Multi-agent learning

Emergence of Conventions

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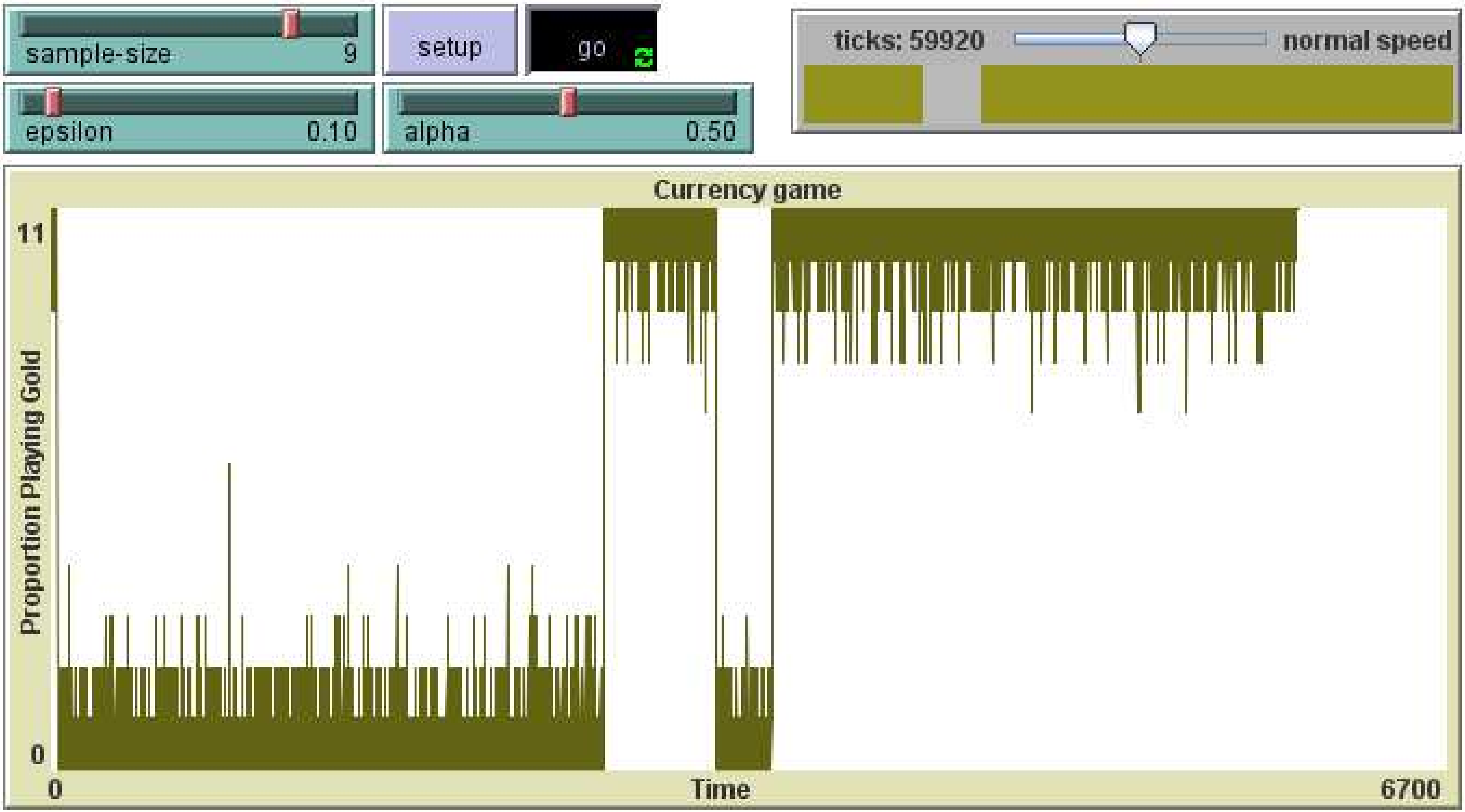
  - Agents act rationally most of the time, but sometimes they do not for whatever reason (error of perception, idiosyncrasy, invention, experimentation).
  - The presence of perturbations implies that the dynamic never settles but remains in flux.
  - If minimal random behaviour is allowed, "sub-optimal" stable states ascend into "global optimal" stable states.

Example: the emergence of common currency



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2. **Perturbed Markov processes.** (Regular perturbed Markov process, punctuated equilibrium, stochastically stable state.)

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   - Compare k so-called z-trees, where k is the number of Recurrence classes in the non-perturbed Markov process (Peyton Young, 1993).

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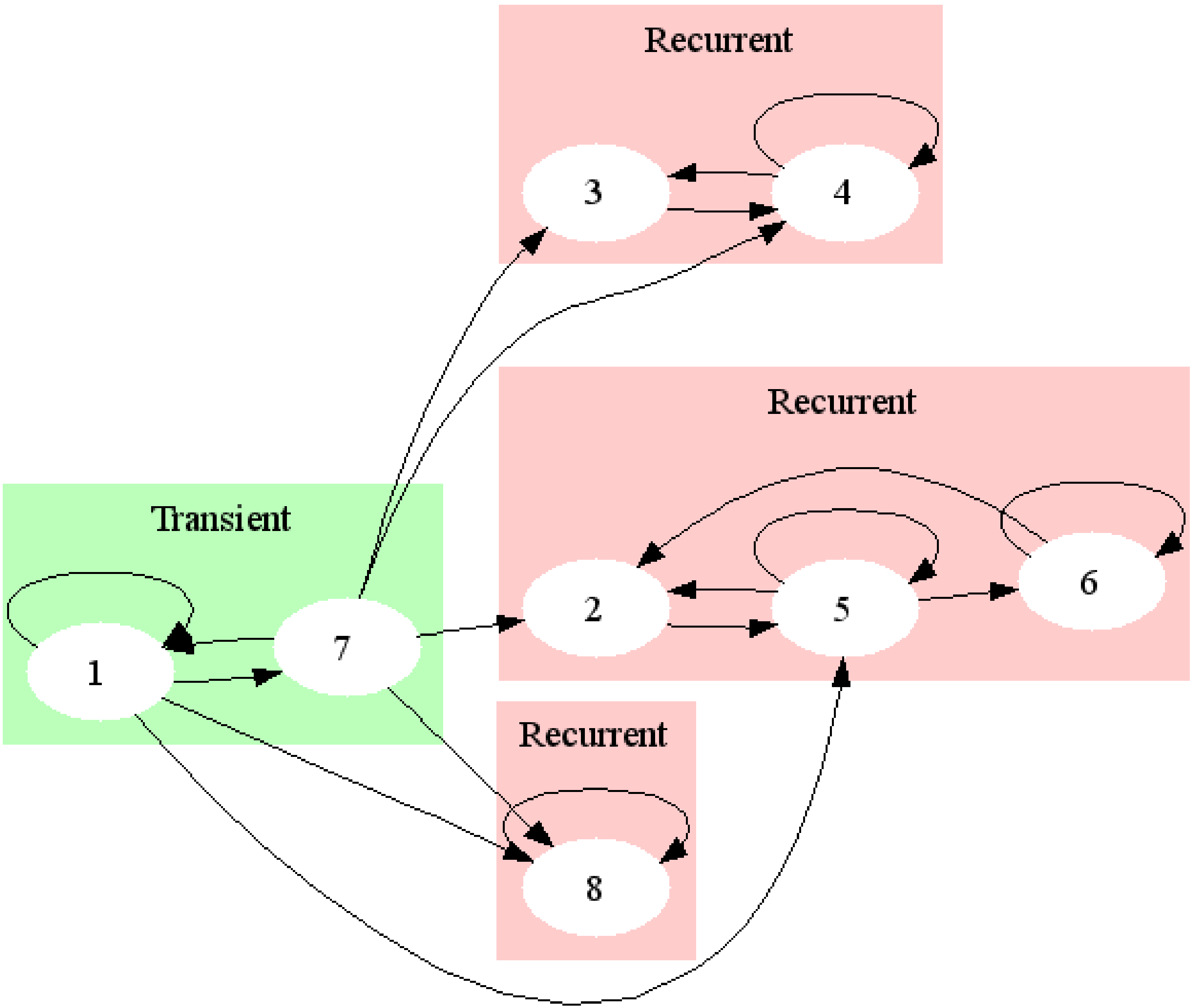
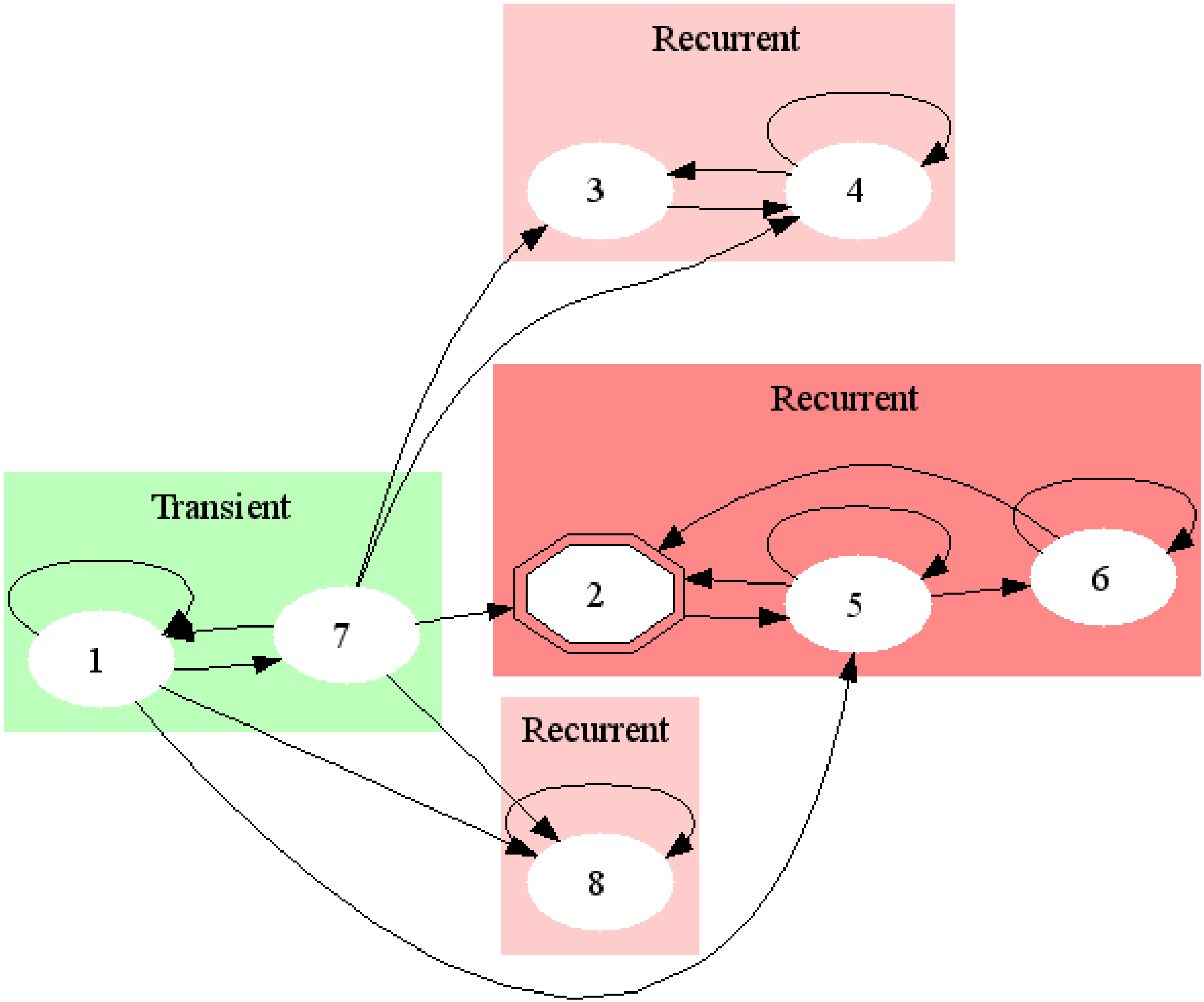
- Schelling's model of segregation (1969).

Part 1: Markov processes

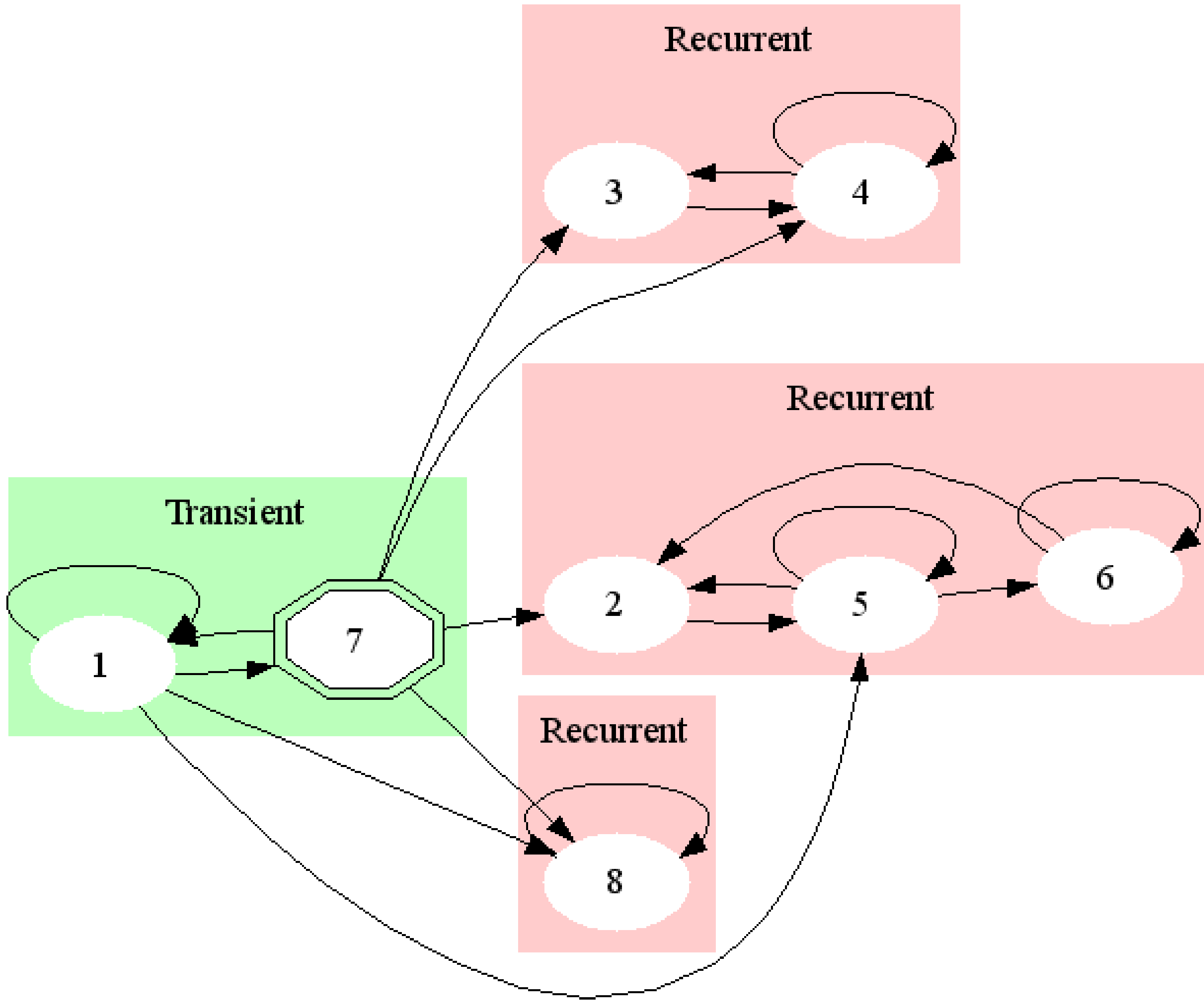
Communication classes



Start state matters



Start state matters... but here it does not



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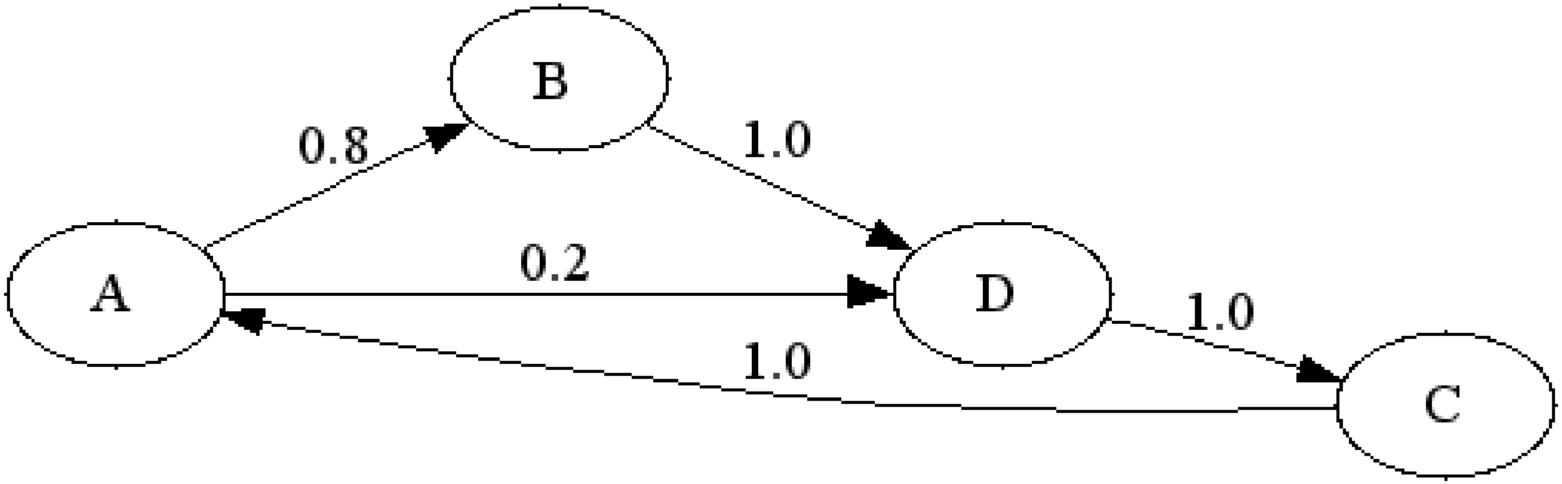

