SIKS tutorial "Agent Systems"

Multi-agent learning

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Objective of this presentation

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To consider processes of **adaptation**

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To consider processes of **adaptation** in **repeated two-player games**

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- Or bandwidth (think competition between broadband providers).
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- Players **adapt** to each other's actions.
- For some modes of adaptation, interesting dynamics will occur.

Outline

1. Introduction
   - Multi-agent learning (MAL)
   - Teaching
2. Cournot dynamics
   - Cournot competition
   - Cournot equilibrium
3. Alternative Cournot dynamics
   - Work of David Rand (1978)
   - Work of Tönu Puu (1991)
   - Conclusions

Outline

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What is multi-agent learning?

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- Vs. one other learning agent (playing $1 - 1$).
- Vs. multiple learning agents (playing $n - n$).
- Vs. very many other learning agents. (Large populations, but playing $1 - 1$.)

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- Formalise and study **emergence** in multi-agent systems.
- Explain how Nash equilibria may come about.

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Teaching

A game in normal form

	L	R
T	$1, 0$	$3, 2$
B	$2, 1$	$4, 0$

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- However: action profile (T, R) with payoff profile $(3, 2)$ Pareto dominates the equilibrium.
- Both can achieve the Pareto optimum if the row player teaches T, and the column player recognises this, and follows.

Coordination game

Who should be teaching here?

	L	R
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Antoine Augustin Cournot (1801-1877)

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- Agents maximise profit given their competitors' decisions.

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- At (arbitrary or periodic) times the quantities x and y are revealed (simultaneously), and both agents adapt their production to the new situation (simultaneously).

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- The Cournot equilibrium is not necessarily Pareto dominant.

Cournot dynamics

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6. If yes, is it possible that some equilibria are missed by adaptation / iteration?

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“Exotic Phenomena in Games and Duopoly”

David Rand. Journal of Math. Ec., 1978, vol. 5(2), pp. 173-184.

Journal of Mathematical Economics 5 (1978) 173–184. © North-Holland Publishing Company

EXOTIC PHENOMENA IN GAMES AND DUOPOLY MODELS

David RAND

Mathematics Institute, University of Warwick, Coventry, U.K.

Received March 1977, final version received November 1977

1. Introduction

In this short note we draw attention to some very complex behaviour which occurs in very simple games and, in particular, in Cournot duopoly and its generalisations [Cournot (1834) and Wald (1951)].

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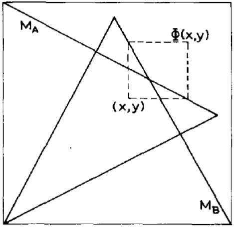

Fig. 1

It will be clear that the results which follow will not be greatly affected if we replace the curves by any segments which approximate them provided the slope is always greater than some constant greater than one. Any difficulties about the structural stability of the results arises from the non-differentiability of the curves at $z_A = (1-\varepsilon_1, \frac{1}{2})$ and $z_B = (\frac{1}{2}, 1-\varepsilon_2)$. Otherwise the results are perfectly stable.

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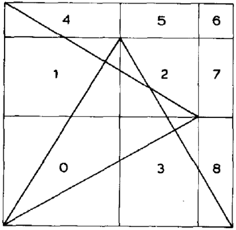


Fig. 2

Thus $[i_0 i_1 \dots i_n] = [i_0] \cap \Phi^{-1}(i_1 i_2 \dots i_n])$. Clearly it is possible that $[i_0 i_1 \dots i_n] = \emptyset$.

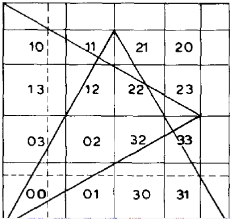
Equivalently the rectangles can be constructed by drawing in the vertical lines through the two points of intersection of l and M_B and the two points of intersection of h and M_A. This cuts

“Exotic Phenomena in Games and Duopoly”

David Rand. Journal of Math. Ec., 1978, vol. 5(2), pp. 173-184.

sets. Moreover, since ε_1 and ε_2 are chosen very small there exists a constant F satisfying $0<F<1$ such that for any such sequence $|i_0 i_1 \dots i_n| < F|i_1 i_2 \dots i_n|$ where $|i|$ denotes the maximum of the height and width of $[i]$. Thus, by induction $|i_0 i_1 \dots i_n| < F^n |i_n| < F^n$, which proves that $|i_0 i_1 \dots i_n| \to 0$ as $n \to +\infty$, whence $[i]$ consists of a single point.

