Multi-agent learning

Conditional Regret

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Thursday 19th February, 2015

Conditional regret: example

Shapley's game:

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G = \begin{array}{c} \\ R \\ Y \\ B \end{array}
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Rounds: 1 2 3 4 5 6 7 8 9 10

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- Average regret: $(3 - 0)/10$.

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The complete conditional regret matrix is:

$$\mathbf{R} = \begin{array}{c} \\ R \\ Y \\ B \end{array} \begin{array}{c} \begin{array}{ccc} R & Y & B \end{array} \\ \begin{pmatrix} 0.0 & 0.1 & 0.0 \\ 0.3 & 0.0 & 0.0 \\ -0.1 & 0.1 & 0.0 \end{pmatrix} \end{array}$$

Row: original actions;
column: alternative actions.

Conditional regret matrix

The conditional regret matrix at time t is

$$R^t(\omega) =_{Def} \begin{pmatrix} (u(1,y^t) - u(1,y^t))e_1^t & \dots & (u(k,y^t) - u(1,y^t))e_1^t \\ \vdots & \ddots & \vdots \\ (u(1,y^t) - u(k,y^t))e_k^t & \dots & (u(k,y^t) - u(k,y^t))e_k^t \end{pmatrix}$$

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- The average conditional regret vector at time t, $\bar{r}^t$, is just the sum of the columns of $\bar{R}^t$.

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- Consider a set of, for row, desirable payoffs $C \subseteq \mathbb{R}^m$.
- If the row player has a strategy $\sigma : H \to \Delta(X)$ such that

$$\lim_{t\to\infty} d(\bar{v}^t, C) = 0 \quad \text{a.s.}$$

then C is said to be approachable.

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- From the separating hyperplane theorem it follows that there is a vector $q^* \in \mathbb{R}^m$ such that $q^*\mathbf{W} \leq 0$.

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- The sequence of r.v.'s $w^1, w^2, \dots = \alpha \cdot v^1, \alpha \cdot v^2, \dots$ is bounded.

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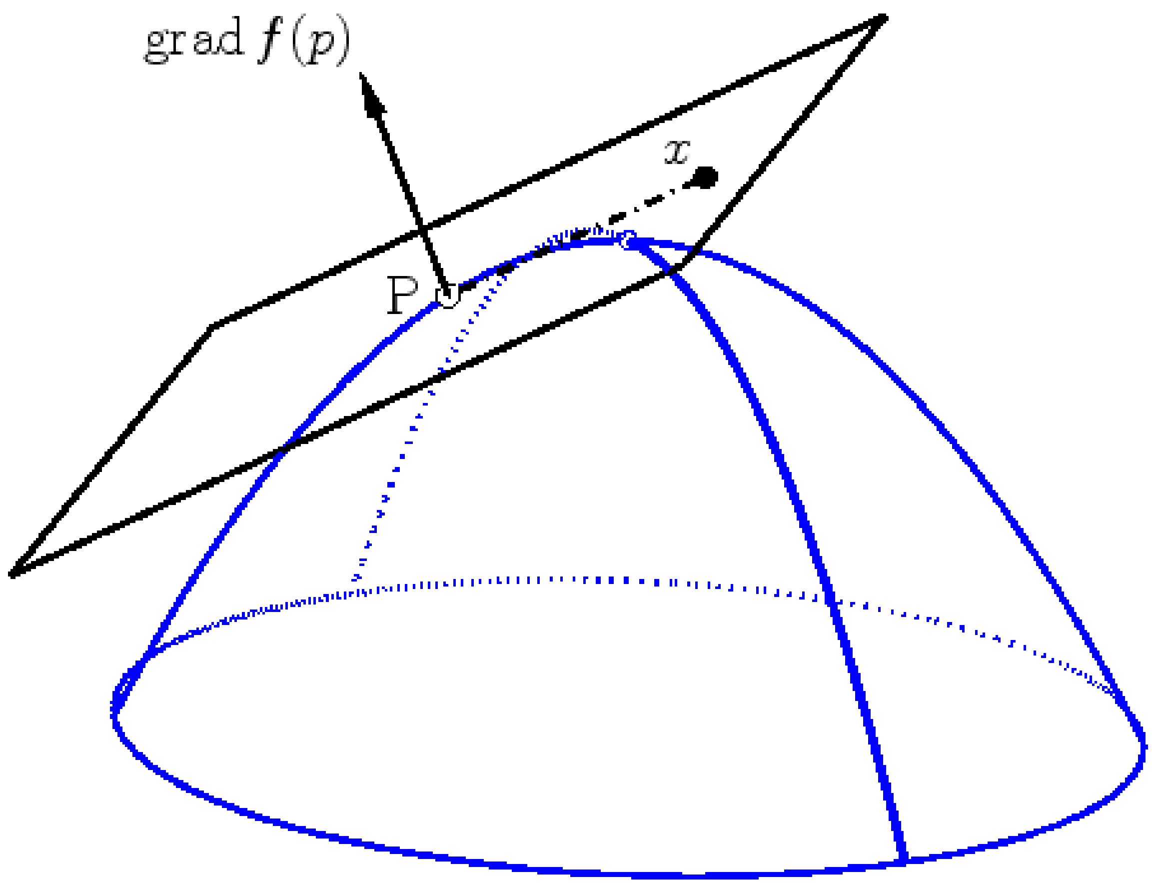
- Therefore $\lim_{t\to\infty} d(\bar{v}^t, C) = 0$ a.s.

