Multi-agent learning

Bayesian play

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Thursday 21st May, 2020

Plan for today

Preparation

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- Bayes' rule.

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- Examples of Bayesian play.

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- Demo.

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Formalism

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- Domination of measures (a.k.a. absolute continuity).

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- Domination of measures (a.k.a. absolute continuity).
- Theorem of Blackwell and Dubins (1992).

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- Domination of measures (a.k.a. absolute continuity).
- Theorem of Blackwell and Dubins (1992).
- Notion of ϵ-closeness.
- Theorem of Kalai and Lehrer (1993): If a player gives all potential play paths a small positive probability ("grain of truth"), then, eventually, his/her subjective beliefs will be ϵ-close to the actual realisation of play.

Literature

Key publication

Kalai & Lehrer (1993). "Rational learning leads to Nash equilibrium". *Econometrica*, Vol. **61**, No. 5, pp 1019-1045.

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Kalai & Lehrer (1993). "Rational learning leads to Nash equilibrium". *Econometrica*, Vol. **61**, No. 5, pp 1019-1045.

Scholarly resources

Young (2004): *Strategic Learning and it Limits*, Oxford UP. Ch. 7: "Bayesian Learning".

Shoham *et al.* (2009): *Multi-agent Systems*. Ch. 7: "Learning and Teaching". Sec. 7.3: "Rational Learning".

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Shoham *et al.* (2009): *Multi-agent Systems*. Ch. 7: "Learning and Teaching". Sec. 7.3: "Rational Learning".

Practical computer science / AI application

Zeng & Sycara (1996): *Bayesian Learning in Negotiation* in: Working Notes of the AAAI Spring Symposium on Adaptation, Co-Evolution and Learning in Multiagent Systems, Stanford, CA.

Part I: Elementary probability and Bayes' theorem

Conditional probability

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Typical exercise: "given $\Pr\{E\}$, $\Pr\{F\}$, and $\Pr\{F|E\}$, compute $\Pr\{E|F\}$". (E.g., E = "influenza", F = "fever".)

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$$\begin{aligned}\Pr\{F\} &= \Pr\{F|E\}\Pr\{E\} \\ &\quad + \Pr\{F|E^c\}\Pr\{E^c\},\end{aligned}$$

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so that

$$\Pr\{E|F\} = \frac{\Pr\{F|E\}\Pr\{E\}}{\Pr\{F|E\}\Pr\{E\} + \Pr\{F|E^c\}\Pr\{E^c\}}.$$

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- $f_Y(y|X=x)$ is the posterior density of Y given $X=x$.
- $f_X(x)$ and $f_Y(y)$ are marginal densities of X and Y.

Part Ia: Bayes' rule applied to guess opponent's reply rules

Use Bayes' rule to compute $\Pr\{g = g_i \mid h\}$ for every i

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- Because of identical priors and normalization, effectively $\Pr\{g = g_i \mid h\} \propto \Pr\{h \mid g = g_i\}$.

Part II: Demo and examples

Demo and examples

Demo and examples

1. **Demo**. Learn reactive reply rules, such as

   - All-C: always cooperate.
   - Unforgiving ("unforgiving"): cooperate until opponent defects, then defect forever.
   - C-90%: cooperate 90% of the time (randomly).
   - Tit-for-tat: mimic opponent's moves.
   - Josh 10%: play tit-for-tat 90% of the time, defect 10% of the time.
   - Majority: respond with the action most played by the opponent.
   - Eatherly: mirror the (projected) mixed strategy of the opponent.
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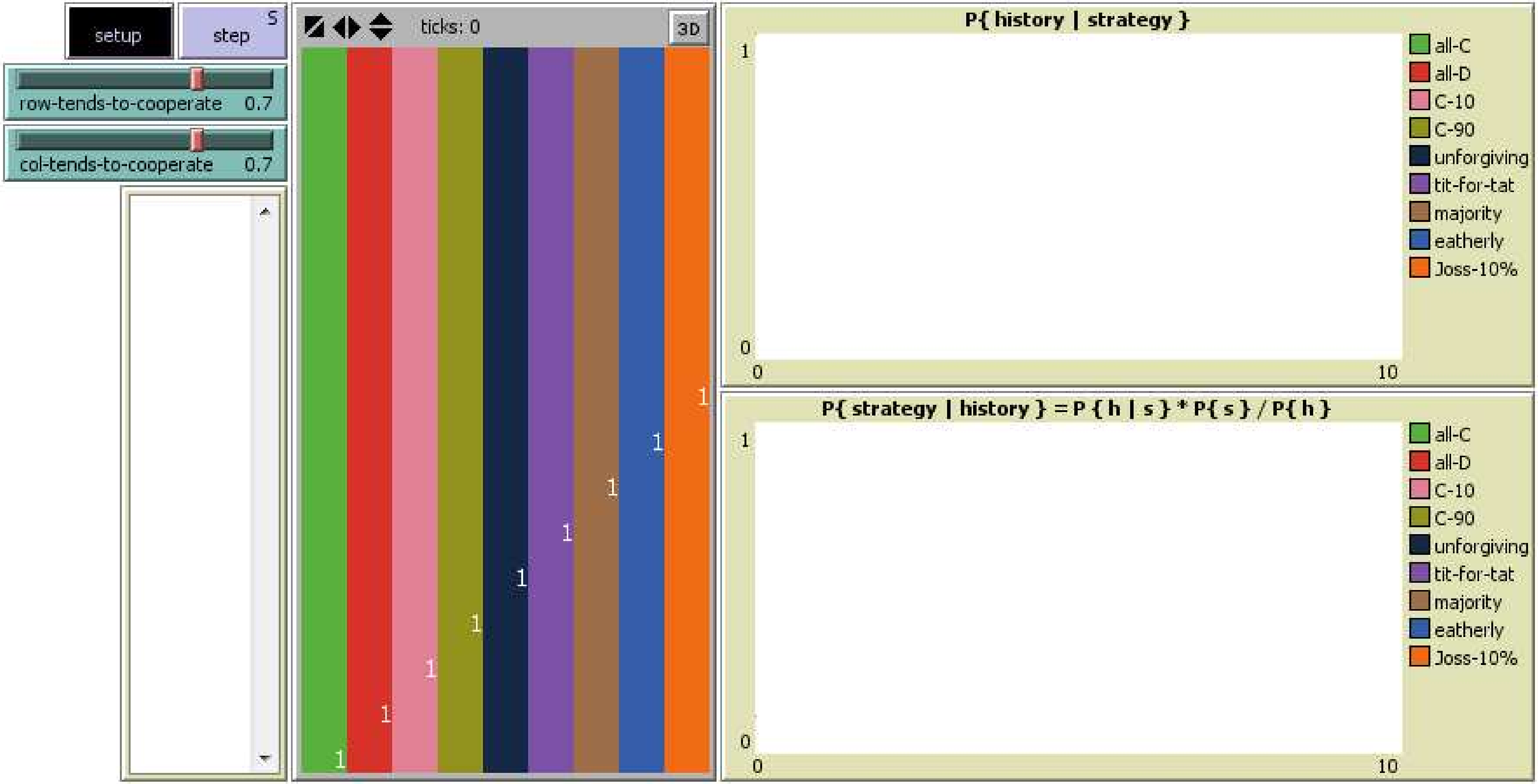
2. **Examples**. Learn reply rules in the repeated prisoners' dilemma; learn reply rules in the coordination game.

Example: learning on reactive strategies



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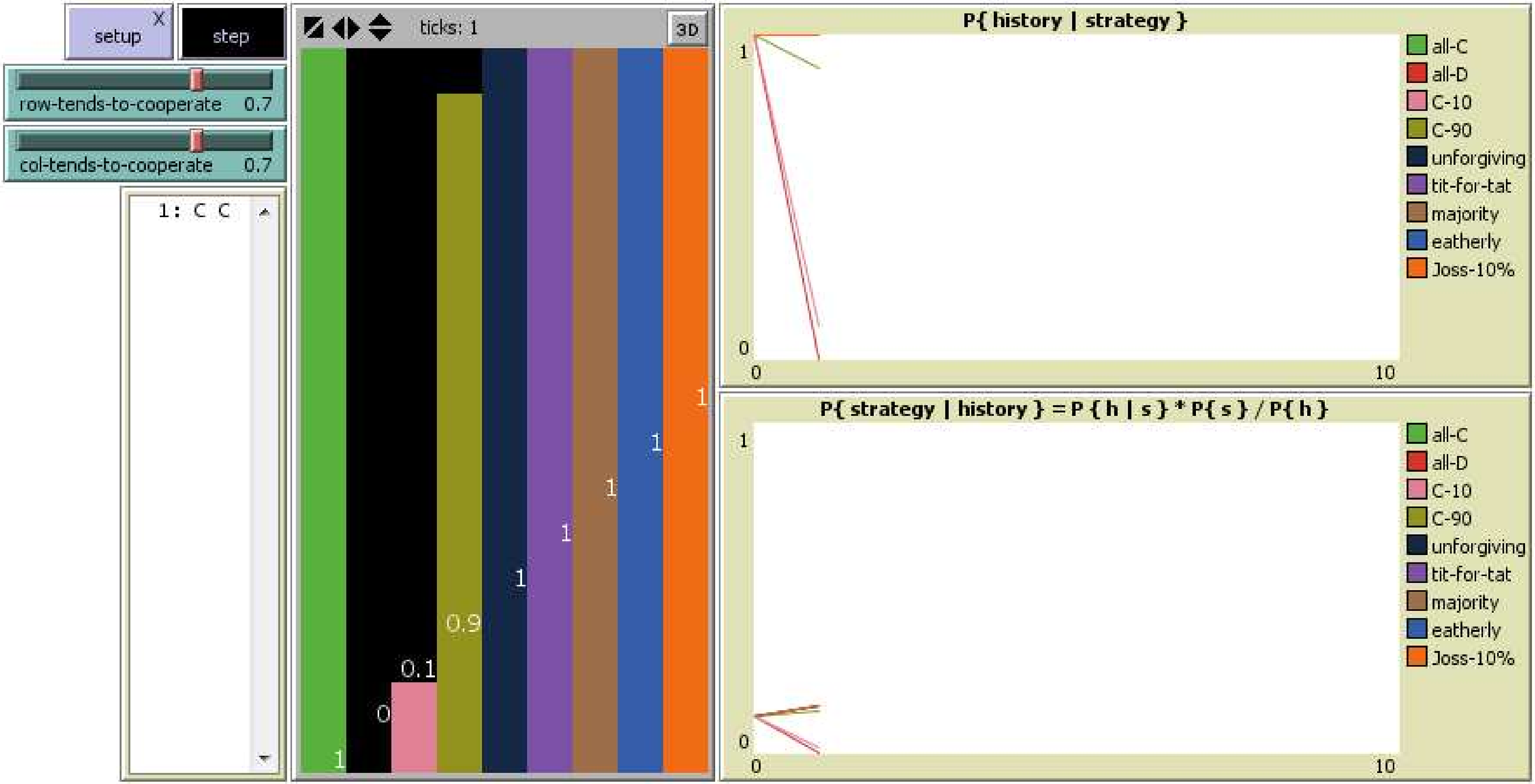