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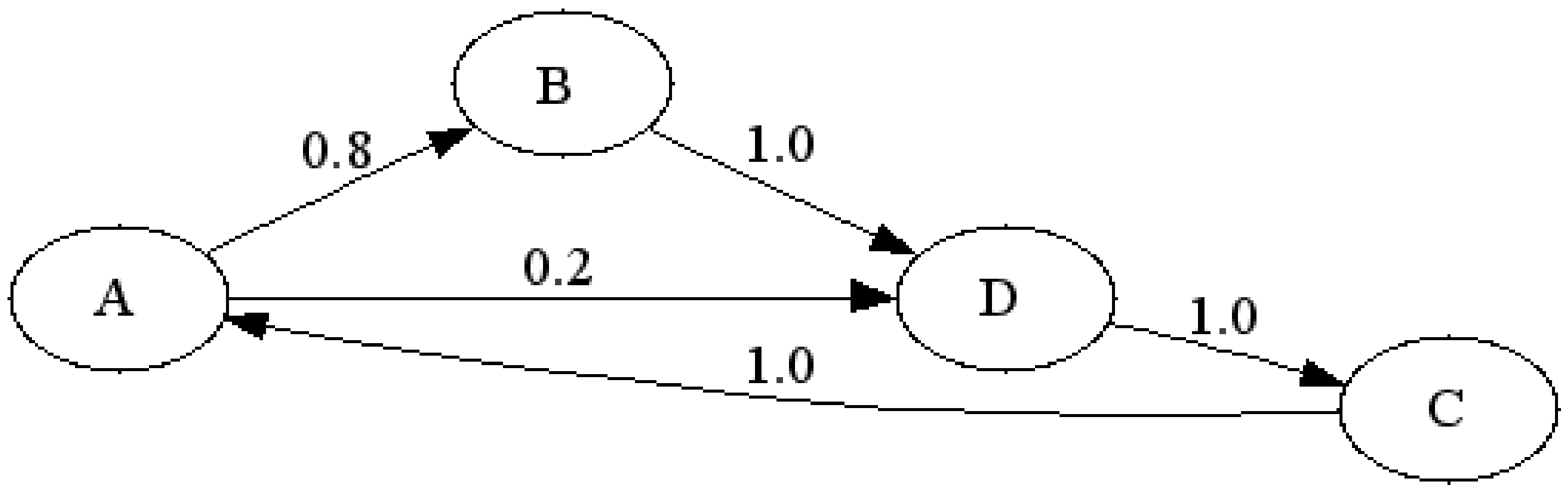

$$P(A) = P(A|A')P(A') + P(A|B')P(B') + P(A|C')P(C') + P(A|D')P(D')$$

Suppose that visiting probabilities are stationary ($A = A'$, $B = B'$, $\dots$):

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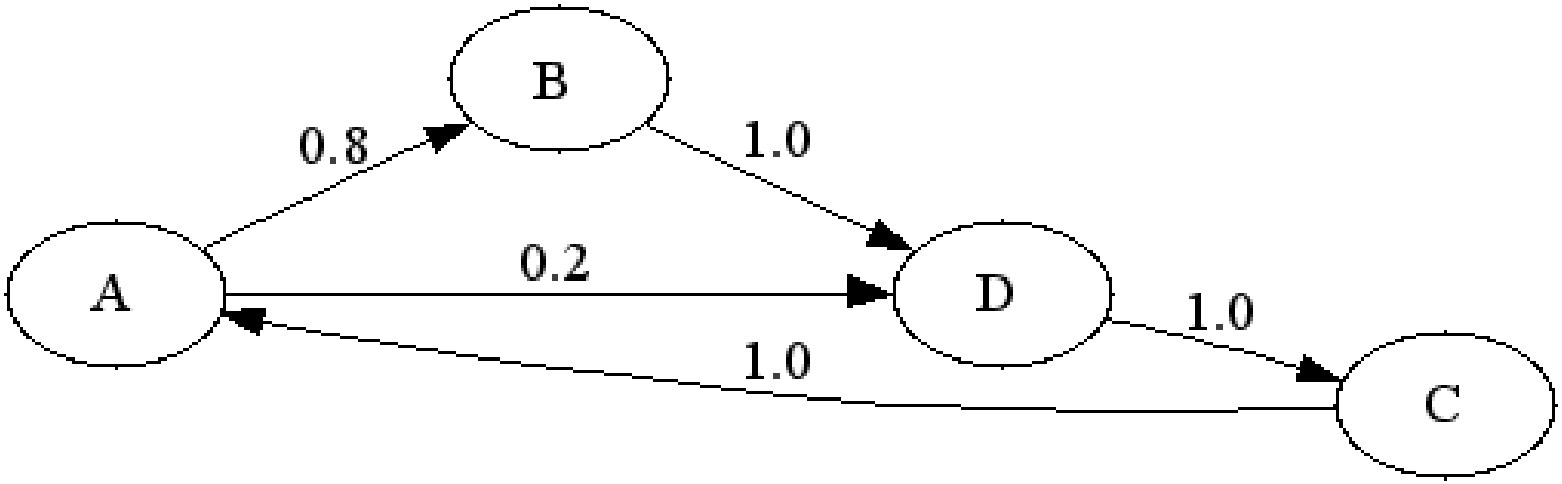

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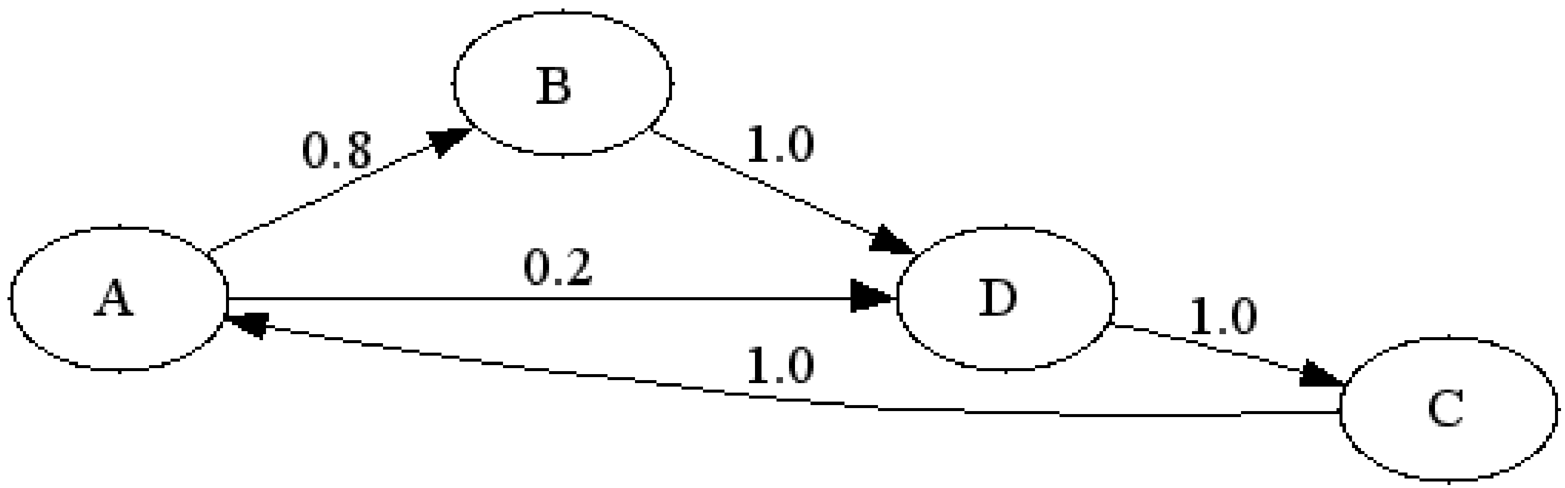

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- Solve n equations in n unknowns. What if S is large?

$$\begin{pmatrix} 0.1 & 0.2 & 0.0 & 0.1 & 0.0 & 0.1 & 0.0 & 0.3 & 0.0 & 0.2 \\ 0.1 & 0.2 & 0.0 & 0.1 & 0.0 & 0.1 & 0.0 & 0.3 & 0.0 & 0.2 \\ 0.1 & 0.2 & 0.0 & 0.1 & 0.0 & 0.1 & 0.0 & 0.3 & 0.0 & 0.2 \\ 0.0 & 0.1 & 0.1 & 0.2 & 0.0 & 0.1 & 0.0 & 0.3 & 0.0 & 0.2 \\ 0.5 & 0.2 & 0.0 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.2 \\ 0.1 & 0.2 & 0.0 & 0.1 & 0.0 & 0.1 & 0.0 & 0.3 & 0.0 & 0.2 \\ 0.0 & 0.1 & 0.1 & 0.2 & 0.0 & 0.1 & 0.0 & 0.3 & 0.0 & 0.2 \\ 0.1 & 0.2 & 0.0 & 0.1 & 0.0 & 0.1 & 0.0 & 0.3 & 0.0 & 0.2 \\ 0.3 & 0.1 & 0.2 & 0.0 & 0.1 & 0.0 & 0.0 & 0.0 & 0.3 & 0.0 \\ 0.1 & 0.2 & 0.0 & 0.1 & 0.0 & 0.1 & 0.0 & 0.3 & 0.0 & 0.2 \end{pmatrix}$$

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0.0 & 0.1 & 0.1 & 0.2 & 0.0 & 0.1 & 0.0 & 0.3 & 0.0 & 0.2 \\
0.5 & 0.2 & 0.0 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.2 \\
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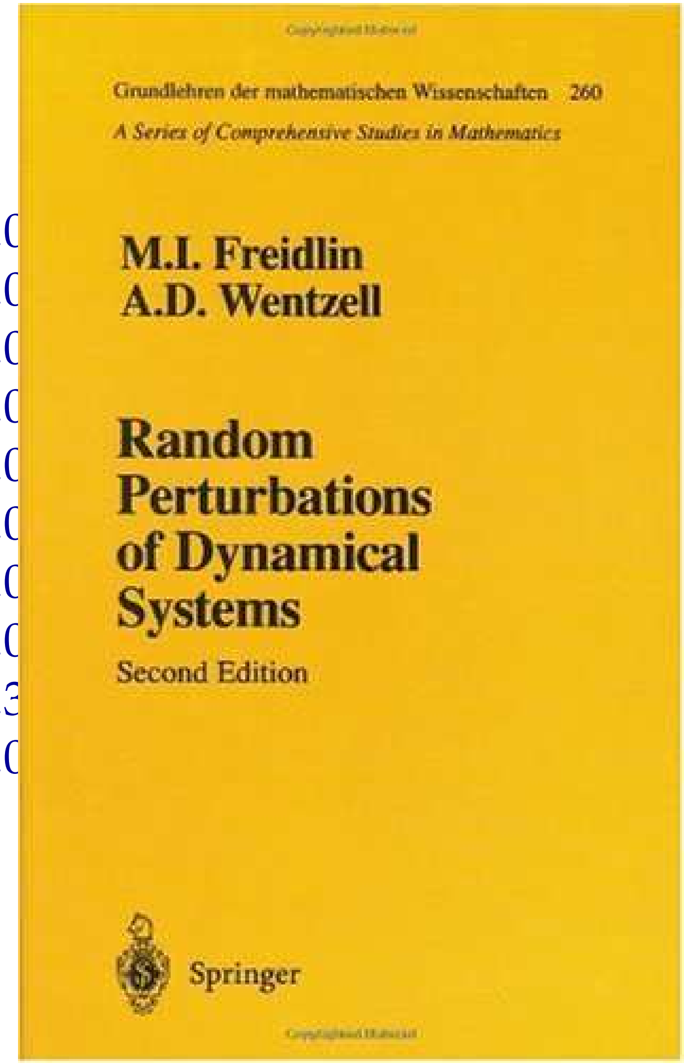

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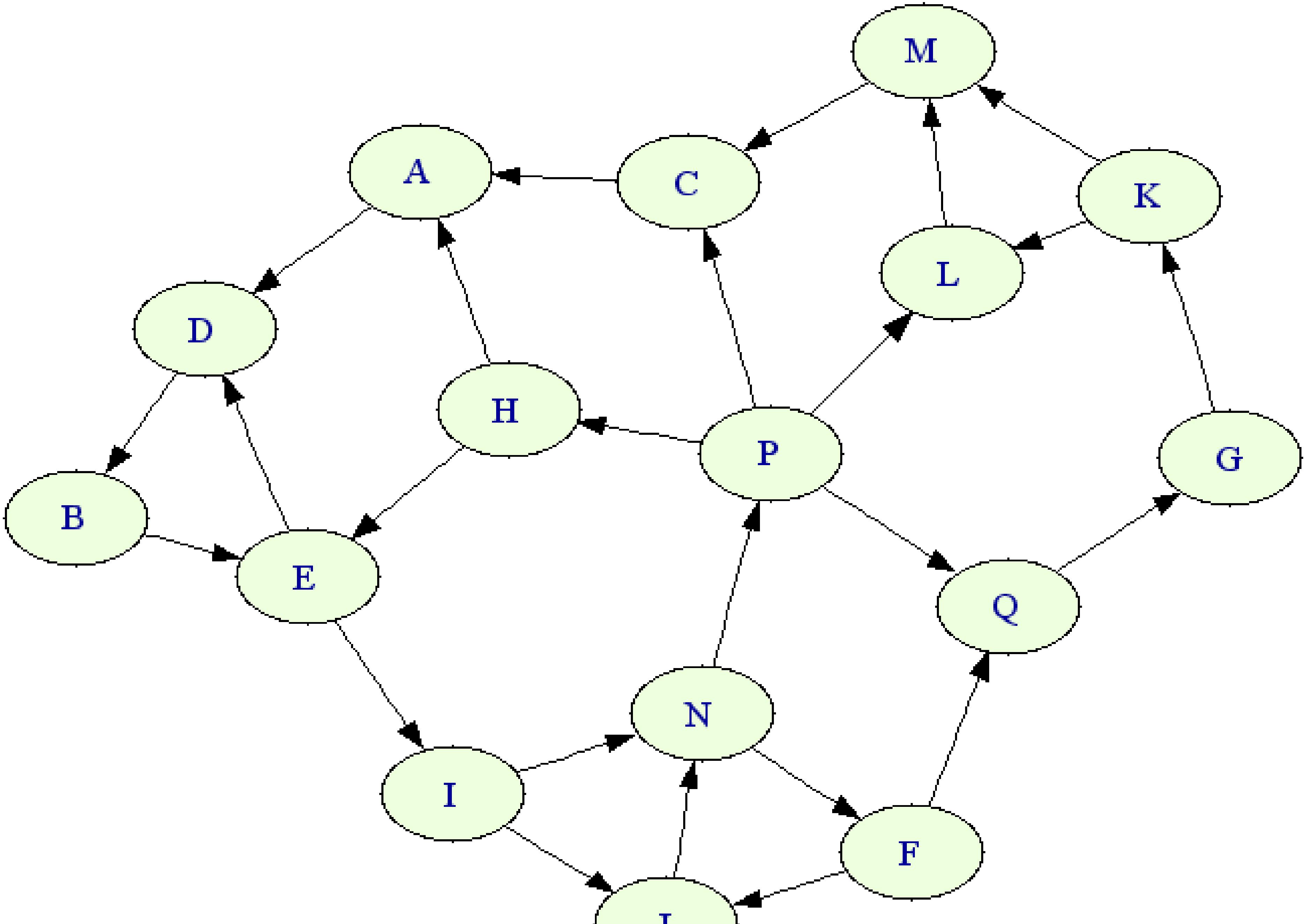
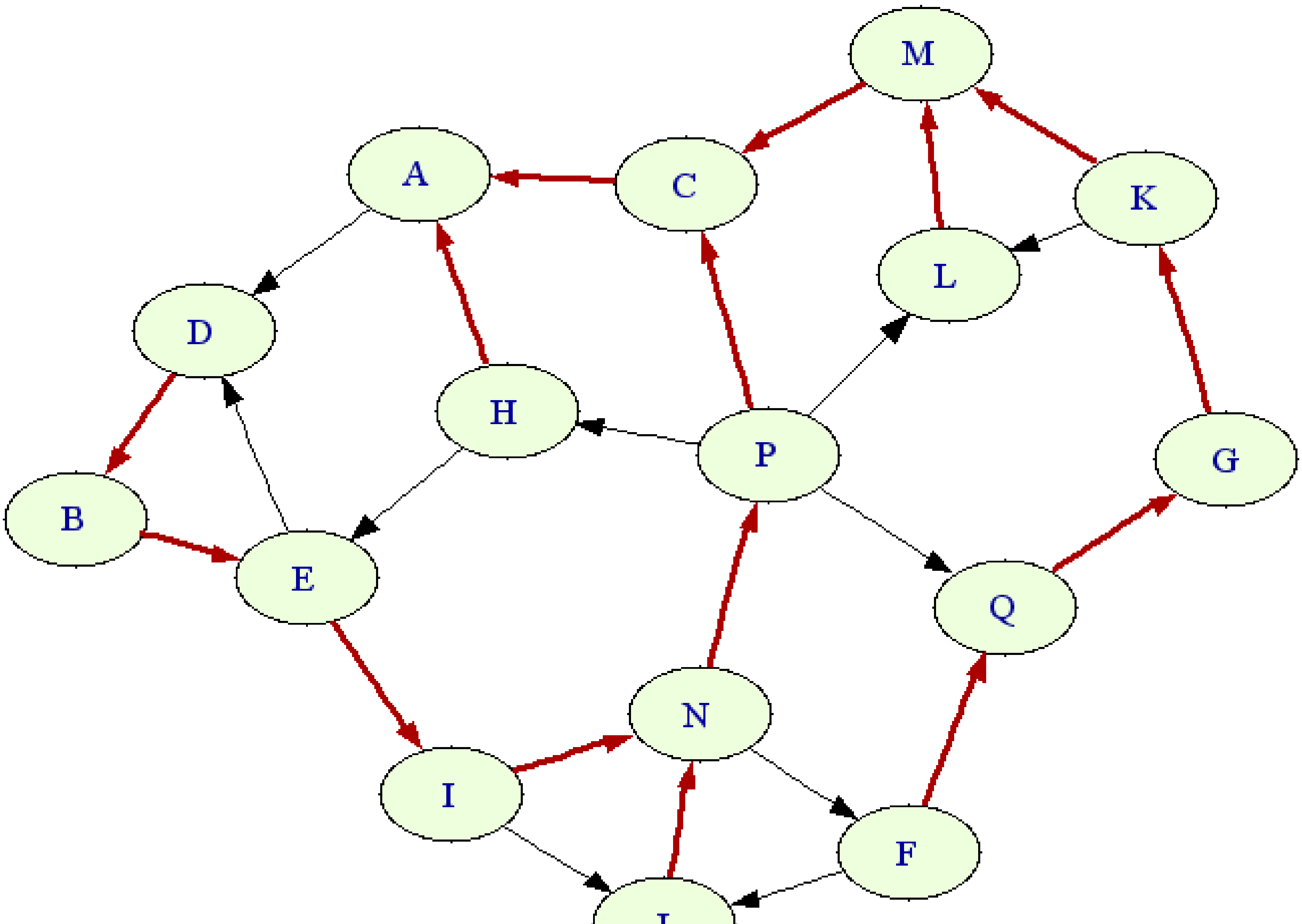
An irreducible (and finite) Markov process



