Multi-agent learning

No-regret learning

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Wednesday 19th May, 2021

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   - Smoothed fictitious play. Give a soft-max response to the (recent) empirical frequency of opponents' actions.
   - Hypothesis testing with smoothed best responses. Give a best response to maintained beliefs about *patterns of play*.

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Three parts.

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Part I: Basic concepts

No-regret: example

Payoffs Player A	0	0	0	1	1	0	0	0	1	0	0	
Actions Player A	L	R	L	L	R	R	L	R	R	R	R	?
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Had L played:	6	$6 - 3$	$(6-3)/11$
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  So no-regret does not take the interactive nature of play into account.

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$$\Leftrightarrow \quad \lim_{t\to\infty} [\, \bar{r}^t_x(\omega)\,]_+ = 0.$$

Part II: proportional regret matching

The strategy of proportional regret matching

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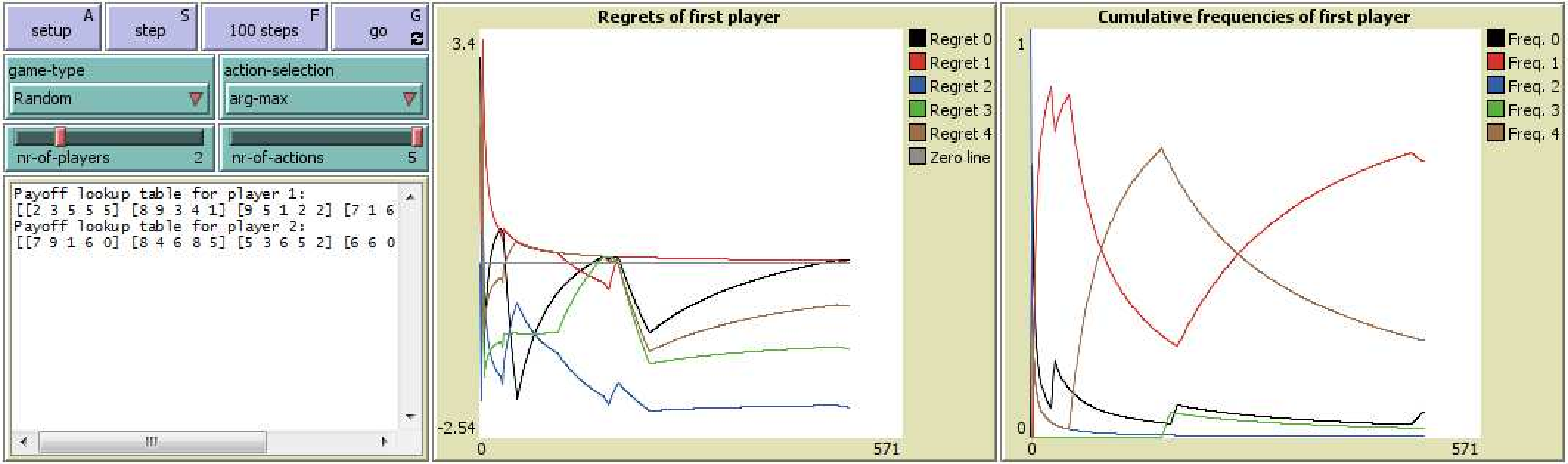
Cumulative payoff matching:

	Accumulated payoff	*Mixed strategy*
Action L:	1	$1/3$
Action R:	2	$2/3$

Regret matching in a 5-person 5-action game

Payoff matrix uninformative. Omitted . . .



Netlogo simulation of regret matching in a 5-person 5-action game.

Regret matching in Shapley's game

	R	Y	B
R	$(1, 0)$	$(0, 0)$	$(0, 1)$
Y	$(0, 1)$	$(1, 0)$	$(0, 0)$
B	$(0, 0)$	$(0, 1)$	$(1, 0)$

Column is "fashion leader", row is "fashion follower". Column wants to wear a different color than row.

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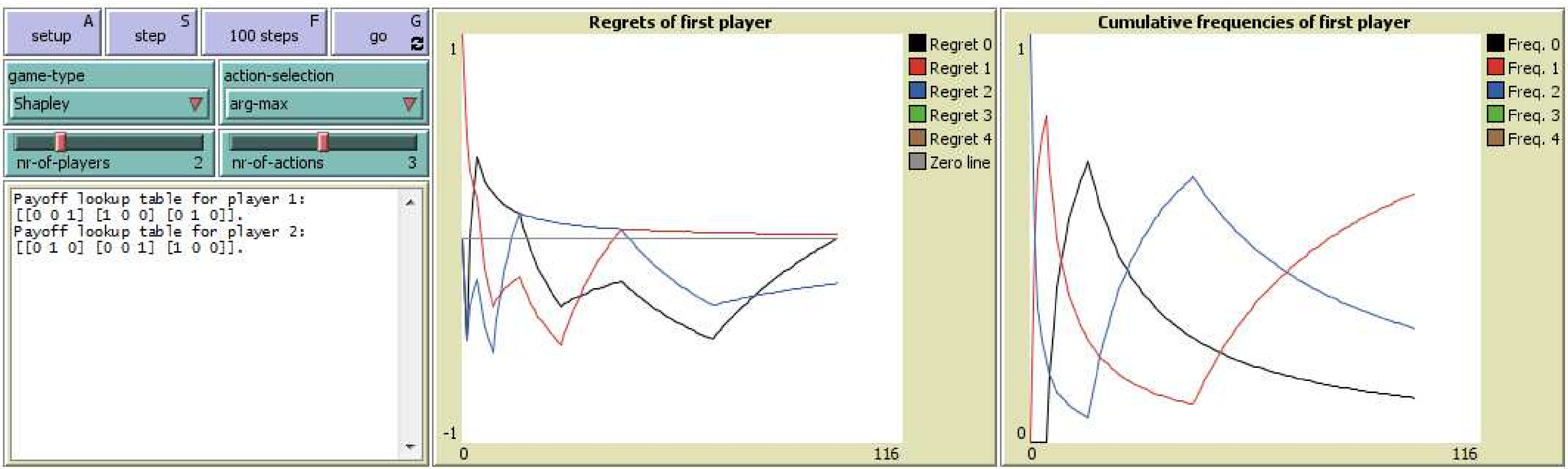

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\Delta r_x^t &=_{Def} \text{ incremental regret for not playing } x : r_x^t \ - \ r_x^{t-1}
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\Delta r_x^t &=_{Def} \text{ incremental regret for not playing } x : r_x^t \;-\; r_x^{t-1} \\
E[\Delta r_x^t] &= \text{ expected incremental regret for not playing } x
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  Vector versions: r^t, $\bar{r}^t$, $[\bar{r}^t]_+, \dots, E[r^t], E[\Delta r^t]$.

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  i.e., the regret vector must approach the negative orthant with probability one.

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E[\Delta r^{t+1}] &= (\; q_1^{t+1} \cdot 0 + q_2^{t+1} \cdot \alpha^{t+1} \;,\; q_1^{t+1} \cdot -\alpha^{t+1} + q_2^{t+1} \cdot 0 \;) \\
&= \alpha^{t+1} (\; q_2^{t+1} \;,\; -q_1^{t+1} \;).
\end{aligned}
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Why does regret matching work?

A strategy q such that $\lim_{t\to\infty}[\bar{r}^t]_+ = 0$: 1st attempt

Take $q_1^t = q_2^t = 1/2$ for all t.

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However, the two terms merely neutralise each other—which is not what we want: we want *all* regrets to be non-positive.

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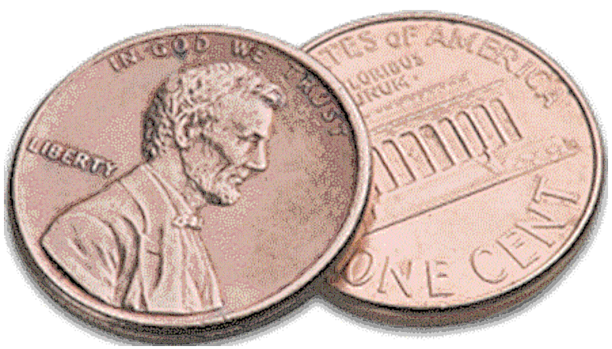

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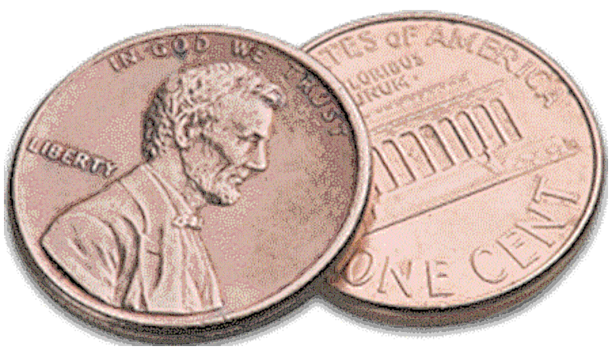

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  - Won't work: suppose you meet an opponent who happens to switch every round as well . . .
- Won't work in general: your corrections may by coincidence be out of phase with the path of play of your opponent. Peyton Young:

> "Recall that no-regret must hold even when Nature is malevolent."
>
> (p. 26)

Decrease of expected regret

The objective is to find a (mixed) strategy $g : H \to \Delta(\{1,2\})$ such that

$$E[\,\bar{r}^{t+1} \mid r^t, \dots, r^1] < \bar{r}^t \tag{1}$$

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because then Blackwell's approachability theorem can be applied to conclude $\lim_{t\to\infty}[\bar{r}^t]_+ = 0$. Since $\Delta E[r^{t+1}]$ is known, we have

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Decrease of expected regret

The objective is to find a (mixed) strategy $g : H \to \Delta(\{1,2\})$ such that

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\end{aligned}$$

So, the objective is to find a strategy such that $\alpha^{t+1}(-q_2^{t+1} ,\, q_1^{t+1}) < \bar{r}^t$.

A strategy q such that $\lim_{t\to\infty}[\bar{r}^t]_+ = 0$: 3rd attempt

A strategy q such that $\lim_{t \to \infty} [\bar{r}^t]_+ = 0$: 3rd attempt

- Recall: our objective is $[\bar{r}^t]_+ \to 0$.

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The last equation amounts to proportional regret matching.

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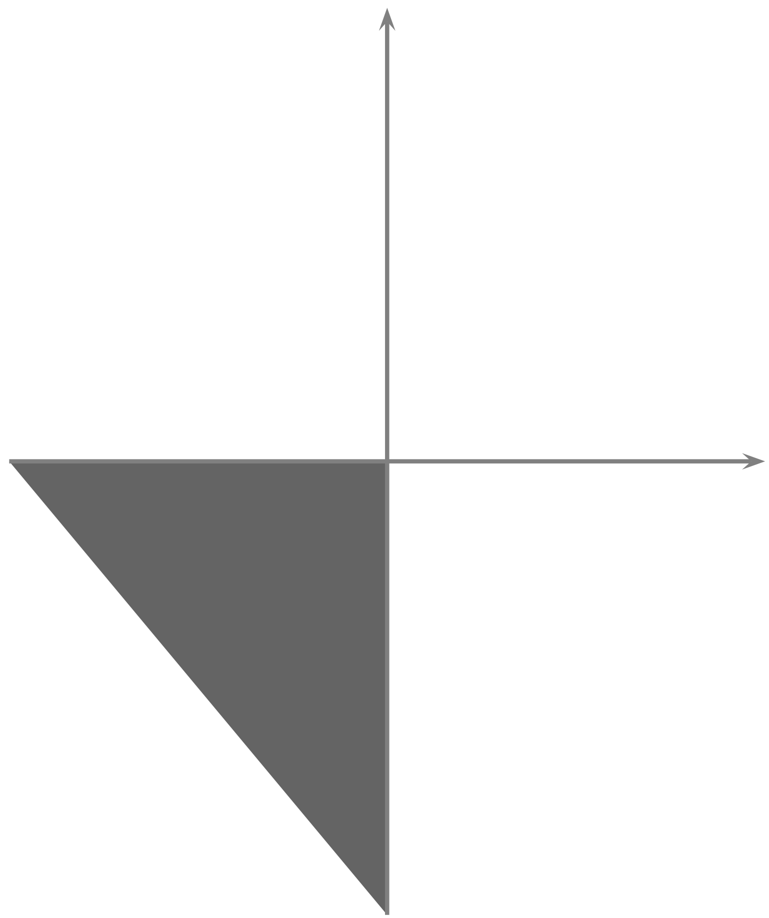
  - If all regret is non-positive, then play an action at random.

Stochastic dynamics of regret matching



- Expected incremental regret, $E[\Delta r^{t+1}]$ is made orthogonal to the current regret, independently of the unknown α^{t+1}.

  Because at A does not know what B will play next, this is crucial.

- $E[\bar{r}^{t+1}]$ is a convex combination of $\bar{r}^t_+$ and $E[\Delta r^{t+1}]$.

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- Ultimately, the result follows from Blackwell's approachability theorem (Strategic Learning and its Limits, 2004, Ch. 4).

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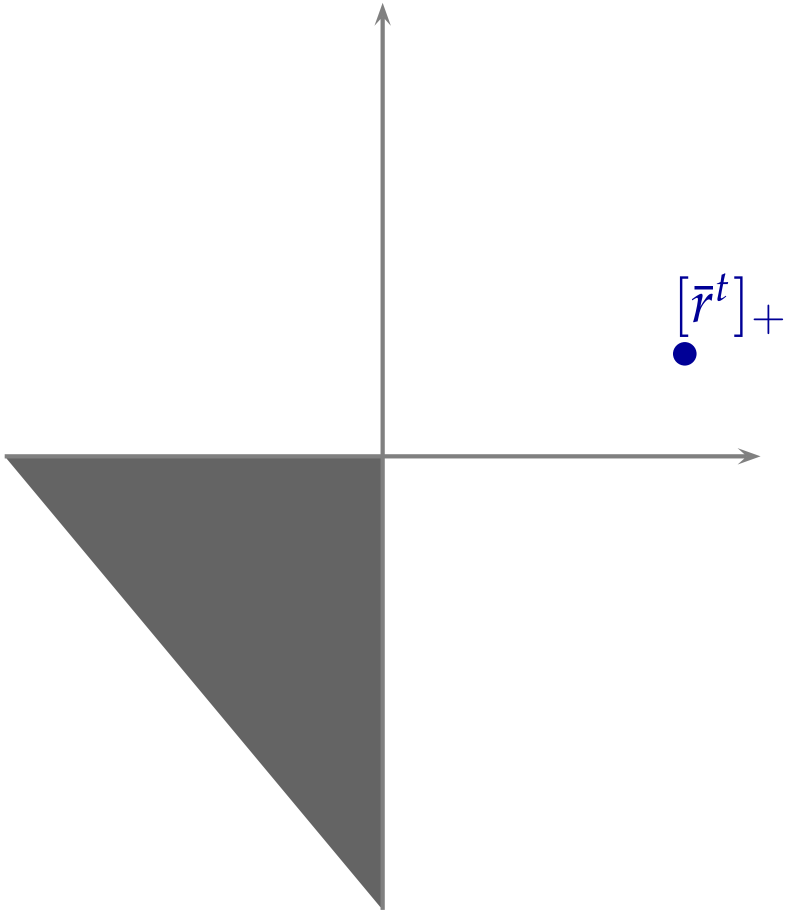

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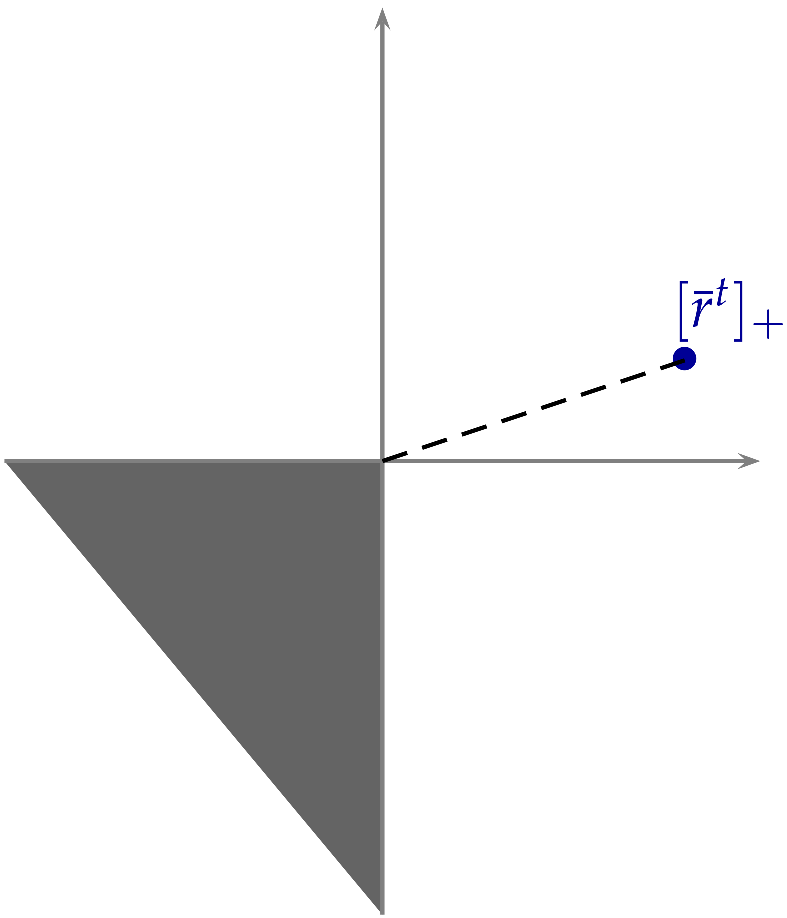

