Multi-agent learning

Equilibria

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4. **Summary**

Recap of notation

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- $S = S_1 \times \cdots \times S_n$ is the set of all possible strategy profiles.
- Profile s is sometimes written as $s = (s_i, s_{-i})$, where s_{-i} is s_i's counter-strategy profile.

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Example

Battle of the sexes:

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D (0.4)	$(0, 0)$	$(2, 1)$

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- $$\begin{aligned} u_1(s) &= \sum_x [s_1(x_1)s_2(x_2)]\,u_1(x) \\ &= s_1(U)s_2(L)u_1(U,L) + s_1(U)s_2(R)u_1(U,R) \\ &\quad + \ s_1(D)s_2(L)u_1(D,L) + s_1(D)s_2(R)u_1(D,R) \\ &= 0.6 \times 0.2 \times 2 + \cancel{0.6 \times 0.8 \times 0} + \cancel{0.4 \times 0.2 \times 0} + 0.4 \times 0.8 \times 1 \end{aligned}$$

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Nash equilibria defined in terms of pure strategies

Best response

Definition (Best response). Strategy s_i is said to be a best response to the counterprofile s_{-i} if

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$$\begin{aligned} u_i(s) &= \textstyle\sum_x s(x) u_i(x) \\ &= \textstyle\sum_{x_i, x_{-i}} s(x_i, x_{-i}) u_i(x_i, x_{-i}) \\ &= \textstyle\sum_{x_i, x_{-i}} s_i(x_i) s_{-i}(x_{-i}) u_i(x_i, x_{-i}) \\ &= \textstyle\sum_{x_i} \sum_{x_{-i}} s_i(x_i) s_{-i}(x_{-i}) u_i(x_i, x_{-i}) \\ &= \textstyle\sum_{x_{-i}} \sum_{x_i} s_i(x_i) s_{-i}(x_{-i}) u_i(x_i, x_{-i}) \end{aligned}$$

- With alternative action x_i':

$$u_i(x_i', s_{-i}) = \textstyle\sum_{x_{-i}} s_{-i}(x_{-i}) u_i(x_i', x_{-i}).$$

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All i maintain some strategy s_i. The strategy profile s is a Nash equilibrium if no one can profit by changing s_i unilaterally.

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Probability distributions over the strategy space

Strategies ⇒ strategy profile ⇒ joint distribution

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- Suppose n players, strategies $s_1, \dots, s_n$ are given:

$$
\begin{array}{cc|cccc}
 & s_{-i} & y_1^{-i} & y_2^{-i} & \dots & y_n^{-i} \\
 & s_i & q_1 & q_2 & \dots & q_n \\
\hline
x_1^i & p_1 & p_1 q_1 & p_1 q_2 & \dots & p_1 q_n \\
x_2^i & p_2 & p_2 q_1 & p_2 q_2 & \dots & p_2 q_n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x_m^i & p_m & p_m q_1 & p_m q_2 & \dots & p_m q_n
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where n is the number of different counter-profiles.

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- Players act independently.

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\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x_m^i & p_m & p_mq_1 & p_mq_2 & \dots & p_mq_n
\end{array}
$$

  where n is the number of different counter-profiles.

- Players act independently.

- The strategy $s_i = (p_1, \dots, p_m)$ and the counter strategy profile $s_{-i} = (q_1, \dots, q_n)$ together define a product distribution $s \in \Delta(X)$:

$$s(x_1, \dots, x_n) =_{Def} s(x_1) \times \dots \times s(x_n).$$

Joint distribution $\not\Rightarrow$ marginal strategies

Suppose a (possibly non-product) distribution $q \in \Delta(X)$ is given.

$$
\begin{array}{cc|cccc}
 & q_{-i} & y_1^{-i} & y_2^{-i} & \cdots & y_n^{-i} \\
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- If players follow q, they need not act independently. (Example: off-diagonal is zero.)
- The marginals form strategies: $s_i = q_i$, $s_{-i} = q_{-i}$.
- But now generally

$$s(x_i, x_{-i}) \neq s(x_i)s(x_{-i}).$$

Joint distribution vs. joint strategy profile

Example. Consider:

	L (0.2)	R (0.8)
U (0.6)		
D (0.4)		

Joint distribution vs. joint strategy profile

Example. Consider:

	L (0.2)	R (0.8)
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Joint distribution vs. joint strategy profile

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	L (0.2)	R (0.8)
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Joint distribution vs. joint strategy profile

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Joint distribution vs. joint strategy profile

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In this case the joint distribution, namely $q = (0.12, 0.48, 0.08, 0.32)$, is induced by marginal distributions $s_1 = (0.6, 0.4)$ and $s_2 = (0.2, 0.8)$: $q = s_1 \times s_2$.