One possible A-tree (i.e., tree for state A)



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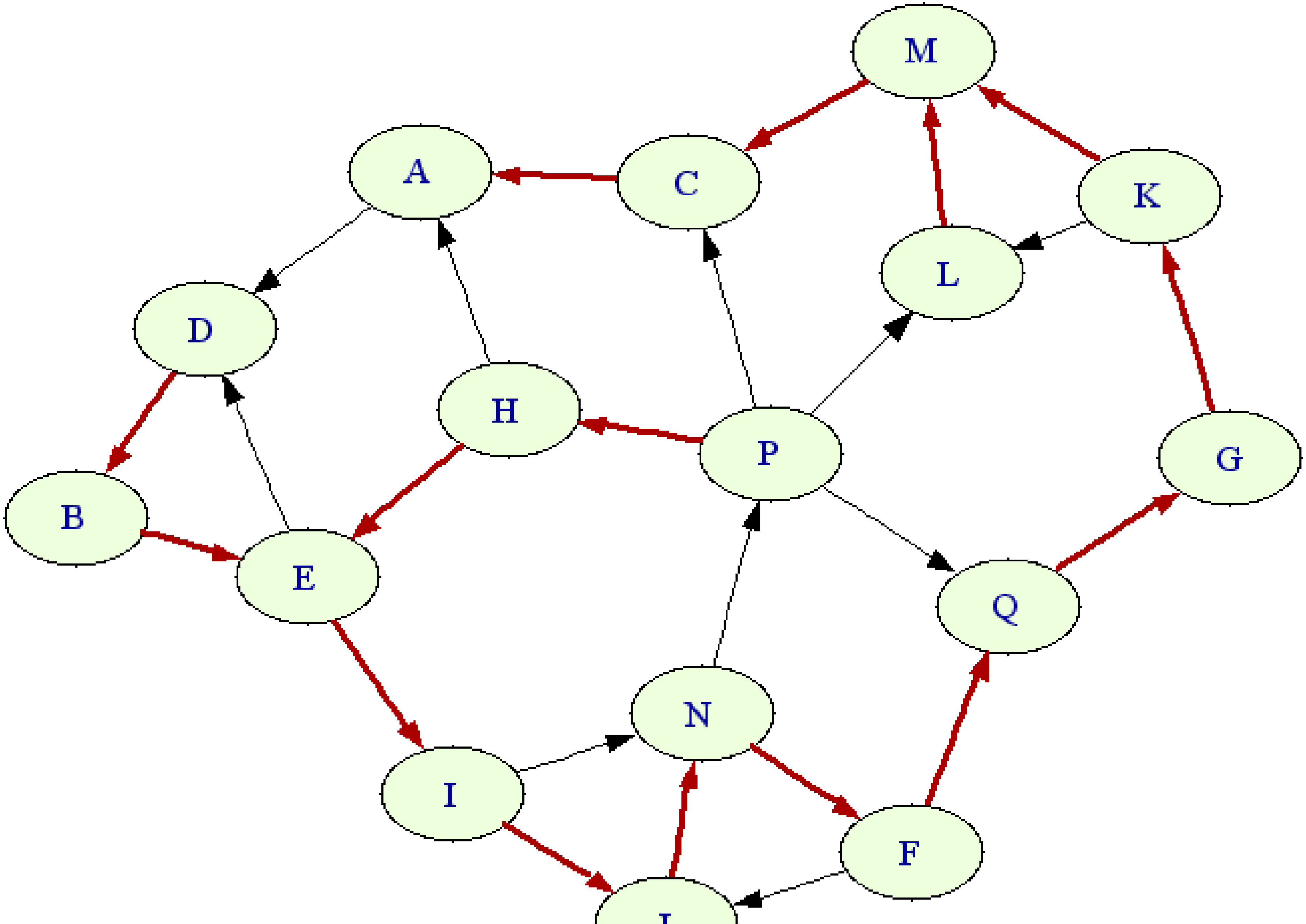

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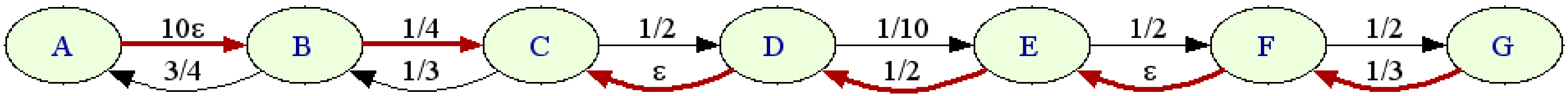
Theorem (Freidlin & Wentzell, 1984). Let P be an irreducible finite Markov process. Then, for all states, the likelihood of that state is proportional to the stationary probability of that state.

Counting s-trees with Freidlin & Wentzell: example



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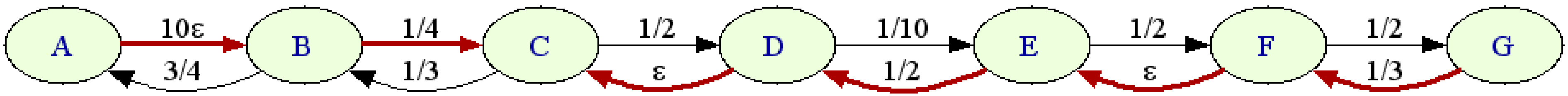

Freidlin & Wentzell (1984):

$$\mu(s) = \frac{v(s)}{\sum_{t \in S} v(t)}, \text{ where } \quad v(t) =_{Def} \sum_{T \in T_s} \ell(T_s)$$

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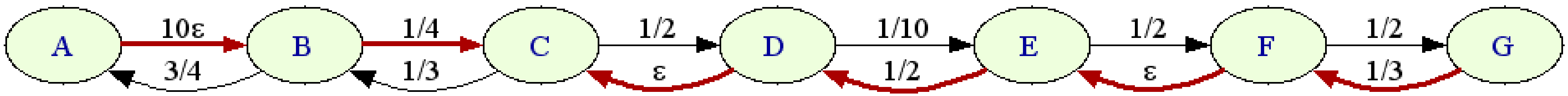

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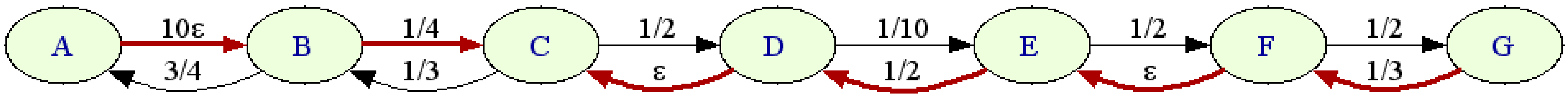

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Similarly:

State:	A	B	C	D	E	F	G
Distribution:	$\epsilon^2/24$	$5\epsilon^3/9$	$5\epsilon^3/12$	$5\epsilon^2/24$	$\epsilon^2/24$	$\epsilon/48$	$\epsilon/32$

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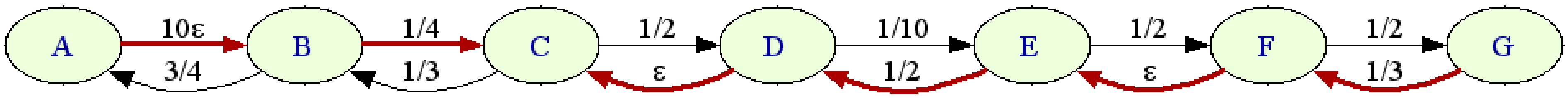

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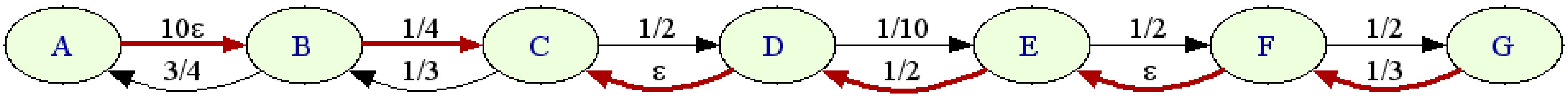
If $\epsilon \to 0$, all ϵ-connections will be squeezed

Counting s-trees with Freidlin & Wentzell: example



Freidlin & Wentzell (1984):

$$\mu(s) = \frac{v(s)}{\sum_{t \in S} v(t)}, \text{ where } \quad v(t) =_{Def} \sum_{T \in T_s} \ell(T_s)$$

The (unique) C-tree is coloured red. $\ell(T_C) = 10\epsilon \cdot 1/4 \cdot \dots = 5\epsilon^3/12$.

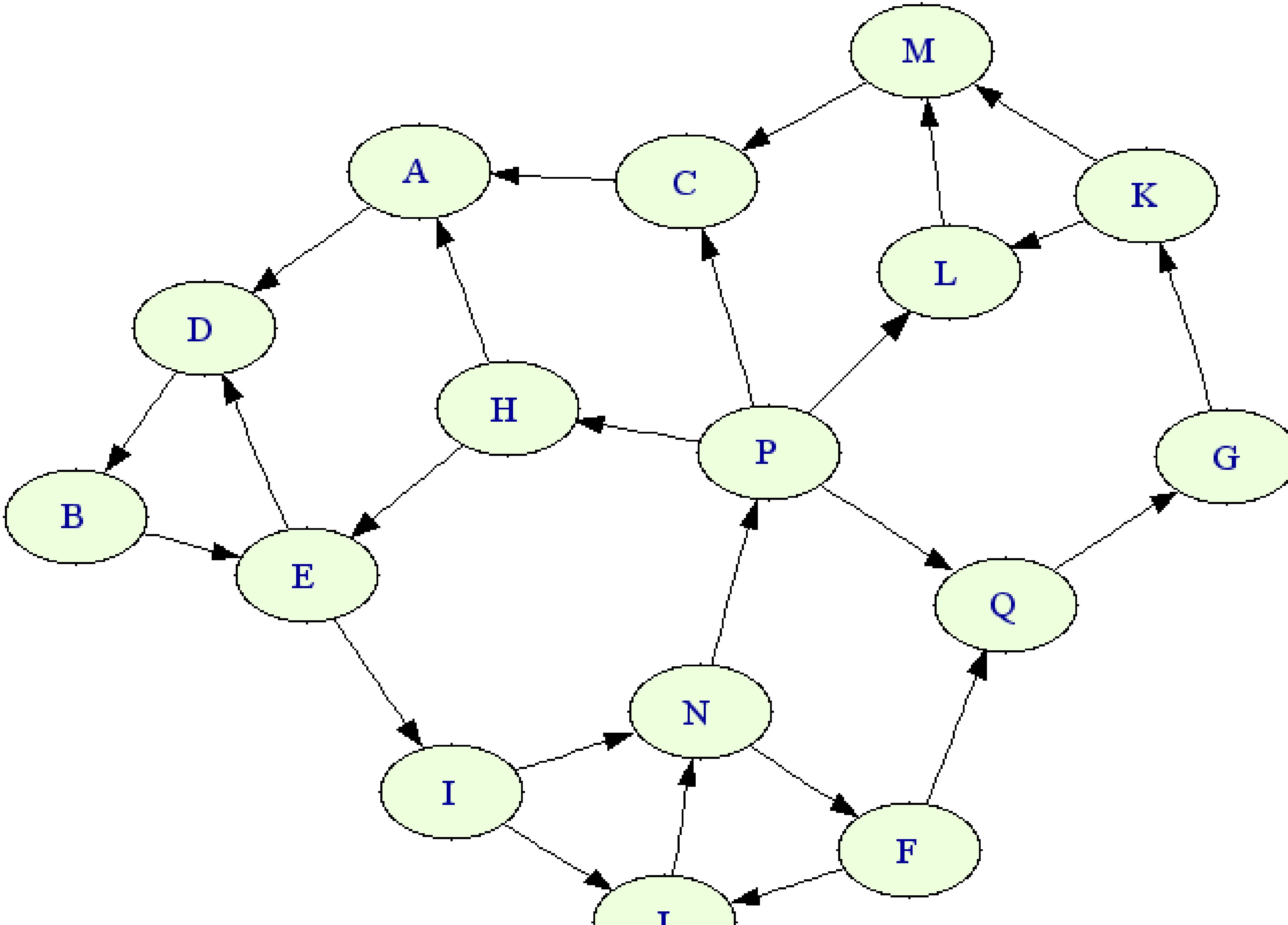
Similarly:

State:	A	B	C	D	E	F	G
Distribution:	$\epsilon^2/24$	$5\epsilon^3/9$	$5\epsilon^3/12$	$5\epsilon^2/24$	$\epsilon^2/24$	$\epsilon/48$	$\epsilon/32$

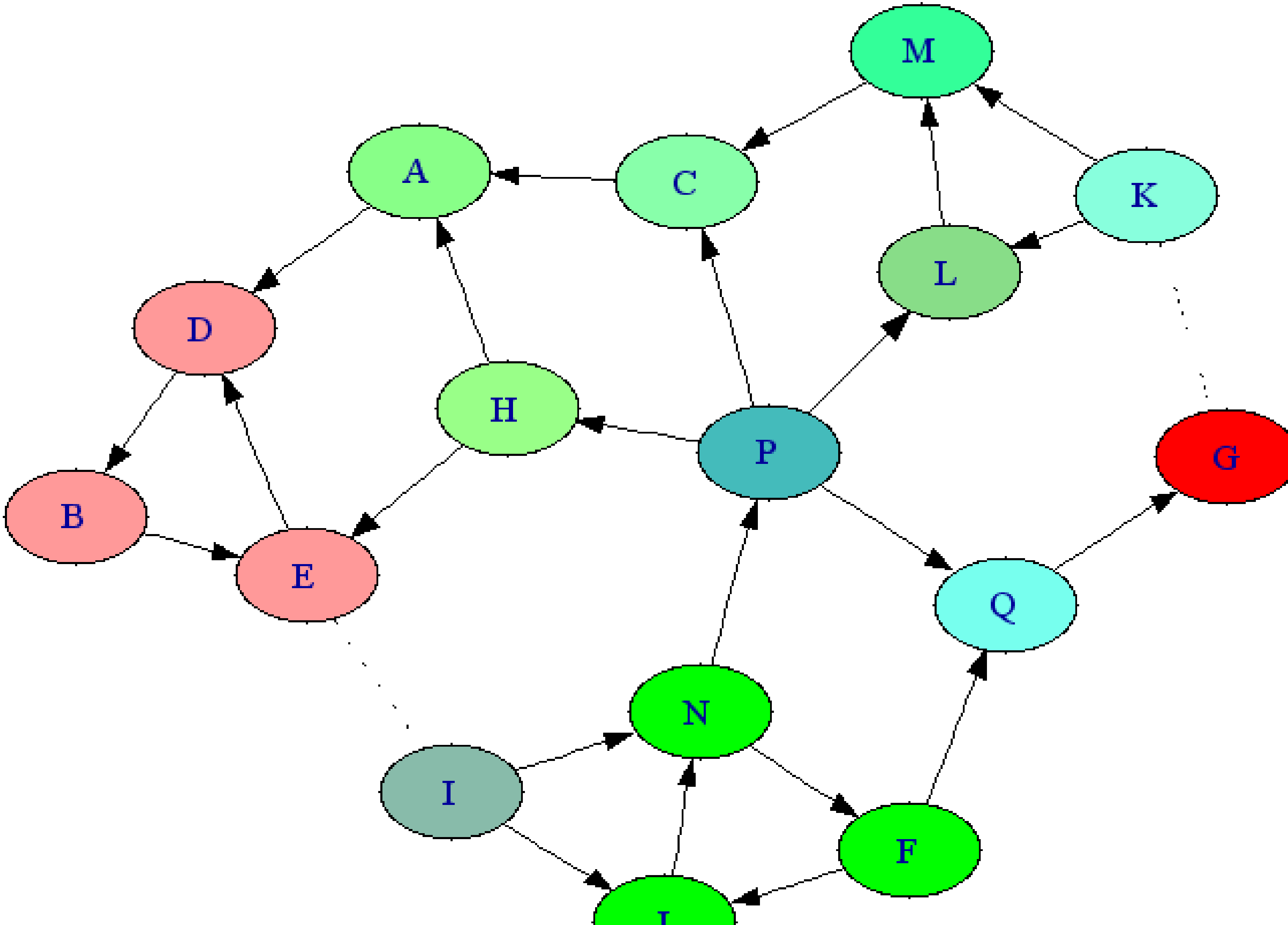
If $\epsilon \to 0$, all ϵ-connections will be squeezed, and F and G “win”.