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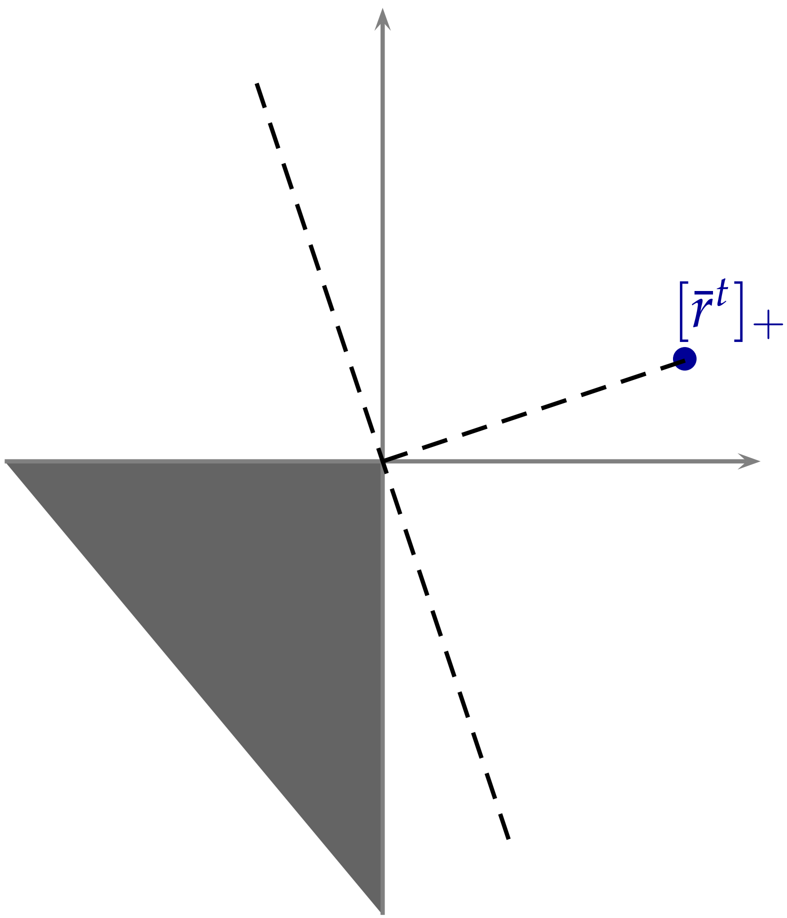

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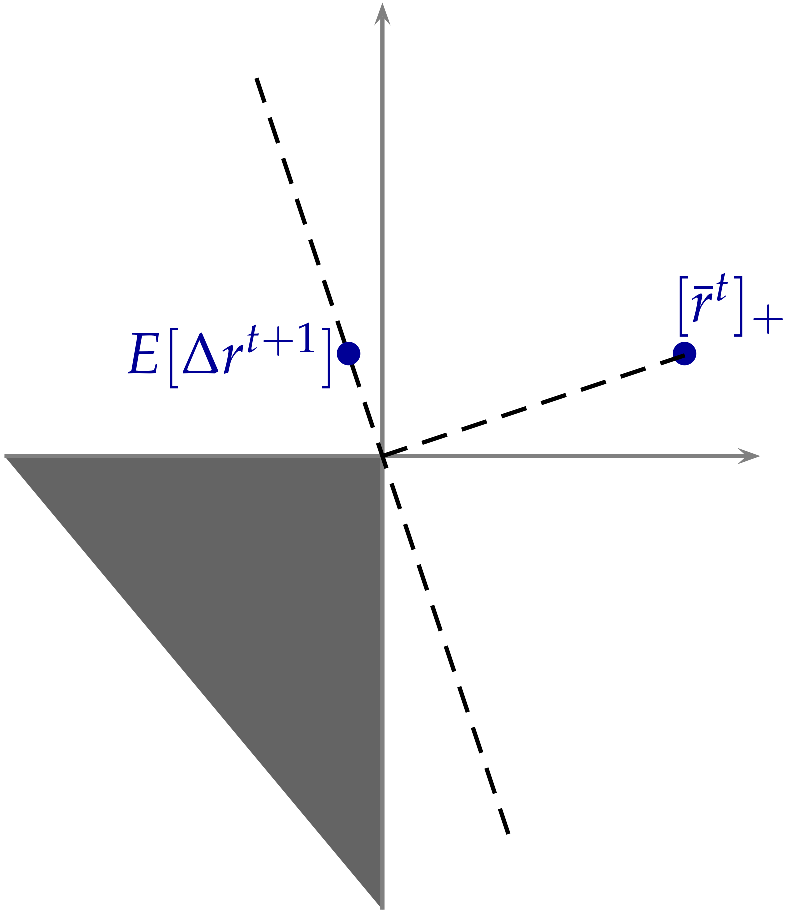

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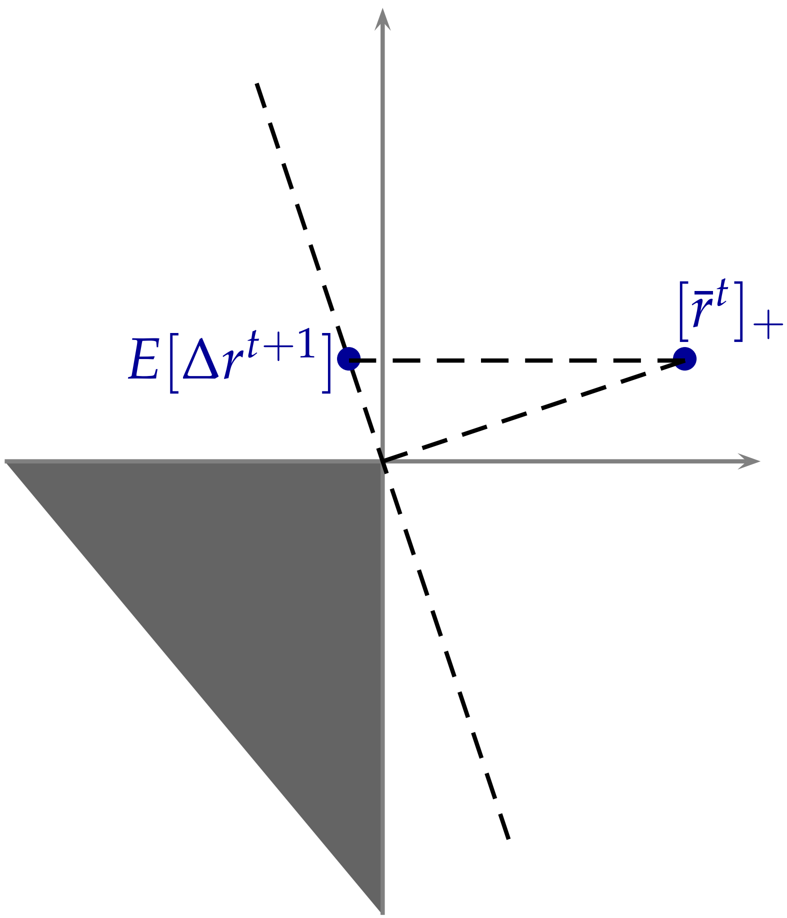

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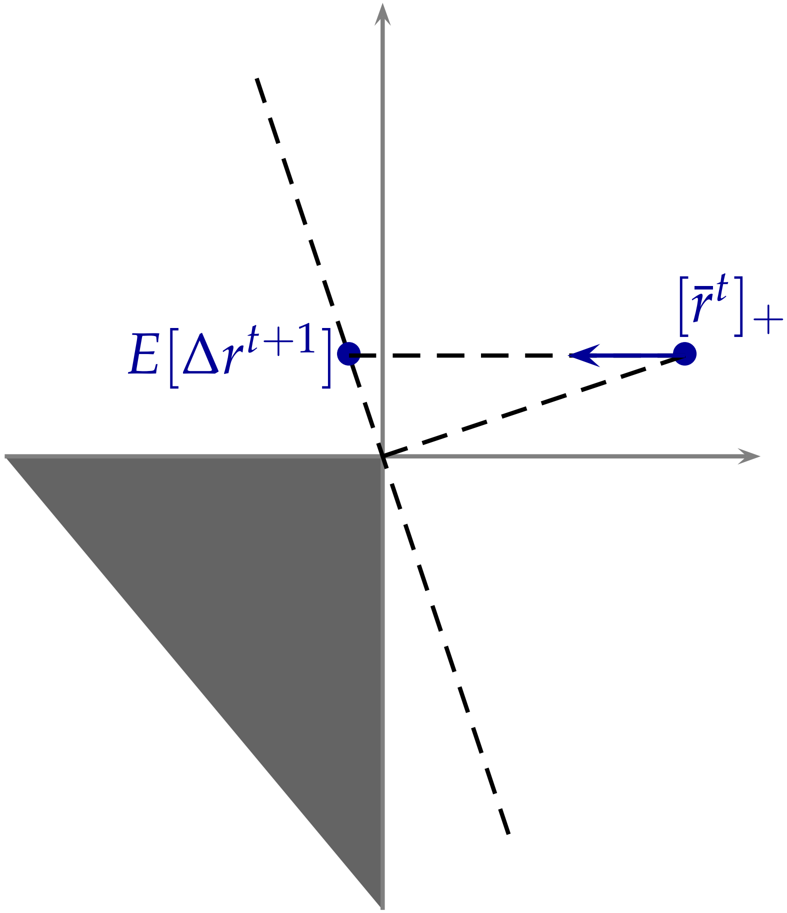

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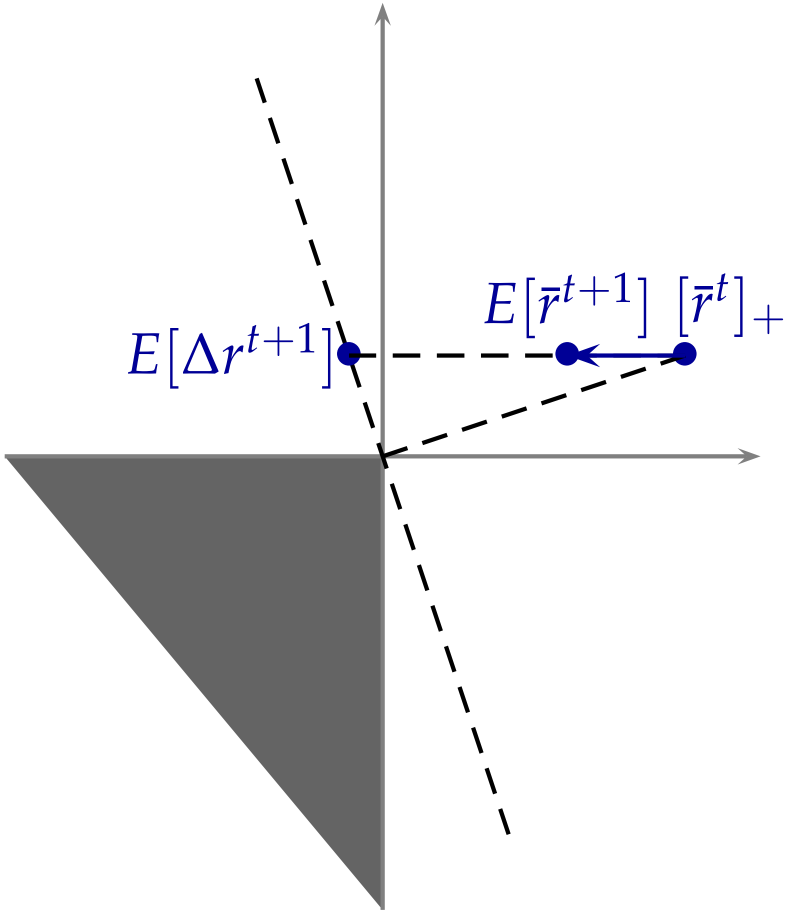

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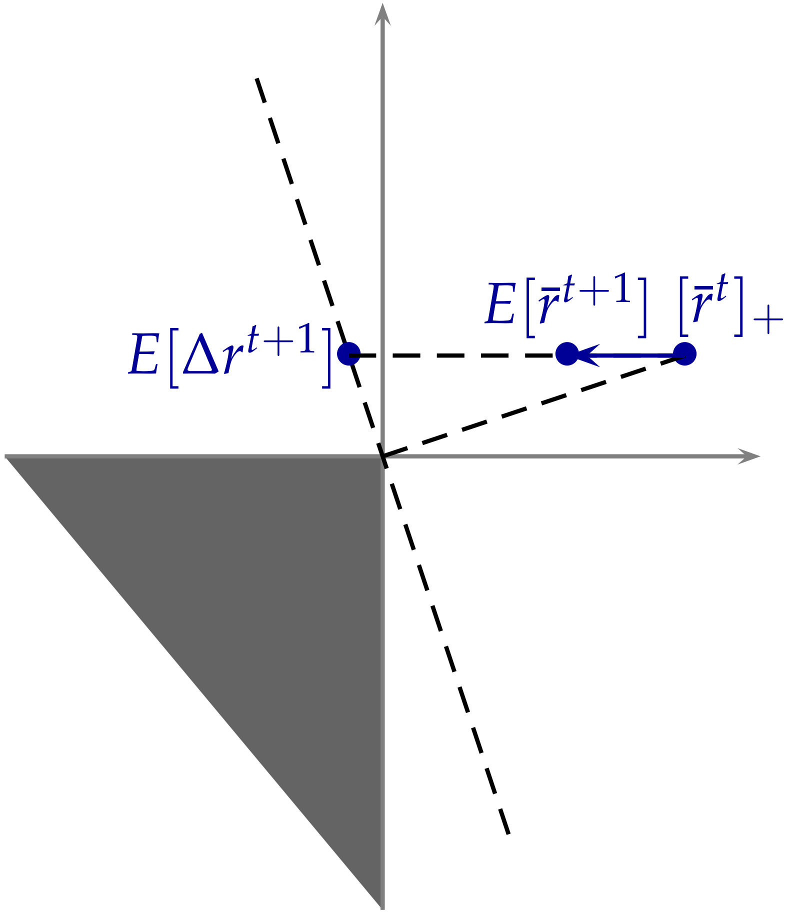

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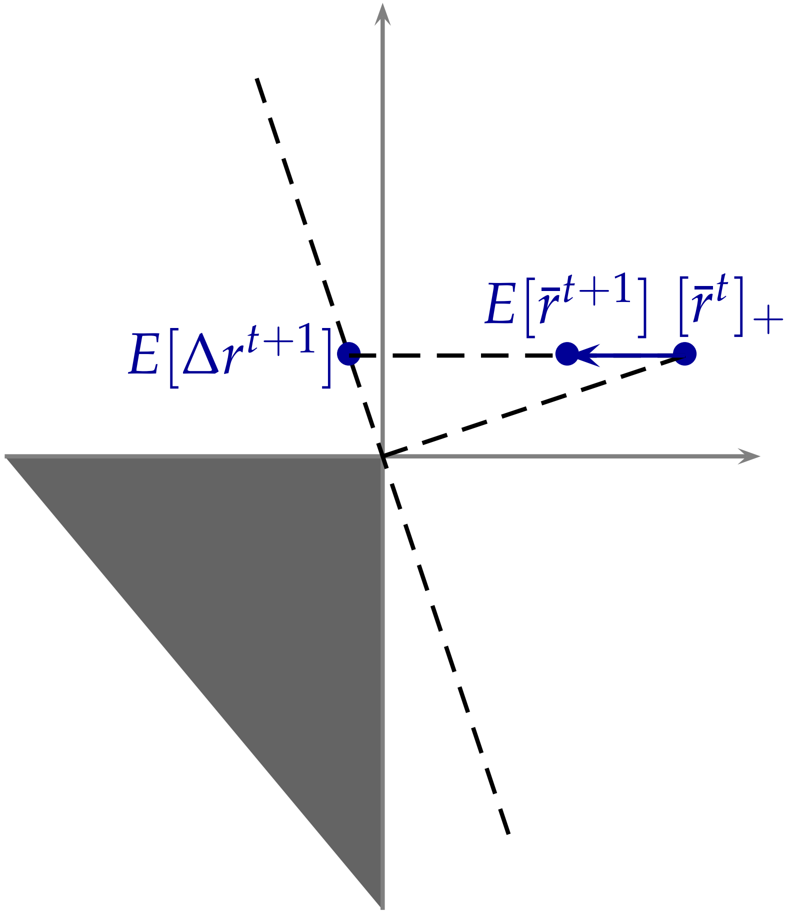