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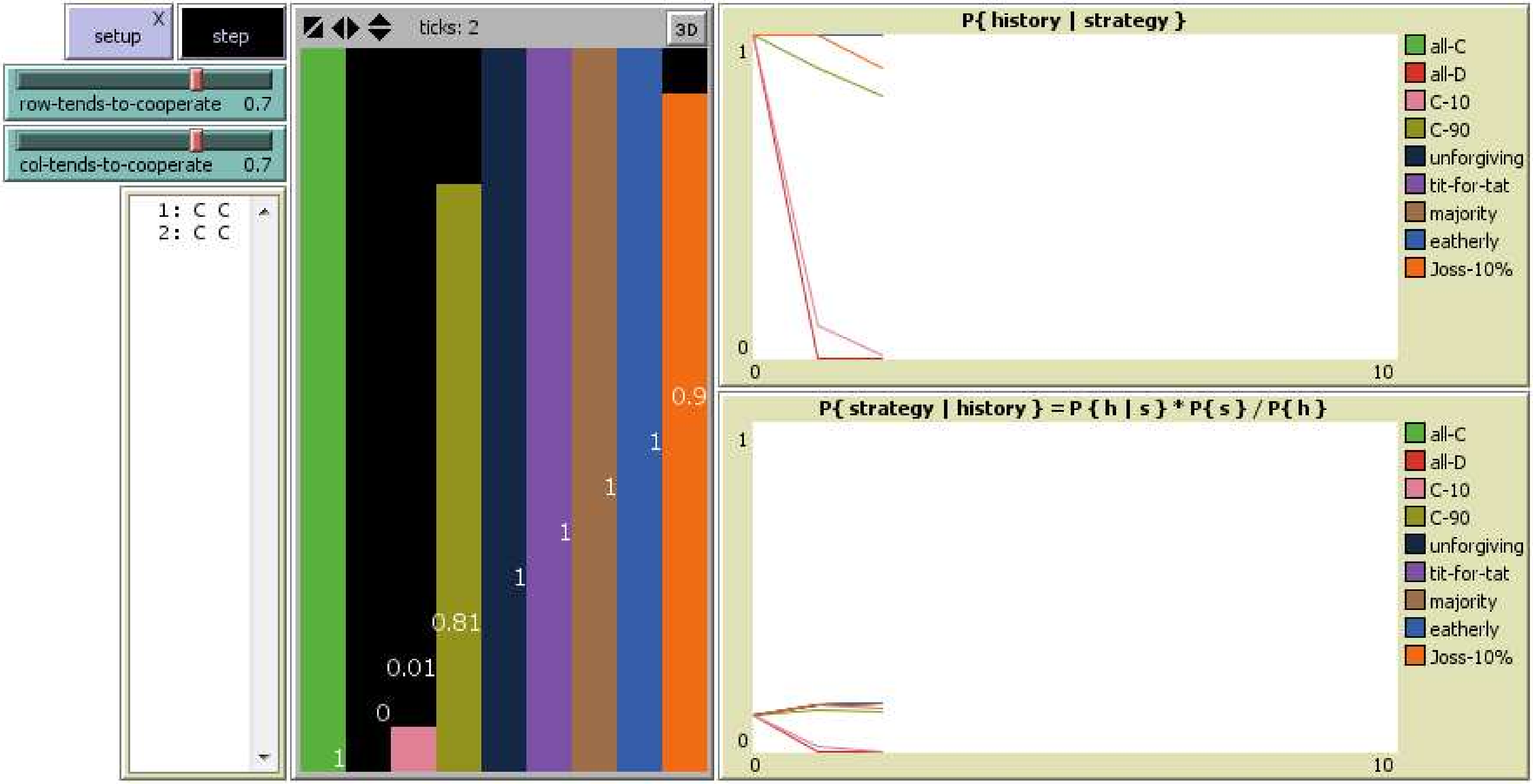

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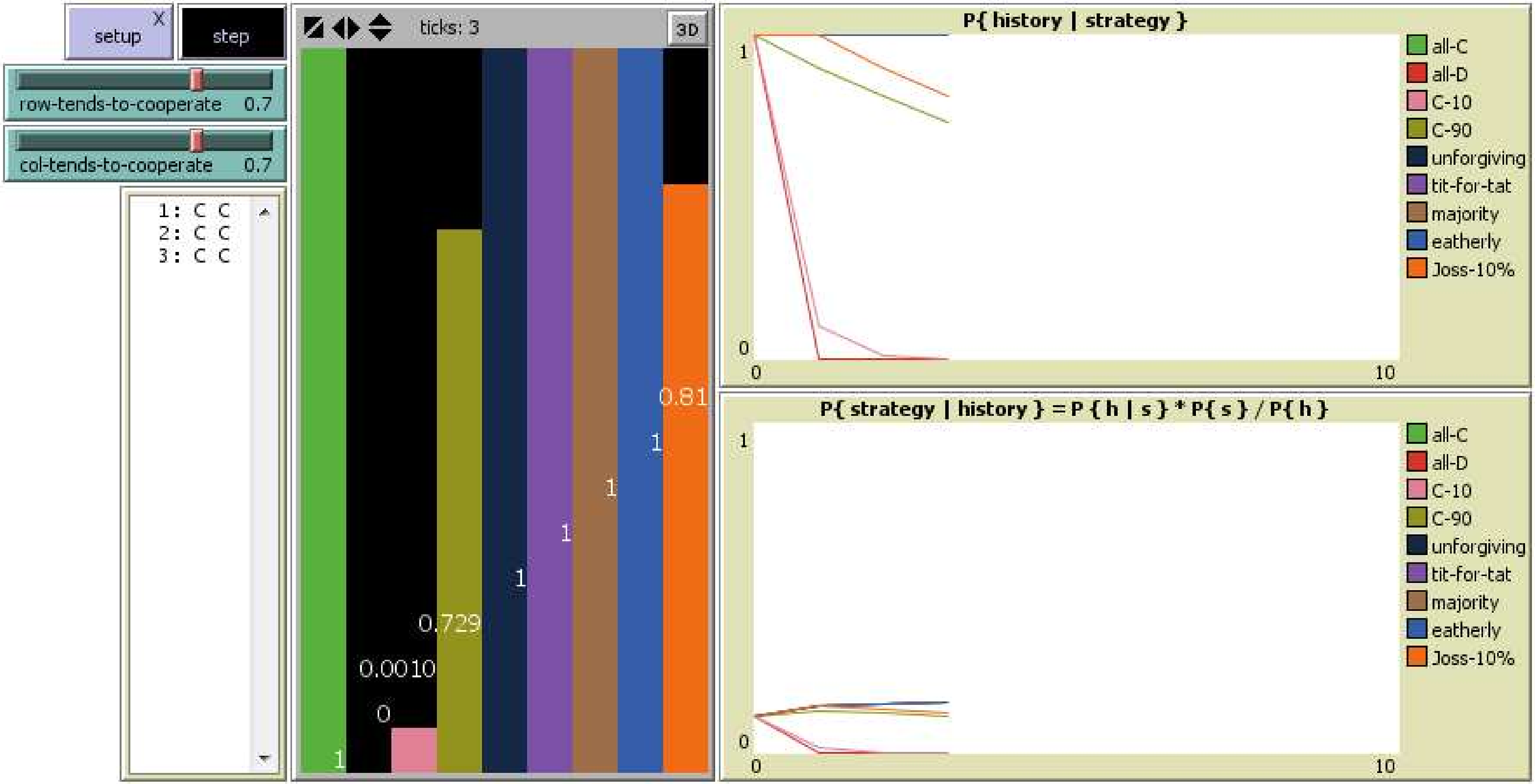