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Contrast this with q':

	L	R
U	0.13	0.47
D	0.07	0.33

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No marginal distributions exist that induce the joint distribution. In particular, s_1 and s_2 don't do it, i.e, $q' \neq s_1 \times s_2$.

Correlated equilibrium

Correlated equilibrium (Intuition)

Chicken game

You:	*Other:* Dare	Sway
Dare	$(-10, -10)$	$(5, 0)$
Sway	$(0, 5)$	$(-1, -1)$

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Three Nash equilibria:

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- $((3/8, 5/8), (3/8, 5/8))$

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Expected payoff $-5/8$ for both in the last equilibrium.

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Correlated equilibrium (Idea). Let a probability distribution

$$q : X \to [0, 1]$$

be given. This q can be seen as a coordinating device.

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Correlated equilibrium (Idea). Let a probability distribution

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Think of a traffic light:

Correlated equilibrium (Intuition)

Chicken game

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Dare	$(-10,-10)$	$(5,0)$
Sway	$(0,5)$	$(-1,-1)$

Three Nash equilibria:

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Think of a traffic light:

$q =$

You:	Other: Green	Red
Green	0.00	0.55
Red	0.40	0.05

Correlated equilibrium (Intuition)

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Each time, the system is in one of these four states.

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- At each realisation of q, every party i comes to know only its coordinate (i.e., action, Green or Red), x_i, of the system state x.

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- At each realisation of q, every party i comes to know only its coordinate (i.e., action, Green or Red), x_i, of the system state x.

Definition. A distribution $q \in \Delta(X)$ is called a correlated equilibrium if no party has an incentive to deviate from its own coordinate x_i, assuming that others do not deviate from x_{-i} as well.

Correlated equilibrium (formula)

Idea:

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Suppose $q \in \Delta(X)$ is given.

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Multiplying by $q(x_i)$ gives, for all i, x_i and x_i':

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For all i, x_i and x'_i:

$$\sum_{x_{-i}} q(x_{-i} | x_i) u_i(x'_i, x_{-i}) \leq \sum_{x_{-i}} q(x_{-i} | x_i) u_i(x_i, x_{-i}).$$

Multiplying by $q(x_i)$ gives, for all i, x_i and x'_i:

$$\sum_{x_{-i}} q(x_i, x_{-i}) u_i(x'_i, x_{-i}) \leq \sum_{x_{-i}} q(x_i, x_{-i}) u_i(x_i, x_{-i}).$$

The latter is often used as the formula to verify a CE.

How to verify a correlated equilibrium

To verify a correlated equilibrium

We will show that

$$q = \begin{array}{r|cc|} & \multicolumn{2}{c|}{\textit{Other:}} \\ \textit{Player 1:} & \text{Green} & \text{Red} \\ \hline \text{Green} & 0.00 & 0.55 \\ \text{Red} & 0.40 & 0.05 \\ \hline \end{array}$$

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Player 1:	*Other:* Green	Red
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is a correlated equilibrium of

Player 1:	*Other:* Green	Red
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- Suppose Player 1 sees Green. Would it be better for him to act as if he sees Red?

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$$\text{Green} : \frac{0}{0.55}(-10) + \frac{0.55}{0.55}5 = 5$$

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- Suppose Player 1 sees Red. Would it be better for him to act as if he sees Green?

$$\text{Red} : \frac{0.40}{0.45}0 + \frac{0.05}{0.45}(-1) = -0.11$$

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- $(5 + (-0.11))/2 = 2.45 >$ payoffs from two out of three NE.

The problem to find all correlated equilibria

Find all correlated equilibria

Problem: find all correlated equilibria for

	Other:	
You:	Green	Red
Green	$(-10, -10)$	$(5, 0)$
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Solution: set

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- $0 \leq \alpha, \beta, \gamma, \delta \leq 1$

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(Columns: *Other:*)

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Find all correlated equilibria

$u_1(\text{act like } G \mid \text{signal } G)$

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Similarly for u_2 (the column player).