Part 2:
Perturbed Markov processes

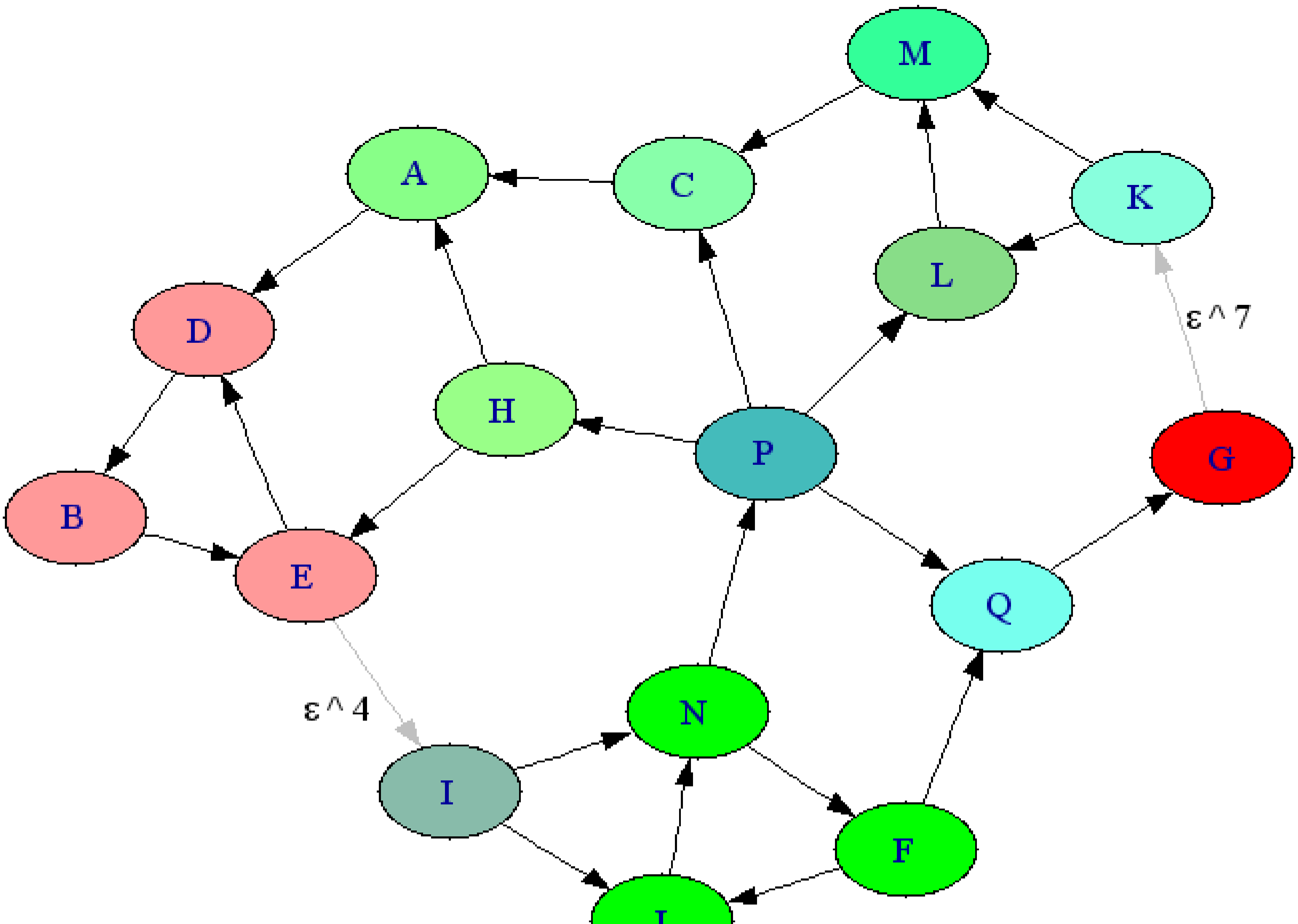
A Markov process may be ergodic (path-independent)



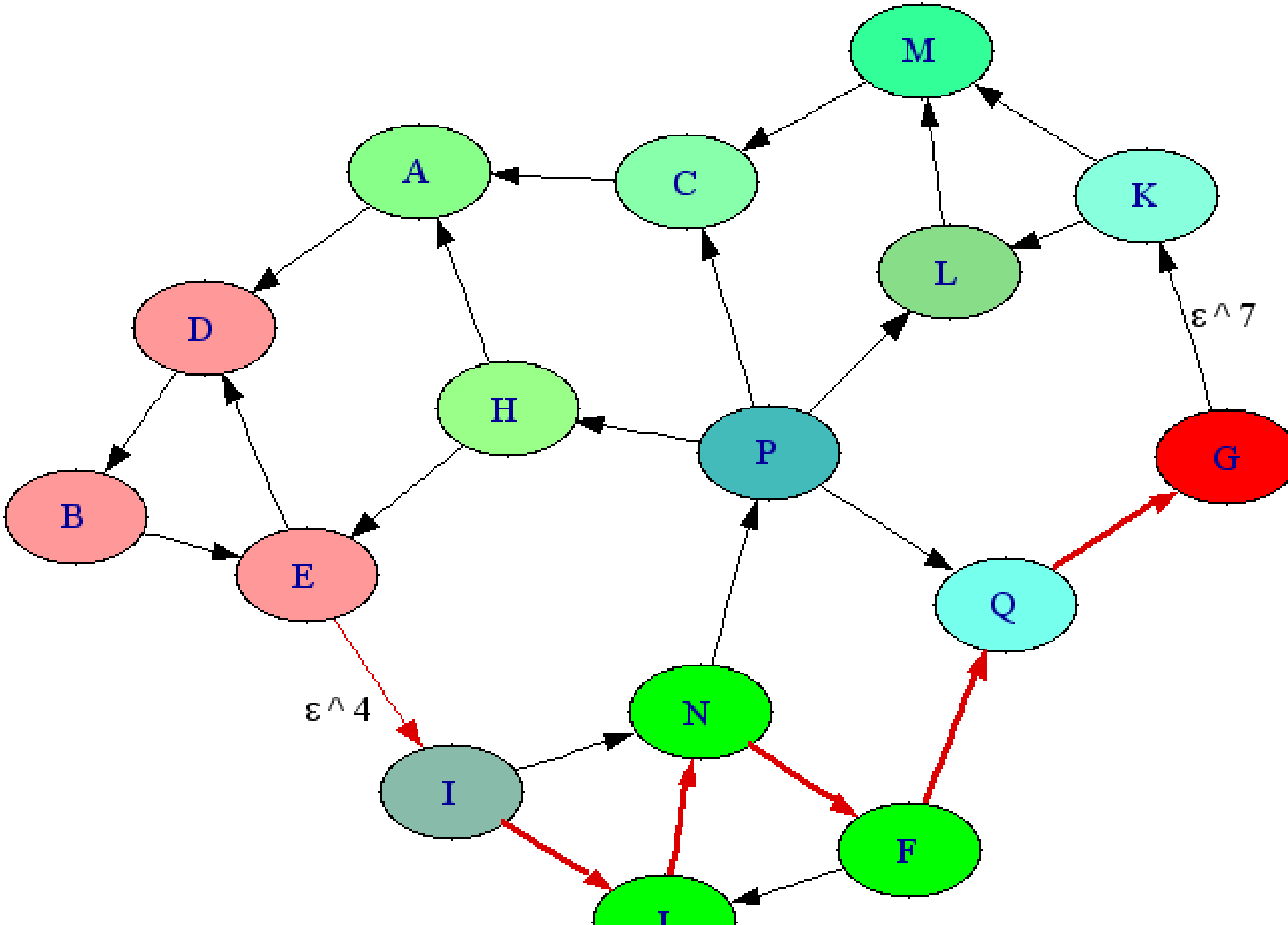
A Markov process may be non-ergodic (path-dependent)



Make it ergodic by perturbing with $\epsilon^{r(s,s')}$



Compute s-trees from P^0-Recurrence classes only



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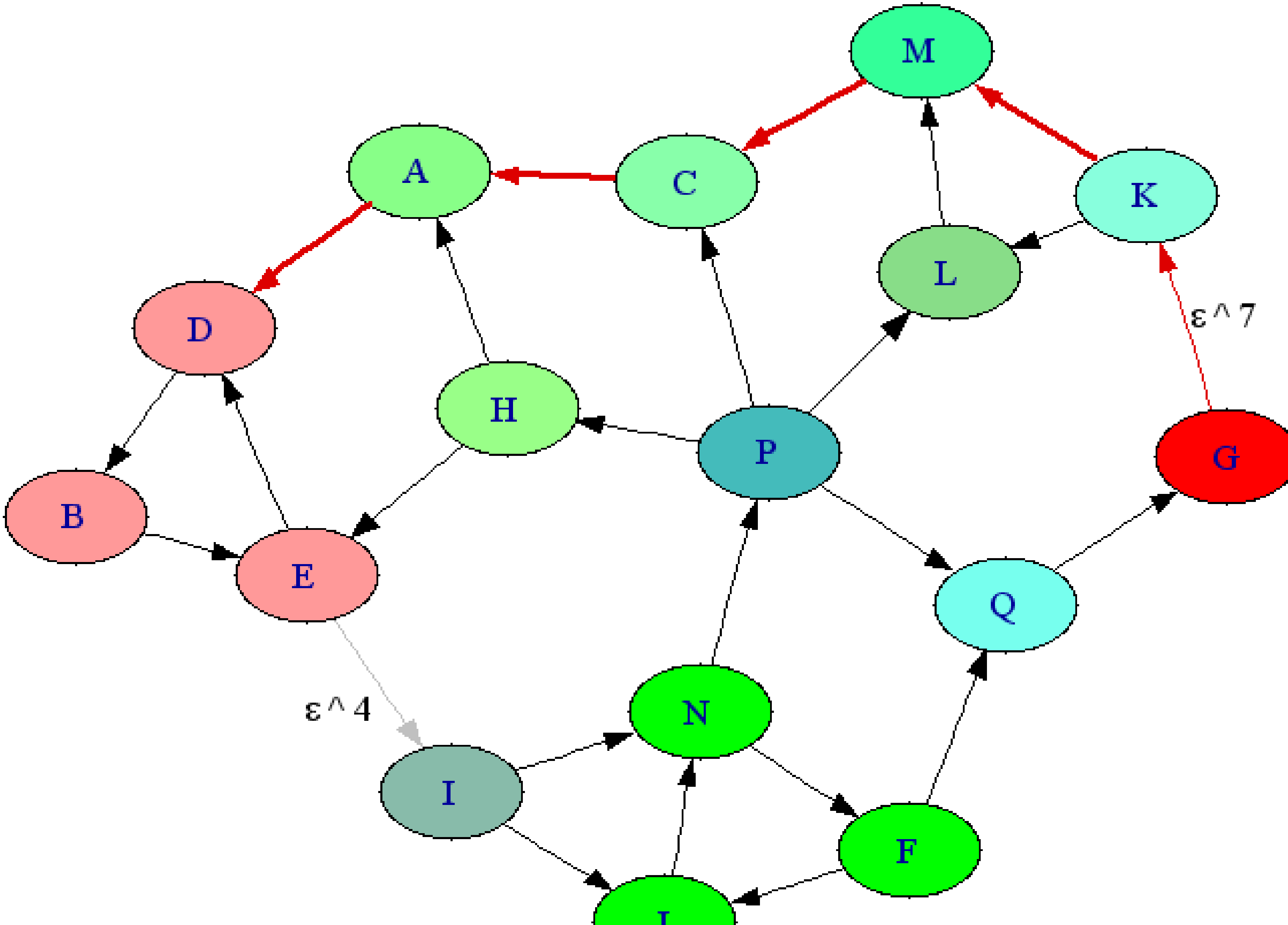
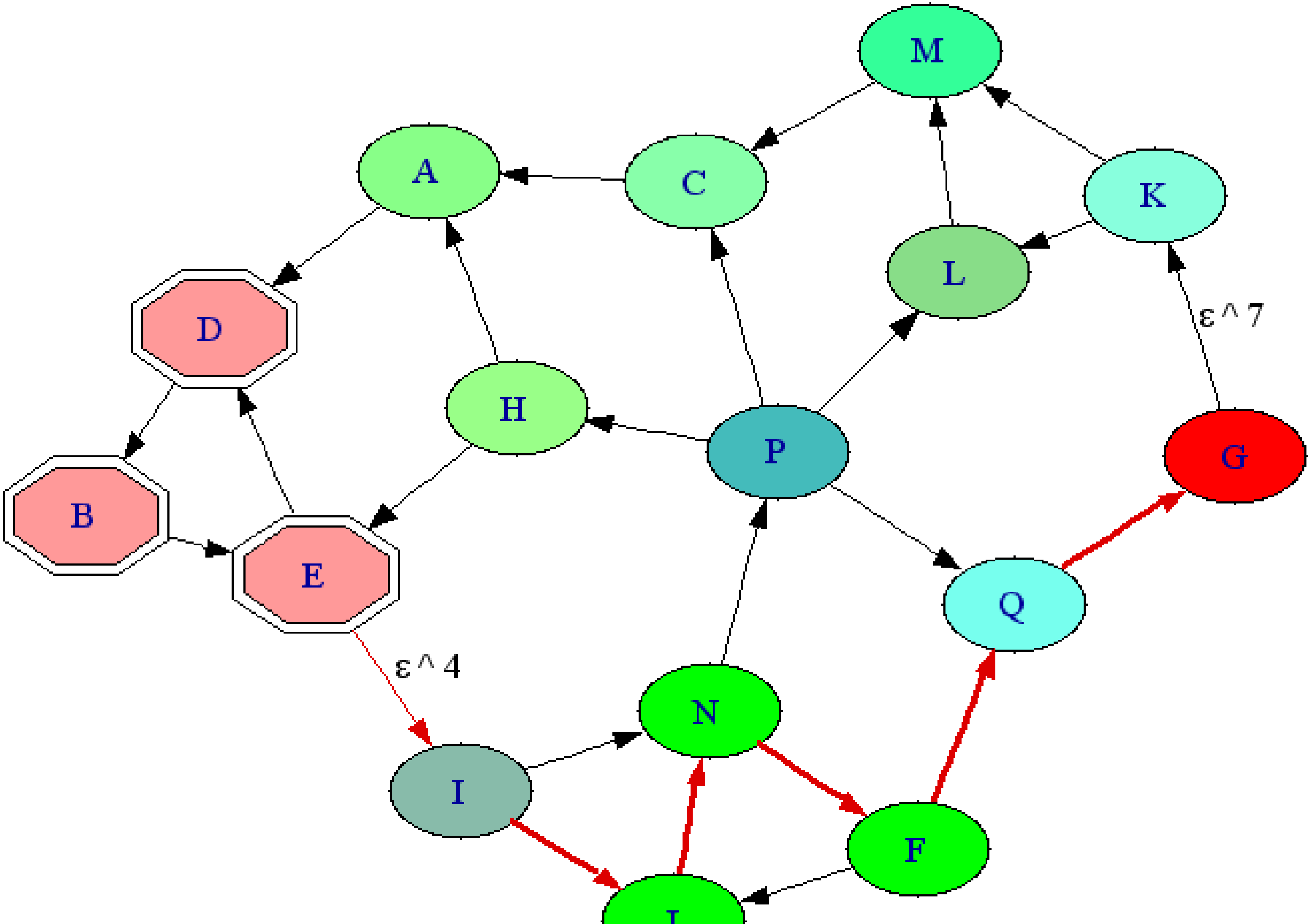


Class $\{B, D, E\}$ possesses lowest stochastic potential, viz.



Example of P^0 and P^ϵ

$$\lim_{\epsilon \to 0} \begin{pmatrix} 0.0 & 0.2 & 0.2 & 0.1 & 0.5 \\ 0.3 & \epsilon^7 & 0.1 & 0.1 & 0.5 - \epsilon^7 \\ 0.1 & 0.2 & 0.2 & 0.0 & 0.5 \\ 0.7 & 0.1 & 0.2 & 0.0 & 0.0 \\ 0.1 & 0.2 - \epsilon^2/2 & 0.2 & \epsilon^2 & 0.5 - \epsilon^2/2 \\ 0.0 & 0.0 & 0.1 & 0.0 & 0.9 \end{pmatrix}$$

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Notice that some P^0-positive probabilities "have to give way" because row probabilities must add up to 1.

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Definition. A state s is said to be stochastically stable if

$$\mu^0(s) > 0$$

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Then the stochastically stable states

$$S^* = \{s \in S \mid \mu^0(s) > 0\}$$

are precisely those that are contained in the Recurrence class(es) of P^0 with minimum stochastic potential.