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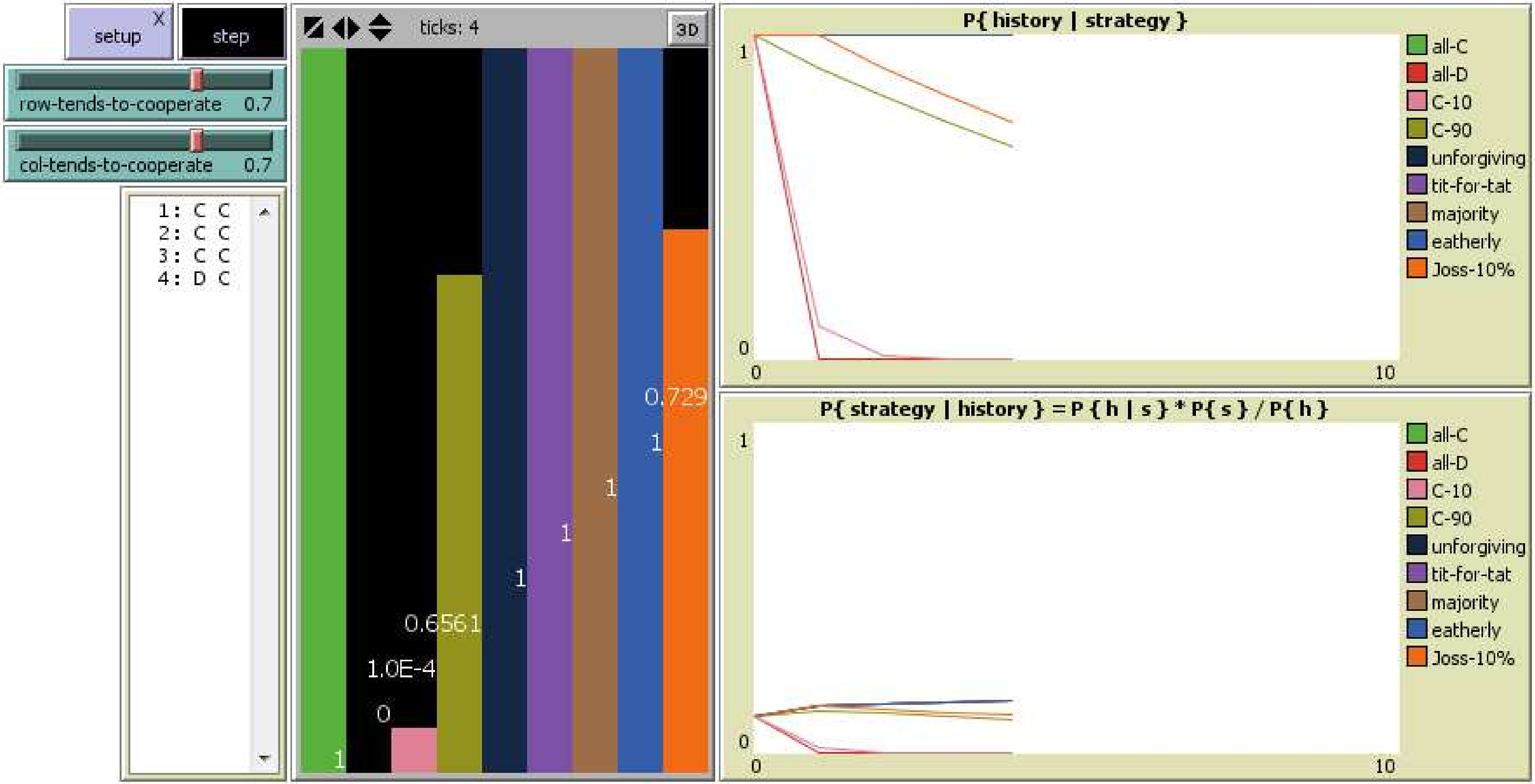

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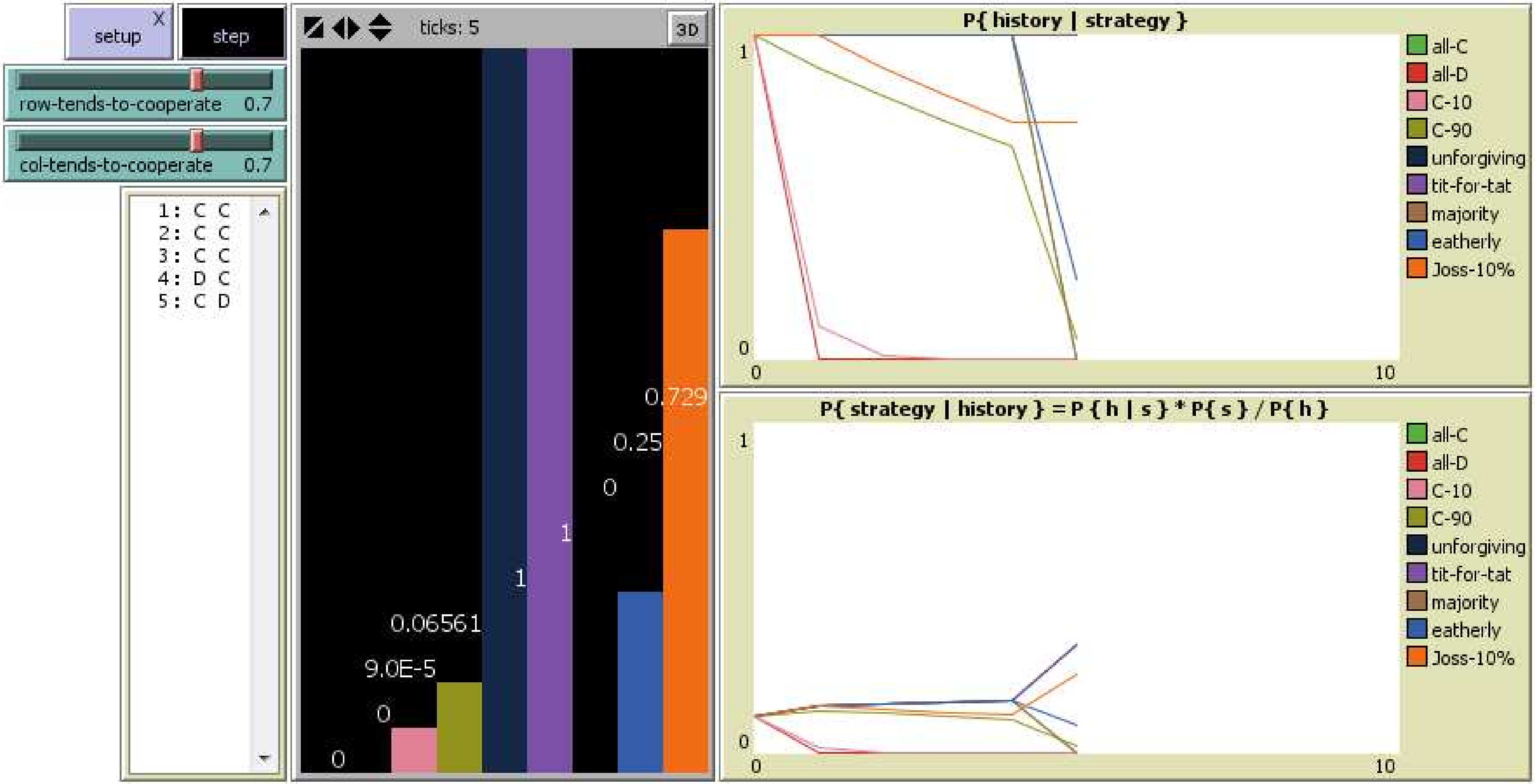