Find all correlated equilibria

We end up with:

$$\begin{cases} 0 \leq \alpha, \beta, \gamma, \delta \leq 1 & 5\gamma - 3\delta \geq 0 \\ \alpha + \beta + \gamma + \delta = 1 & -5\alpha + 3\gamma \geq 0 \\ -5\alpha + 3\beta \geq 0 & 5\beta - 3\delta \geq 0 \end{cases}$$

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This is a solid convex polyhedron in $\mathbb{R}^3$:

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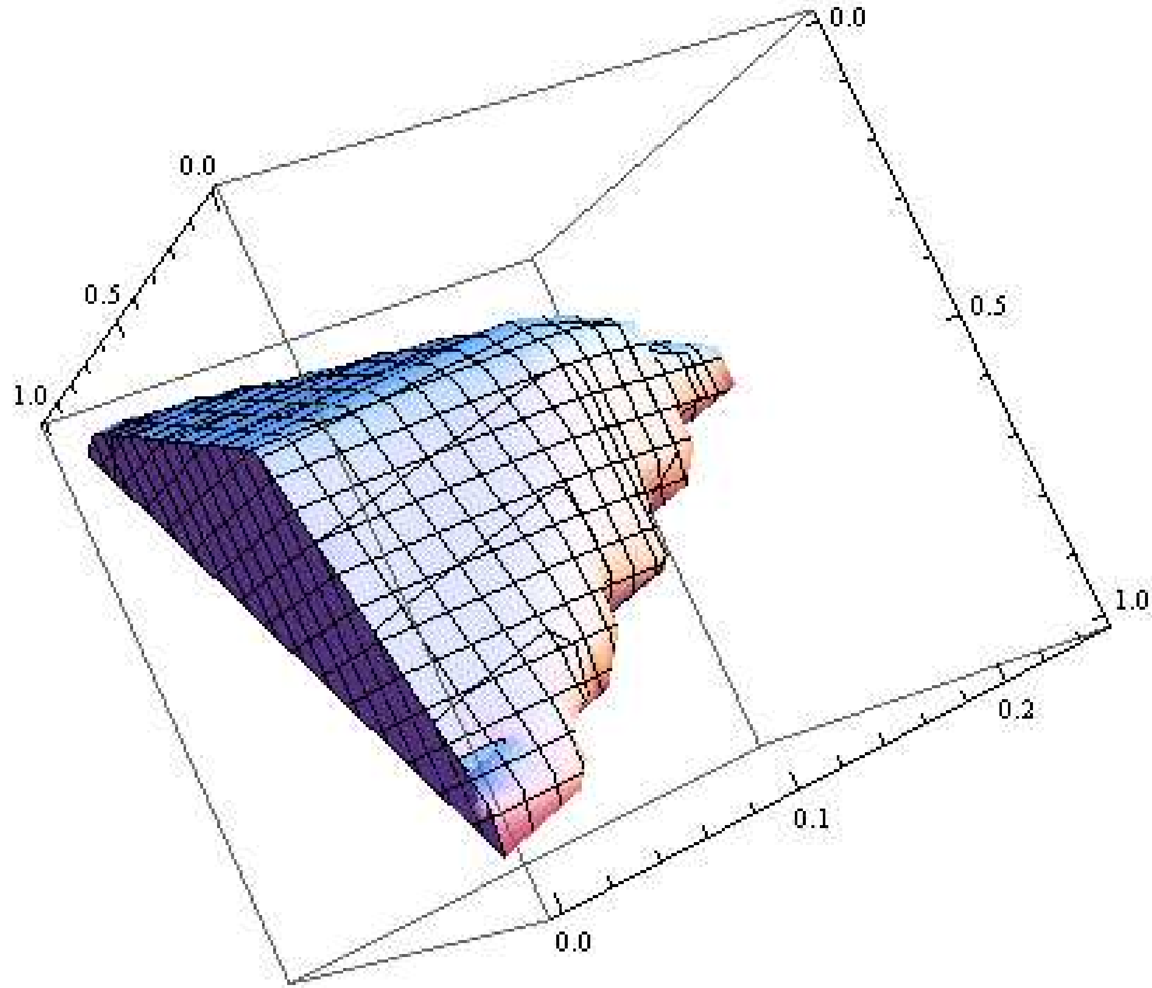
Correlated equilibrium

Admissible values for α, β and γ in the traffic light problem:



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Gives:

$$(\alpha, \beta, \gamma, \delta) = \left(0, \frac{3}{11}, \frac{3}{11}, \frac{5}{11}\right).$$

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Gives:

$$(\alpha, \beta, \gamma, \delta) = \left(0, \frac{3}{11}, \frac{3}{11}, \frac{5}{11}\right).$$

Answer: at most $5/11 = 45\%$ of the time.