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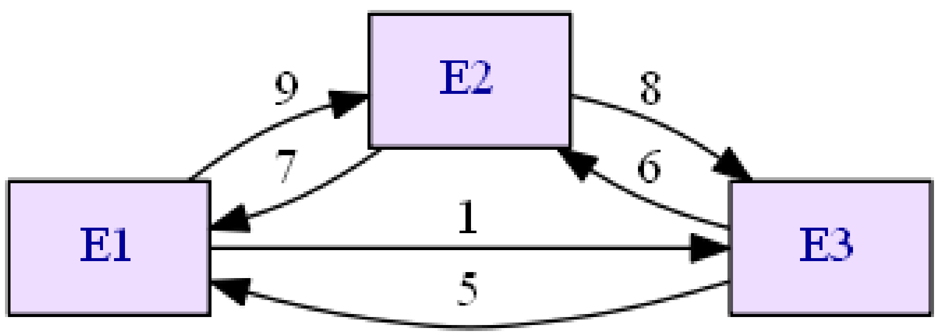
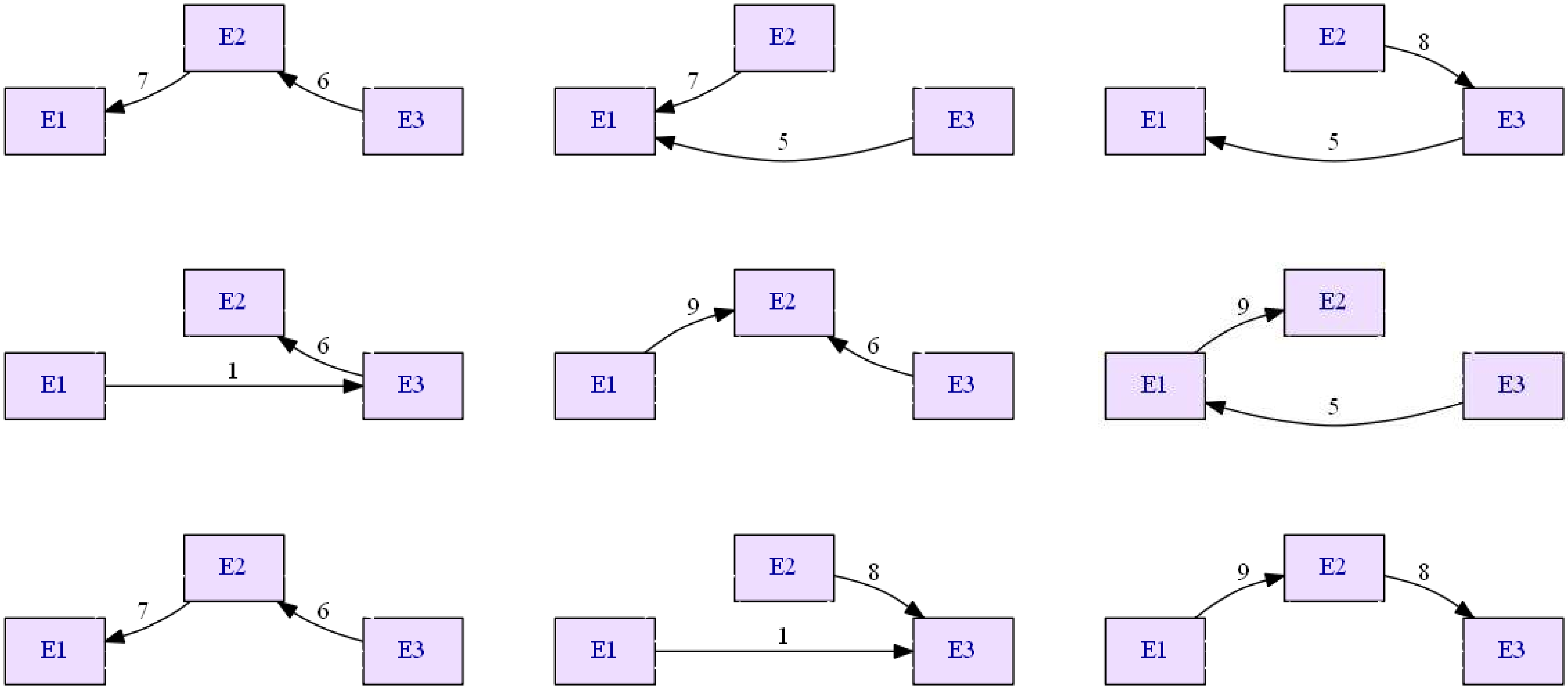
- Minimum path resistances here are 1, 5, 6, 7, 8, 9.



Nine j-trees generated by three recurrence classes



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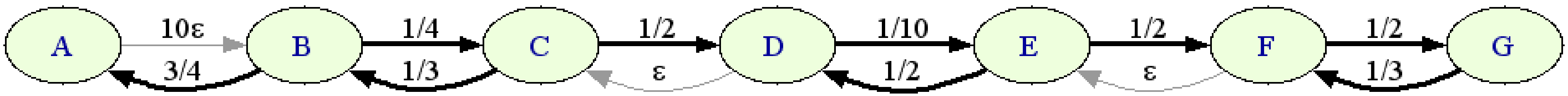

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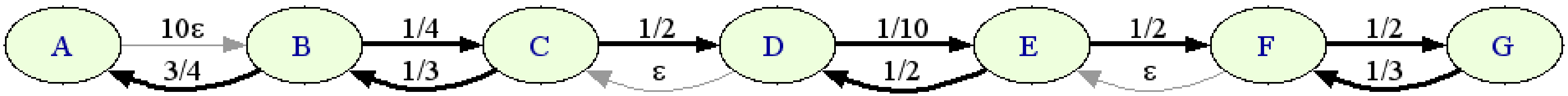

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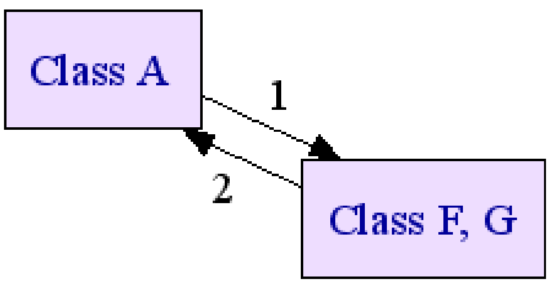

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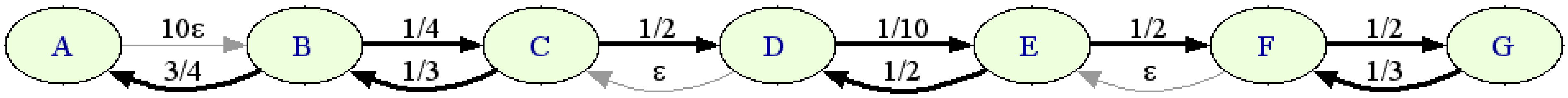

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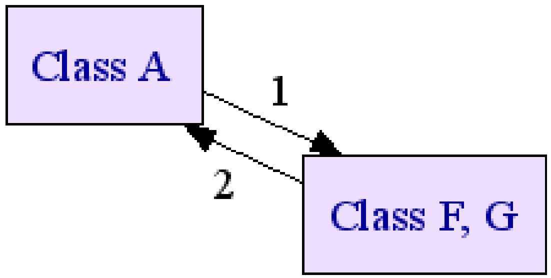

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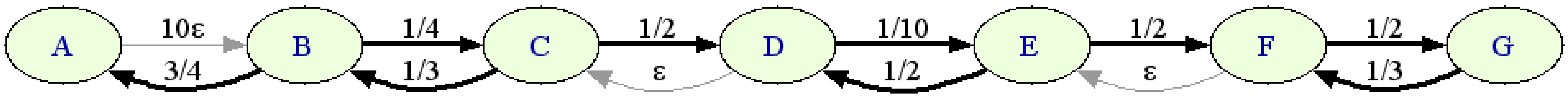

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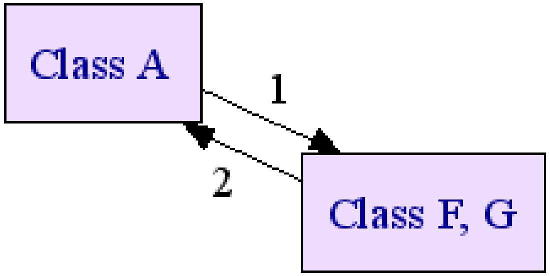

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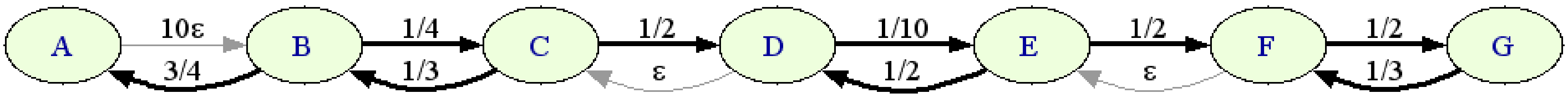

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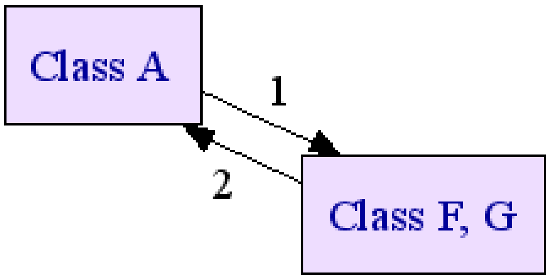

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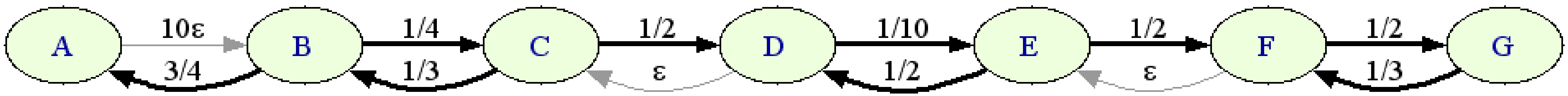

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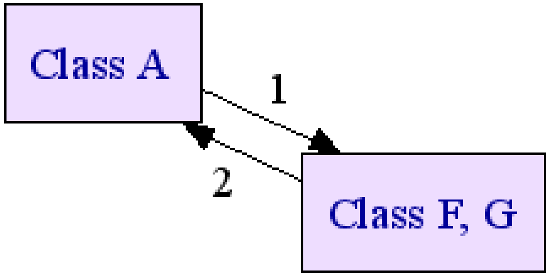

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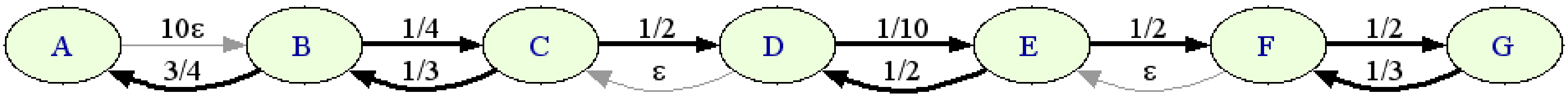

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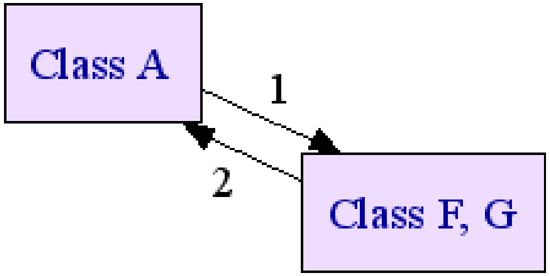

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6. Conclusion: E_2 is stochastically stable, E_1 is not.

Part 3: Applications

Idiosyncratic play in technology adoption

"How, then, might institutional change occur? Because best-response play renders both conventions absorbing states, it is clear that in order to understand institutional change, some kind of nonbest-response play must be introduced. Suppose there is a probability ϵ that when individuals are in the process of updating, each may switch their type for idiosyncratic reasons. Thus, $1 - \epsilon$ represents the probability that the individual pursues the best-response updating process described above. The idiosyncratic play accounting for nonbest responses need not be irrational or odd; it simply represents actions whose reasons are not explicitly modeled. Included is experimentation, whim, error, and intentional acts seeking to affect game outcomes but whose motivations are not captured by the above game."

From *Microeconomics: behavior, institutions, and evolution* (Bowles, 2003).

Technology adoption

		Other:	
		Operating system A	Operating system B
You:	Operating system A	(a, a)	$(0, 0)$
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Total number of players : N, for example $N = 5$

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$$\text{Sample size} : n, \text{ for example } n = 4$$

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Technology adoption

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\text{Total number of players } A : K&, \text{ for example } K = 3 \\
\text{Sample size} : n&, \text{ for example } n = 4 \\
\text{Number in sample playing } A : k&, \text{ for example } k = 2.
\end{aligned}$$

$$\begin{aligned}
P(\ \#A's = 2 \mid AABBB\ ) &= \text{hypergeometric}(N, K, n, 2) \\
&= \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}} = \frac{\binom{3}{2}\binom{5-3}{4-2}}{\binom{5}{4}} = \frac{\binom{3}{2}\binom{2}{2}}{\binom{5}{4}} = \frac{3 \cdot 1}{5} - = \frac{3}{5}.
\end{aligned}$$

Technology adoption

		Other: Operating system A	Operating system B
You:	Operating system A	(a, a)	$(0, 0)$
	Operating system B	$(0, 0)$	(b, b)

$$\begin{aligned} \text{Total number of players} &: N, \text{ for example } N = 5 \\ \text{Total number of players } A &: K, \text{ for example } K = 3 \\ \text{Sample size} &: n, \text{ for example } n = 4 \\ \text{Number in sample playing } A &: k, \text{ for example } k = 2. \end{aligned}$$

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This process is path-dependent (non-ergodic): for example always $BABBB, BABBB, \text{etc.} \to BBBBB$. With $b \gg a$ even $BAABB, \text{etc.} \to BBBBB$.

The tipping effect

$$\begin{aligned} \text{Total number of players} : N, &\quad \text{for example } N = 5 \\ \text{Sample size} : n, &\quad \text{for example } n = 3 \\ \text{Total number of players currently playing } A : K, &\quad \text{for example } K = 2 \end{aligned}$$

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- How many idiosyncratic transitions must be made to move from E_1 to E_2? What is the resistance from E_1 to E_2? From E_1 to E_2?
- What is (are) the stochastically stable state(s)?