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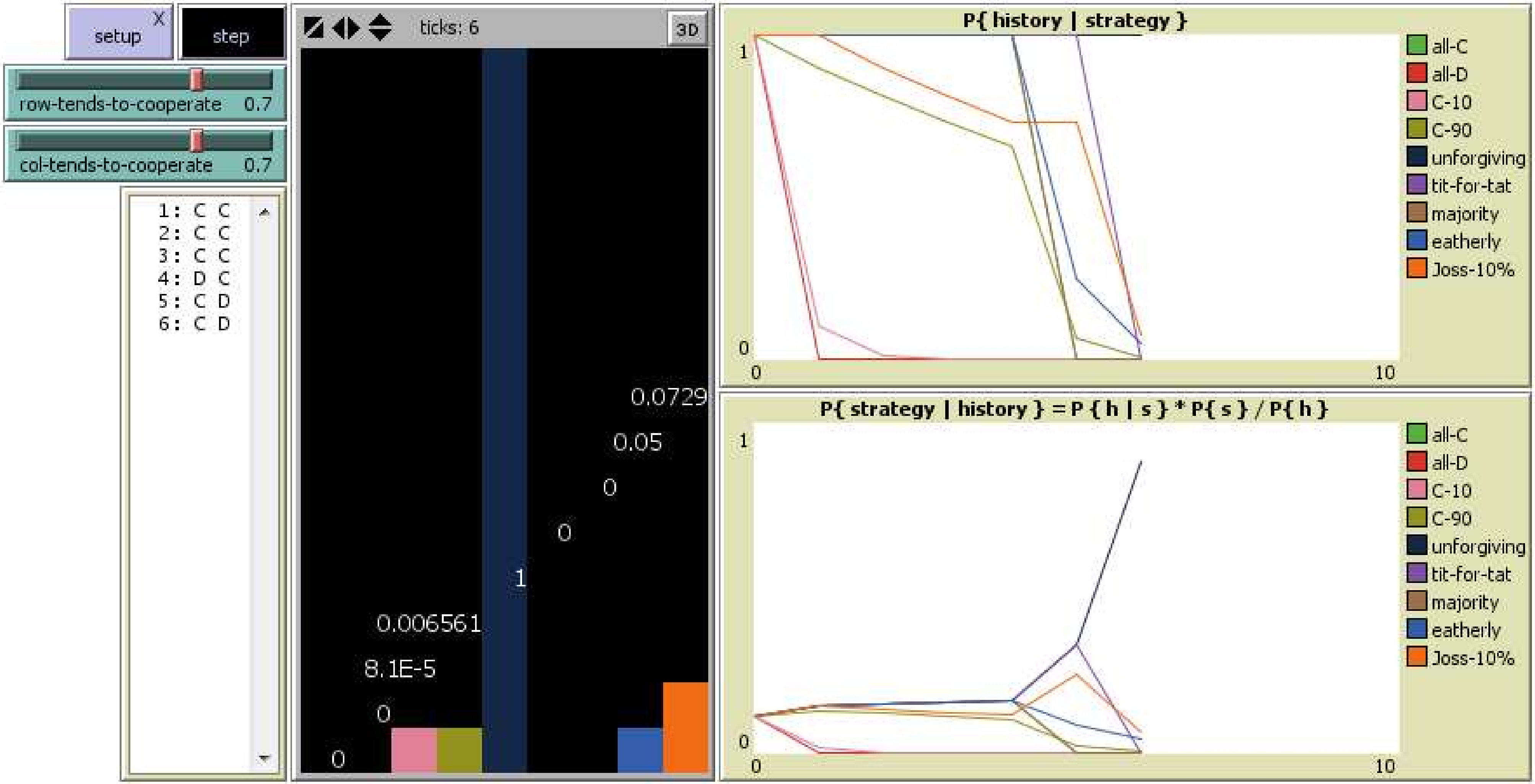

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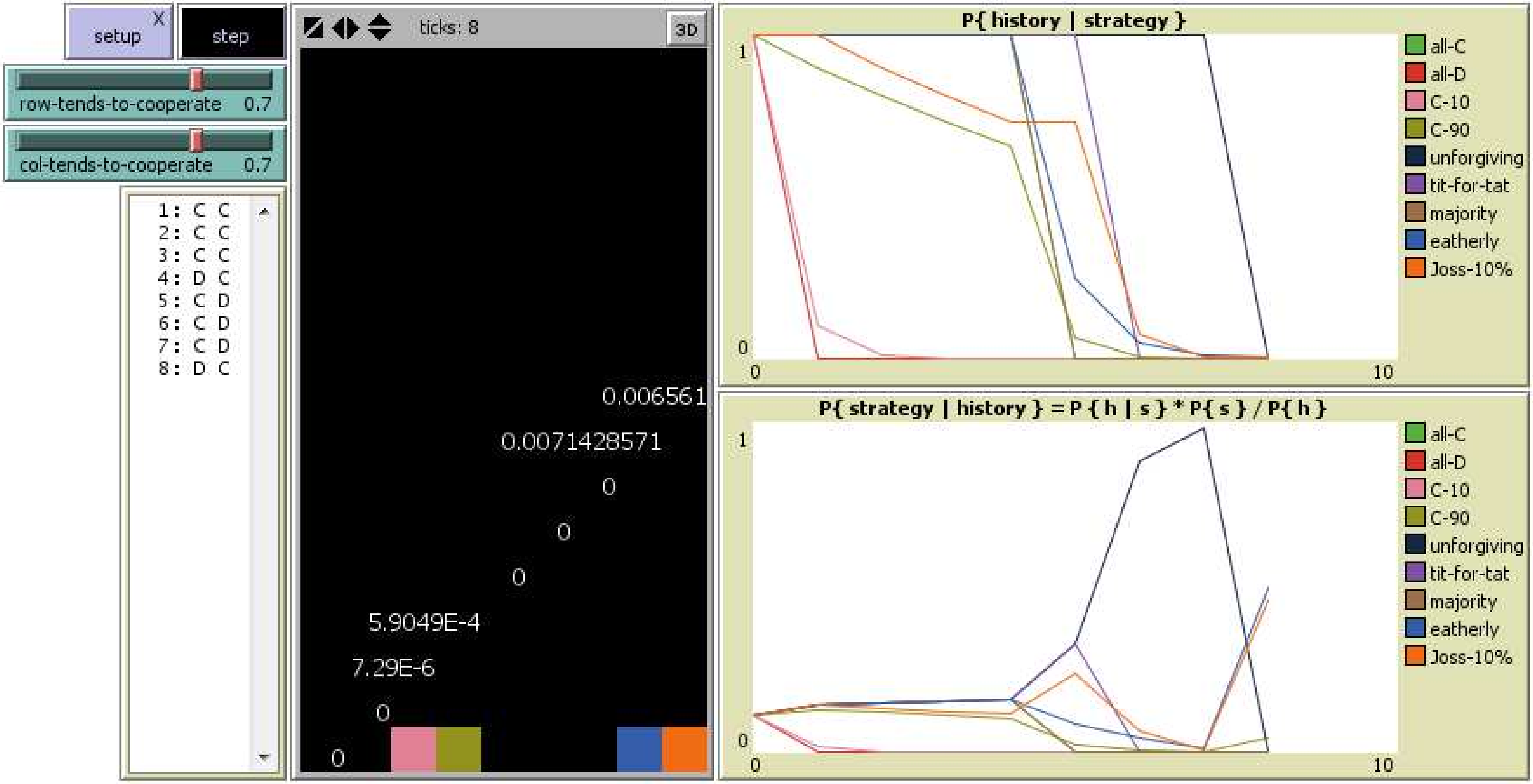

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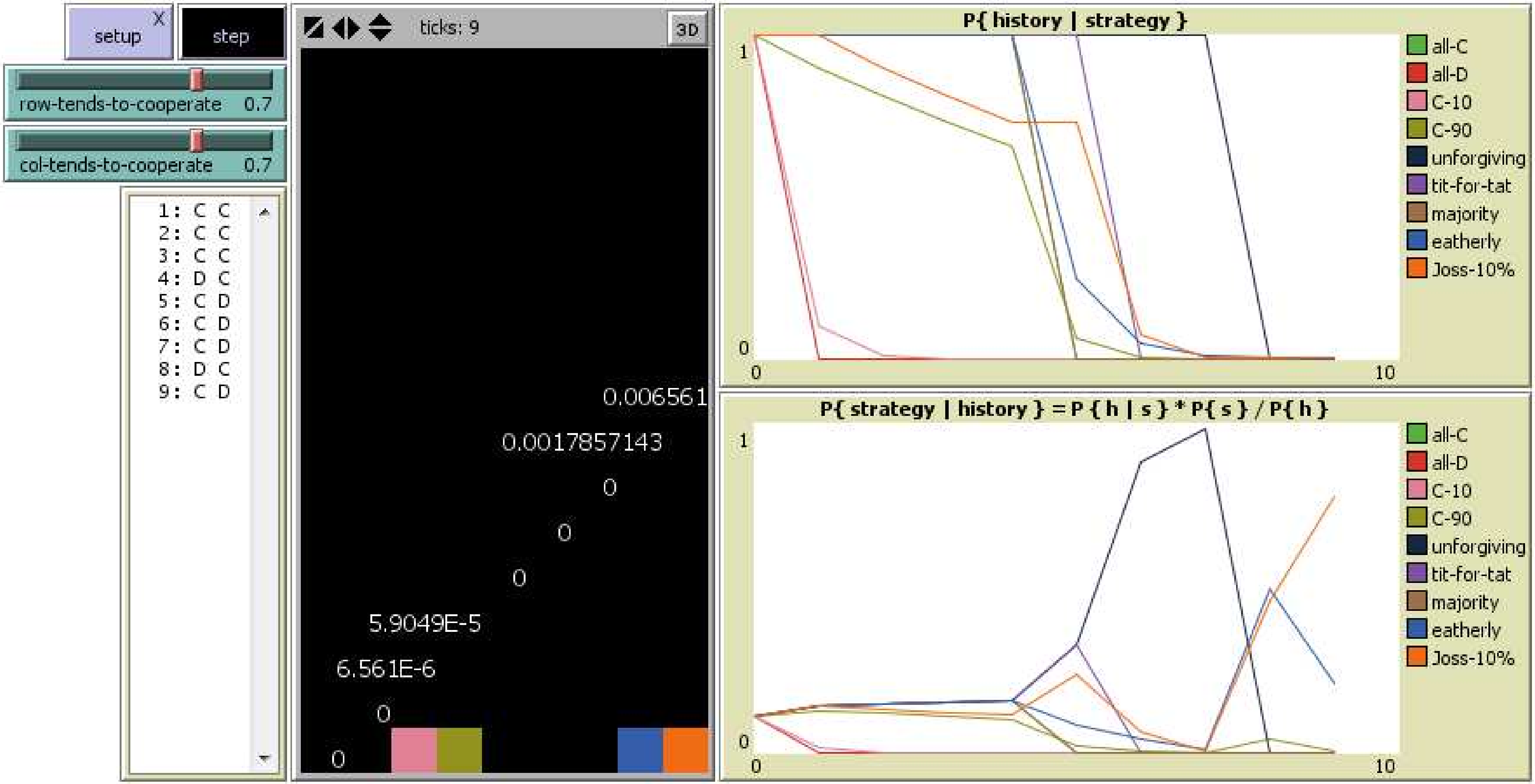