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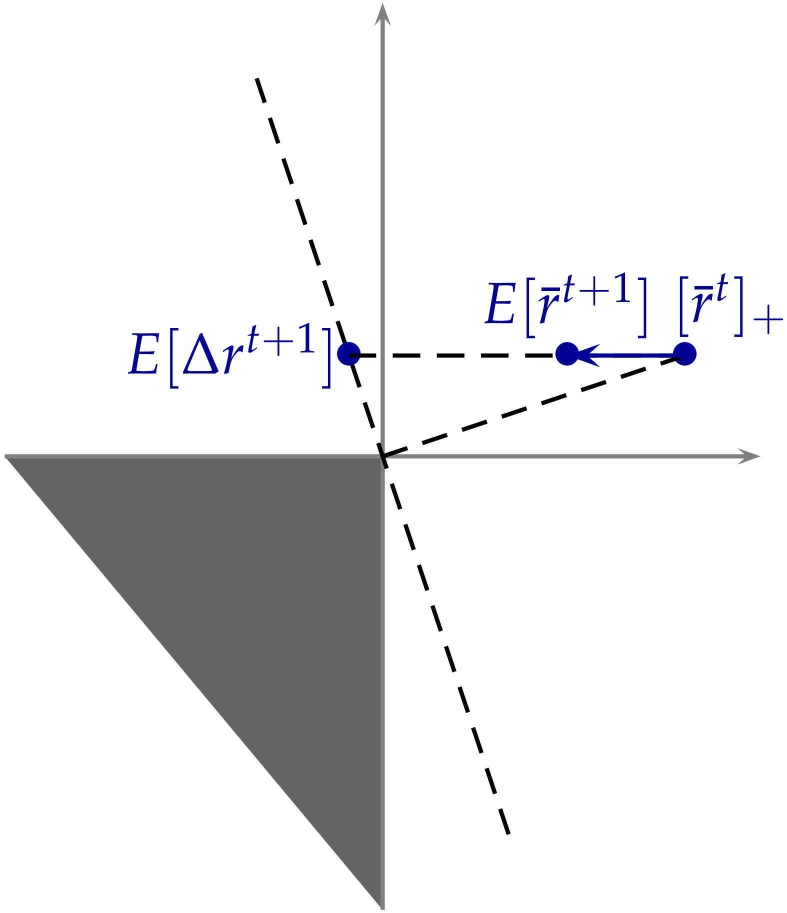

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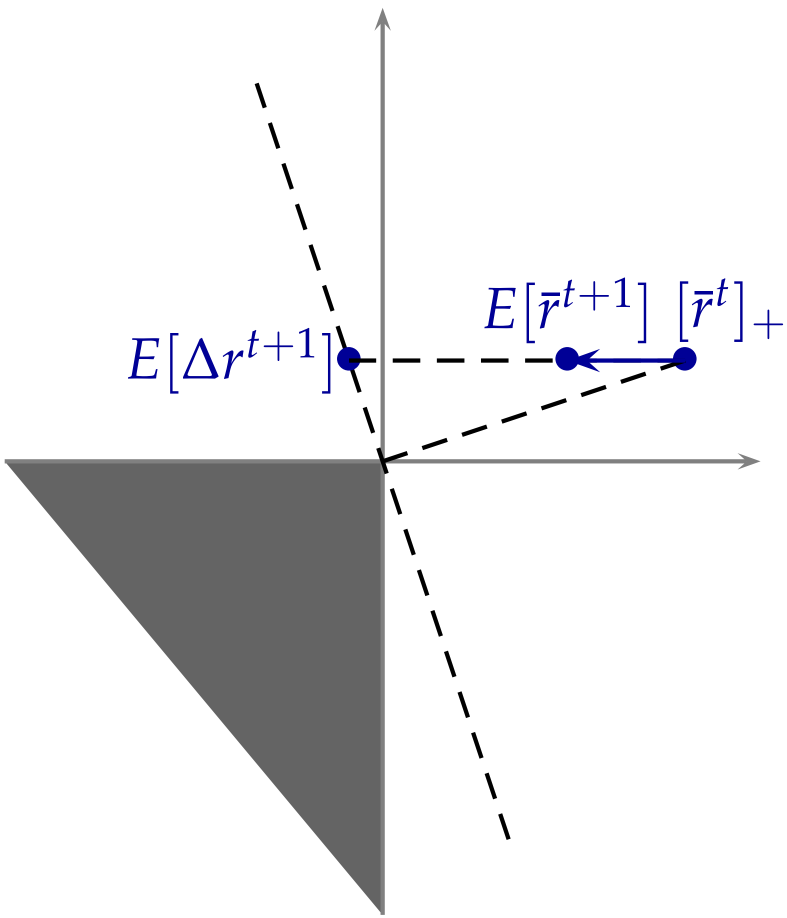

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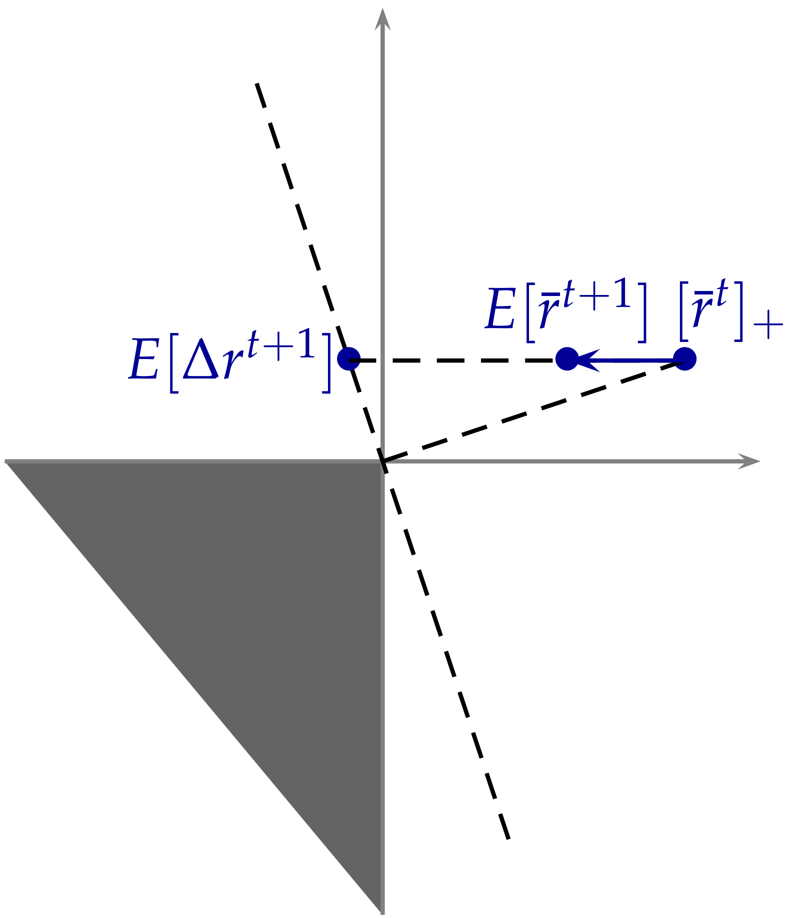

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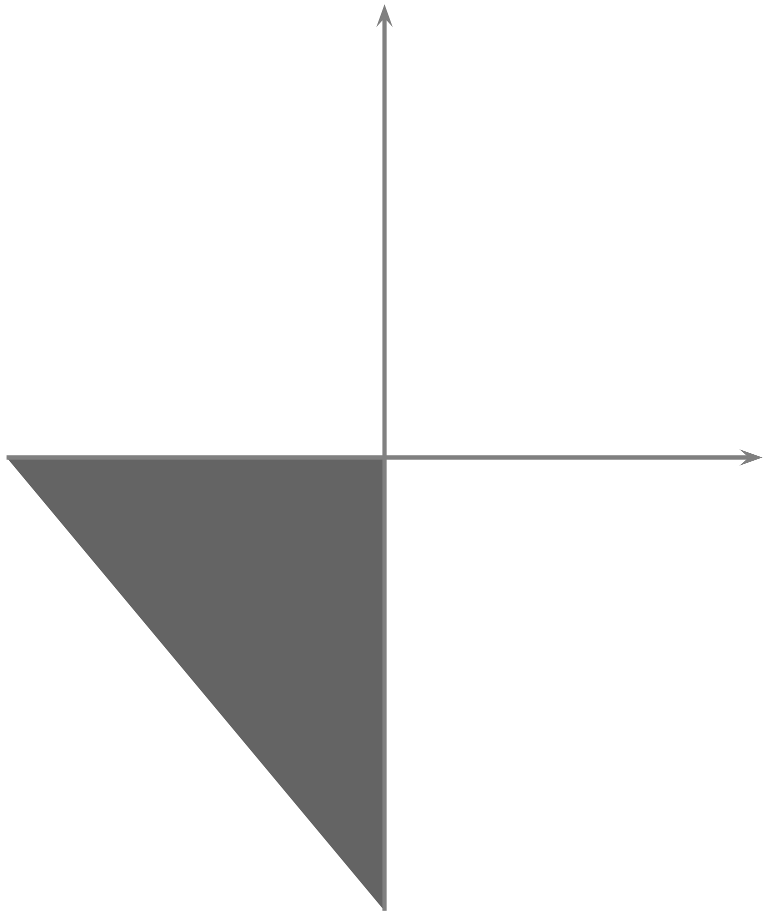

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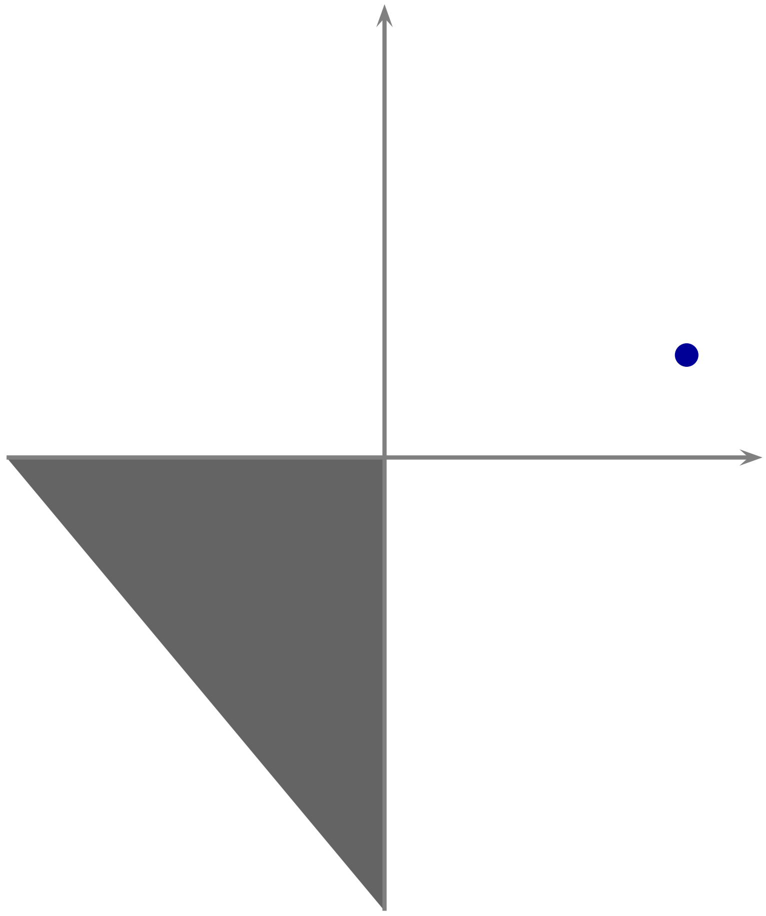

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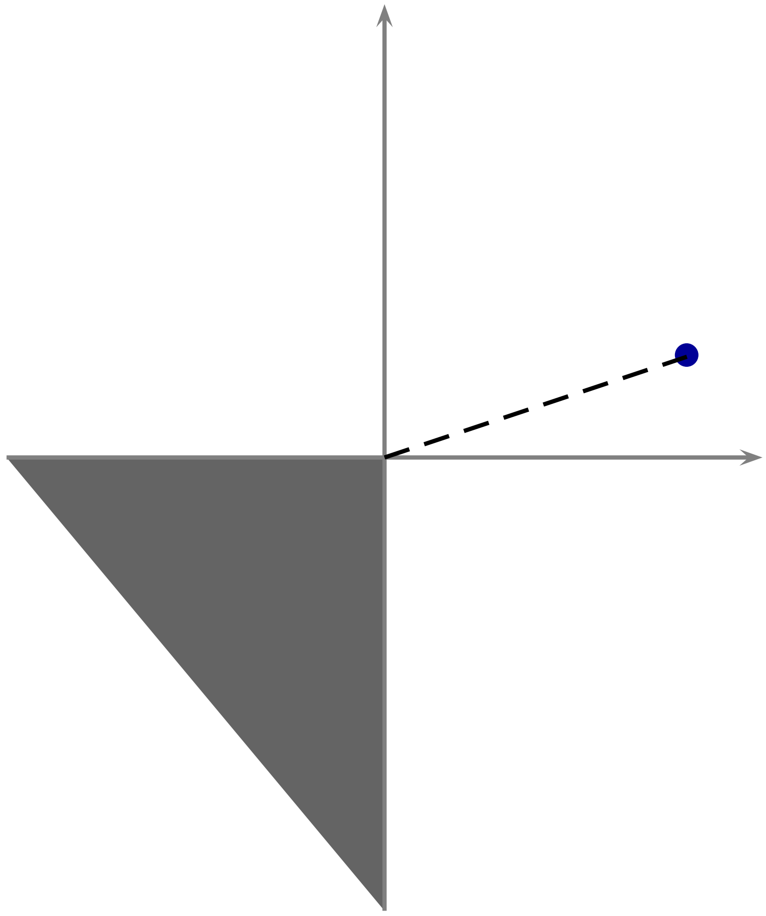

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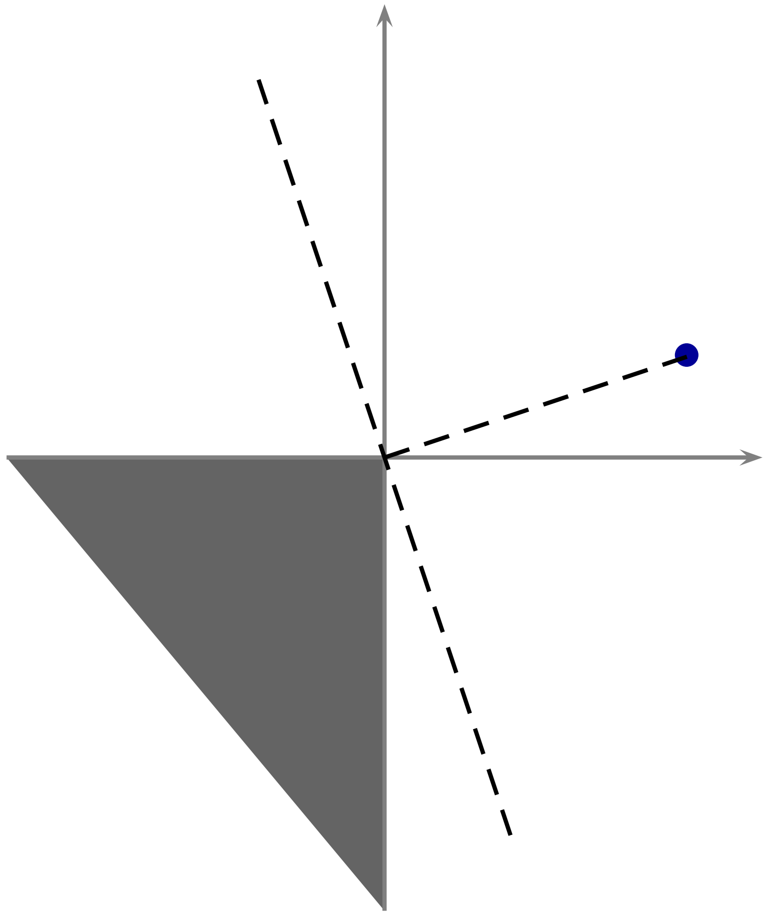

