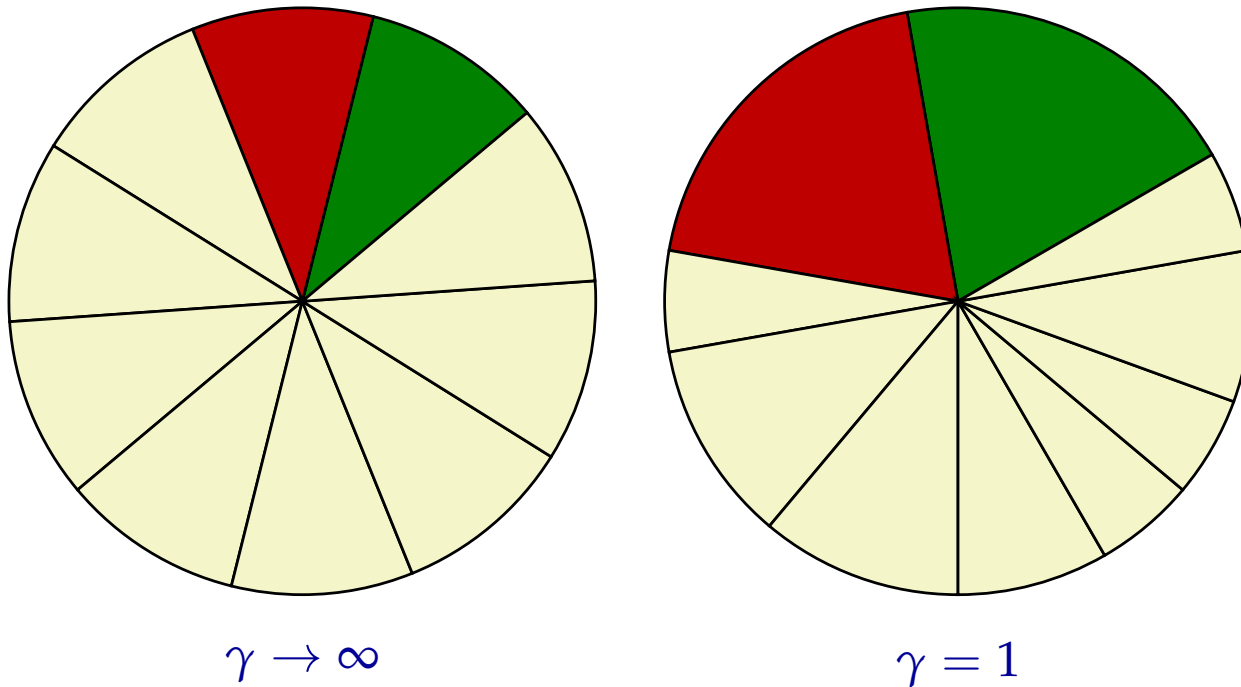
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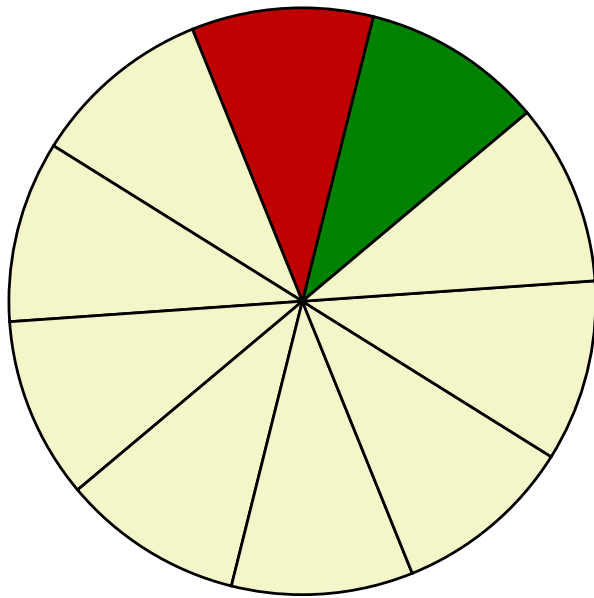


