Multi-agent learning

Introduction

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- Multi-agent learning is a discipline on the interface of multi-agent systems and machine learning.
- MAL differs from single-agent learning. Besides learning, it involves trying to influence other agents by executing the right actions (called teaching).
- MAL is a young and active research area. Most contributions are from 2000 and later.

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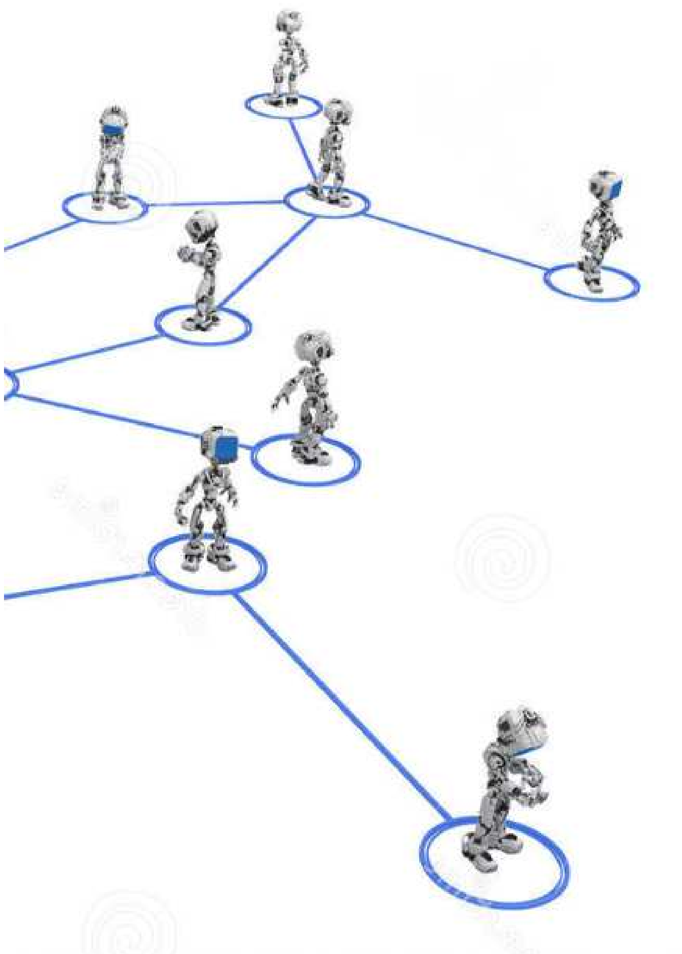

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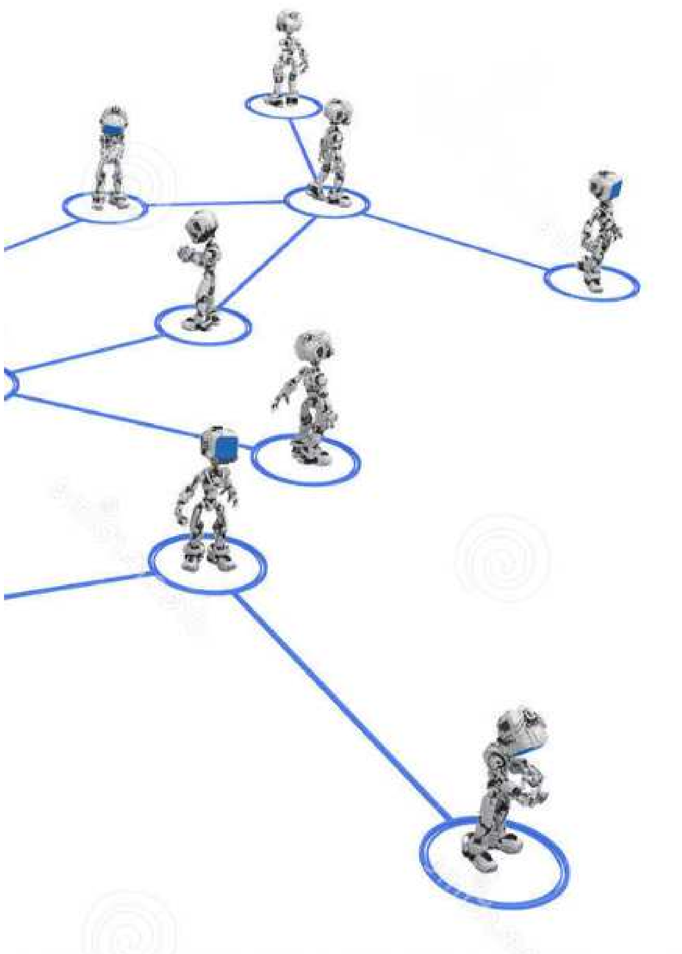