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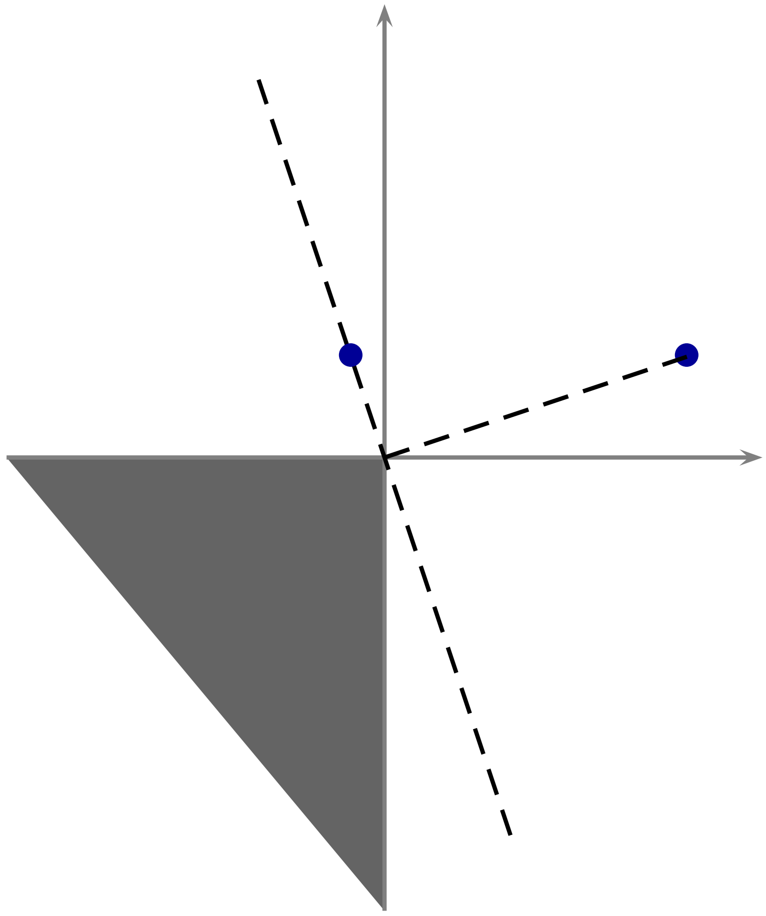

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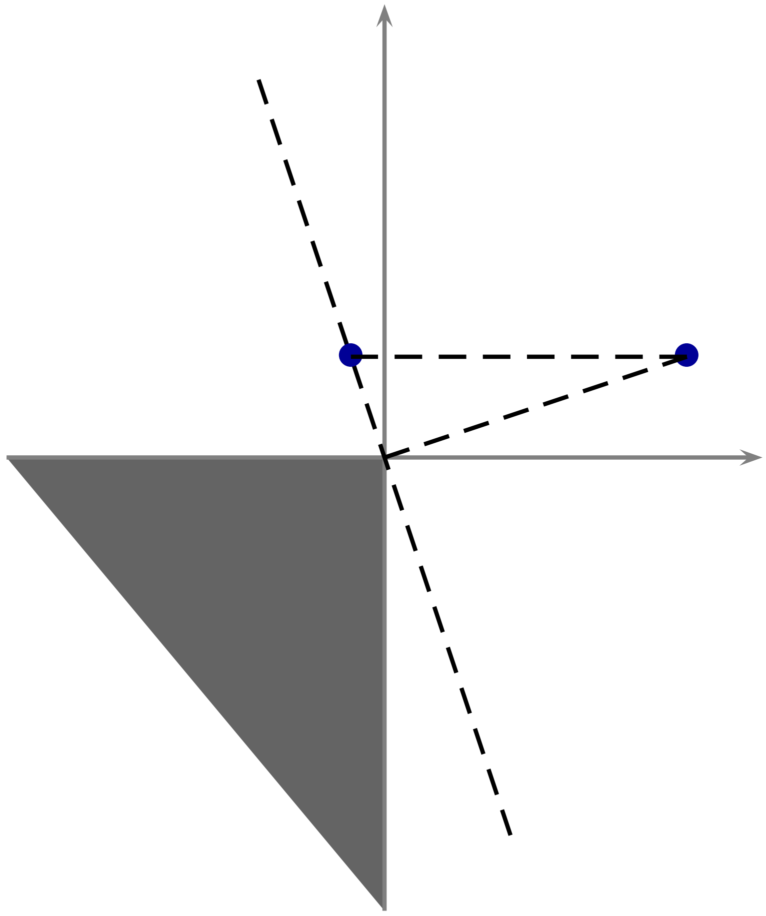

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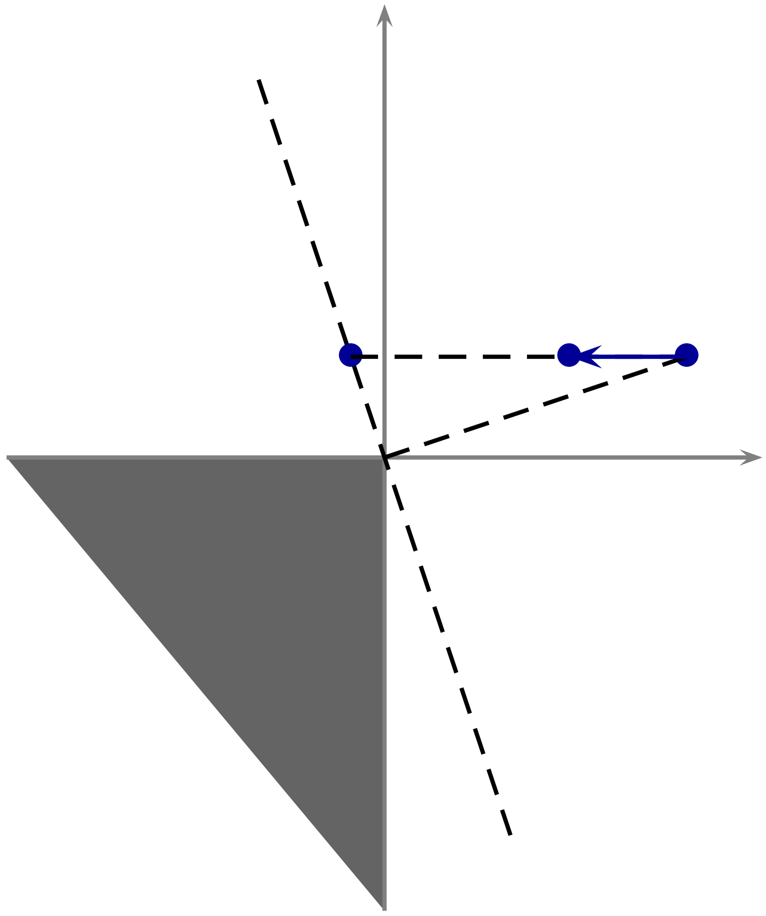
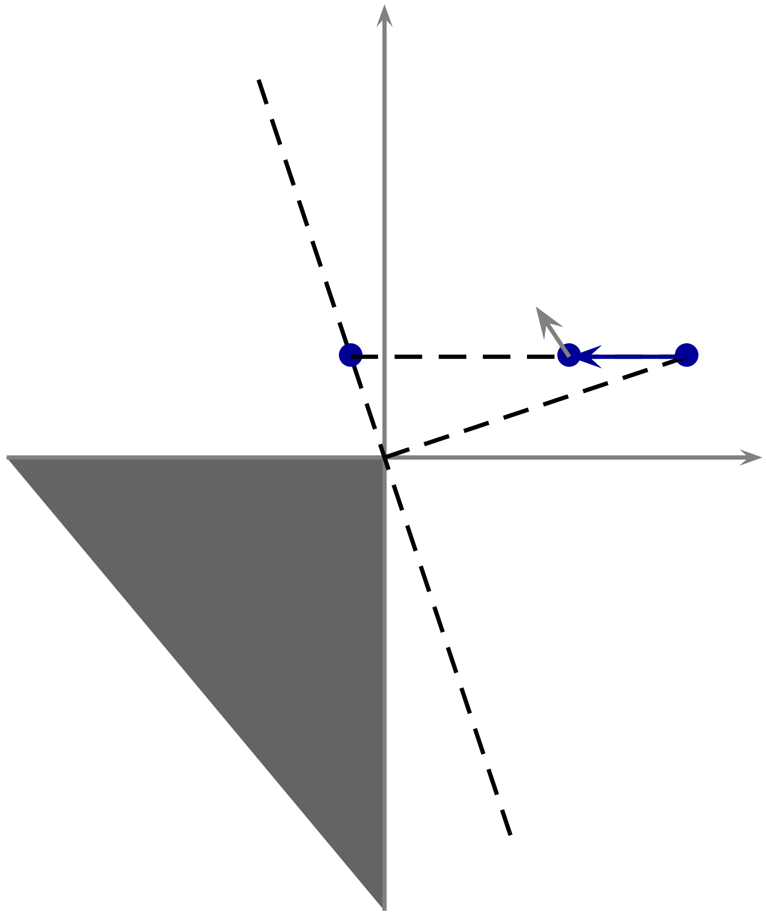


Stochastic dynamics of regret matching



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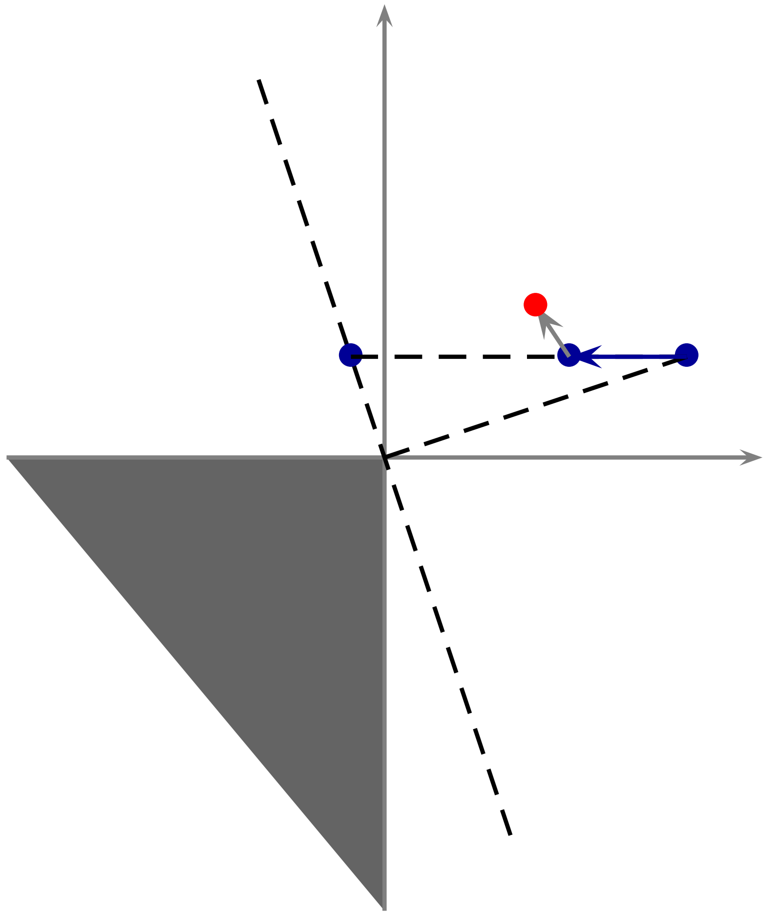

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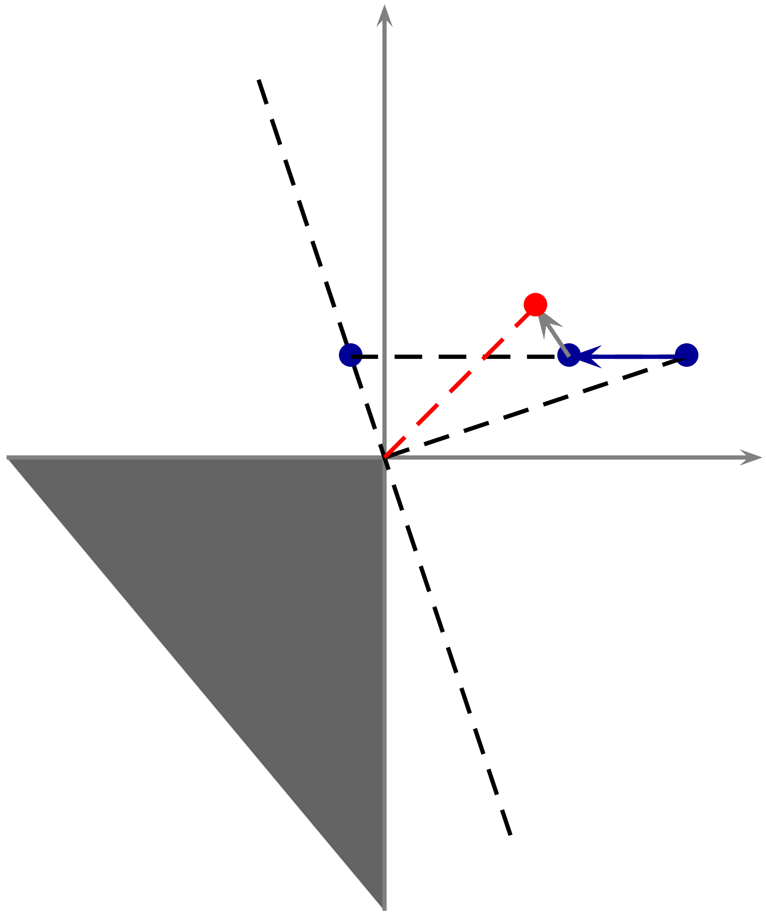

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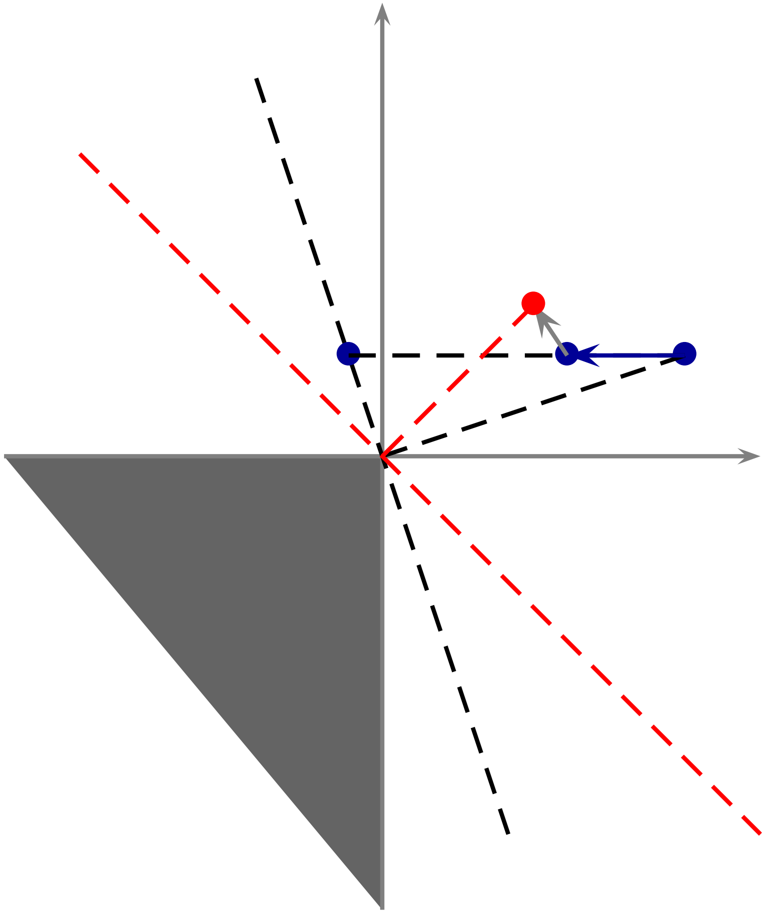