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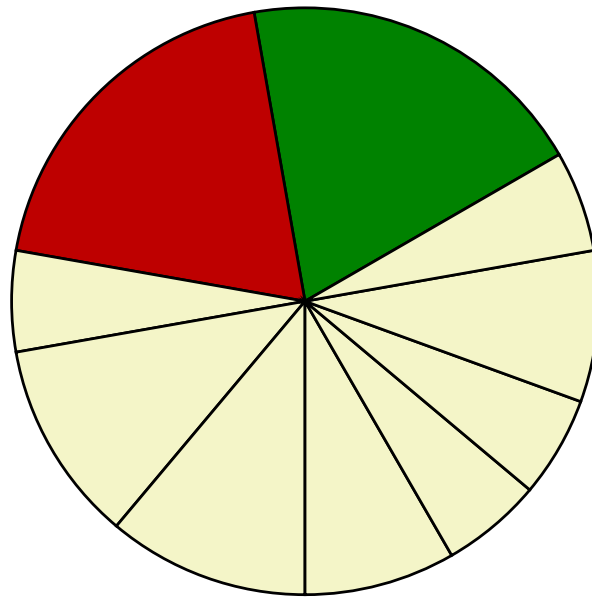
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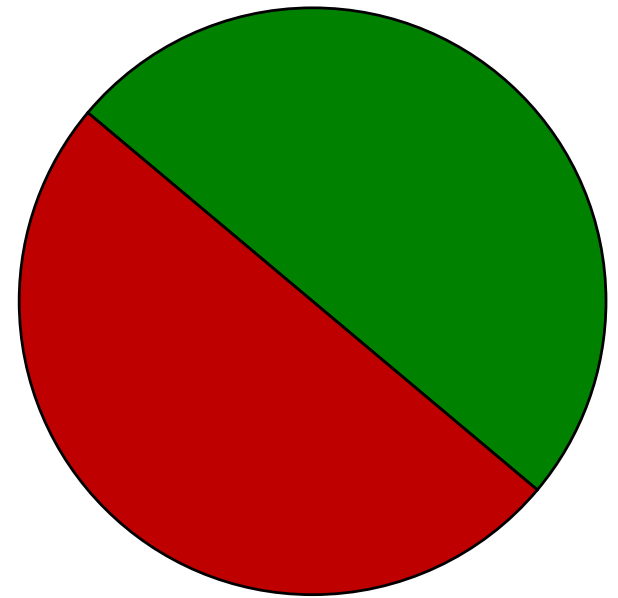
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Fudenberg & Levine, 1995. "Consistency and cautious fictitious play," *Journal of Economic Dynamics and Control*, Vol. 19 (5-7), pp. 1065-1089.

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Definition. Let X be action profiles, and $q \in \Delta(X)$. Then q is a **coarse correlated equilibrium** (CCE) if no one wants to opt out prior to a realisation of q in the form of an action profile.

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But there is another l.a. with **no** regret and convergence to **zero**-CCE!

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A theorem on exponentiated regret matching (Mas-Colell *et al.*, 2001) ensures that individual players have no-regret with probability one, and the empirical distribution of play converges to the set of coarse correlated equilibria (PY, p. 37 for RM, p. 60 for ERM).

FP

FP vs. Smoothed FP

FP vs. Smoothed FP vs. Exponentiated regret matching

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Hart, S., and Mas-Colell, A. (2000). "A simple adaptive procedure leading to correlated equilibrium". *Econometrica*, 68, pp. 1127-1150.