Tangent plane

Blackwell's theorem make use of the concept of tangent plane:



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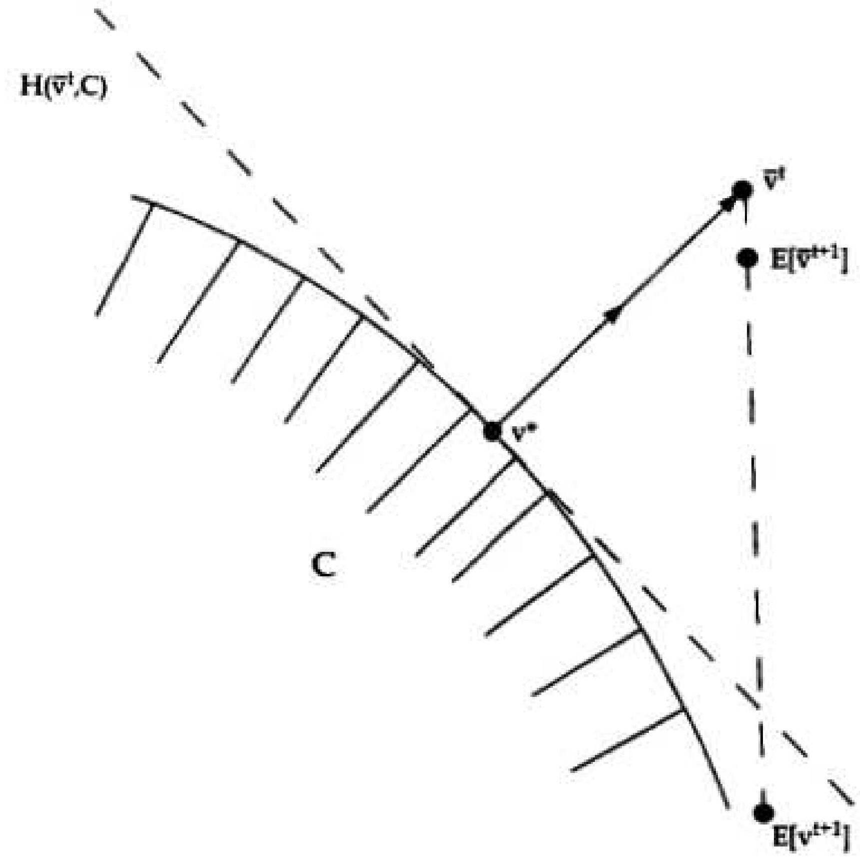
Theorem (Blackwell, 1956) Let G be a finite two-player game with payoffs in $\mathbb{R}^m$. A closed non-empty convex set C in $\mathbb{R}^m$ is approachable by the row player if and only if every tangent half-space containing C is approachable.