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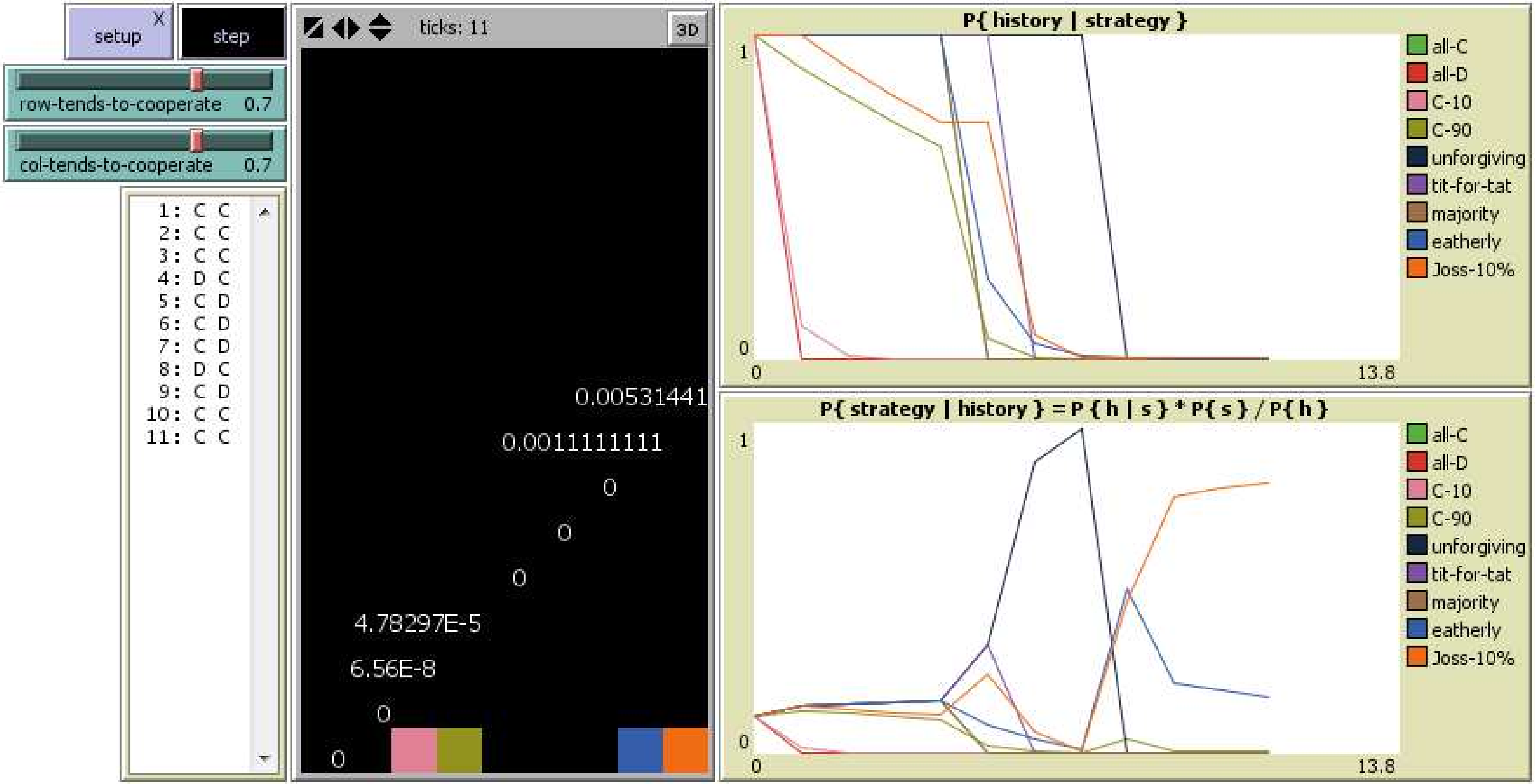

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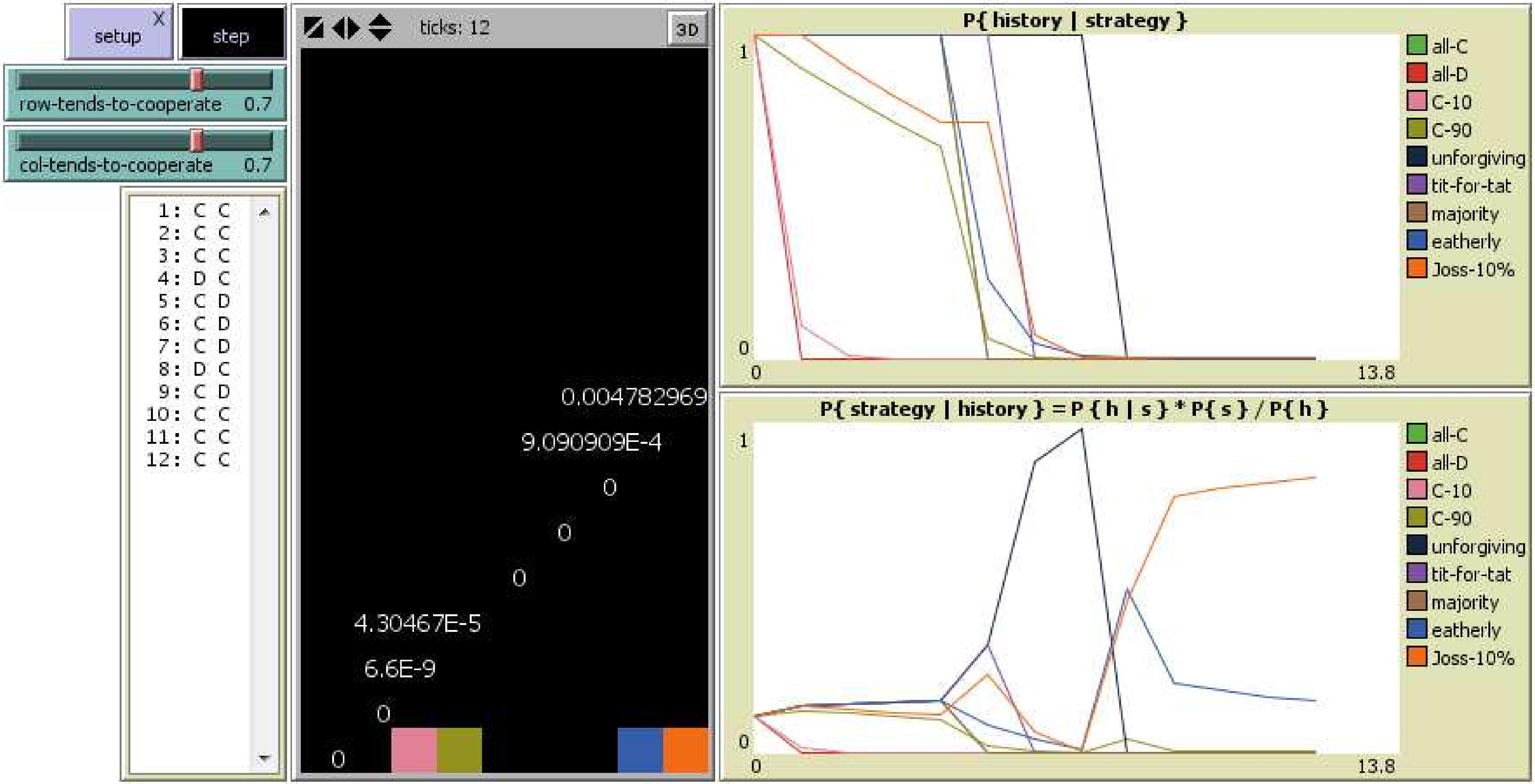

Example: learning on reactive strategies

Explanation of Pr$\{h \mid s_2 = \text{Joss-10\%}\}$

Event	Description
$s_2 = \text{Joss-10\%}$	Player 2's strategy is Joss-10%
$X_1^{t-1} = C$	Player 1 cooperated in the previous round
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Round:	1	2	3	4	5	6	7
Player 1:	C	C	C	D	D	C	C
Player 2 x (and h):	C	D	C	C	D	C	C
$\Pr\{X_2^t = x \mid s_2 = \text{Joss-10\%}\}$:	0.9	0.1	0.9	0.9	1	0	0
$\Pr\{h_2^t = h \mid s_2 = \text{Joss-10\%}\}$:	0.9	0.09	0.081	0.0729	0.0729	0	0

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$$\begin{aligned}\Pr\{h_2^t = C, h \mid X_1^{t-1} = C\} &= (\Pr\{X_2^t = C \mid X_1^{t-1} = C, \xi^t\}\Pr\{\xi^t\} + \\ &\qquad \Pr\{X_2^t = C \mid X_1^{t-1} = C, \bar{\xi}^t\}\Pr\{\bar{\xi}^t\})\Pr\{h\} \\ &= (0 \cdot 0.1 + 1 \cdot 0.9)\Pr\{h\} \\ &= 0.9\Pr\{h\}.\end{aligned}$$

Example: the disrupted unforgiving reply rule

Define

$$g_t =_{Def} \text{Play unforgiving before } t\text{, defect unconditionally at } t \text{ and later.}$$

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Disrupted unforgiving strategy: opponent's perspective

Same game (and same realisation of play) but now how beliefs of Player 2 evolve.