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Is it possible to let the row driver wait all the time without compromising a correlated equilibrium?

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Gives:

$$(\alpha, \beta, \gamma, \delta) = (0, 0, 1, 0)\,.$$

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Gives:

$$(\alpha, \beta, \gamma, \delta) = (0, 0, 1, 0)\,.$$

Answer: yes

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Answer: yes, in that case $\gamma = 1$, i.e., the column driver then has to be given green light all of the time.

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Gives:

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{9}{98}, \frac{15}{98}, \frac{1}{2}, \frac{25}{98} \right).$$

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Answer: no. To maintain an equilibrium, the row driver has to give way $15/98 \approx 15\%$ of the time.

Coarse correlated equilibria

Coarse correlated equilibrium (definition)

$q =$

You:	*Other:* Green	Red
Green	0.00	0.55
Red	0.40	0.05

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Definition. A distribution $q \in \Delta(X)$ is called a coarse correlated equilibrium or Hannan set, if, prior to announcing $x \in X$, no party has an incentive to deviate from its own coordinate x_i, assuming that others do not deviate from x_{-i} as well.

Correlated equilibrium (formula)

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Now, in a CEE, everyone blindly accepts what is given. (Think traffic light!)

For all players i and alternative actions x'_i:

$$\begin{aligned}\sum_{x_{-i}} q_{-i}(x_{-i})u_i(x'_i, x_{-i}) &\leq \sum_{x_i, x_{-i}} q(x_i, x_{-i})u_i(x_i, x_{-i}) \\ &= \sum_{x} q(x)u_i(x) \\ &= u_i(q).\end{aligned}$$

This is the same formula as for a Nash equilibrium, only the joint distribution q is not necessarily a distribution induced by strategies $\{s_i\}_i$.

Find CCE for the traffic light problem

For all i and x_i': $\sum_{x_{-i}} q_{-i}(x_{-i})u_i(x_i', x_{-i}) \leq u_i(q)$.

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We will have to solve

$$\begin{cases} (\alpha + \gamma) \cdot -10 + (\beta + \delta) \cdot \ \ 5 \leq -10\alpha + 5\beta + 0\gamma + 1\delta, \\ (\alpha + \gamma) \cdot \ \ \ 0 + (\beta + \delta) \cdot -1 \leq -10\alpha + 5\beta + 0\gamma + 1\delta, \\ (\alpha + \beta) \cdot -10 + (\gamma + \delta) \cdot \ \ 5 \leq -10\alpha + 0\beta + 5\gamma + 1\delta, \\ (\alpha + \beta) \cdot \ \ \ 0 + (\gamma + \delta) \cdot -1 \leq -10\alpha + 0\beta + 5\gamma + 1\delta. \end{cases}$$

Find CCE for the traffic light problem (continued)

Which is

$$\begin{pmatrix} 10 & 0 & 5 & -2 \\ 5 & 3 & 0 & 1 \\ 10 & 5 & 0 & -2 \\ 5 & 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

provided $0 \leq \alpha, \beta, \gamma, \delta \leq 1$ and $\alpha + \beta + \gamma + \delta = 1$.