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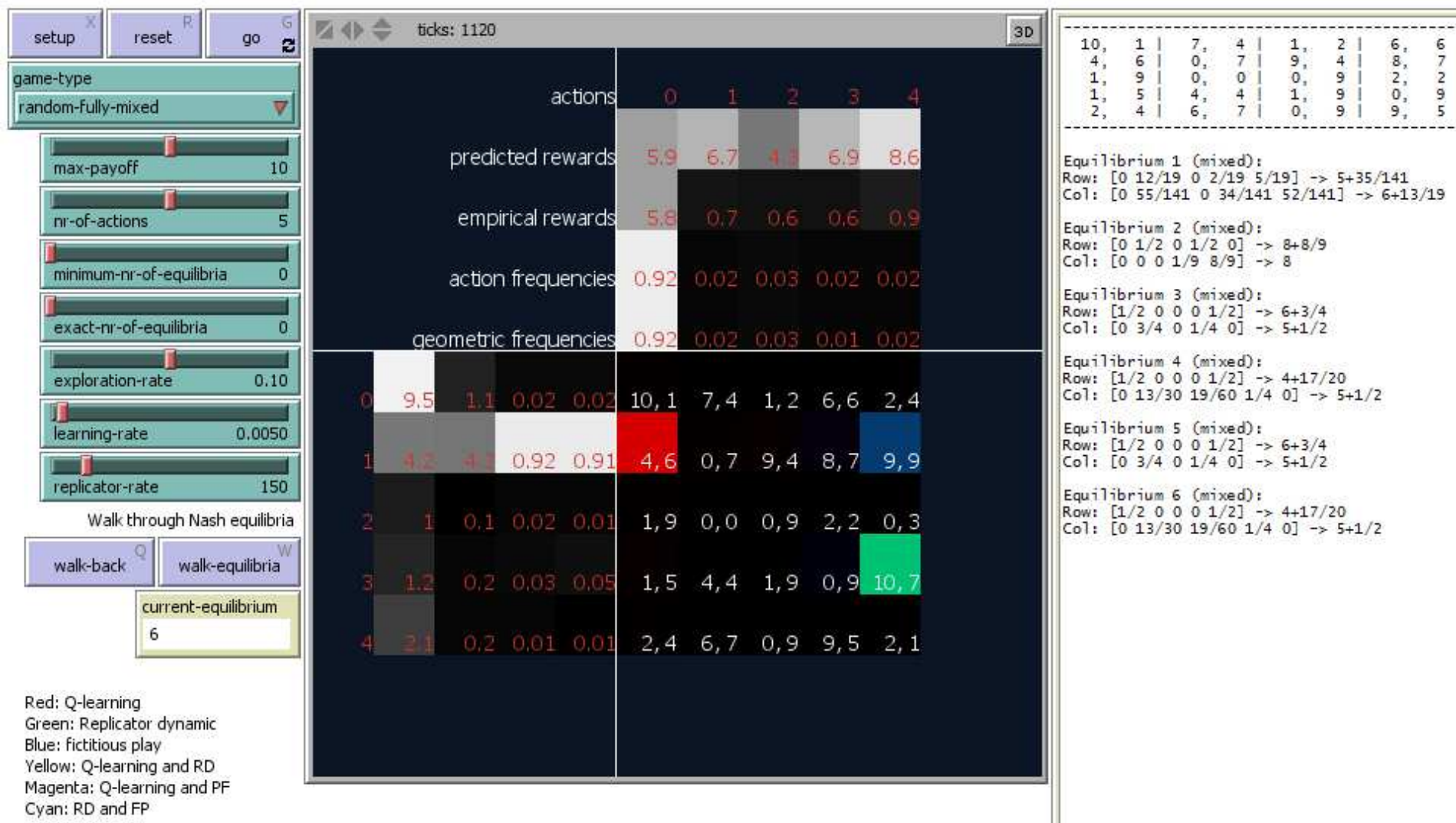
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Fictitious play compared to other algorithms



Part III:

Finite memory and inertia

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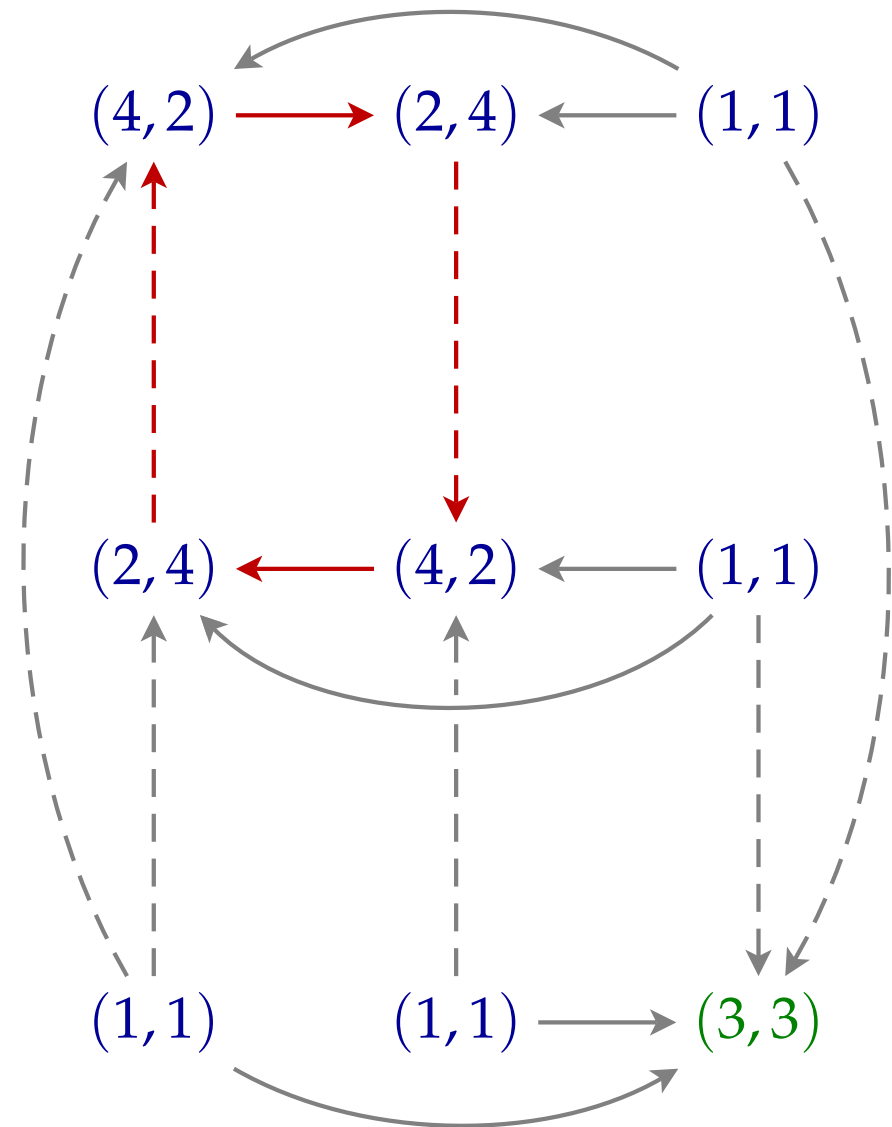
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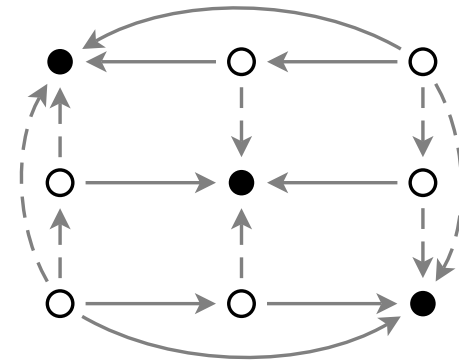
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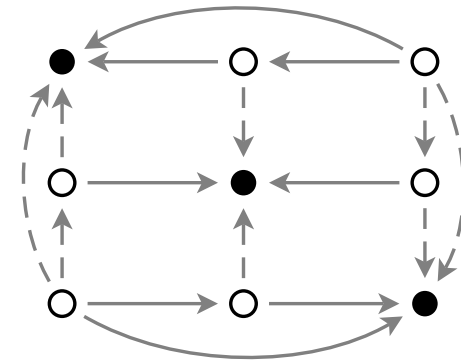
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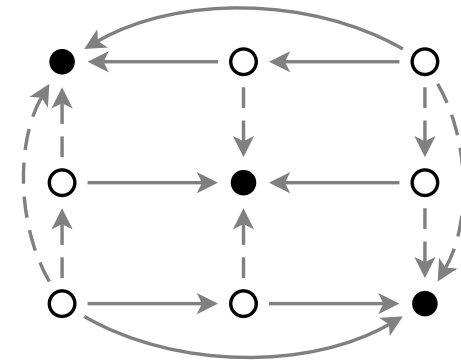