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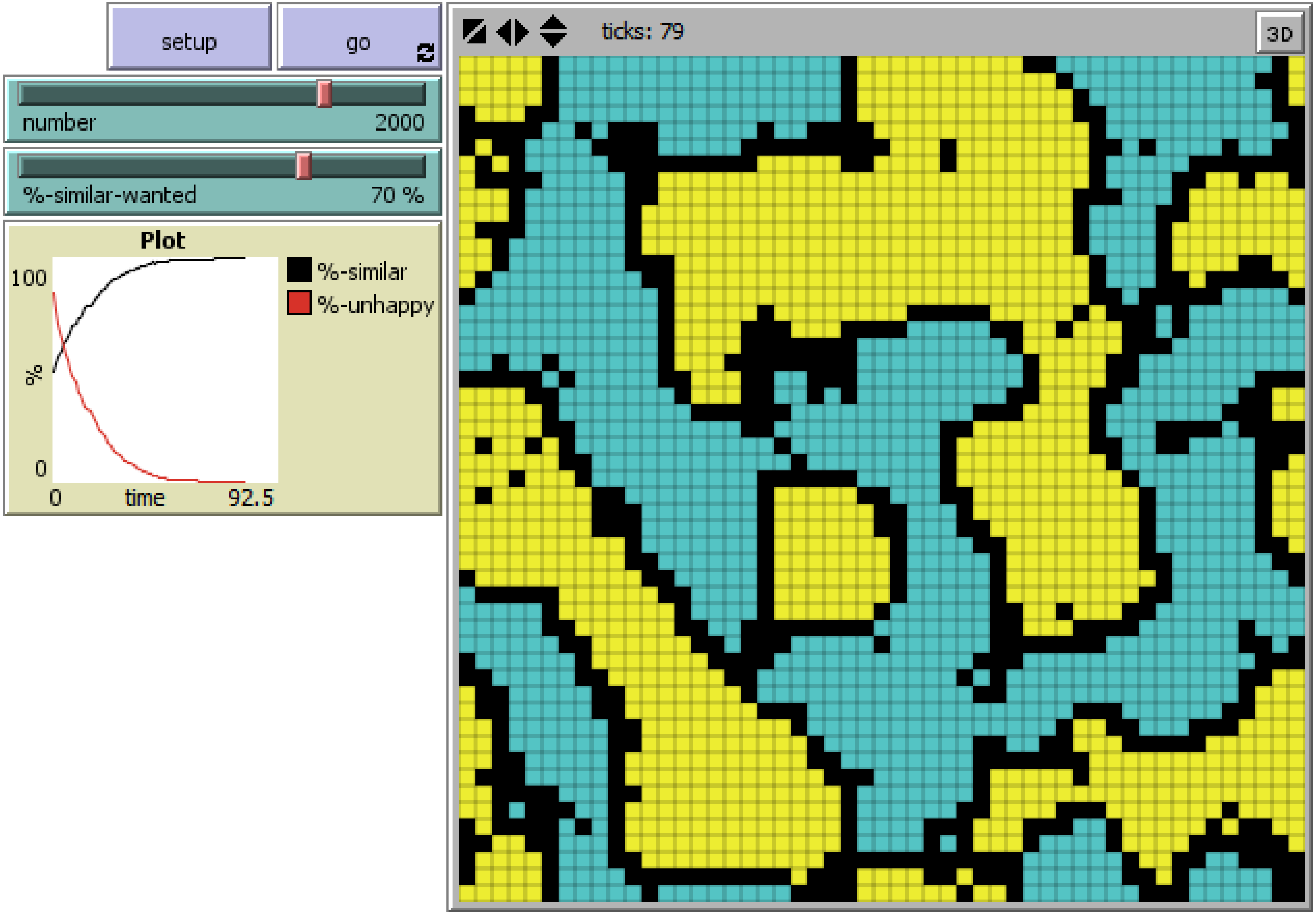
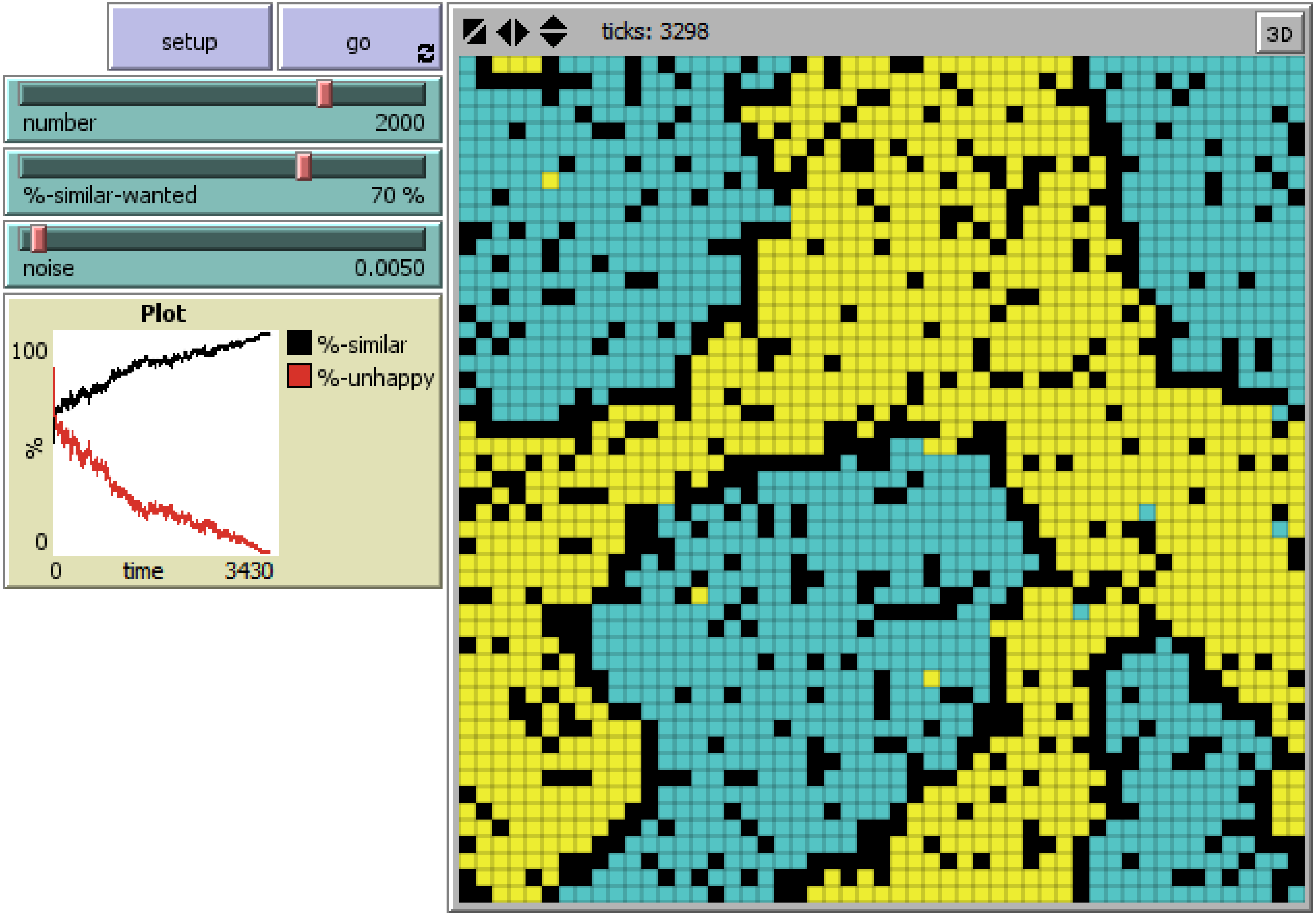
Part 4: Schelling's model of segregation

Schelling's model in 2D (torus)



Schelling's model in 2D: P^ϵ for small ϵ



Schelling's model in 1D (circle)

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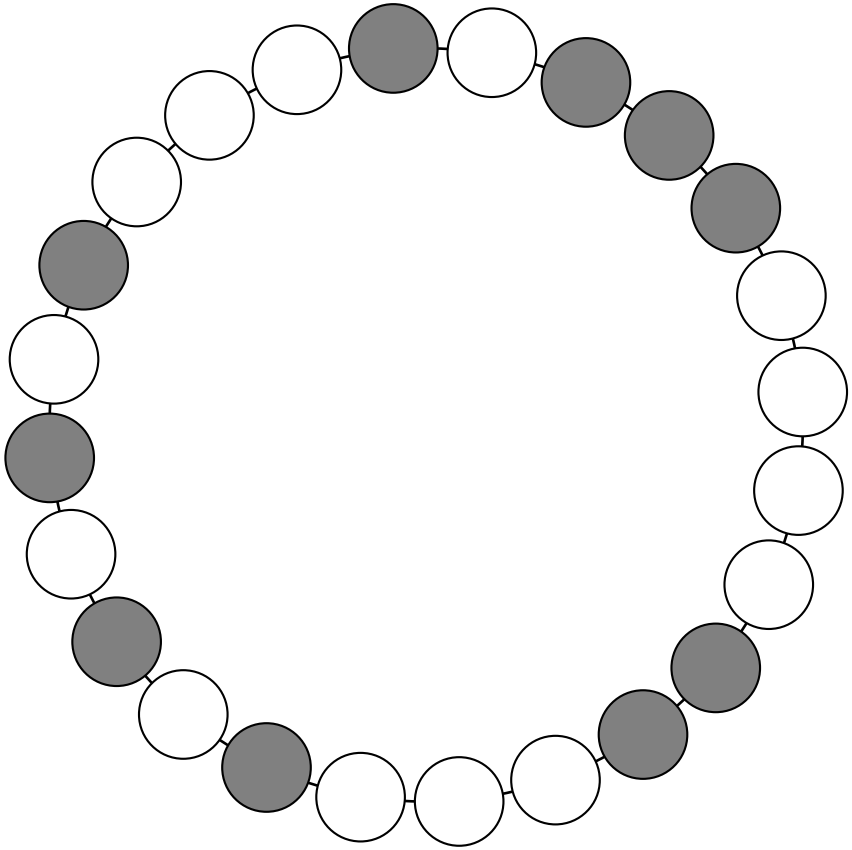
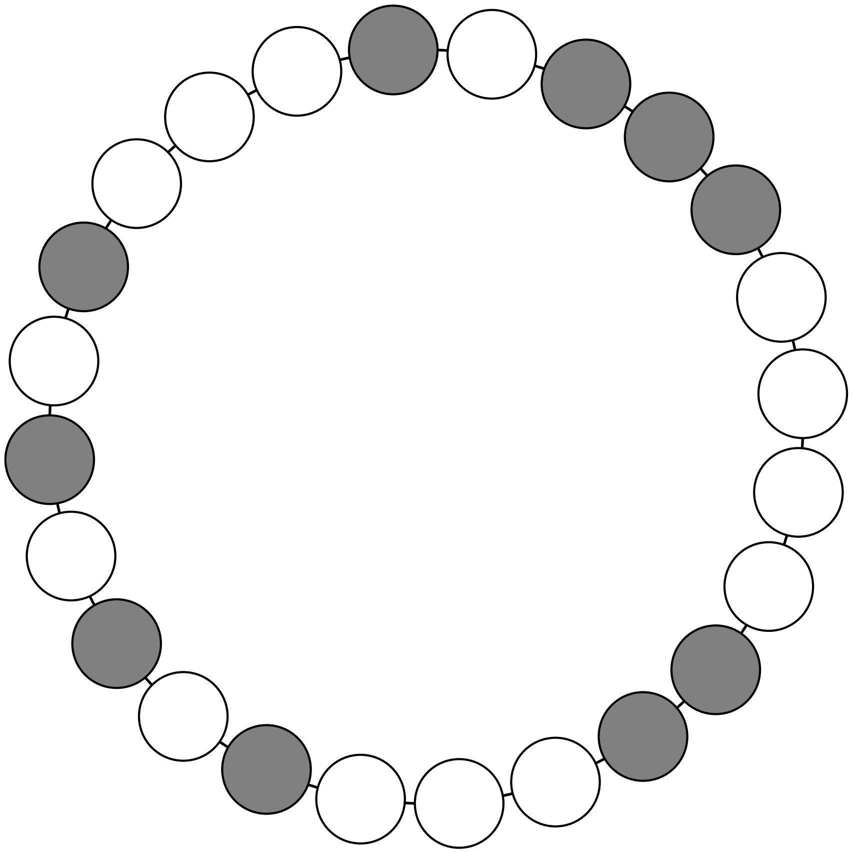


Schelling's model in 1D (circle)



- Schelling (1969, 1971, 1978).

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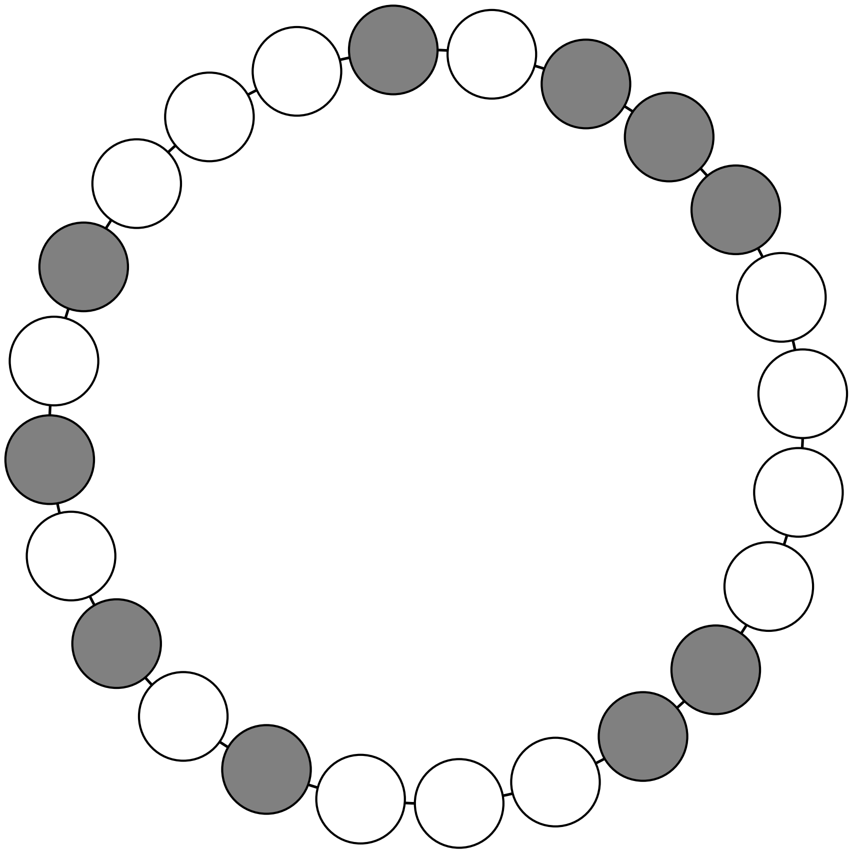

other people are content.

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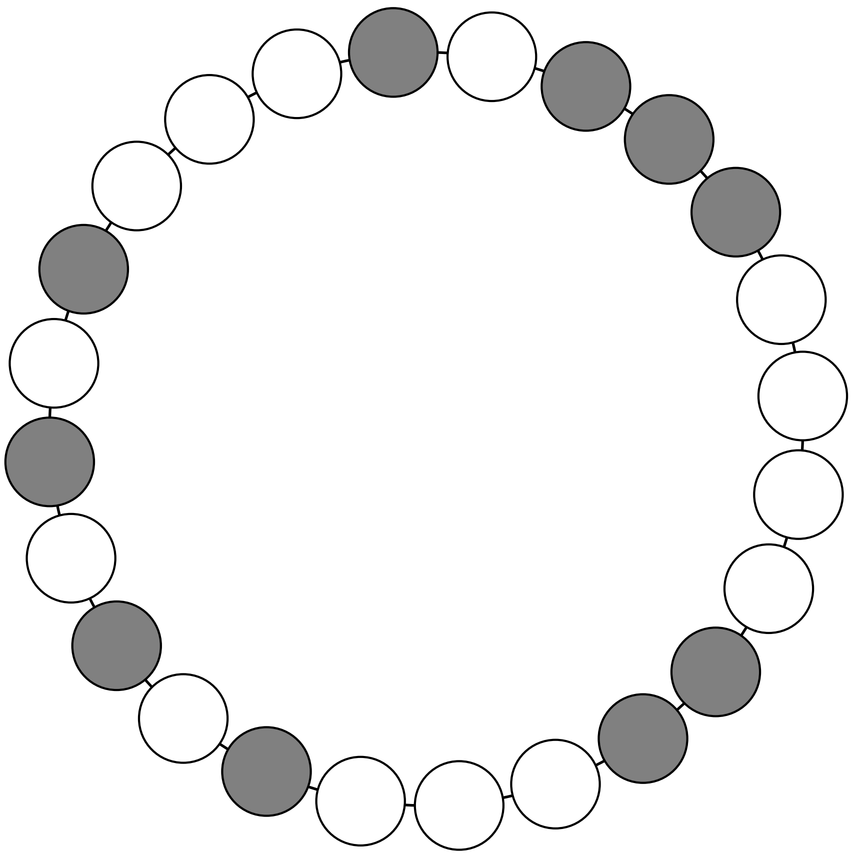

- Schelling (1969, 1971, 1978).
- Isolated people are discontent, other people are content.
- Some swaps and and their total profit:

Trade	Profit
DD $\rightarrow$ CC	2
DC $\rightarrow$ CC	1
CD $\rightarrow$ DC	0
CC $\rightarrow$ CD	-1
CC $\rightarrow$ DD	-2

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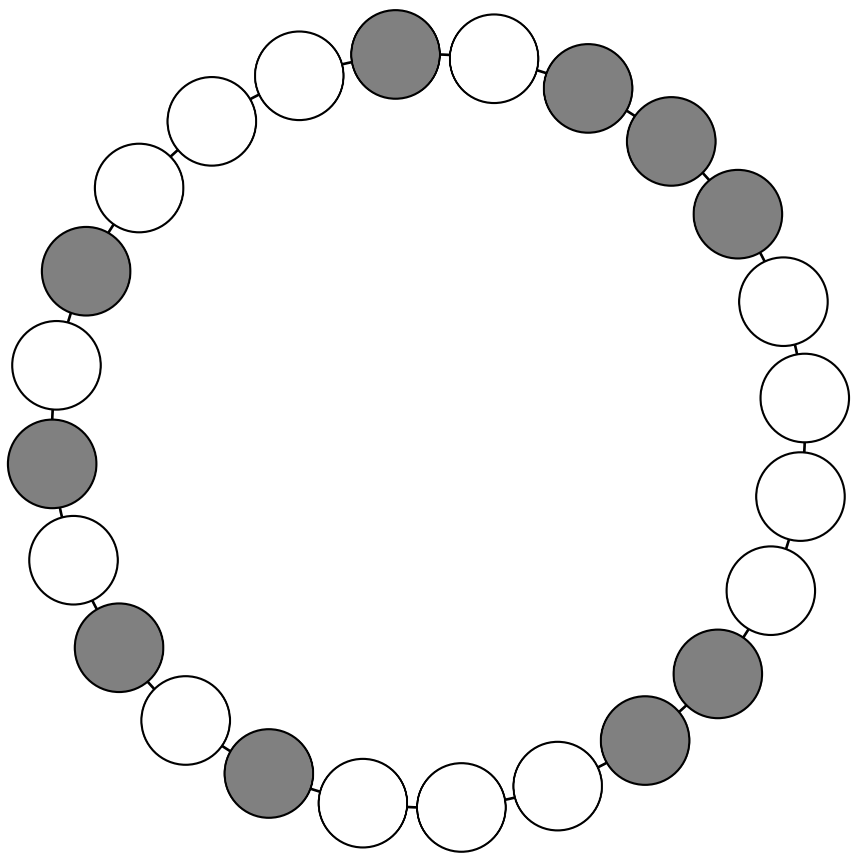

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- This "problem" can be "solved" in "hundreds" of ways. (Analytically, stochastically, whatever.)