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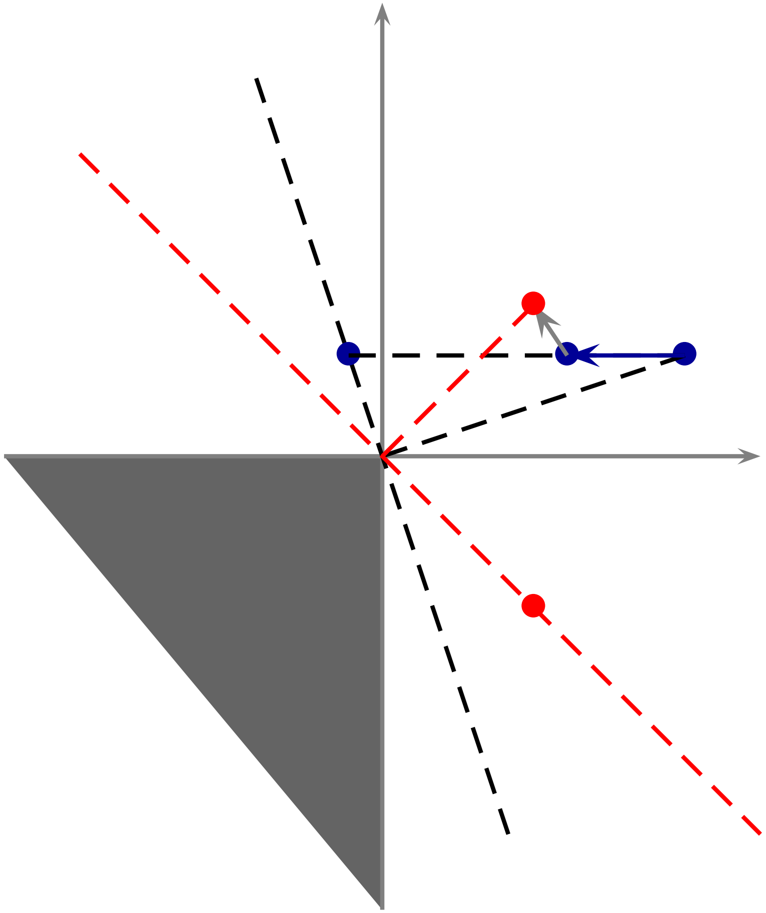

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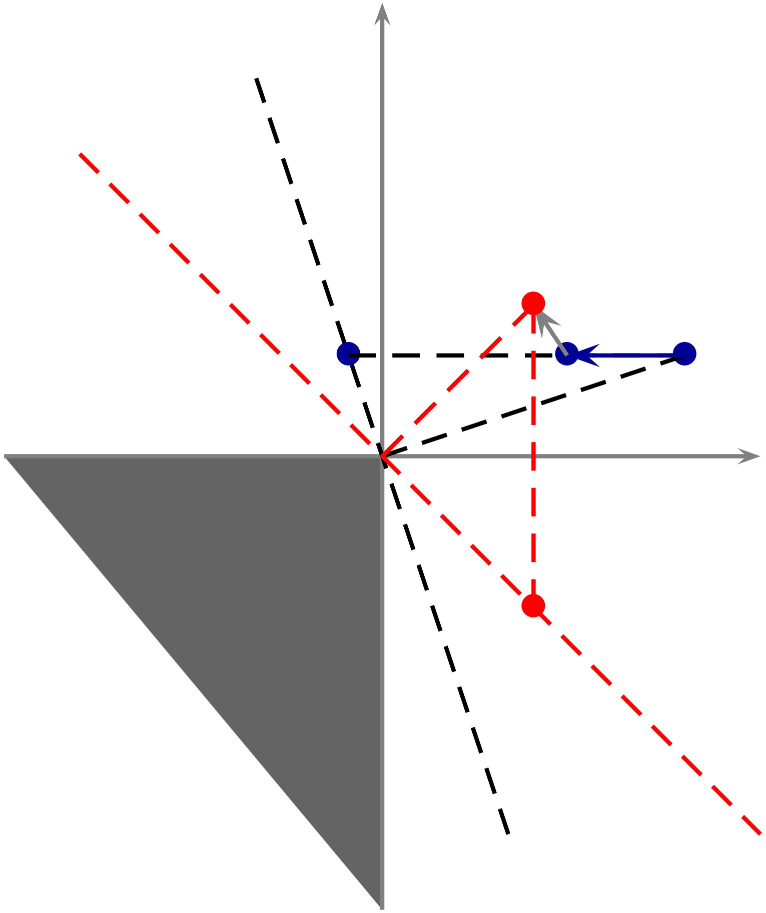

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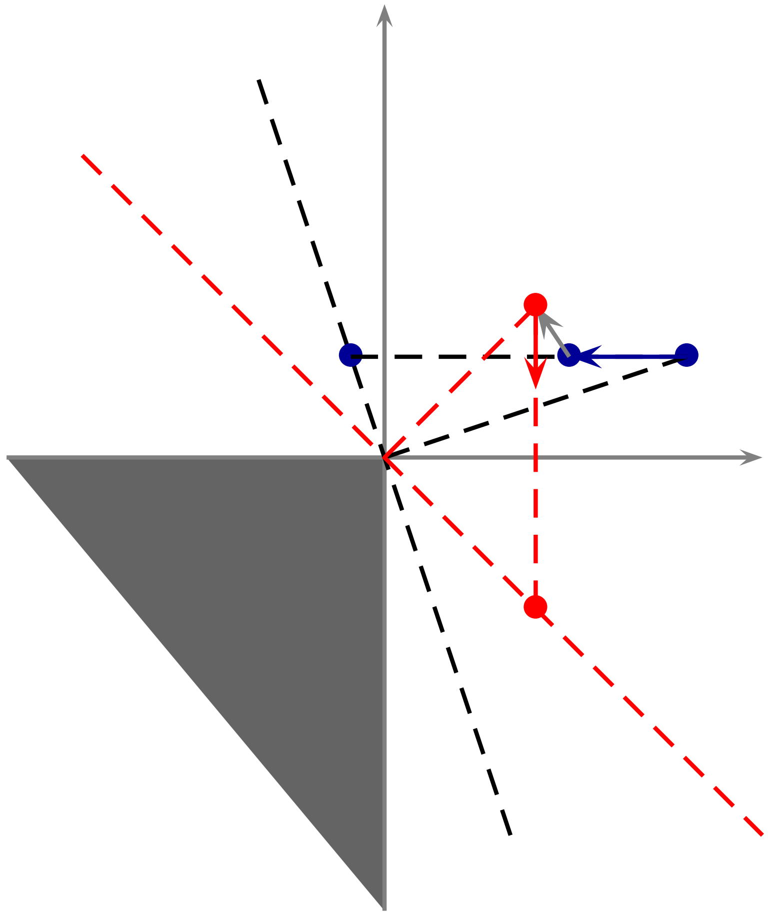

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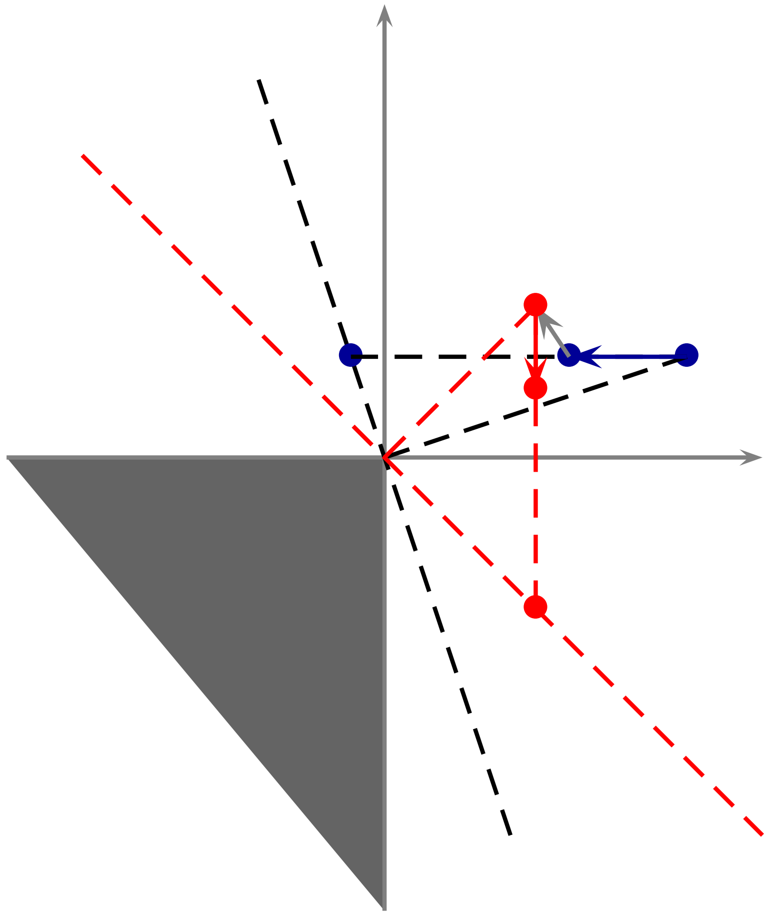

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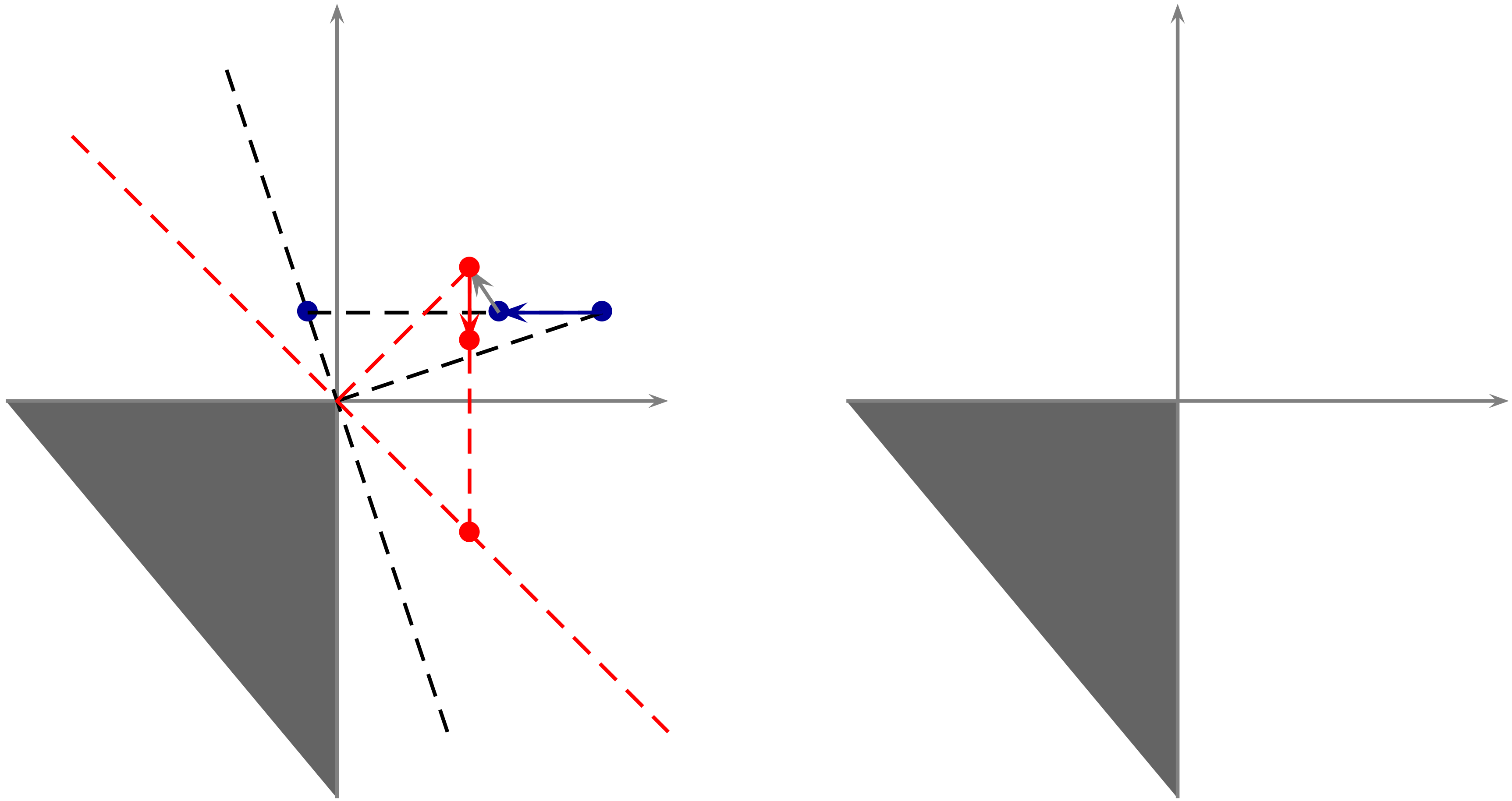

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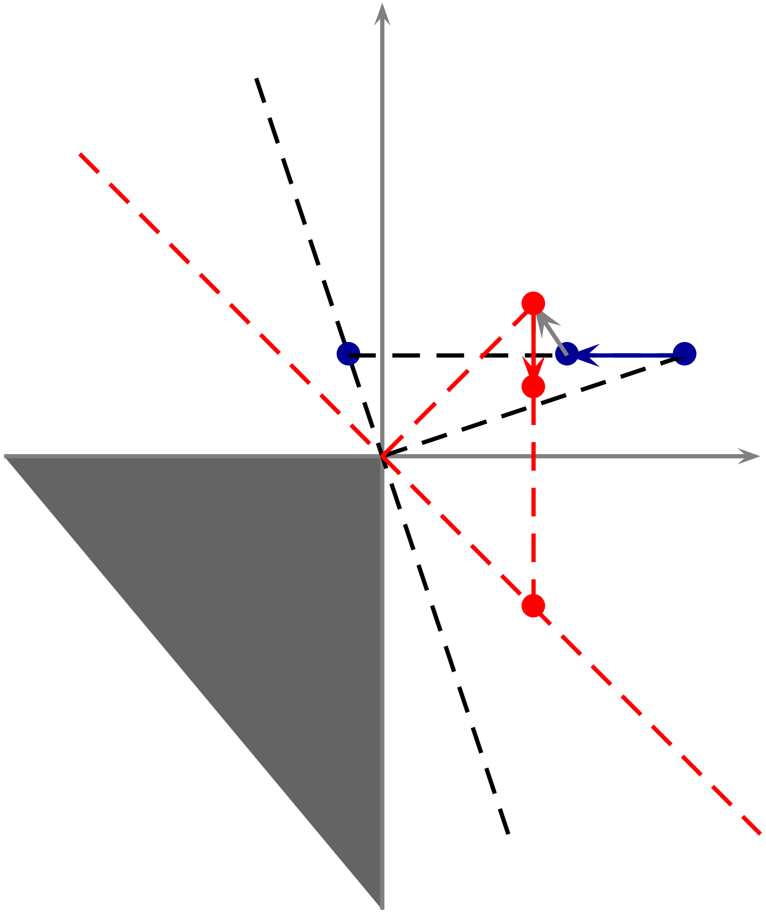

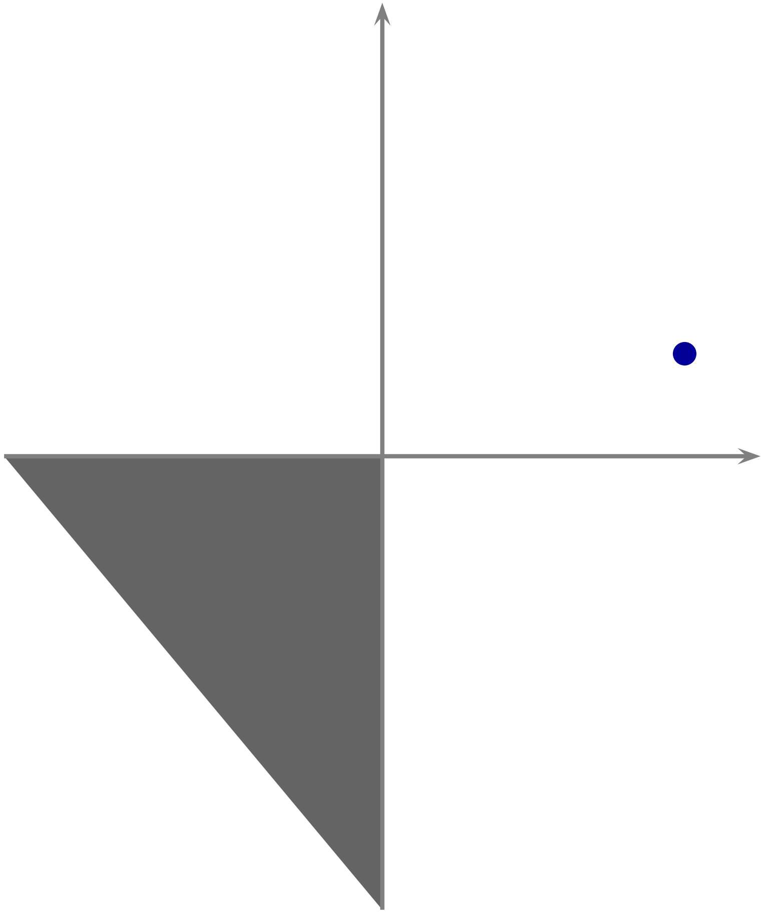

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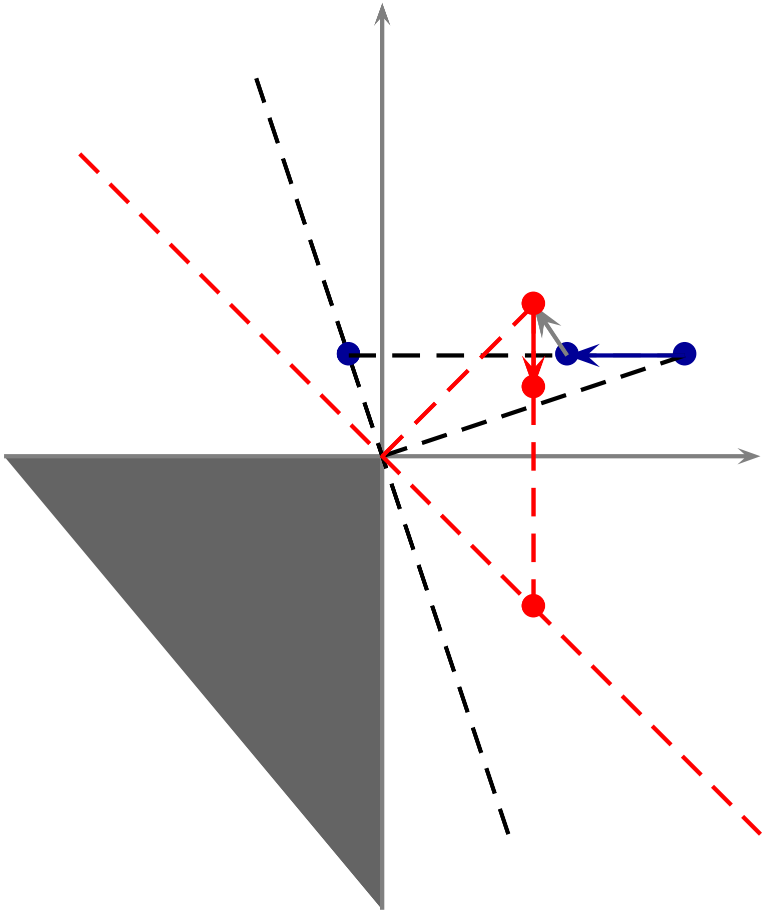