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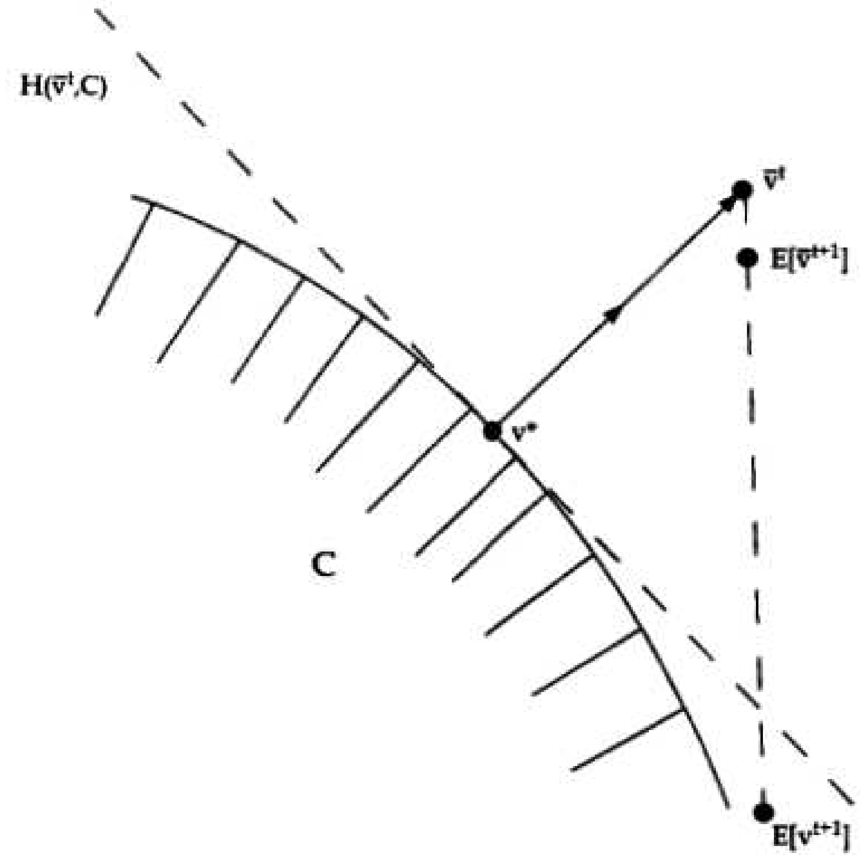