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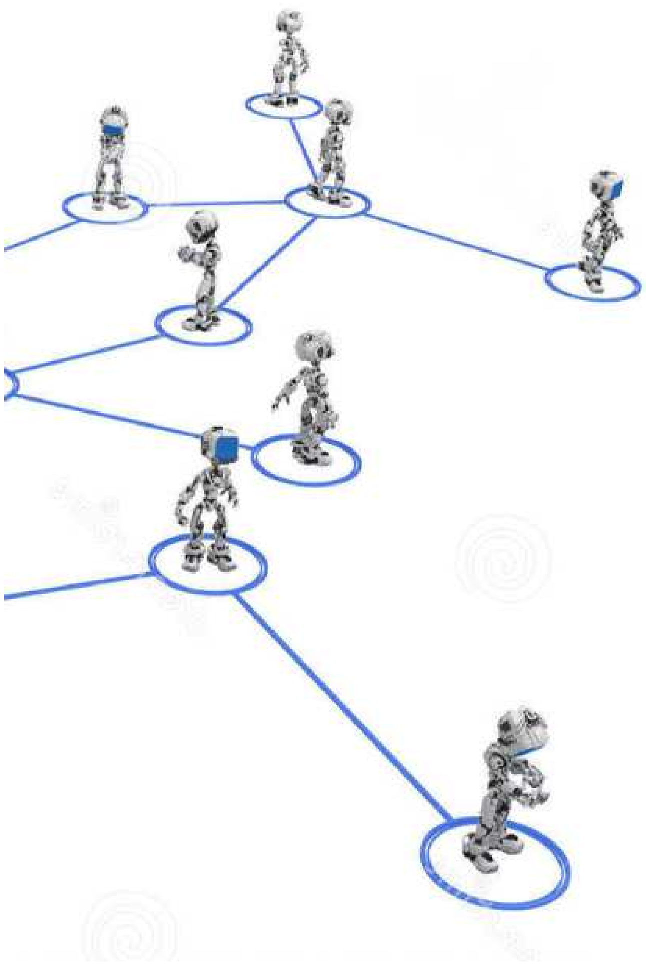

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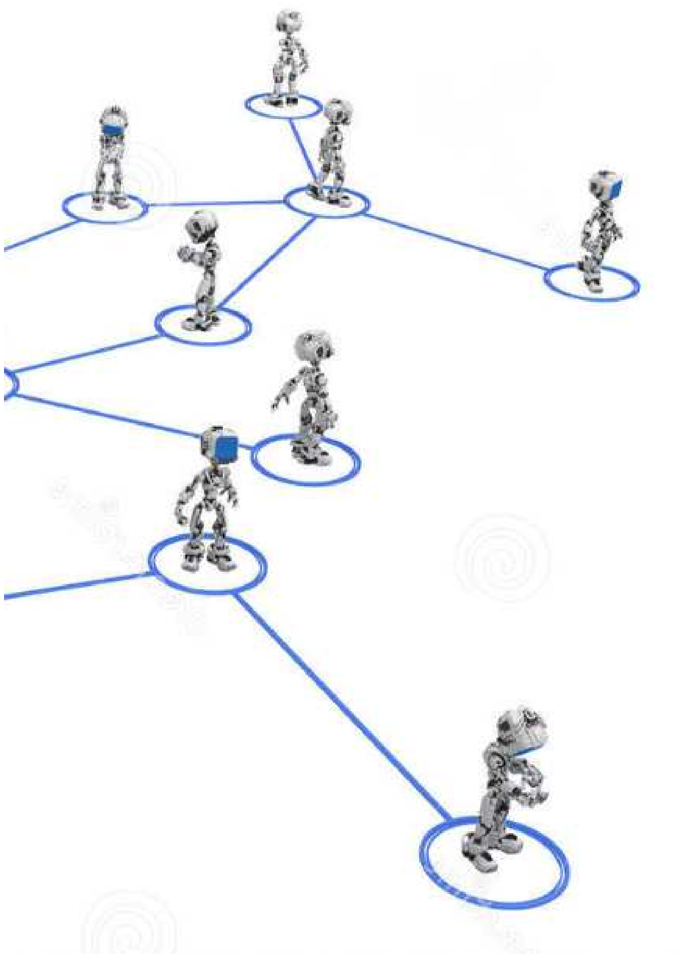

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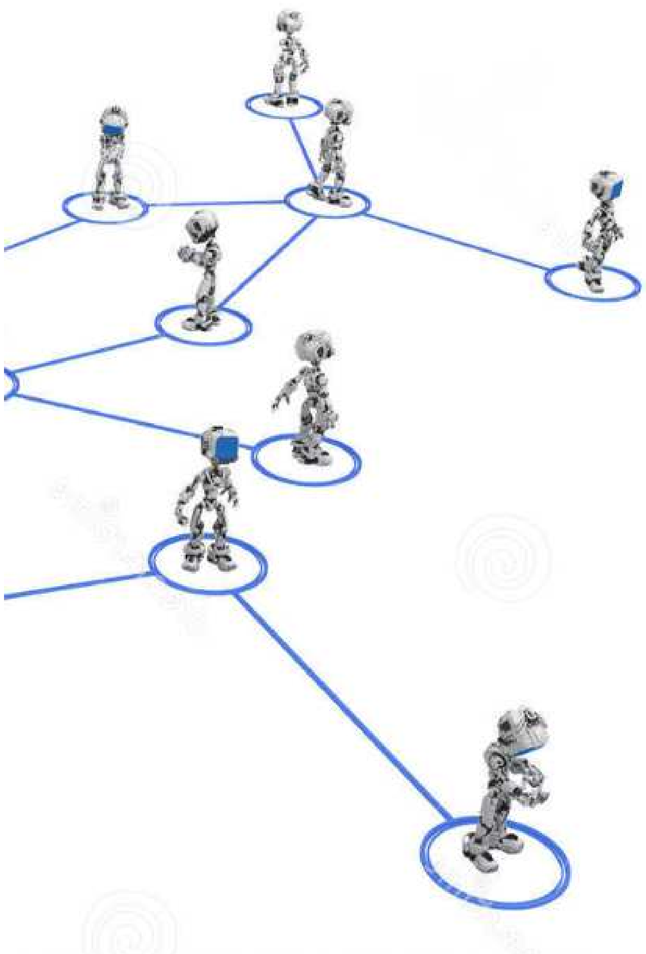