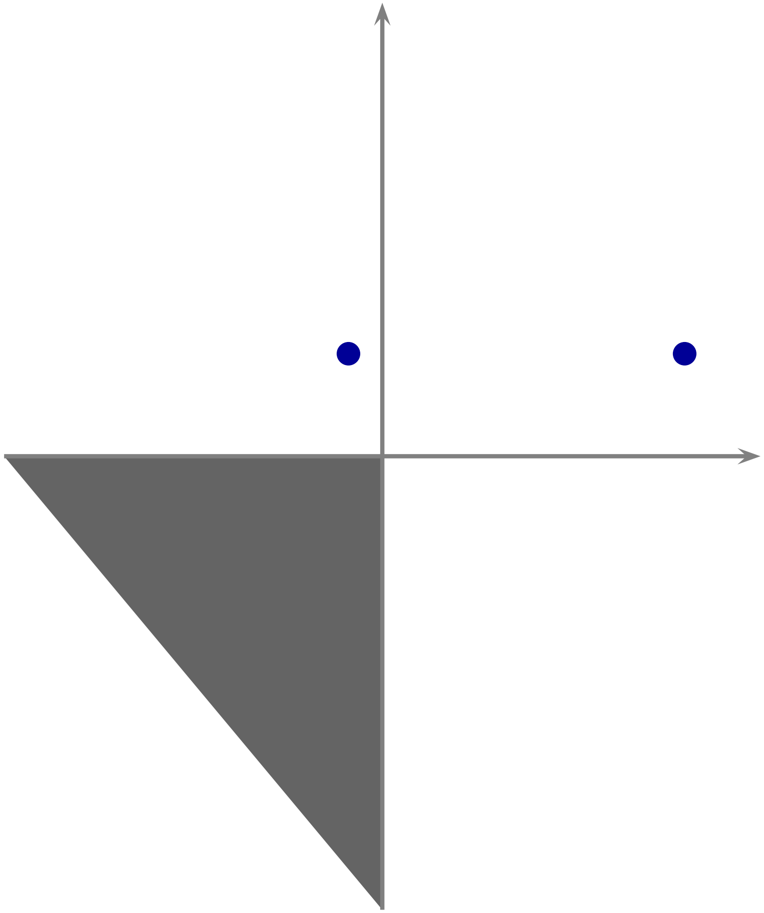

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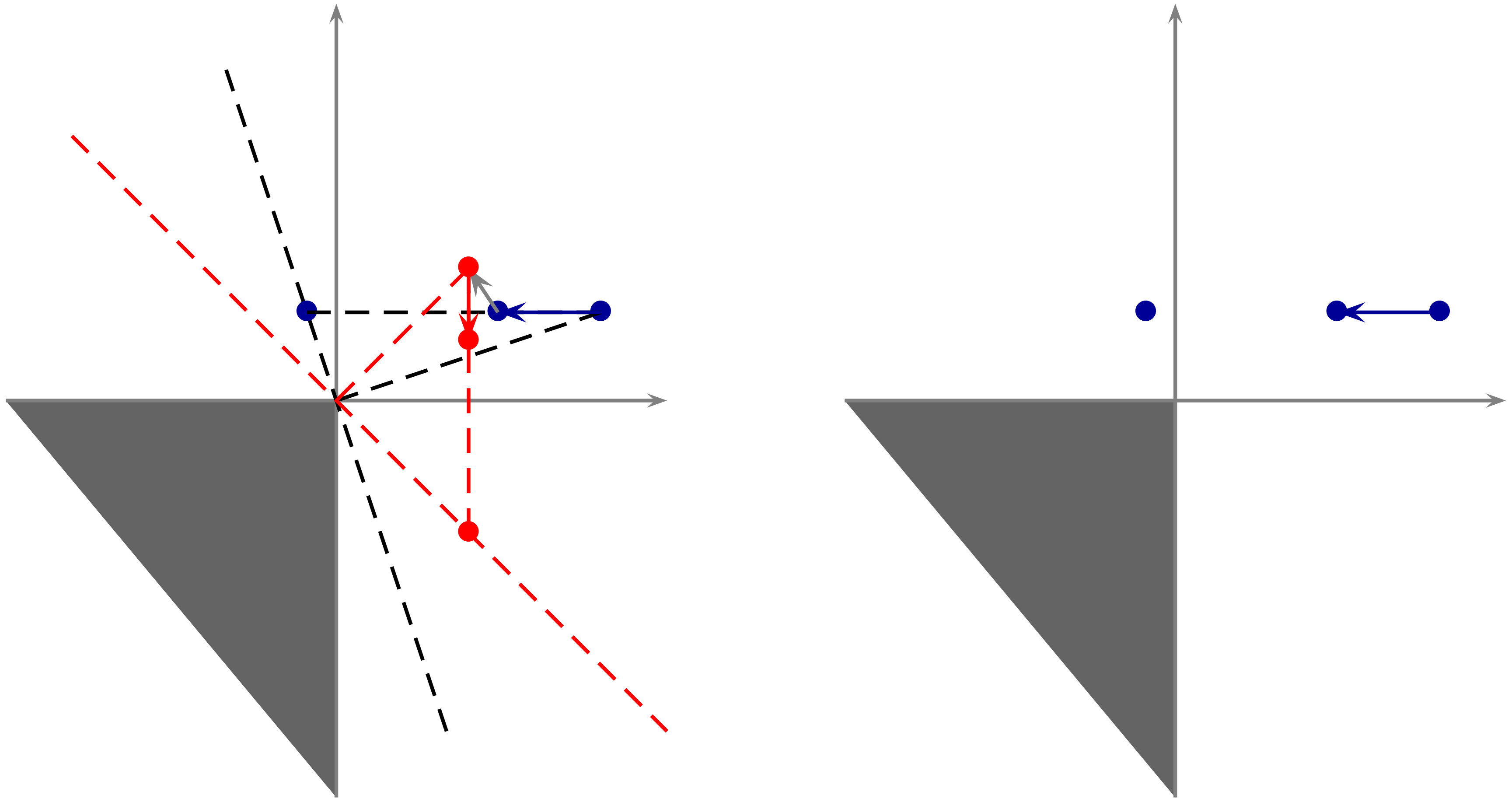

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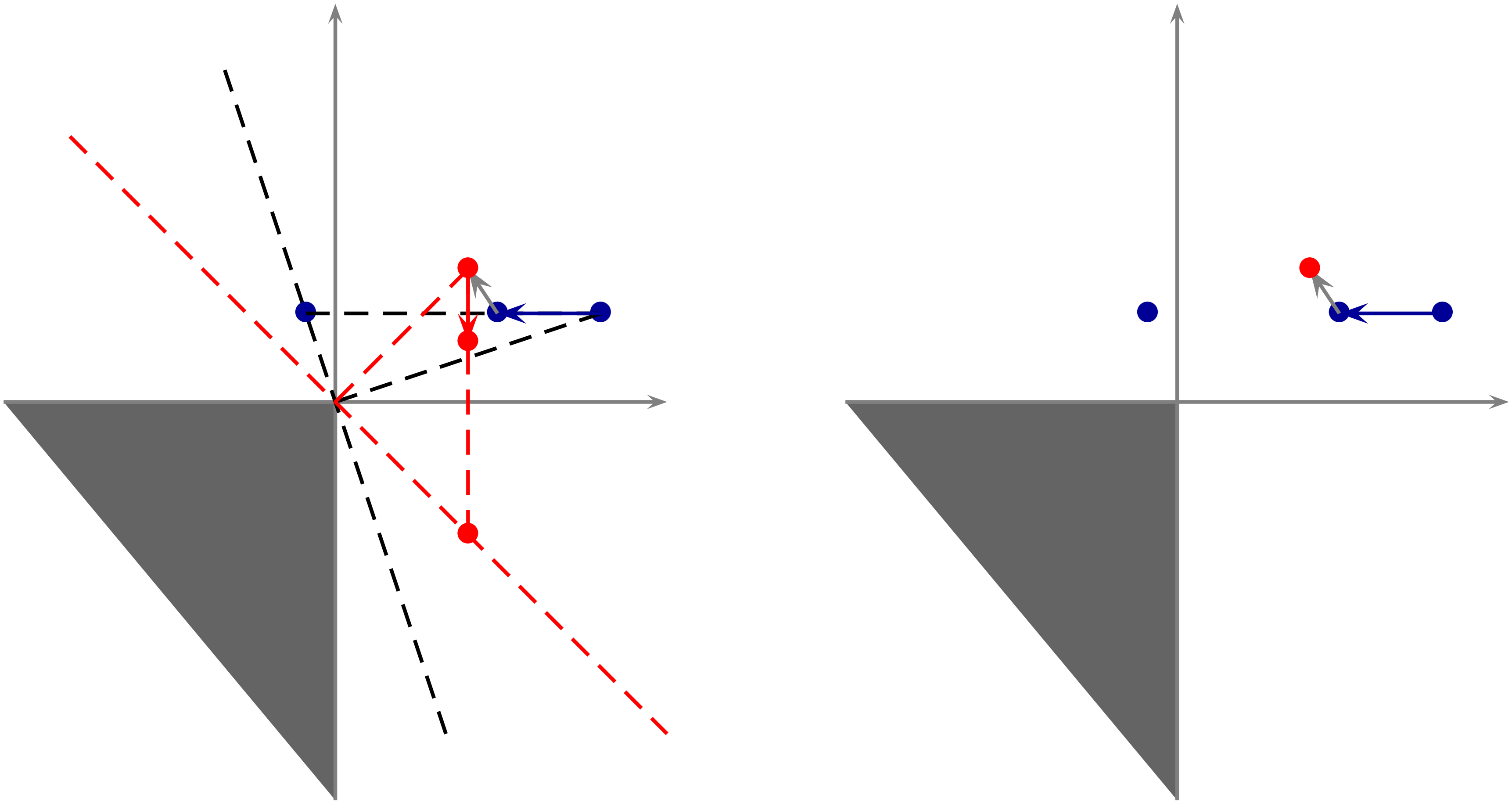

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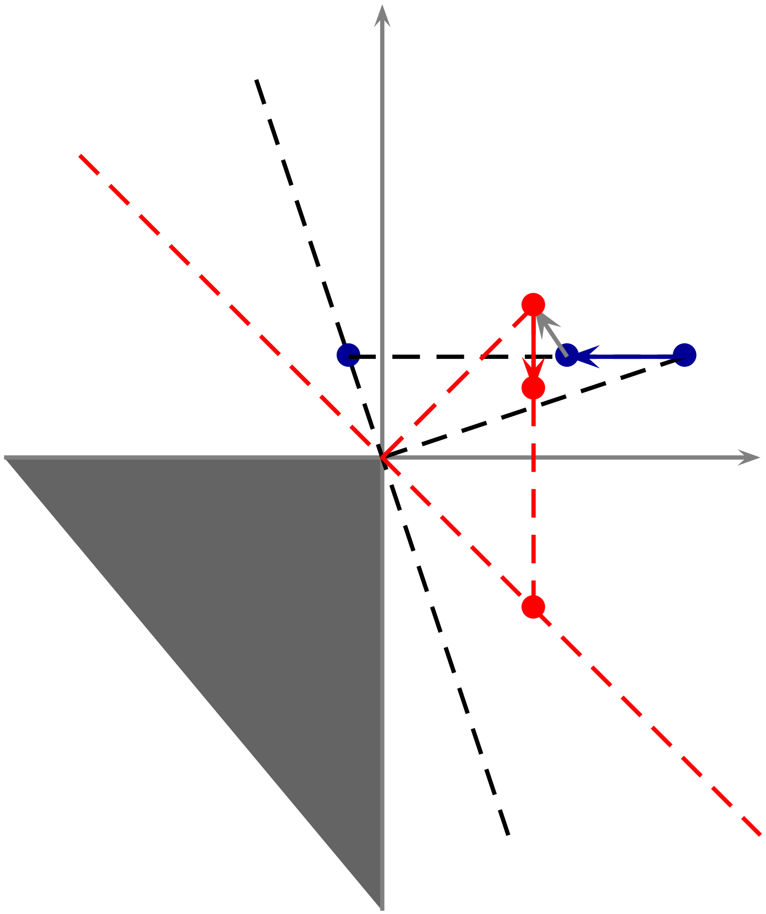

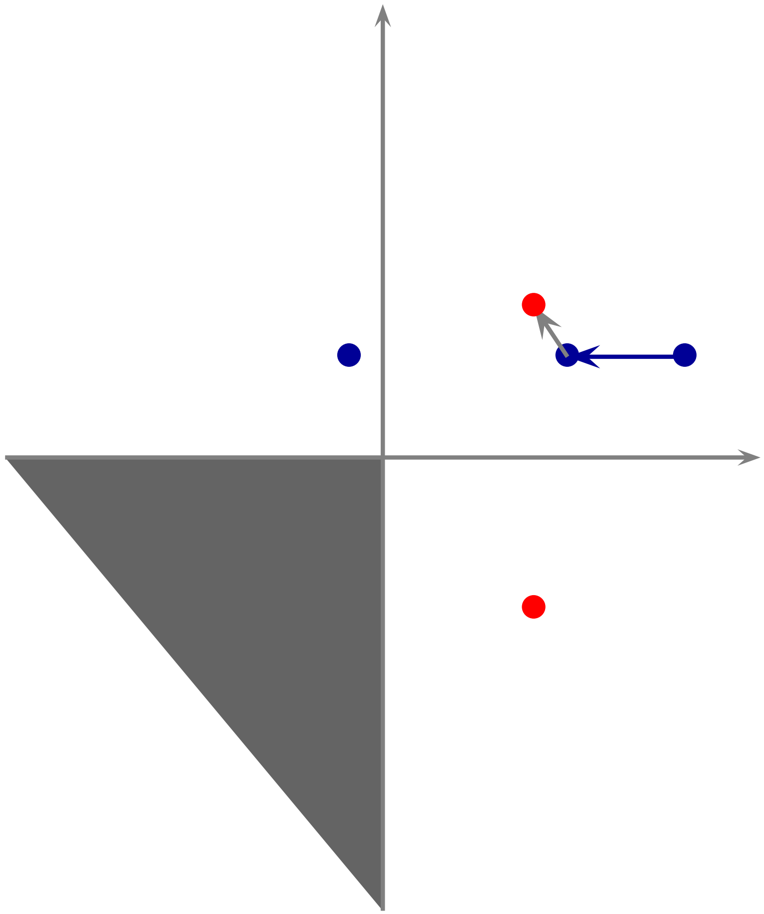

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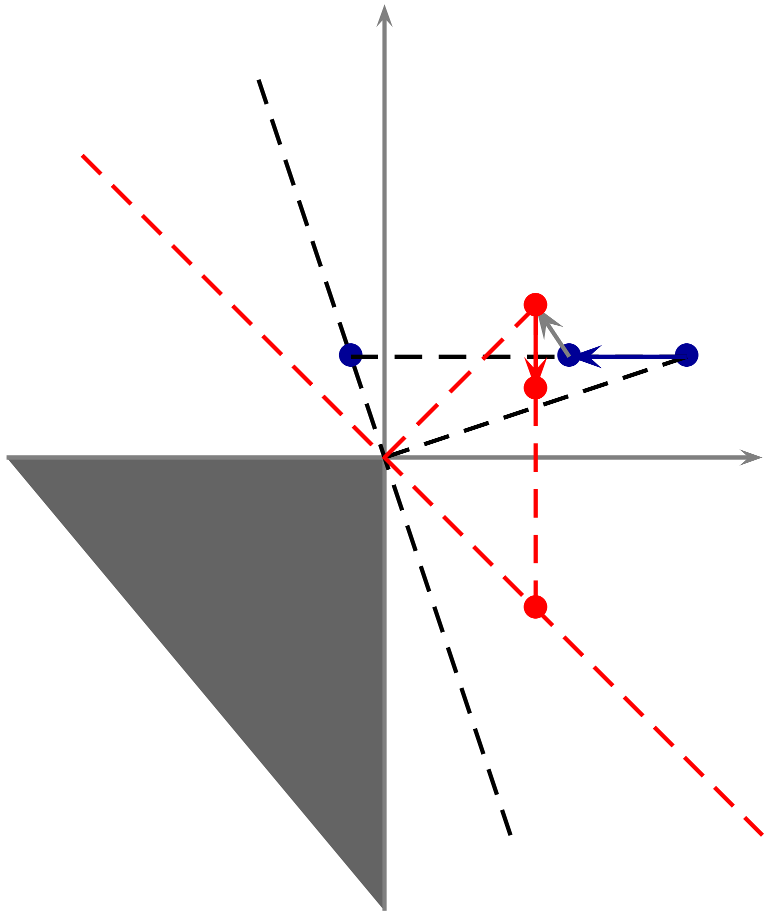