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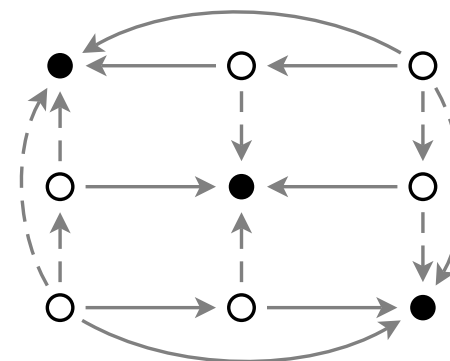


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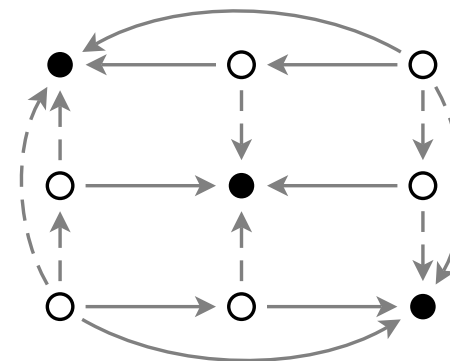


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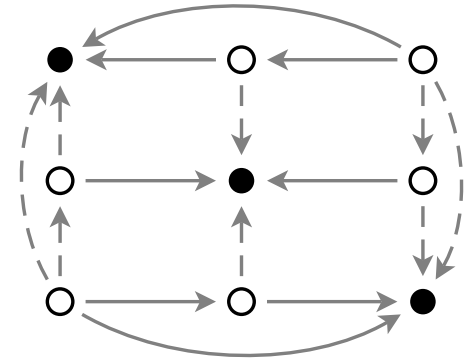
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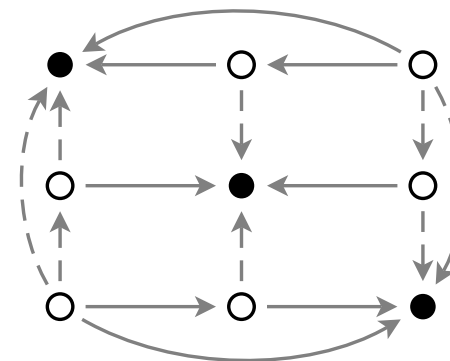
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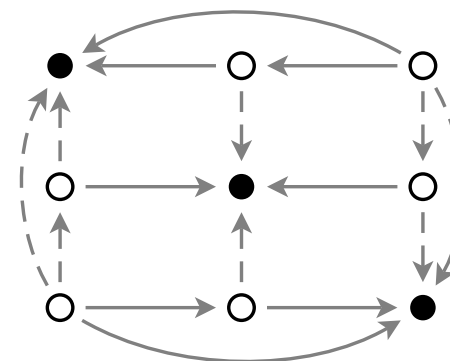
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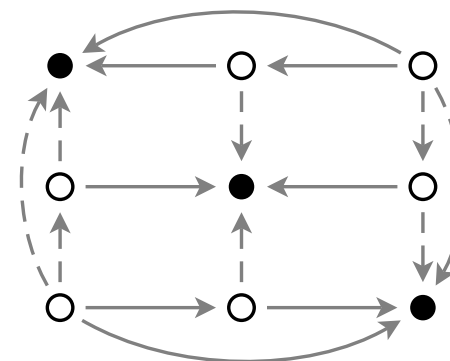
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6. It can be shown that, due to weak acyclicity, inertia, and (4), the process eventually lands in an absorbing state which, due to (5), is a repeated pure Nash equilibrium. \square