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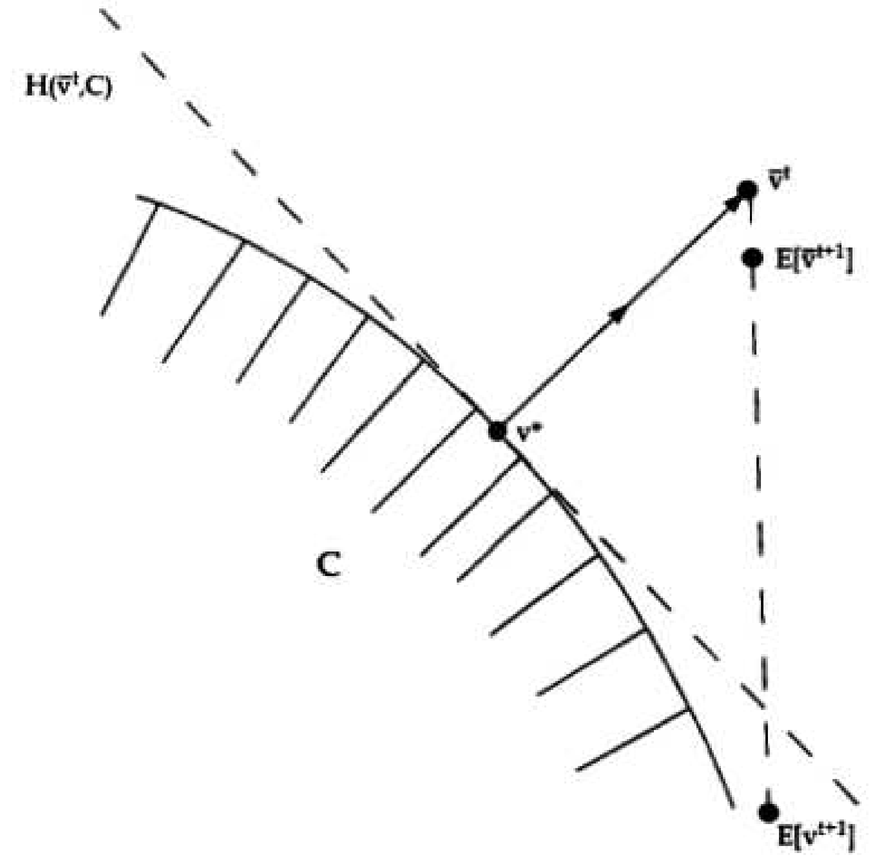

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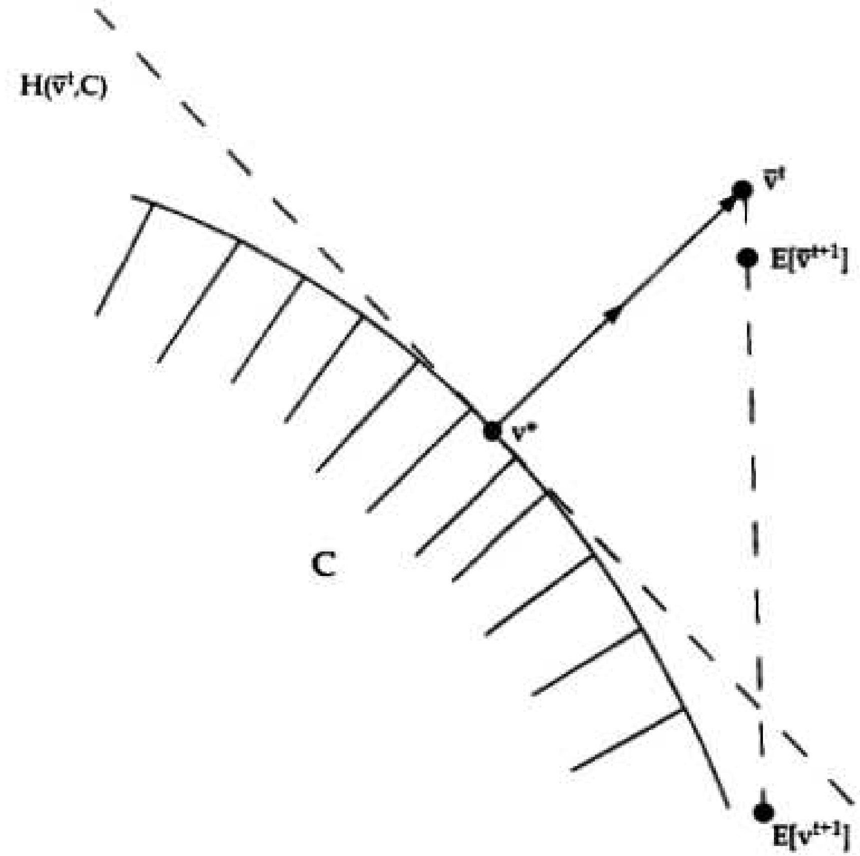