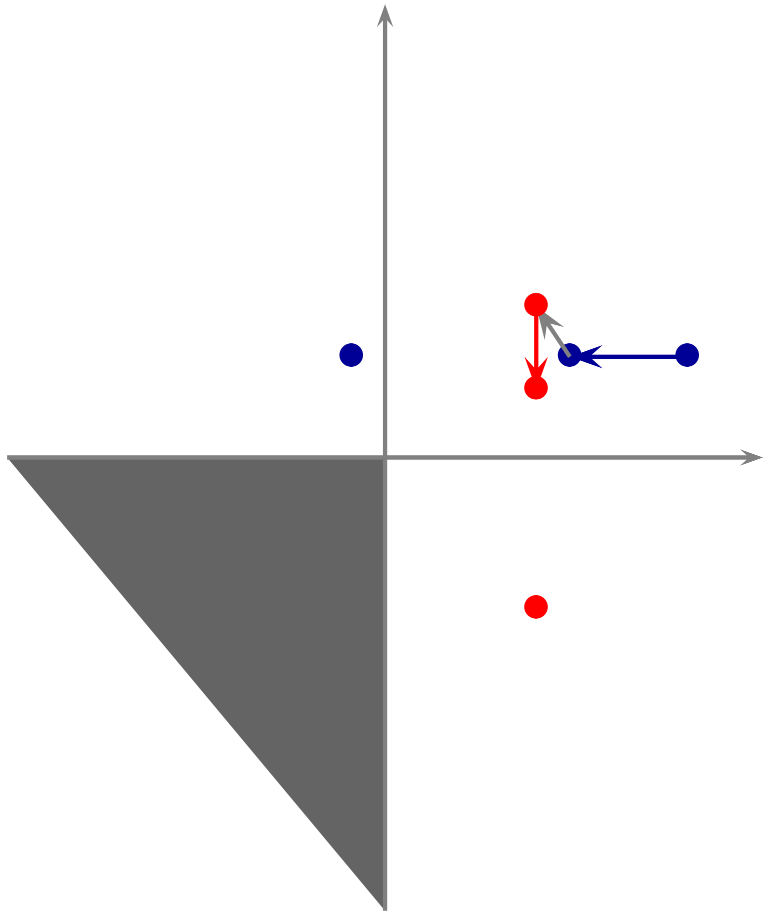

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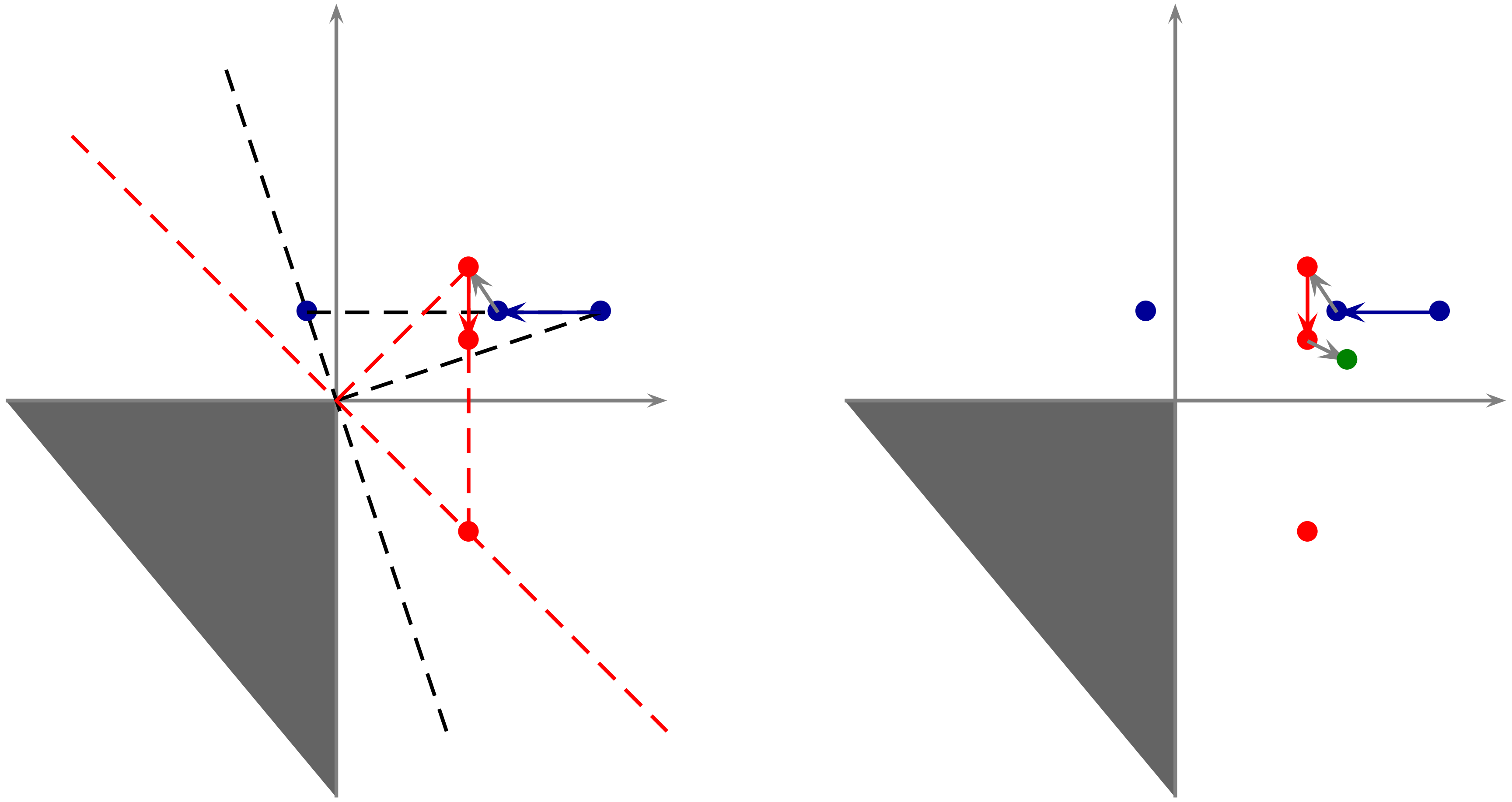

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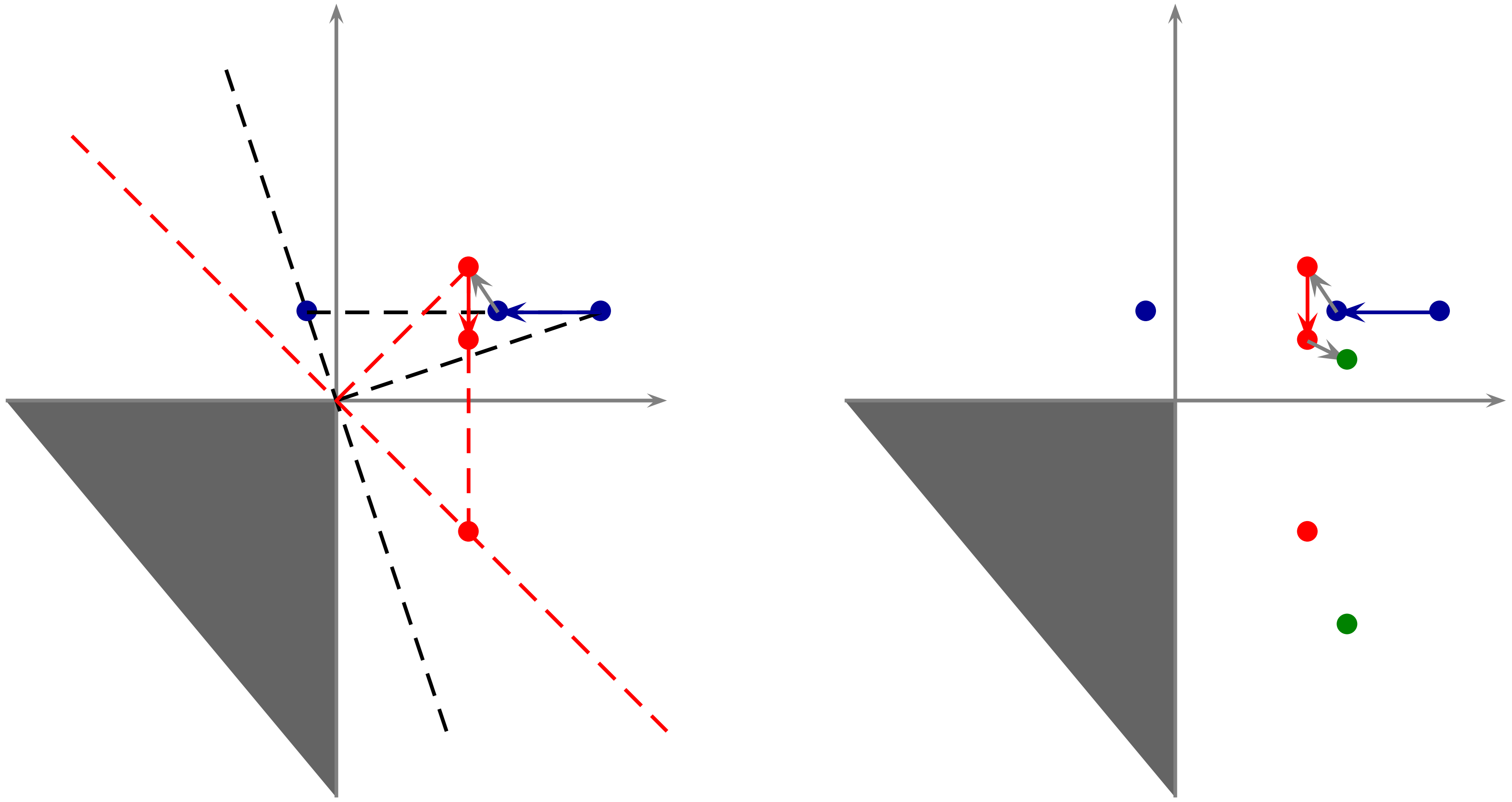

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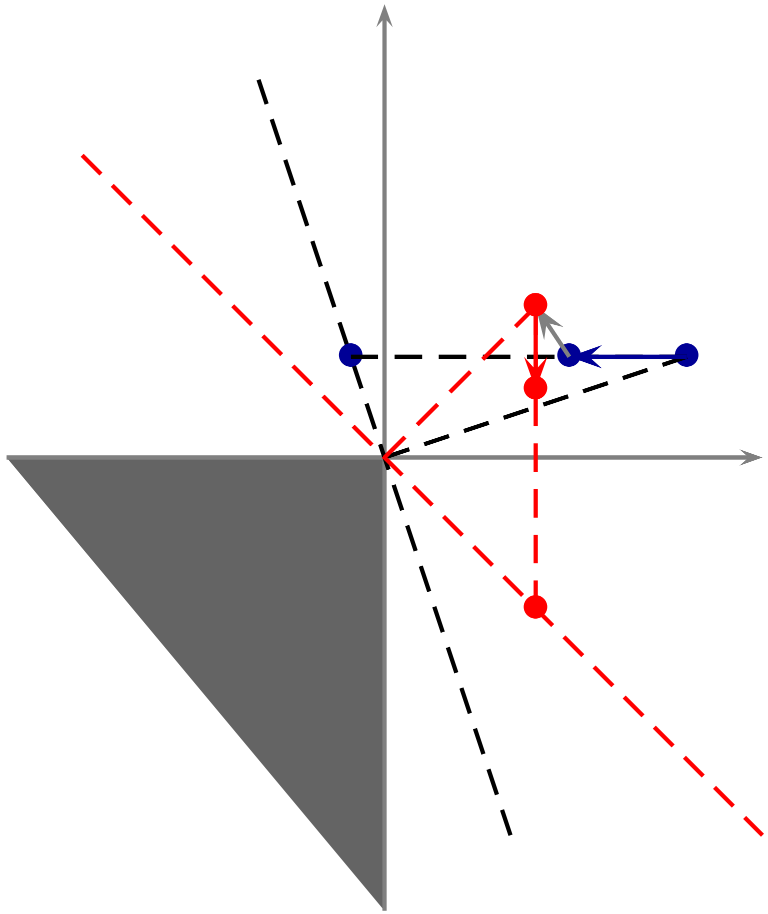