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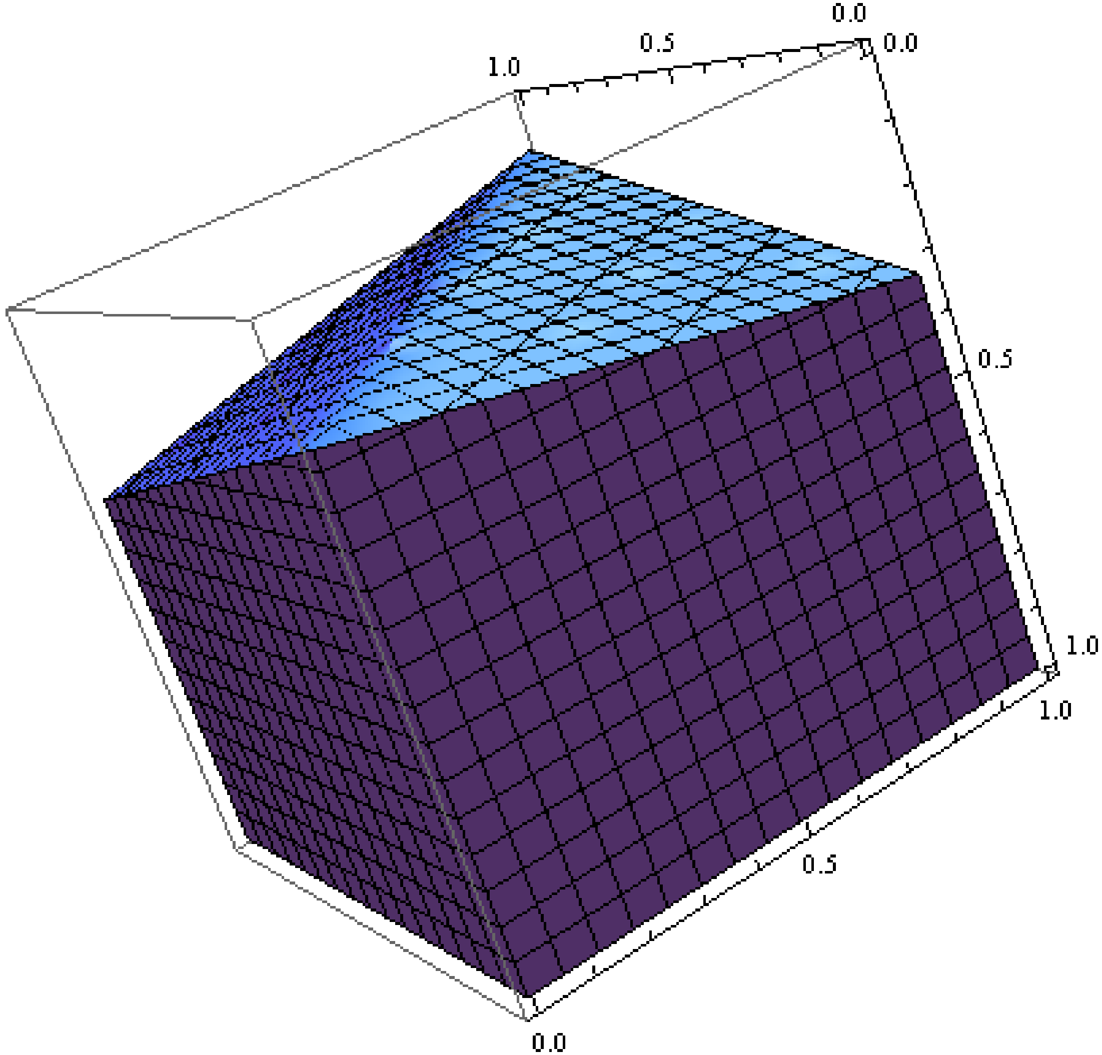
Substitute $1 - (\alpha + \beta + \gamma)$ for δ. Then

$$\begin{pmatrix} 12 & -2 & 3 \\ 6 & 2 & -1 \\ 12 & 3 & -2 \\ 6 & -1 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \geq \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

provided $0 \leq \alpha, \beta, \gamma \leq 1$.

Find CCE for the traffic light problem (continued)

Admissible values for α, β and γ in the traffic light problem:



Hierarchy of equilibria

Nash equilibrium ⇒ correlated equilibrium

If strategies are independent, we have

$$s_{-i}(x_{-i}|x_i) = s_{-i}(x_{-i})$$

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Immediately,

$$\text{for all } i \text{ and } x_i' : \sum_{x_{-i}} s_{-i}(x_{-i}) u_i(x_i', x_{-i}) \leq u_i(s) \qquad \text{(Nash)}$$

$$\Rightarrow \quad \text{for all } x_i,\, i \text{ and } x_i' : \sum_{x_{-i}} s(x_{-i}|x_i) u_i(x_i', x_{-i}) \leq u_i(s) \qquad \text{(CE)}$$

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The latter is the conditional formulation of a correlated equilibrium. (See slide where formula for CE is introduced.)

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The latter is the conditional formulation of a correlated equilibrium. (See slide where formula for CE is introduced.)

Therefore, every Nash equilibrium is a correlated equilibrium.

Summary

NE: for all i and x'_i : $\sum_{x_{-i}} s_{-i}(x_{-i})\, u_i(x'_i, x_{-i}) \leq \sum_x s(x) u_i(x)$

CE: for all x_i, i and x'_i : $\sum_{x_{-i}} q(x_i, x_{-i})\, u_i(x'_i, x_{-i}) \leq \sum_{x_{-i}} q(x_i, x_{-i}) u_i(x_i, x_{-i})$

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- We already derived NE $\Rightarrow$ CE. Therefore, NE $\Rightarrow$ CE $\Rightarrow$ CCE.