We assume now that the situation is as in fig. 3, so that $\Phi(I^2 \backslash J)$ is contained in the union of the rectangles [10], [13], [03], [00], [01], [30] and [31].



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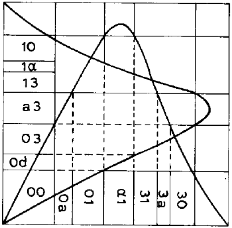
where $i_0, i_1, \ldots, i_n \in \Omega$.

To simplify the exposition we shall also assume that h meets M_A in the interior of $[10] \cup [40]$ and l meets M_B in the interior of $[30] \cup [80]$. Let $\hat{\Omega}$ denote the set {01,1d,13,a3,03,0d,00,0a,01,d1,31,3a,30}. Then this just says that if x belongs to the union of [4], [5], [6], [7], [8], [α] and [β] then $\Phi(x) \in [ij]$ where $ij \in \hat{\Omega}$ (see fig. 6).



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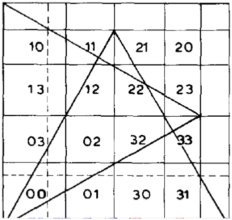

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instead of the two lines $y=C_A$, $x=C_B$ as in Example 1 to cope with the fact that the components of the preimages of rectangles intersecting this area may be larger in size than the original rectangle. This prevents us from obtaining a direct analogue of Lemma 3.2. Now continue this decomposition by constructing the subsets

$$[i_0 i_1 \dots i_n] = \{x \in I^2 | \Phi^j(x) \in [i_j] \text{ for } 0 \leqq j \leqq n\},$$

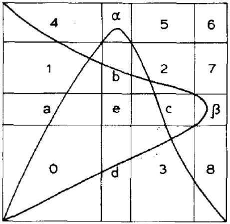

Fig. 5

Outline

1. Introduction
   - Multi-agent learning (MAL)
   - Teaching
2. Cournot dynamics
   - Cournot competition
   - Cournot equilibrium
3. Alternative Cournot dynamics
   - Work of David Rand (1978)
   - Work of Tönu Puu (1991)
   - Conclusions

Alternative assumptions

Proposed around 1991

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- Agents have different **production costs** α and β per unit.
- **Adaptation** proceeds gradually, through learning:

$$\text{new} = (1-\delta)\cdot \text{old} + \delta \cdot \text{input}.$$

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- Compute response functions as usual.

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- If $\alpha/\beta \in [4/25, 25/4]$ then trajectory remains bounded.
- If $0.16 \leq \alpha/\beta \leq 0.171$ or $5.828 \leq \alpha/\beta \leq 6.25$ then bounded but not stable $\Rightarrow$ periodicity, semi-periodicity, or chaos.

Four types of system behaviour

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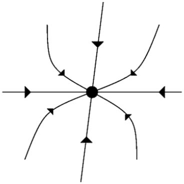

Stable

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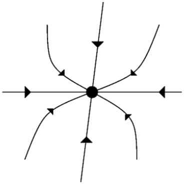

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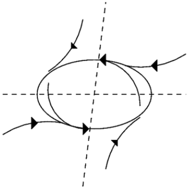

Periodic

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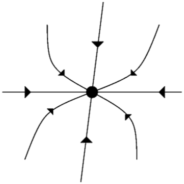

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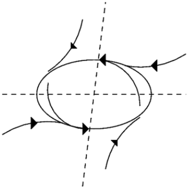

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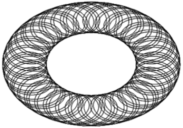

Quasiperiodic

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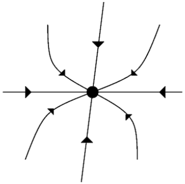

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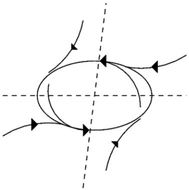
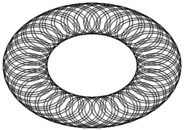


Periodic



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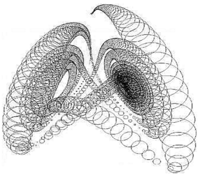

Chaotic

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