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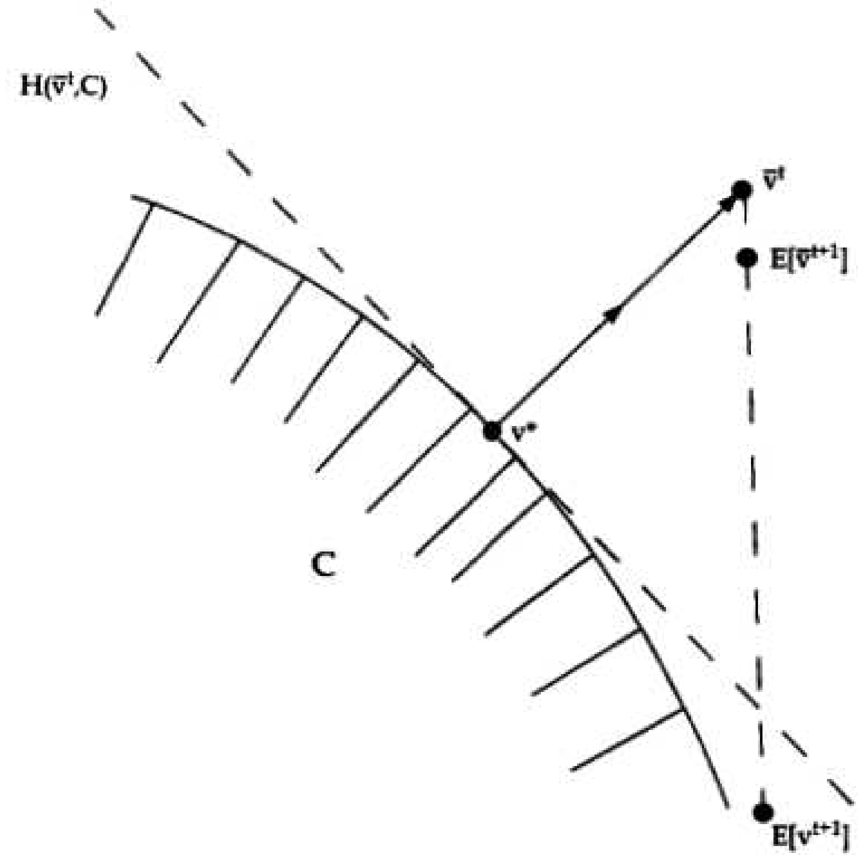

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Eliminating regret

(Result left hanging in Ch. 2)

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- Suppose $\bar{r}^t \notin C$. Then the closest point in C is $\bar{r}^t_-$.
- Blackwell: randomise play such that $E[r^{t+1}] \perp (\bar{r}^t - \bar{r}^t_-)$, so $E[r^{t+1}] \perp \bar{r}^t_+$.

Eliminating regret

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So take $q^T \sim \bar{r}^t_+$, i.e., take q^T proportional to $\bar{r}^t_+$.

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$$\sum_h q_h (\bar{r}_{hj})_+ \; - \; q_j \sum_h (\bar{r}_{jh})_+ \; = \; 0. \tag{2}$$

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- Every q satisfying this equation is called a left-invariant for $\overline{\mathbf{R}}_+$.

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- Any such q^* has the property that $q\mathbf{N} = 0$, so q^* is the desired left-invariant for $\mathbf{M}$.

Eliminating conditional regret

Theorem (Foster and Vohra, 1999). *Given a finite game* G*, if in each period a player plays a distribution* q *that satisfies*

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Corollary. When every player uses the Foster-Vohra algorithm to suppress conditional regret, the empirical joint frequency of play converges almost surely to the set of correlated equilibria.