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Suppose Player 1 and 2 play the coordination game, and deem the following strategies possible:

$$\begin{aligned} S = \{\; & L^* : \text{stay left forever}\;, \\ & R^* : \text{stay right forever}\;, \\ & F^* : \text{play mixed } 0.5 \text{ forever}\;\}. \end{aligned}$$

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$$\begin{aligned}\Pr\{R^*|h\} &= \frac{\Pr\{h|R^*\}\Pr\{R^*\}}{\sum_{j=1}^3 \Pr\{h|s_j\}\Pr\{s_j\}} \\ &= \frac{0 \cdot 0.3}{0 \cdot 0.6 + 0 \cdot 0.3 + (\frac{1}{2})^2 \cdot 0.1} \\ &= 0.\end{aligned}$$

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Example: Coordination game

Suppose Player 1 and 2 play the coordination game, and deem the following strategies possible:

$$\begin{aligned} S = \{ \; & L^* : \text{stay left forever} \; , \\ & R^* : \text{stay right forever} \; , \\ & F^* : \text{play mixed } 0.5 \text{ forever } \}. \end{aligned}$$

Suppose

	1's prior	2's prior
L^*	0.6	0.2
R^*	0.3	0.5
F^*	0.1	0.3.

After

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Now, $\mathrm{BR}(s_p|h) \cap S \neq \emptyset$, so that from round to round play vs. prediction is a closed system.

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- **Solution:** extend μ of H to μ on a σ-algebra of Ω.

Part III: Formalism

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Often, (2) is meant.

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Often, this is abbreviated to Δ_{-i}^H.

Idea of Kuhn's proof

$$
\begin{array}{rr|cccccc|c}
\text{distribution} & & \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_r & \dots & \\
\textbf{strategies} & \mathbf{G} & \mathbf{g^j_1} & \mathbf{g^j_2} & \mathbf{g^j_3} & \dots & \mathbf{g^j_r} & \dots & \mathbf{g^j} \\
\hline
\text{histories} & h_1 & q^1_1 & q^1_2 & q^1_3 & \dots & q^1_r & \dots & \lambda_1 q^1_1 + \cdots + \lambda_r q^1_r + \dots \\
& h_2 & q^2_1 & q^2_2 & \dots & \dots & q^2_r & \dots & \lambda_1 q^2_1 + \cdots + \lambda_r q^2_r + \dots \\
& h_3 & q^3_1 & \dots & \dots & \dots & q^3_r & \dots & \lambda_1 q^3_1 + \cdots + \lambda_r q^3_r + \dots \\
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histories	h_1	q_1^1	q_2^1	q_3^1	$\dots$	q_r^1	$\dots$	$\lambda_1 q_1^1 + \dots + \lambda_r q_r^1 + \dots$
	h_2	q_1^2	q_2^2	$\dots$	$\dots$	q_r^2	$\dots$	$\lambda_1 q_1^2 + \dots + \lambda_r q_r^2 + \dots$
	h_3	q_1^3	$\dots$	$\dots$	$\dots$	q_r^3	$\dots$	$\lambda_1 q_1^3 + \dots + \lambda_r q_r^3 + \dots$
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Remarks:

1. This table suggests that S is countable, but S may be uncountable. (For example, if the prior is a β-distribution.)

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3. It must be checked that every $\lambda_1 q^j_1 + \dots + \lambda_r q^j_r + \dots$ is a legitimate mixed strategy.

Idea of Kuhn's proof

distribution		λ_1	λ_2	λ_3	$\dots$	λ_r	$\dots$	
strategies	$\mathbf{G}$	$\mathbf{g_1^j}$	$\mathbf{g_2^j}$	$\mathbf{g_3^j}$	$\dots$	$\mathbf{g_r^j}$	$\dots$	$\mathbf{g^j}$
histories	h_1	q_1^1	q_2^1	q_3^1	$\dots$	q_r^1	$\dots$	$\lambda_1 q_1^1 + \dots + \lambda_r q_r^1 + \dots$
	h_2	q_1^2	q_2^2	$\dots$	$\dots$	q_r^2	$\dots$	$\lambda_1 q_1^2 + \dots + \lambda_r q_r^2 + \dots$
	h_3	q_1^3	$\dots$	$\dots$	$\dots$	q_r^3	$\dots$	$\lambda_1 q_1^3 + \dots + \lambda_r q_r^3 + \dots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

Remarks:

1. This table suggests that S is countable, but S may be uncountable. (For example, if the prior is a β-distribution.)

2. If S is uncountable, the sums on the right are integrals.

3. It must be checked that every $\lambda_1 q_1^j + \dots + \lambda_r q_r^j + \dots$ is a legitimate mixed strategy. The sum may then be abbreviated by, e.g., q^j.

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Summary:

$$H \times \Delta_{-i} \xrightarrow{\tau_i} \Delta_{-i}$$

$$\vdots$$

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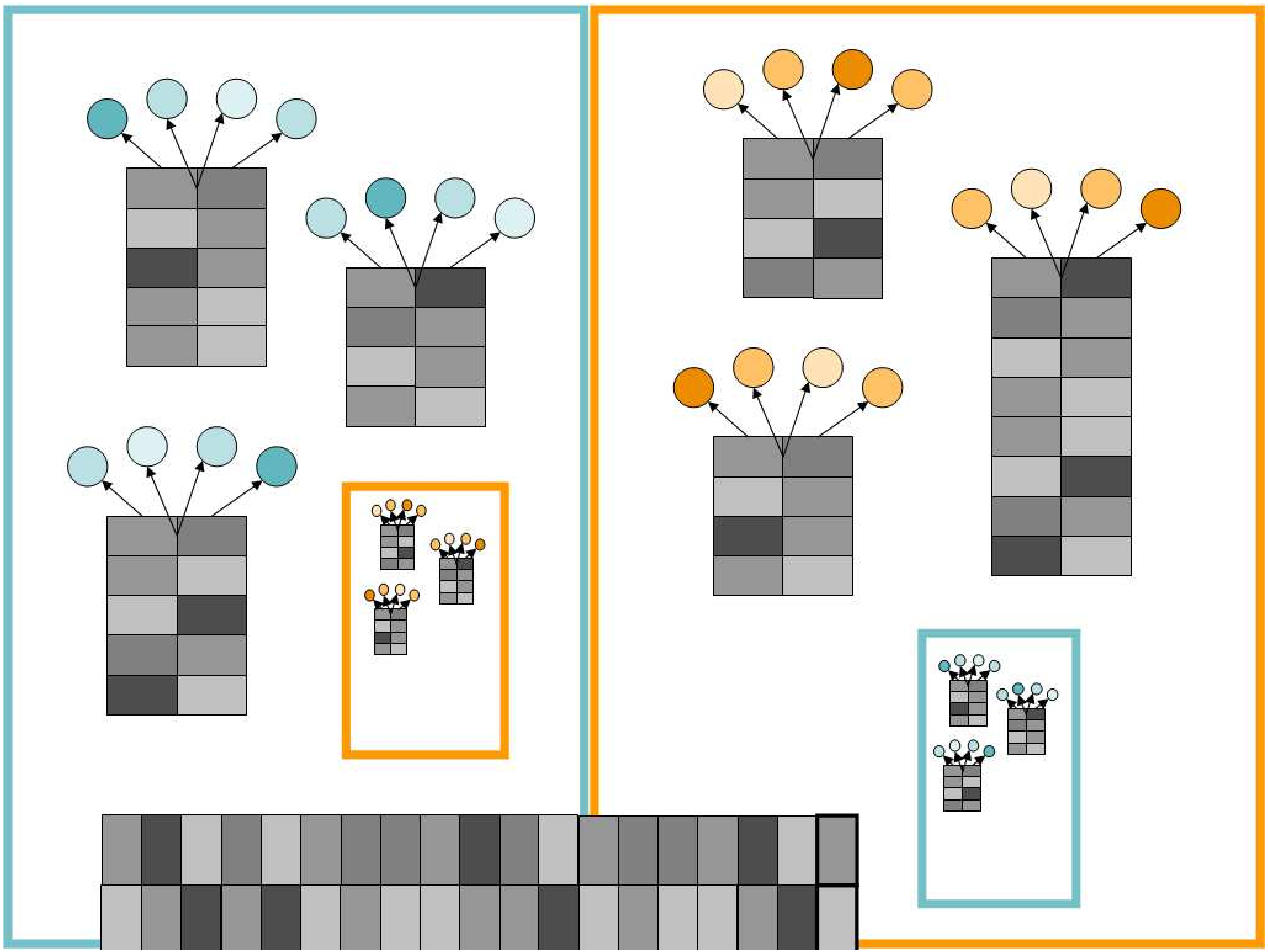
Behav. strategy and belief of opponent's behav. strategy