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Infinitely many disjoint histories of length \mathbf{m} occur, hence infinitely many *independent* events “homogeneous at $t + \mathbf{m}$ ” occur. Apply the (second) **Borel-Cantelli lemma**: if $\{E_n\}_n$ are independent events, and $\sum_{n=1}^{\infty} \Pr(E_n)$ is unbounded, then $\Pr(\text{an infinite number of } E_n \text{ 's occur}) = 1$. \square

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- The “image” $\langle x^{(1)} \rangle, \dots, \langle x^{(l)} \rangle$ of a better reply-path $x^{(1)}, \dots, x^{(l)}$ is followed to a sink : $\geq (\gamma\lambda^{nm-1})^L$, where L is the length of a longest better-reply path.

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Since Z^* is encountered infinitely often, the result follows. \square

Summary



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- There is a family of so-called **better-reply learning rules**, that
 - i) play sub-optimal actions, and
 - ii) can be brought arbitrary close to fictitious play.
- In weakly acyclic n -person games, every better-reply process with finite memory and inertia converges to a pure Nash equilibrium.

What next?

Gradient dynamics:

- Like fictitious play, players model (or assess) each other through mixed strategies.
- Strategies are not played, only maintained.
- Due to **CKR** (common knowledge of rationality, cf. Hargreaves Heap & Varoufakis, 2004), all models of mixed strategies are correct. (I.e., $q^{-i} = s^{-i}$, for all i .)
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Exam problems

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Answer. *d)*: For some but not all games with mixed equilibria only, the empirical frequencies converge.

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Answer d): When $\gamma \downarrow 0$ and $a \rightarrow \infty$, then both converge to fictitious play.

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Answer d): When $\gamma \downarrow 0$ and $a \rightarrow \infty$, then both converge to fictitious play. When $a = 0$ and $\gamma \rightarrow \infty$, then exponentiated regret matching plays randomly, and smoothed fictitious play converges to random play.

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See also the cake pies on Slide 24 and the table on Slide 32.

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Explanation of conditioning on an event

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Problem. A : policy holder is accident prone; A_1 : policy holder will have an accident within one year.

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Problem. A : policy holder is accident prone; A_1 : policy holder will have an accident within one year. Suppose $\Pr\{A_1|A\} = 0.4$, $\Pr\{A_1|\bar{A}\} = 0.2$ and $\Pr\{A\} = 0.3$.

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Answer. $\Pr\{A_1\}$

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Answer. $\Pr\{A_1\} = \Pr\{A_1|A\}\Pr\{A\}$

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Problem. A : policy holder is accident prone; A_1 : policy holder will have an accident within one year. Suppose $\Pr\{A_1|A\} = 0.4$, $\Pr\{A_1|\bar{A}\} = 0.2$ and $\Pr\{A\} = 0.3$. Compute $\Pr\{A_1\}$.

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Similarly,

$$\Pr\{T|h\} = \Pr\{T|F, h\}\Pr\{F|h\} + \Pr\{T|\bar{F}, h\}\Pr\{c|h\}.$$

Noisy fictitious play

Determine the strategy of Row at round eight if the history of play is

$$h = TC, BC, TL, BL, TR, TR, TL.$$

Answer. Projected mixed strategy of Col after seven rounds: $L: 3/7$; $C: 2/7$; $R: 2/7$.

Expected payoffs:

$$\begin{aligned} E[T] &= \frac{3}{7}0 + \frac{2}{7}1 + \frac{2}{7}4 = \frac{10}{7}, \\ E[M] &= \frac{3}{7}1 + \frac{2}{7}0 + \frac{2}{7}5 = \frac{13}{7}, \\ E[B] &= \frac{3}{7}3 + \frac{2}{7}2 + \frac{2}{7}0 = \frac{13}{7}. \end{aligned}$$

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Problem. Row is a 10% noise (plays a random action 10% of the time at random moments) fictitious player in a normal form game with payoffs

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