Young's take on Schelling's model

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- Some trades and their total profit:

Trade	Profit	Probability
DD $\to$ CC	$2 - 2m$	$1 - \mathcal{O}(\epsilon)$
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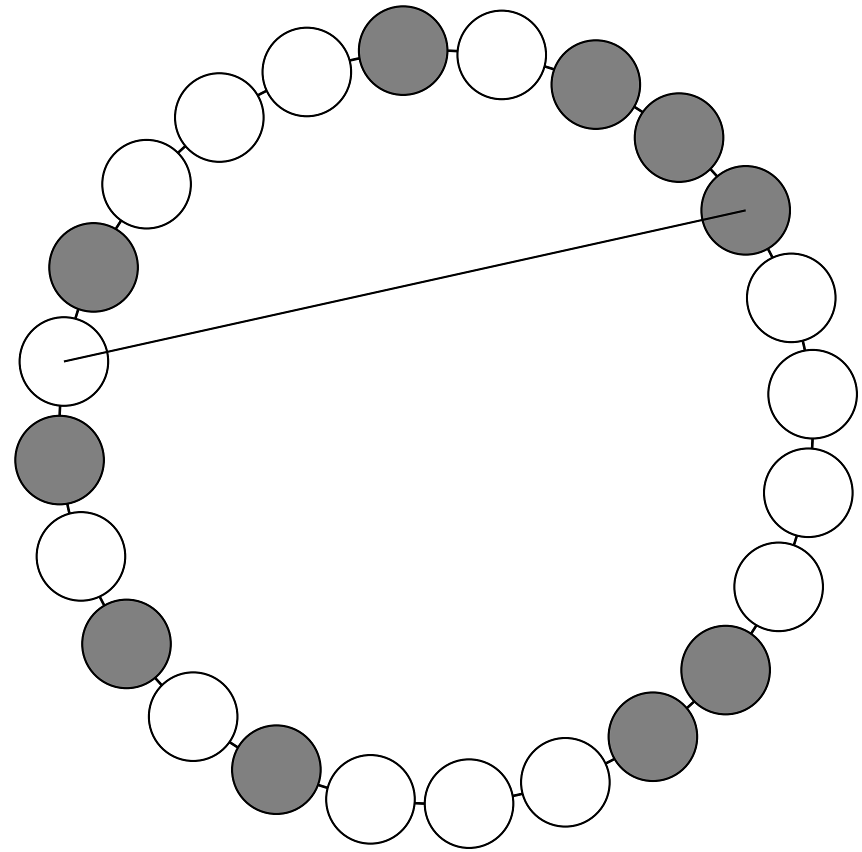
- P^ϵ allows all trades. P^ϵ is ergodic.



An example of a trade with cost $1-2m$ and probability 1 (in P^0) or near 1 (in P^ϵ).

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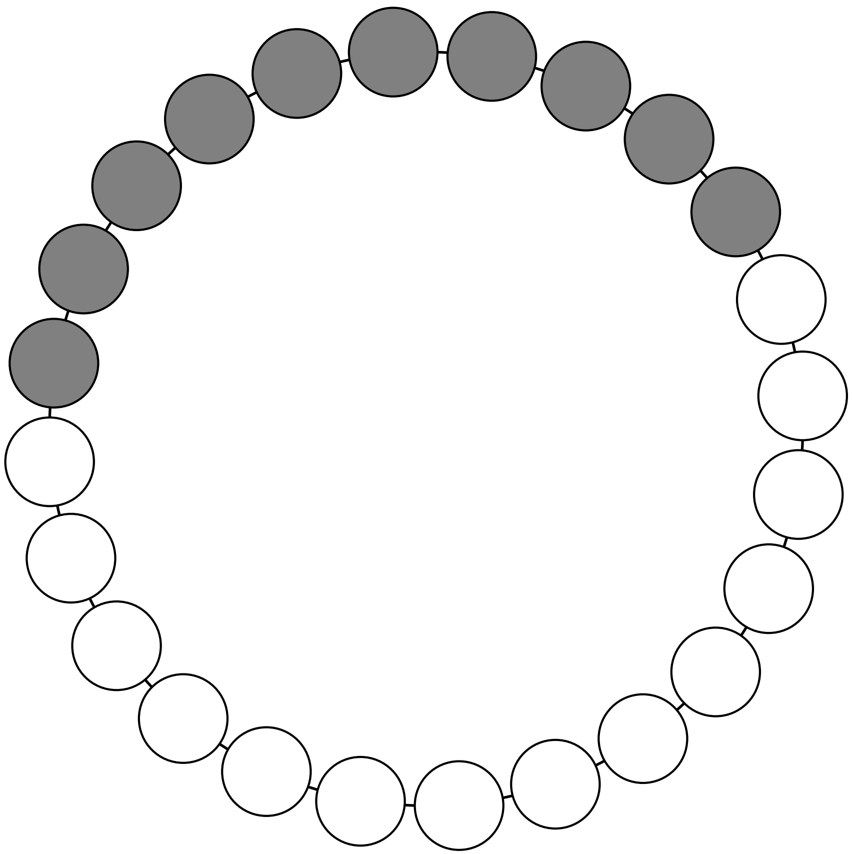

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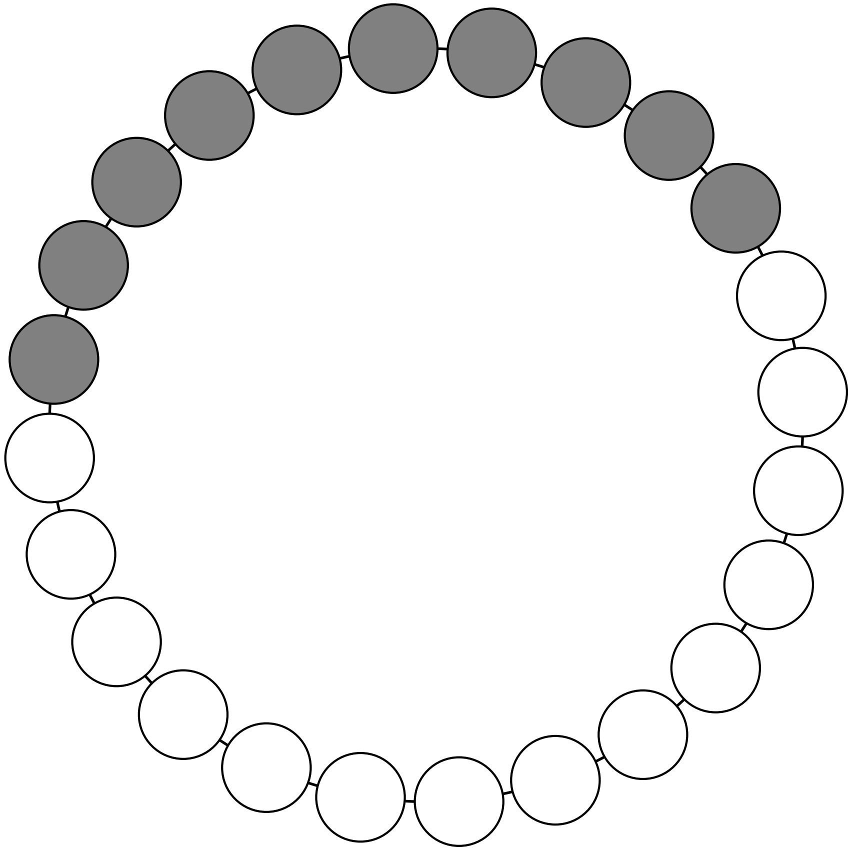

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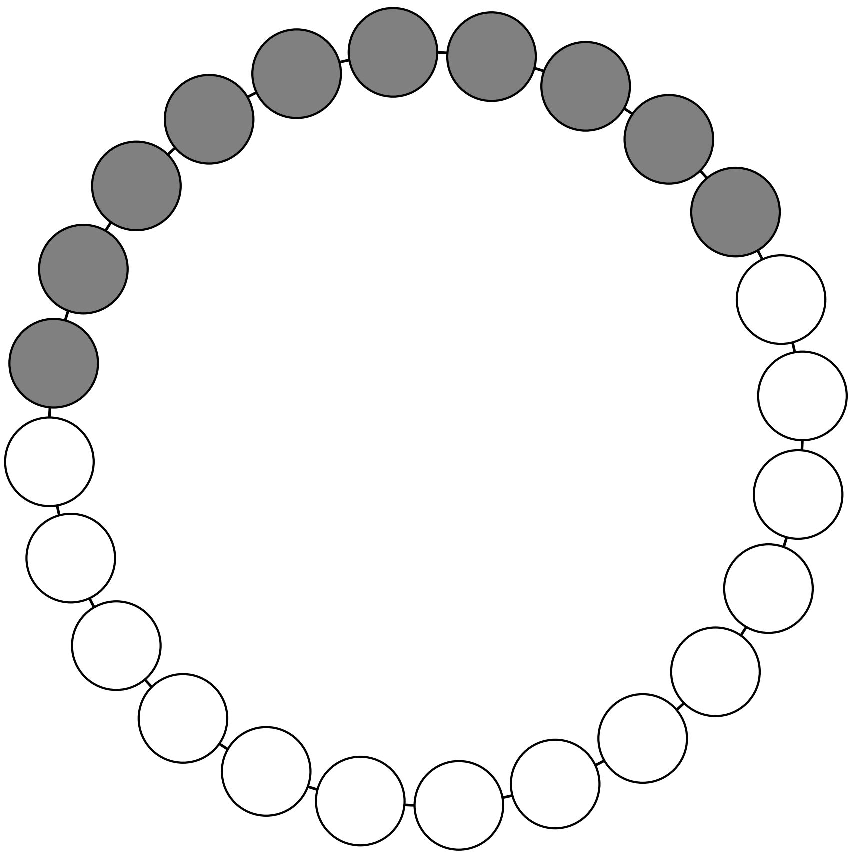

- Completely segregated $=_{Def}$ a state in which all the A's are lined up on one side of the circle and all the B's are on the other.
- Dispersed $=_{Def}$ absorbing nut not completely segregated.

Theorem (H. Peyton Young, 1993). The set of stochastically stable states in the perturbed one-dimensional version of Schelling's model equals the set of completely segregated absorbing states S.

Determine all Recurrence classes of P_0

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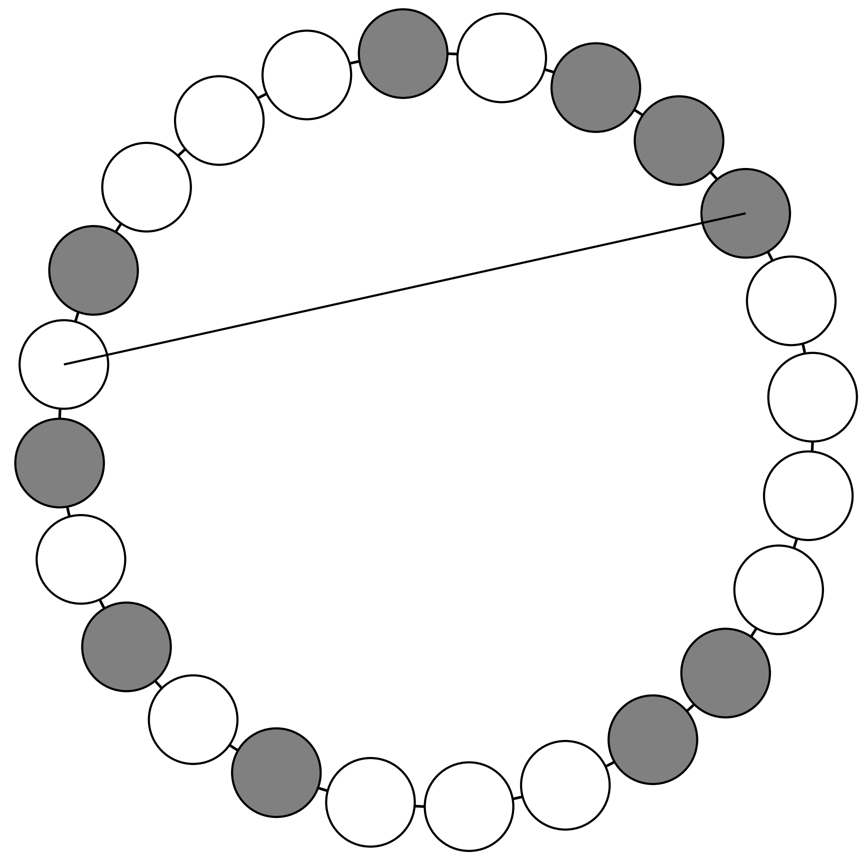

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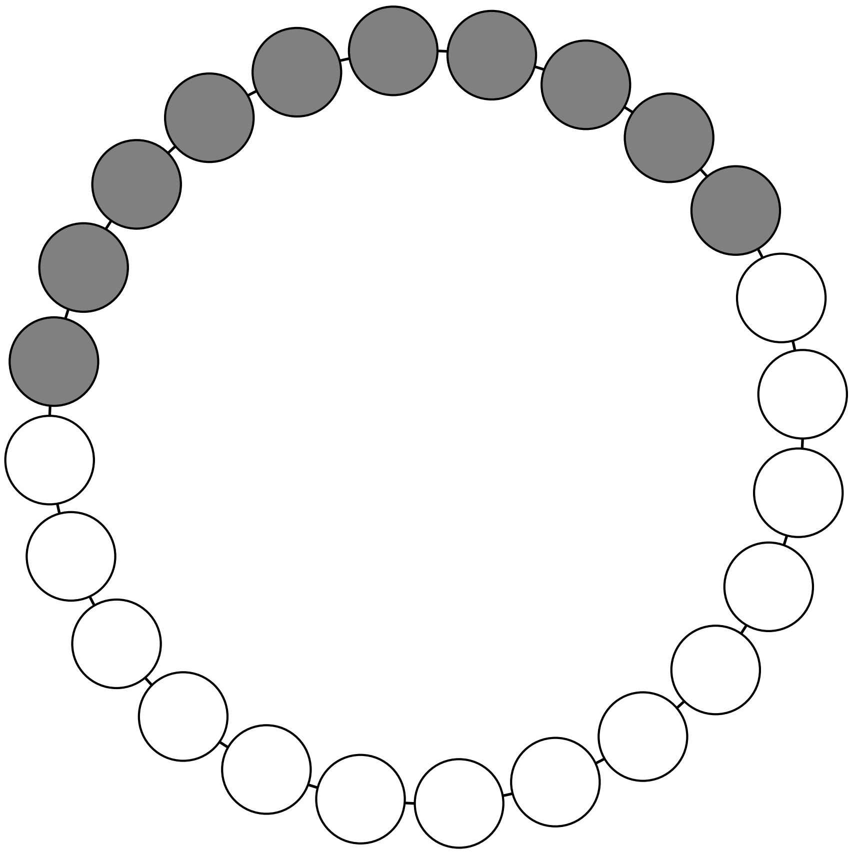

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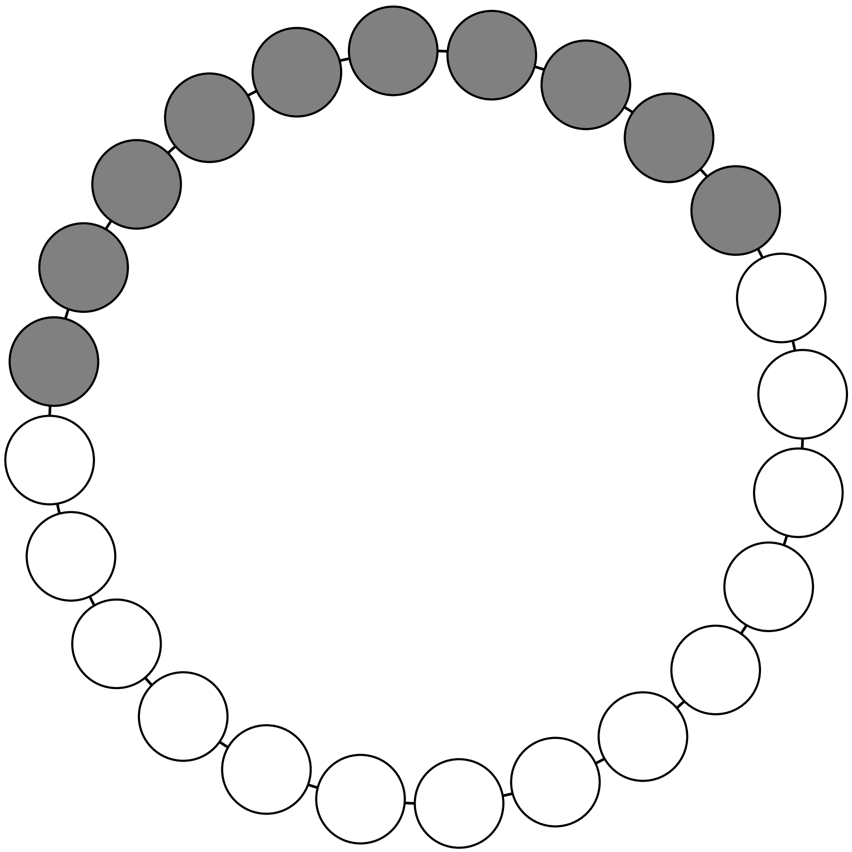

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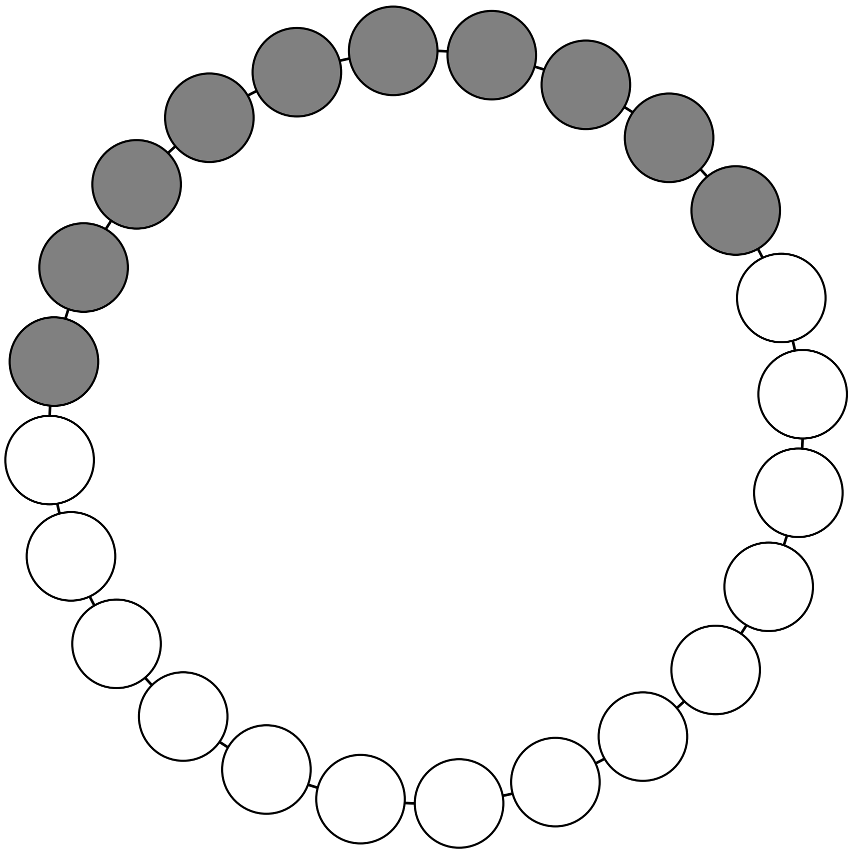