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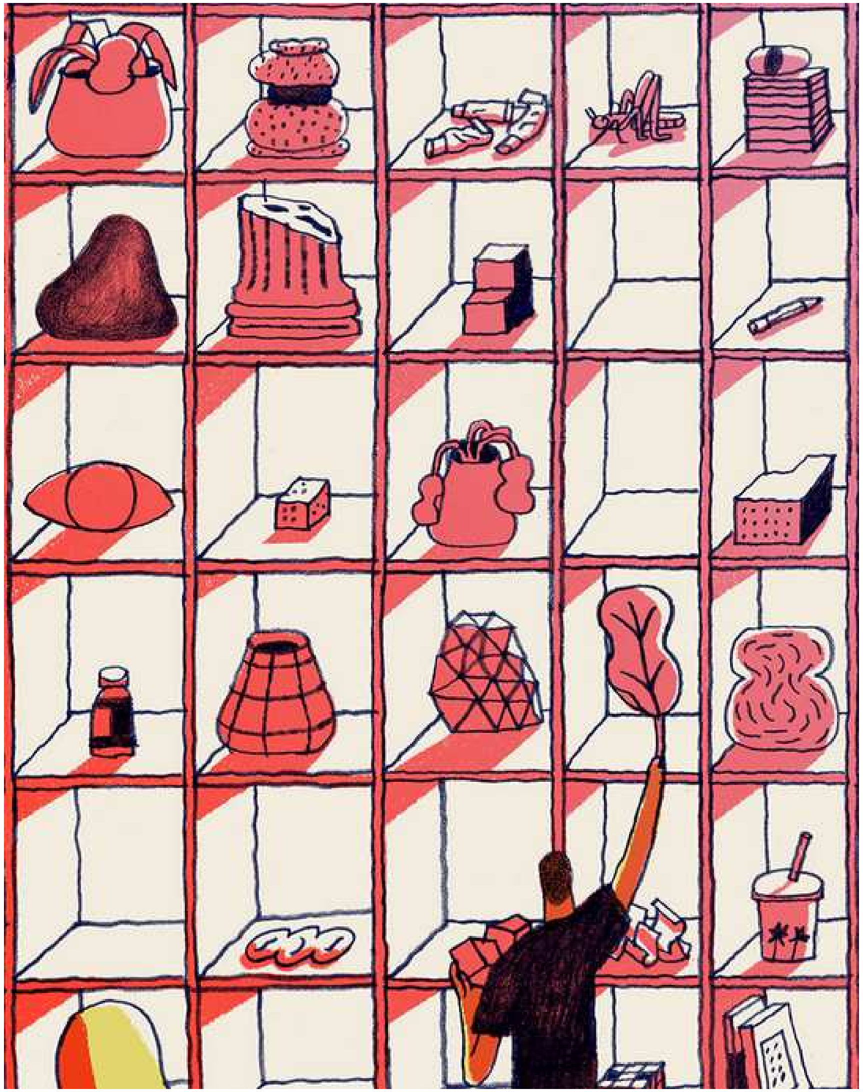
- With Bayesian play, forecasting is much more subtler. There, a forecast is a probability distribution over the reply rules of other players.



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- Beliefs ν_i are maintained through Bayesian updating.

Part IV: True distribution of play vs. subjective distribution of play

The true distribution of play μ on H

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  (exercise) and write the latter set as Δ^H, we get rid of repetitions of histories in the function argument.
- The reply profile $g \in \Delta^H$ induces a probability distribution μ on H, inductively:

  $$\mu\{hx\} = \begin{cases} 1 \cdot g(x) & \text{if } h = \emptyset, \\ \mu\{h\} \cdot g(x|h) & \text{otherwise.} \end{cases}$$

  The probability measure μ on H, on its turn, induces a probability measure on (a σ-algebra of) Ω.

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Standard results in probability theory[1]now imply that μ' can be extended to a proper probability measure on (a σ-algebra of) Ω.

[1]Carathéodory's extension theorem.

Player i's subjective distribution of play μ_i on H

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- The beliefs on reply rules of other players, ν_i, together with i's own reply rule g_i, also forms some sort of "reply profile".

  E.g., $(\nu_2^1, g_2, \nu_2^3, \dots, \nu_2^n)$ represents the further course of play as Player 2 sees it.

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- Similarly, players i's conditionalised predictive learning rule, (ν_i, g_i), generates a subjective distribution of play, μ_i, on Ω:

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_n \end{pmatrix} \text{ is generated by } (\nu, g) = \begin{pmatrix} g_1 & \nu_1^2 & \nu_1^3 & \dots & \nu_1^n \\ \nu_2^1 & g_2 & \nu_2^3 & \dots & \nu_2^n \\ \nu_3^1 & \nu_3^2 & g_3 & \dots & \nu_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu_n^1 & \nu_n^2 & \nu_n^3 & \dots & g_n \end{pmatrix}.$$

Part V: Main results

The subjective view on distribution of play

Given ω, we would like to say something like:

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First stab at the formulation of a theorem:

If $t \to \infty$, then i's subjective view on play at t, namely,

$$\mu_{\mathbf{i}}\{\cdot \mid \text{realised play until } t\},$$

approximates the true distribution of play at t, namely,

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So, if i's beliefs contain a grain of truth on actual play, then μ_i is, roughly put, "as expressive as" μ.

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Definition. Let μ on Ω represent the true distribution of play. Another measure, μ' on Ω, is said to merge with the true distribution of play if, for almost all play path ω

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Theorem (Blackwell and Dubins, 1962). *Let μ represent the true distribution of play, and let μ_i represent i's view on the distribution of play. If $\mu \ll \mu_i$, then i's beliefs merge with the true distribution of play.*

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$$\lim_{t\to\infty} \frac{1}{t} \sum_{s \leq t} \| q^{-i(s+1)} - p^{-i(s+1)} \|^2 = 0$$

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- Very good predictor $\Rightarrow$ good predictor.

Good prediction vs. very good prediction

Let $\omega \in \Omega$ be given. At round t,

- Player i's predictive learning rule yields a forecast $p^{-i} \in \Delta_{-i}$.
- The actual reply rules g induce the distribution proper, $q^{-i} \in \Delta_{-i}$.

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- From (Blackwell and Dubins, 1962) the following can be proven.

Corollary. If $\mu \ll \mu_i$ then i is a very good predictor a.s.

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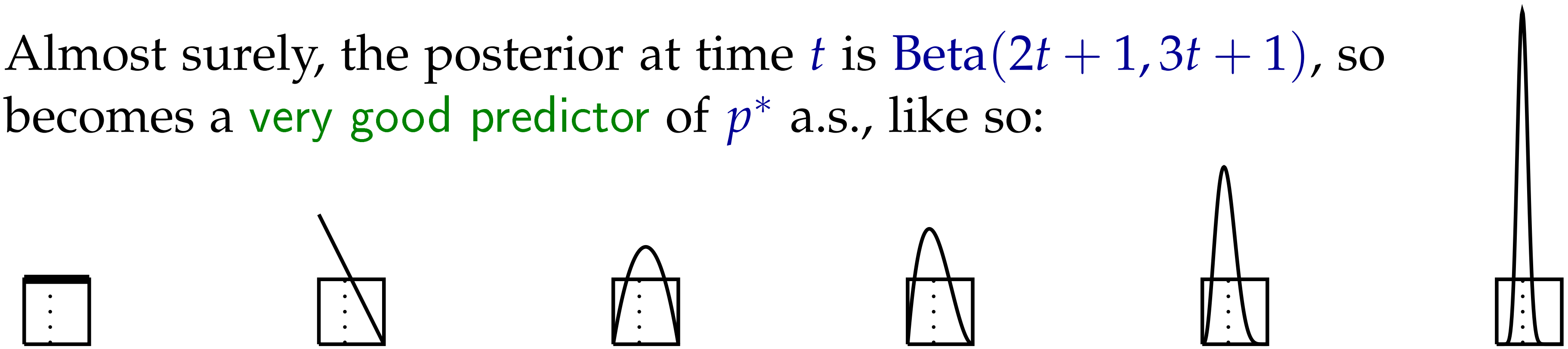

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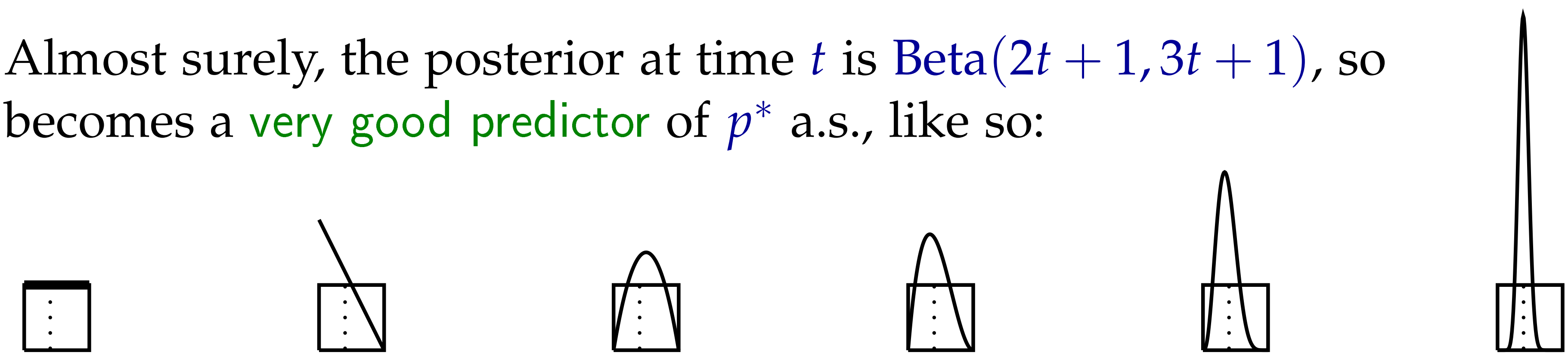