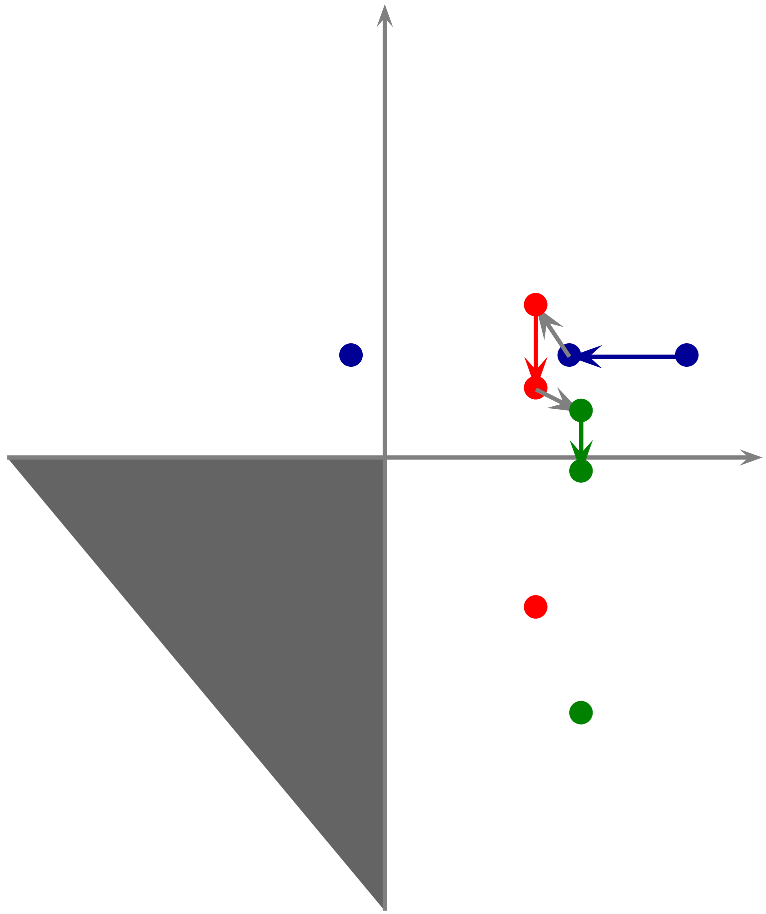

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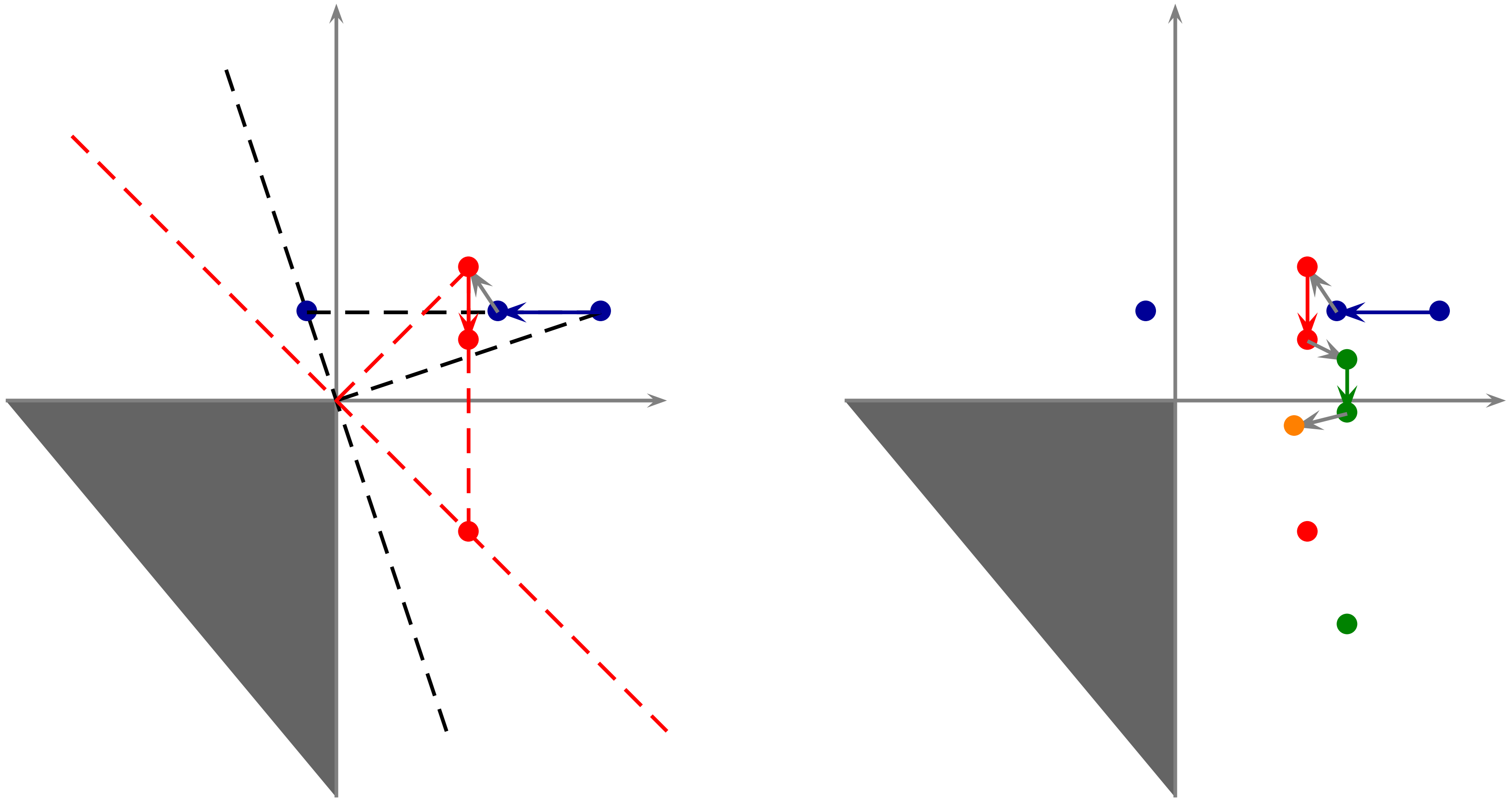

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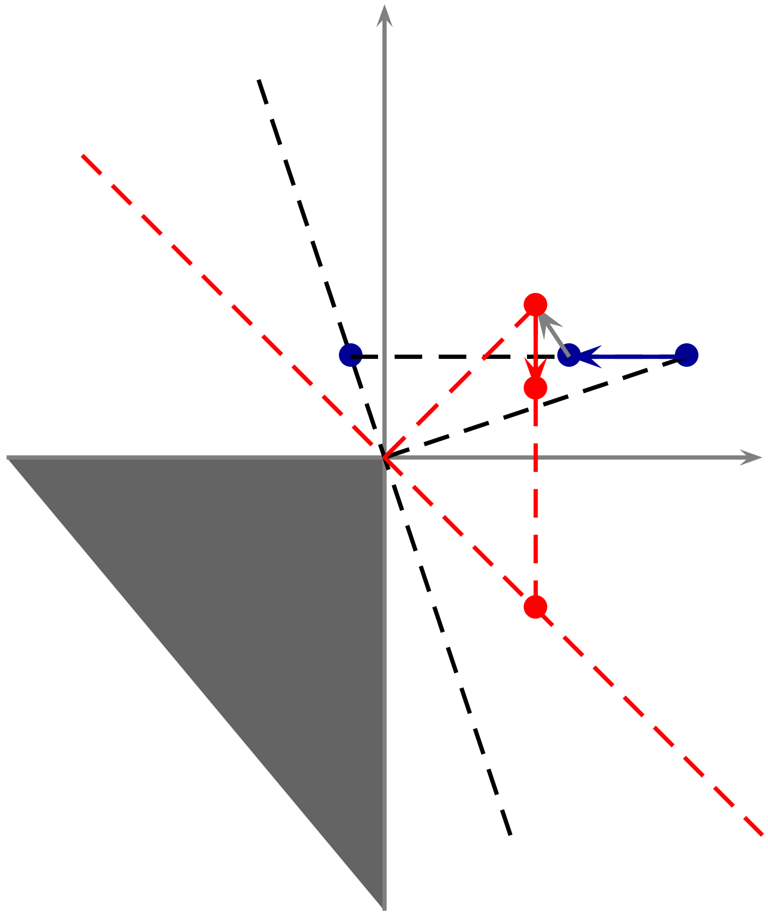

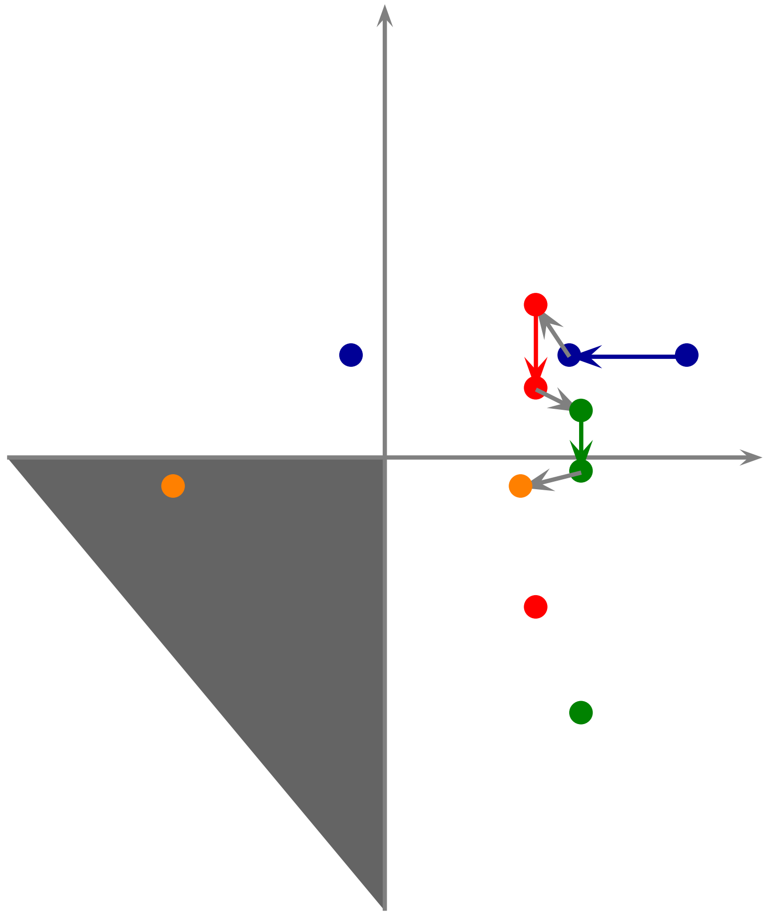

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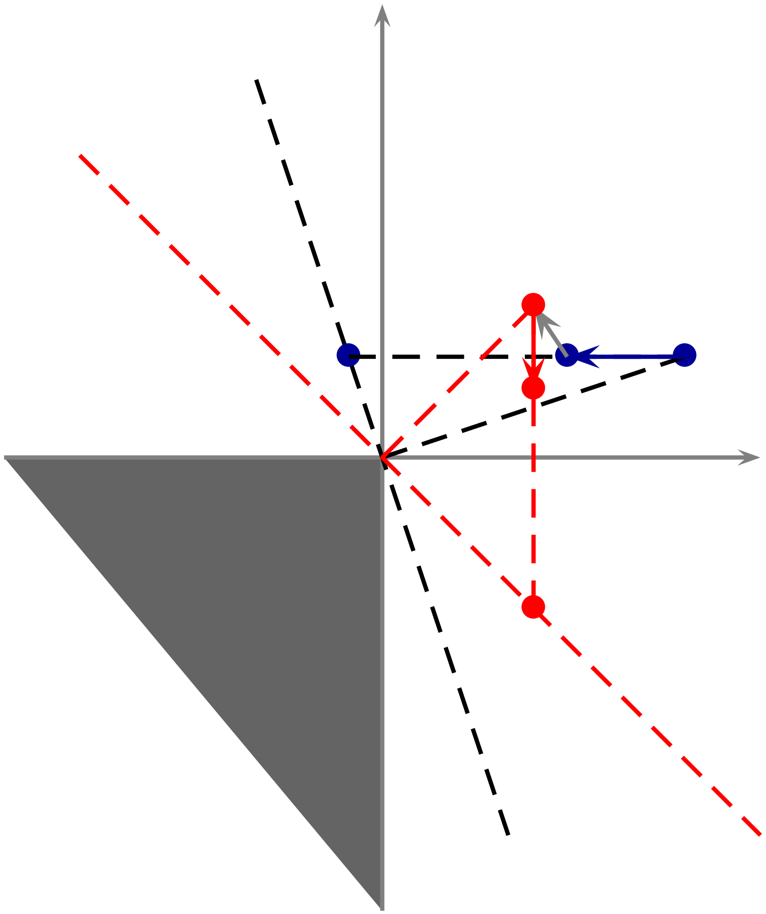

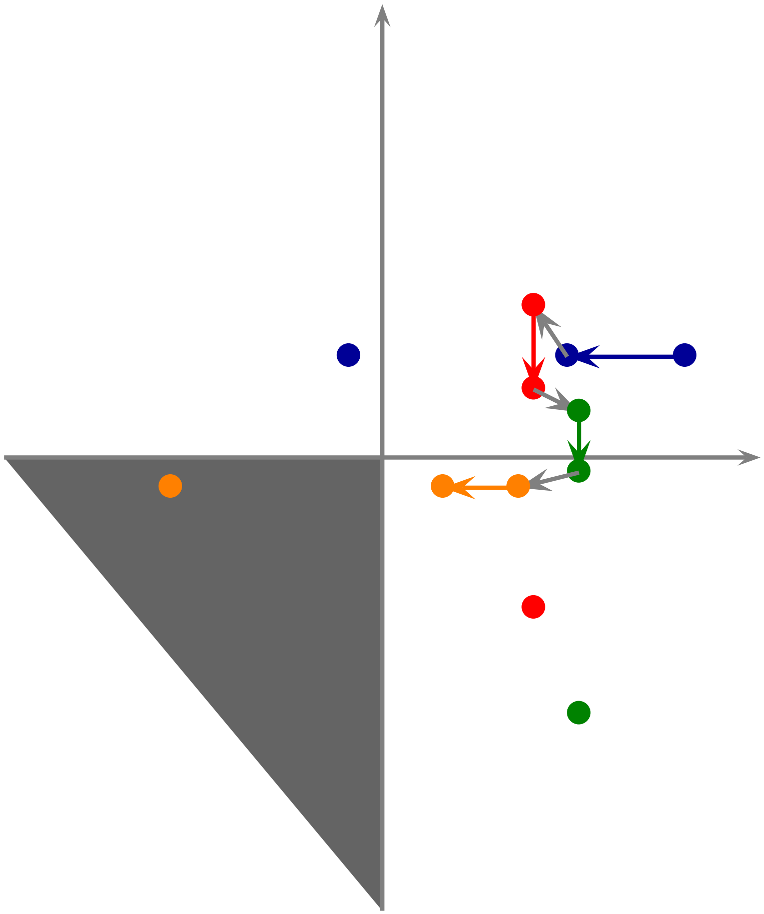

Part III: ϵ-Greedy Off-policy Regret Matching

ϵ-Greedy regret matching (Foster & Vohra, 1998)

ϵ-greedy regret matching. Let $\epsilon > 0$ small.

1. **Explore**. Play randomly $\epsilon\%$ of the time.
2. **Exploit**. Else, play off-policy regret matching.

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Define off-policy regret for x in round t as

$$\bar{r}_x^t =_{Def} \bar{u}_x^t(E) - \bar{u}^t, \qquad \text{where} \quad \bar{u}_x^t(E) = \left[\frac{1}{|E_x|} \sum_{t \in E_x} u(x^t, y^t) \right]$$

and $E_x = \{ t \mid \text{player } A \text{ experimented in round } t \text{ and played } x \}$.

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- Proposed as a forecasting heuristic by Foster and Vohra (1993).
- Does not need to know the actions of its opponents.
- Turns out to estimate regrets.

ϵ-Greedy regret matching (outline of proof)

Theorem (Foster *et al.*, 1998). *For all* $\delta > 0$ *there exists an* $\epsilon > 0$ *such that* ϵ*-greedy regret matching has at most* δ *regret.*

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$$e^t_x = (\text{ player } A \text{ explores at } t \text{ and chooses } x\ ) \ ?\ 1 : 0.$$

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It follows that

$$E[e^t] = \left(\frac{\epsilon}{k}, \ldots, \frac{\epsilon}{k}\right).$$

ϵ-Greedy regret matching (outline of proof)

Define

$$z_x^t =_{Def} \left(\frac{k}{\epsilon} \cdot e_x^t \cdot u(x, y^t) \right) - u(x, y^t).$$

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$$z_x^t =_{Def} \left(\frac{k}{\epsilon} \cdot e_x^t \cdot u(x, y^t) \right) \; - \; u(x, y^t).$$

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$$\begin{aligned} E[z_x^t] &= E\left[\left(\frac{k}{\epsilon} \cdot e_x^t \cdot u(x, y^t) \right) - u(x, y^t) \right] \\ &= \frac{k}{\epsilon} \cdot E\left[e_x^t \cdot u(x, y^t) \right] - E[u(x, y^t)] \end{aligned}$$

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\end{aligned}$$

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Strong law of large numbers for dependent random variables. Let $\{w^t\}^t$ be a bounded sequence of possibly dependent random variables in R^k. Let $z^t = E[\, w^t \mid w^{t-1}, w^{t-2}, \dots, w^1\,] - w^t$, and $\bar{z}^t$ the average of the z^t's. Then $\lim_{t\to\infty} \bar{z}^t = 0$ with probability one.[a]

[a] PY refers to Loève, 1978, Book II, Th. 32.E.1.

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then from the strong law of large numbers for dependent random variables it follows that $\lim_{t\to\infty} \bar{z}^t = 0$ a.s.

Estimated vs. true regret

Now write $\bar{z}_x^t$ as follows (!):

$$\bar{z}_x^t = \underbrace{\frac{1}{t}\sum_{s=1}^{t}\frac{k}{\epsilon}\cdot e_x^s\cdot u(x, y^s) - \bar{u}^t}_{\substack{\text{scaled}\\ \text{empirical regret}}} \;-\; \underbrace{\frac{1}{t}\sum_{s=1}^{t} u(x, y^s) - \bar{u}^t}_{\text{true regret}}$$

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5. If ϵ is set to $\delta/2$, then empirical regret remains within $2 \cdot \delta/2$ from zero. $\square$

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Exam problem

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Problem. The following game is played with regret matching with initial action profile (T, L).

	L	R
T	$1, 4$	$3, 1$
B	$2, 0$	$0, 5$

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t :	1	2	3	4	5
a_{row} :	T	B	B	T	T
a_{col} :	L	L	R	R	**R**
x_{row} :	1	2	0	3	3
x_{col} :	4	0	5	1	1
$\Delta r(T)$:	0	-1	3	0	0
$r(T)$:	0	-1	2	2	2
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$\Delta r(L)$:	0	0	-5	3	3
$r(L)$:	0	0	-5	-2	$\mathbf{1}$
$\Delta r(R)$:	-3	5	0	0	0
$r(R)$:	-3	2	2	2	$\mathbf{2}$

The answer is R, $(2, 1)$. $\square$

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