Example

In an earlier example, the conditional regret matrix after 10 rounds was:[1]

$$\mathbf{R} = \begin{array}{c c} & \begin{array}{ccc} R & Y & B \end{array} \\ \begin{array}{c} R \\ Y \\ B \end{array} & \begin{pmatrix} 0.0 & 0.1 & 0.0 \\ 0.3 & 0.0 & 0.0 \\ -0.1 & 0.1 & 0.0 \end{pmatrix} \end{array}$$

So,

$$\mathbf{N} = \begin{array}{c c} & \begin{array}{ccc} R & Y & B \end{array} \\ \begin{array}{c} R \\ Y \\ B \end{array} & \begin{pmatrix} -0.1 & 0.1 & 0.0 \\ 0.3 & -0.3 & 0.0 \\ -0.1 & 0.1 & 0.0 \end{pmatrix} \end{array}$$

[1]There is a typo in SLaiL, third row.

We need to find a $q \geq 0$ such that $q\mathbf{N} = 0$, $|q| = 1$. If we set $\beta = 0.3$, then q is the fixed-point of

$$\mathbf{I} + \frac{1}{\beta}\mathbf{N} = \begin{array}{c c} & \begin{array}{ccc} R & Y & B \end{array} \\ \begin{array}{c} R \\ Y \\ B \end{array} & \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1 & 0 & 0 \\ 0 & 1/3 & 2/3 \end{pmatrix} \end{array}$$

with $|q| = 1$.
By the theory of Markov chains such a fixed point exists. In this case, it is

$$q^* = (1/3, 1/3, 1/3).$$

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Theorem. If, in a finite game G, a player uses incremental conditional regret matching with a sufficiently small learning parameter, his conditional regrets become non-positive almost surely, independently of the behaviour of the participants.

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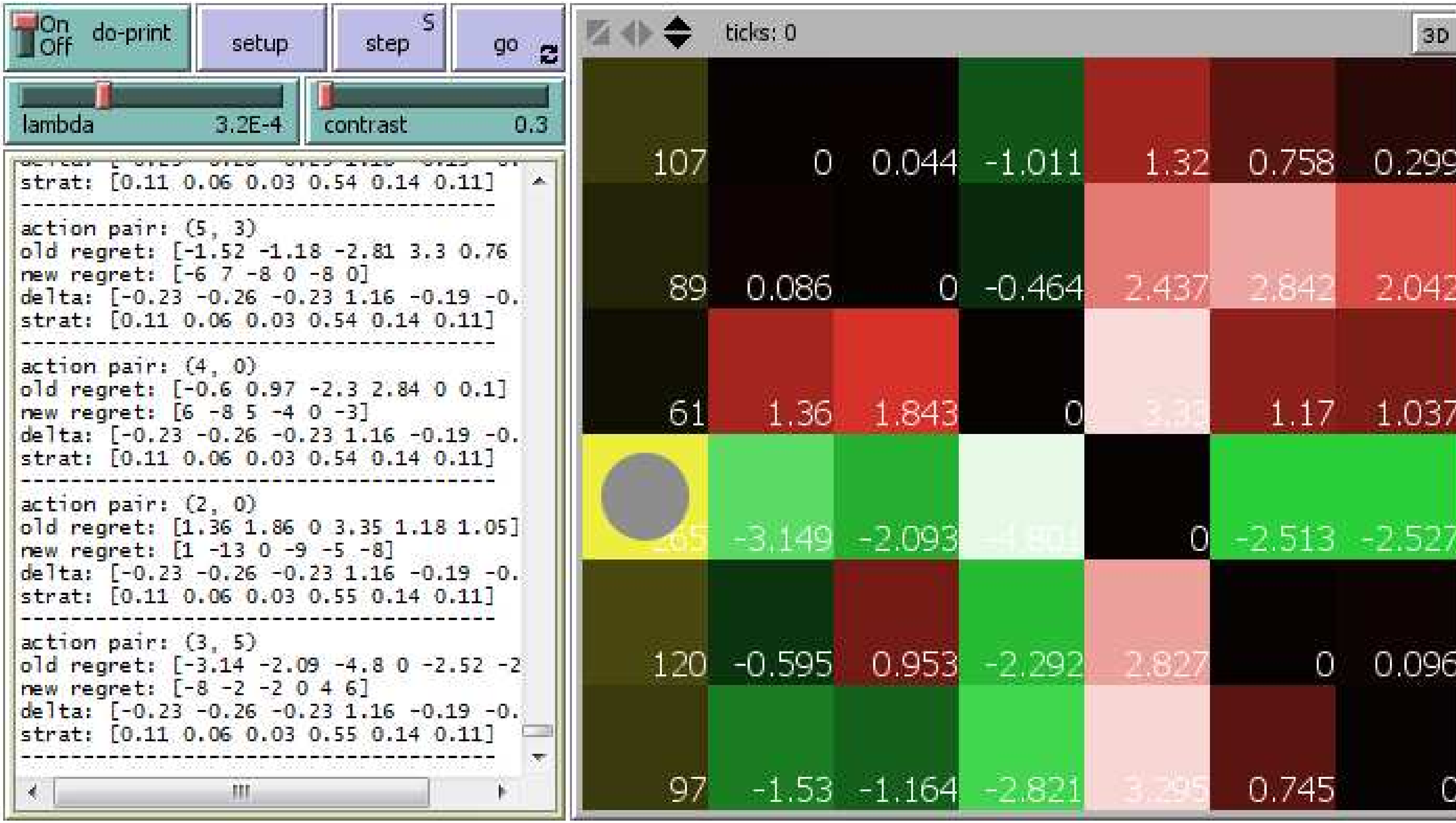
Proof. Based on the standard iterative procedure for find an invariant distribution of a finite Markov chain.