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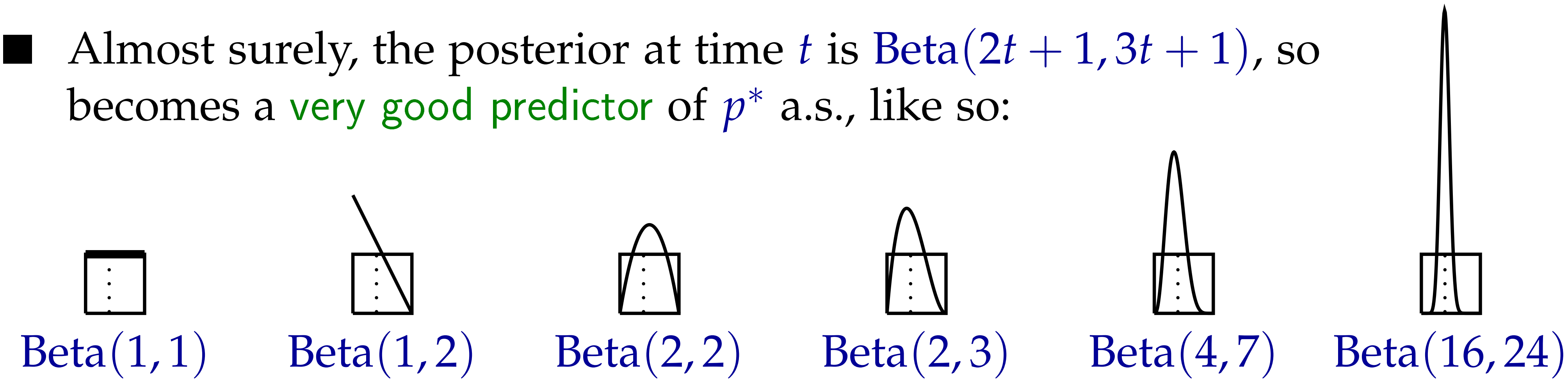

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Definition. Let μ and τ probability distributions. μ and τ are said to be ϵ-close, written $\mu \sim_\epsilon \tau$, if for some $Y \subseteq X$ we have $\mu\{Y\} > 1 - \epsilon$ and $\tau\{Y\} > 1 - \epsilon$ and for all measurable $E \subseteq Y$:

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  Example. Let $\epsilon = 0.01$, $\mu\{E\} = 0.001$ and $\tau\{E\} = 0.009$. Then $|\mu\{E\} - \tau\{E\}| < \epsilon$ while $\tau\{E\} = 9\mu\{E\}$.

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  Example. Let $\epsilon = 0.01$, $\mu\{E\} = 0.001$ and $\tau\{E\} = 0.009$. Then $|\mu\{E\} - \tau\{E\}| < \epsilon$ while $\tau\{E\} = 9\mu\{E\}$.
- Thus, being ϵ-close means that not only does the player assess the future correctly, *it even assesses developments following unlikely histories correctly*.

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Definition. Let μ and μ' be distributions of play and $\epsilon > 0$. Then μ' is said to ϵ-play like μ if u' is ϵ-close to μ.

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- This theorem implies closeness of probabilities also for courses of play that are extremely unlikely.
- The theorem does not state that a player learns to predict other players' off path strategies. (Recall Player 2's beliefs in unforgiving constellation.)

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  - It be analysed *quantitatively* (by indicating the rate of convergence). This is much harder.

In Chapter 22 of the *Handbook on Computational Economics* (2006), Young describes how social dynamics can drift to so-called stochastically stables states, provided individuals act rationally most of the time.

The following slides were not used.

Notation: $B^A = \{f \mid f : A \to B\}$

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Clearly, The mappings are each other's inverse. Both mappings are surjective.
Further, fundamental math defines 0 as $\emptyset$, and $n+1$ as $n \cup \{n\}$:

$$\begin{aligned}
0 &= \emptyset \\
1 &= 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{\emptyset\} \\
2 &= 1 \cup \{1\} = \{\emptyset, \{\emptyset\}\} \\
3 &= 2 \cup \{2\} = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\} \\
&\vdots \quad \vdots
\end{aligned}$$

So

$$\begin{aligned}
B^n &= B^{\{0,1,\dots,n-1\}} \\
&= \{f \mid f : \{0, 1, \dots, n-1\} \to B\}.
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- $X^Y \times X^Z \sim X^{(Y \cup Z)}$, provided Y and Z are disjoint

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The factor $\lambda_r|h$ represents the posterior probability of choosing g^j_r given h:

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Playing against the strategies g_r^j with the probabilities λ_r and playing against an equivalently constructed behaviour strategy g^j generate identical probability distributions on future play paths.

(Kuhn, 1953; Aumann, 1964.)

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1. The theorem om Radon-Nikodym (mentioned above), which (roughly!) says that if $\nu \ll \mu$, then ν can be expressed in terms of μ.
2. Lévy's theorem, which (very roughly!) is about continuity of expectation through a so-called *filter*.[2]

Theorem. (Lévy's Convergence Theorem). *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\{\mathcal{F}_n\}_n$ be a non-decreasing family of σ-algebras contained in $\mathcal{F}$. Let $\mathcal{F}_\infty$ be the smallest σ-algebra around $\cup\{\mathcal{F}_n\}_n$. Let X be a random variable with finite expectation. Then, both P-a.s.*[3] *and in the L_1-sense,*[4]

$$E[X|\mathcal{F}_n] \to E[X|\mathcal{F}_\infty] \text{ as } n \to \infty.$$

[2] Sequence of monotone non-decreasing σ-algebras.
[3] $P\{| \ldots - \ldots | > \epsilon\} \to 0$.
[4] $E| \ldots - \ldots | \to 0$.

The Radon-Nikodym theorem

Given X and σ-finite probability measures μ and ν on X such that $\nu \ll \mu$, then there is a measurable function $f : X \to [0, \infty)$, such that for all μ-measurable sets E,

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- The theorem tells if and how it is possible to change from one probability measure to another.
- Specifically, the probability density function of a random variable is the Radon-Nikodym derivative of the induced measure with respect to some base measure (usually the Lebesgue measure for continuous random variables).

Lévy's Theorem

Let $(\Omega \mathcal{F}, P)$ *be a probability space, and let* $\{\mathcal{F}_n\}_n$ *be a non-decreasing family of* σ*-algebras contained in* $\mathcal{F}$. *Let* $\mathcal{F}_\infty = \sigma(\cup\{\mathcal{F}_n\}_n)$. *Let* X *be a random variable with finite expectation. Then, both* P*-a.s. and in the* L_1*-sense,*

$$E[X|\mathcal{F}_n] \to E[X|\mathcal{F}_\infty] \text{ as } n \to \infty.$$

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Essentially, it is a sufficient condition for the almost sure convergence to imply L1-convergence. The condition $|X_n| < Y$, $\mathrm{E}Y < \infty$ could be relaxed. Instead, the sequence $\{X_n\}_{n=1}^{\infty}$ should be uniformly integrable.

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The theorem is simply a special case of Lebesgue's dominated convergence theorem in measure theory.

Part VI: An Impossibility Result

Suppose players respond rationally

Definition. *Given a predictive learning rule* (f_i, g_i)*, rule* g_i *is said to be* rational *given* f_i *if, for each* $h \in H$*, rule* g_i *maximises expected discounted payoffs over all continuations of* h*.*

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- Also, g_i cannot depend on u_{-i} because i's payoffs do not give information on the payoffs of opponents.

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Definition. A vector of n profiles

$$(\nu_i, g_i) = \begin{pmatrix} g_1 & \nu_1^2 & \nu_1^3 & \dots & \nu_1^n \\ \nu_2^1 & g_2 & \nu_2^3 & \dots & \nu_2^n \\ \nu_3^1 & \nu_3^2 & g_3 & \dots & \nu_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu_n^1 & \nu_n^2 & \nu_n^3 & \dots & g_n \end{pmatrix}$$

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There's no catch: recall that players converge even if they do not play best replys but play, e.g., maxmin.

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   From here, G''s matrix is fixed, and G' is ready to be played.

Example. "Uncertain matching pennies". Property: when λ is sufficiently small, G' still possesses a unique (mixed) Nash equilibirum.

$$M = \begin{array}{c} \\ \mathrm{H} \\ \mathrm{T} \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{H} & \mathrm{T} \end{array} \\ \begin{pmatrix} 1+\epsilon_{11}, -1+\epsilon'_{11} & -1+\epsilon_{12}, 1+\epsilon'_{12} \\ -1+\epsilon_{21}, 1+\epsilon'_{21} & 1+\epsilon_{22}, -1+\epsilon'_{22} \end{pmatrix} \end{array}$$

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- This is far more stronger than the negation of asymptotic closeness, in the sense of Kalai and Lehrer.

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- Explanation:
  - Kalai *et al.*'s main theorem states that players will (almost) correctly predict the on-path portions of the other players' strategies.
  - Kalai *et al.*'s main theorem does not state that players will (almost) learn the true strategies of their opponents.