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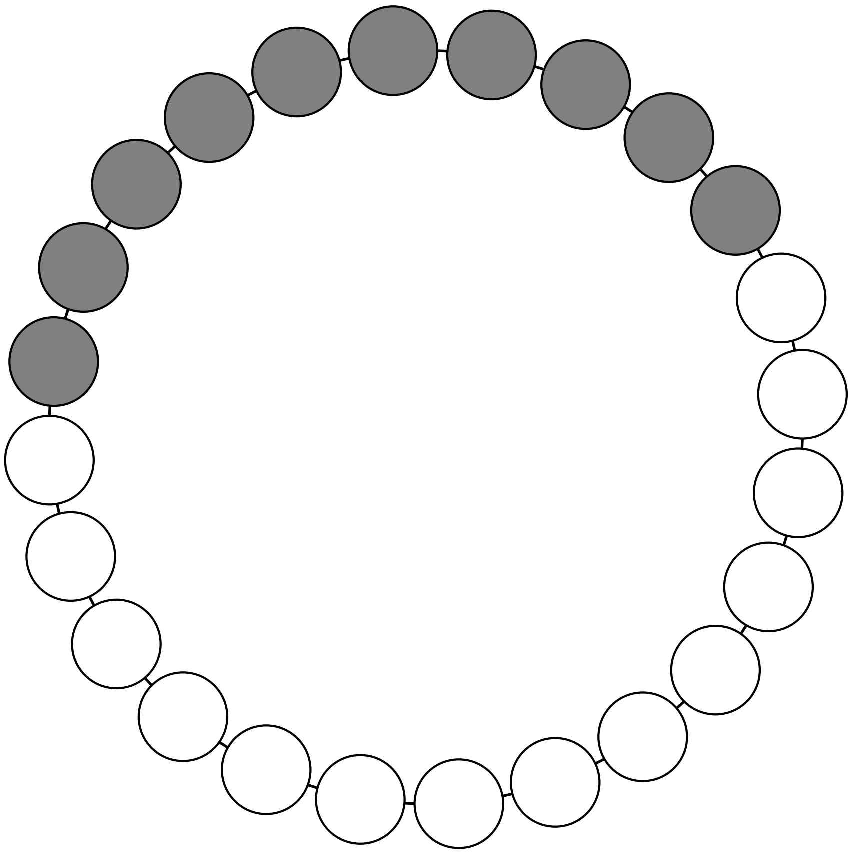

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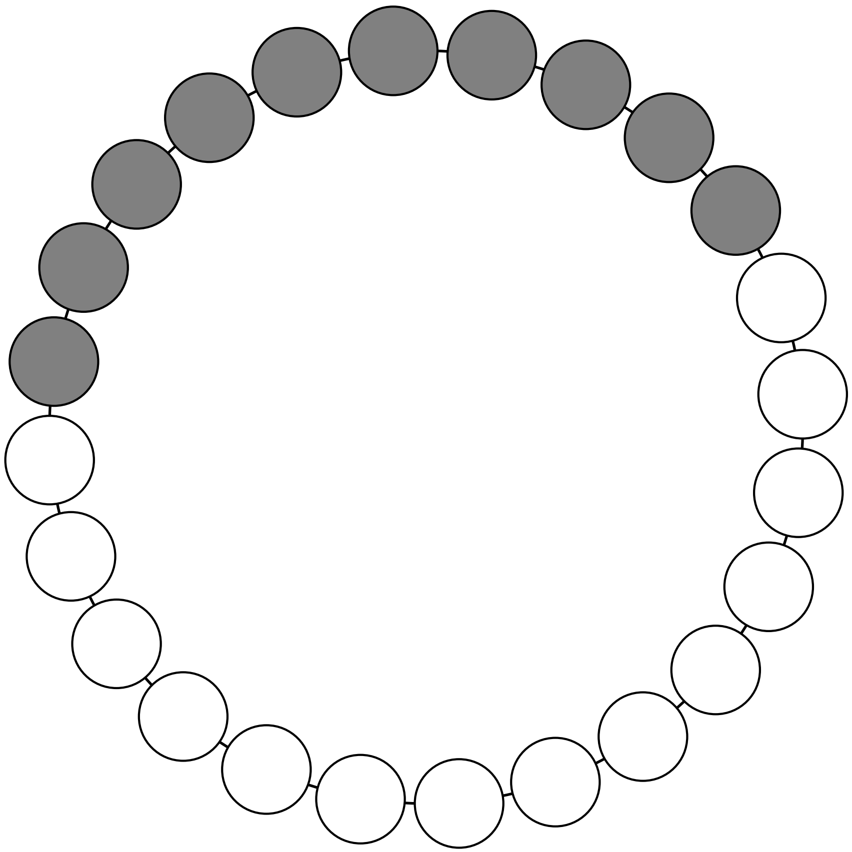

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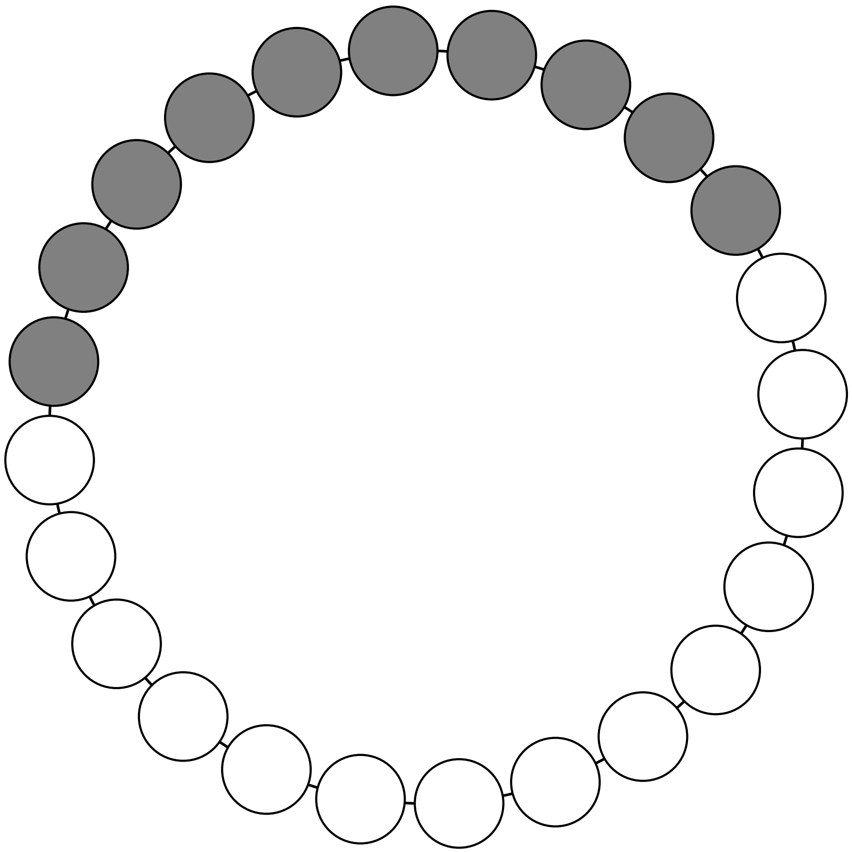

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3. The classes with lowest potential are

$$\mathcal{L} = \{\{s\} \mid s \in S\}.$$

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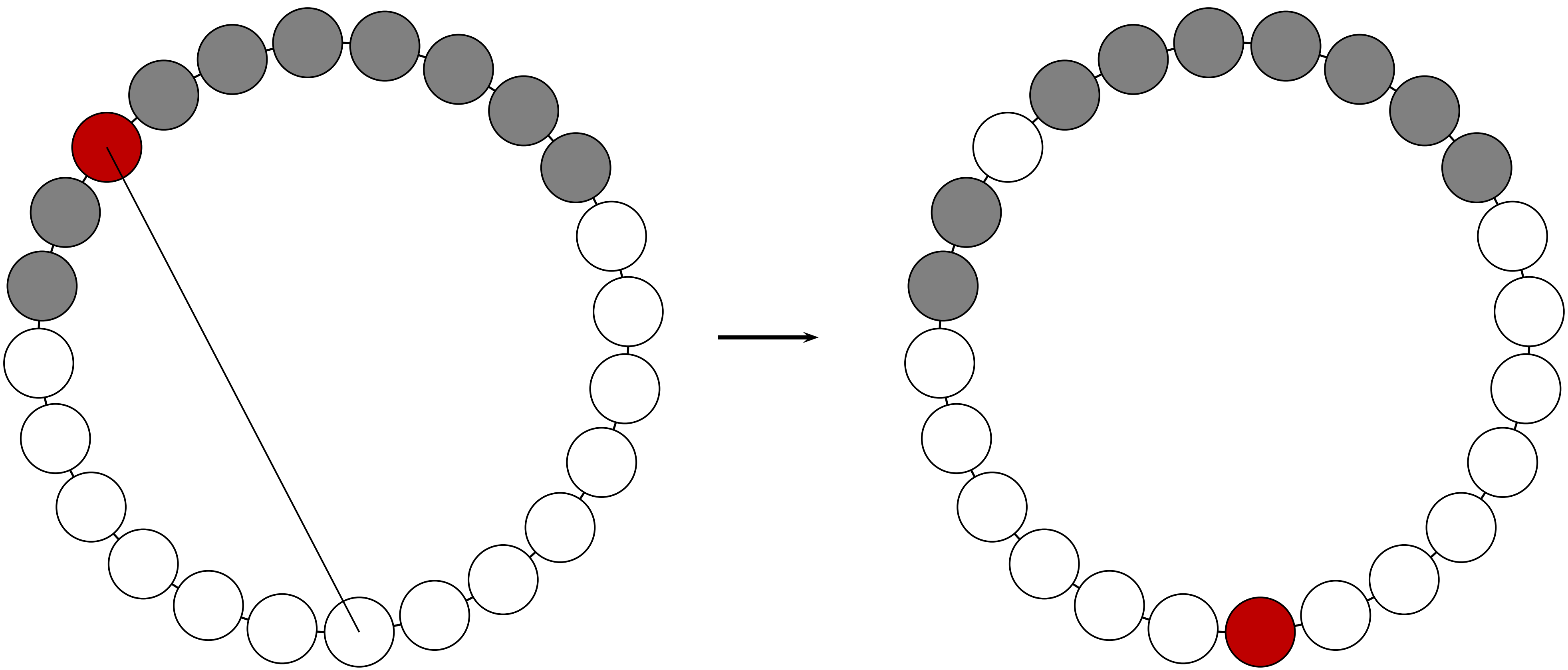

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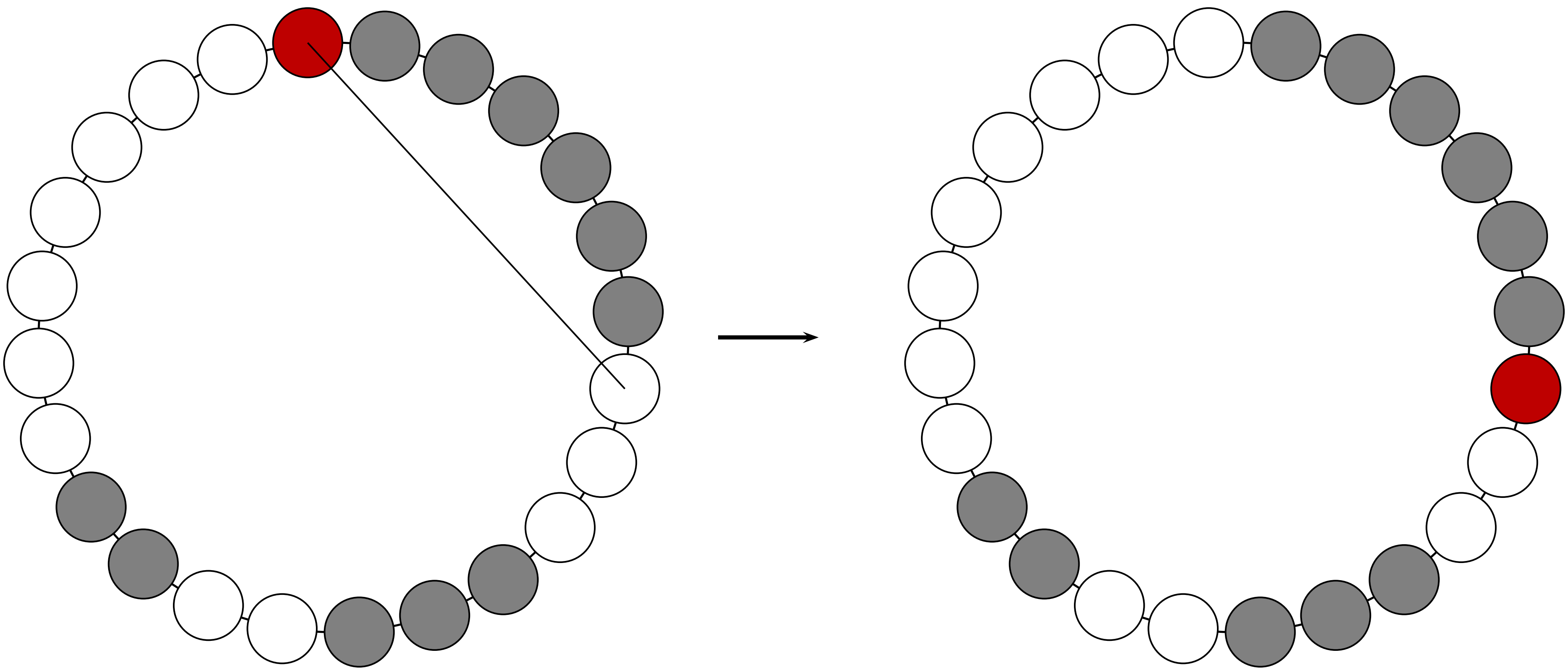

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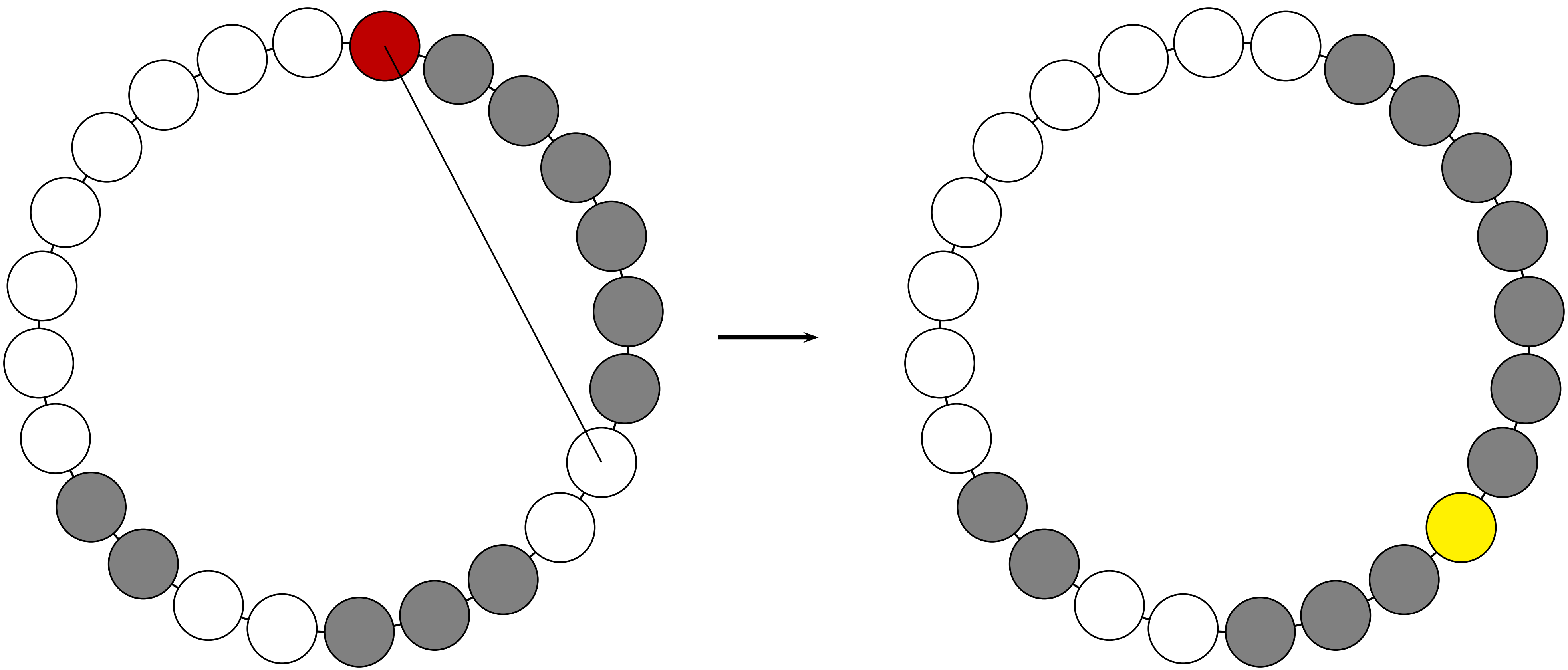

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- From elements in D to elements in S: ok! (Put tail to head of small groups repeatedly. If one large cluster, continue as in previous case.)

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