Corollary. If, in a finite game G, all players use incremental conditional regret matching with a sufficiently small learning parameter, the empirical joint frequency of play converges almost surely to the set of correlated equilibria.

Demo



On
Off
do-print
setup
step
S
go
lambda 3.2E-4
contrast 0.3
strat: [0.11 0.06 0.03 0.54 0.14 0.11]
action pair: (5, 3)
old regret: [-1.52 -1.18 -2.81 3.3 0.76
new regret: [-6 7 -8 0 -8 0]
delta: [-0.23 -0.26 -0.23 1.16 -0.19 -0.
strat: [0.11 0.06 0.03 0.54 0.14 0.11]
action pair: (4, 0)
old regret: [-0.6 0.97 -2.3 2.84 0 0.1]
new regret: [6 -8 5 -4 0 -3]
delta: [-0.23 -0.26 -0.23 1.16 -0.19 -0.
strat: [0.11 0.06 0.03 0.54 0.14 0.11]
action pair: (2, 0)
old regret: [1.36 1.86 0 3.35 1.18 1.05]
new regret: [1 -13 0 -9 -5 -8]
delta: [-0.23 -0.26 -0.23 1.16 -0.19 -0.
strat: [0.11 0.06 0.03 0.55 0.14 0.11]
action pair: (3, 5)
old regret: [-3.14 -2.09 -4.8 0 -2.52 -2
new regret: [-8 -2 -2 0 4 6]
delta: [-0.23 -0.26 -0.23 1.16 -0.19 -0.
strat: [0.11 0.06 0.03 0.55 0.14 0.11]
ticks: 0
3D
107 0 0.044 -1.011 1.32 0.758 0.299
89 0.086 0 -0.464 2.437 2.842 2.042
61 1.36 1.843 0 1.17 1.037
-3.149 -2.093 0 -2.513 -2.527
120 -0.595 0.953 -2.292 2.827 0 0.096
97 -1.53 -1.164 -2.821 0.745 0

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- The effect is that change in behaviour over periods is discontinuous.
- All players need to use this method to eliminate conditional regret.