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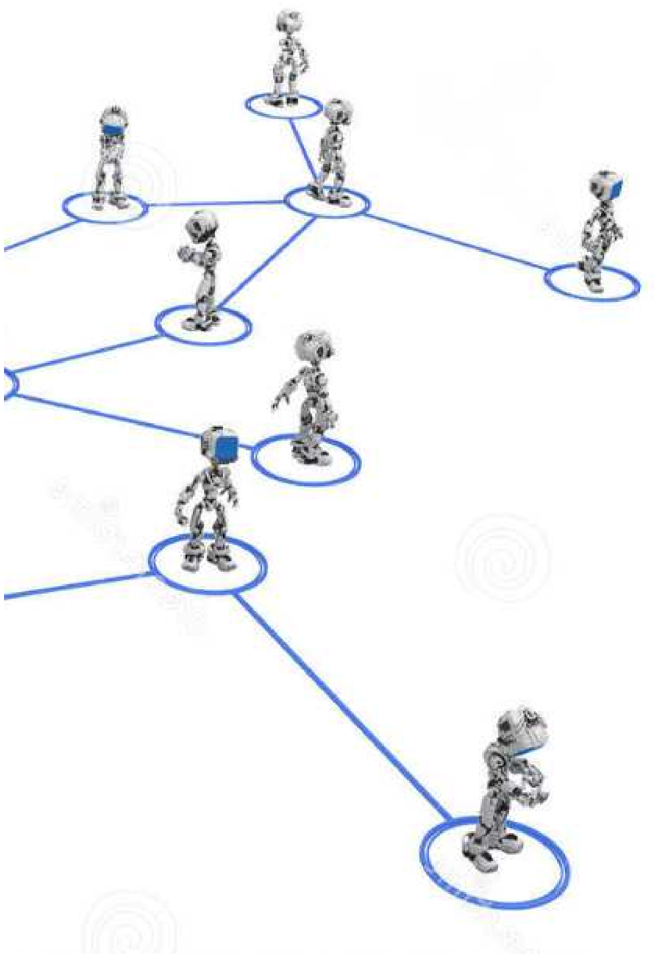

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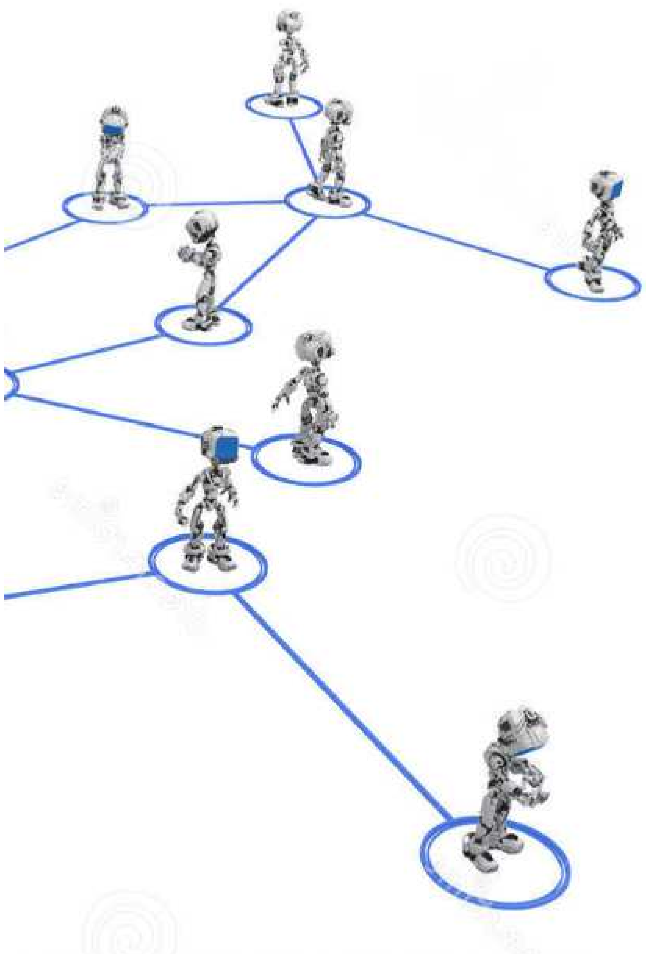

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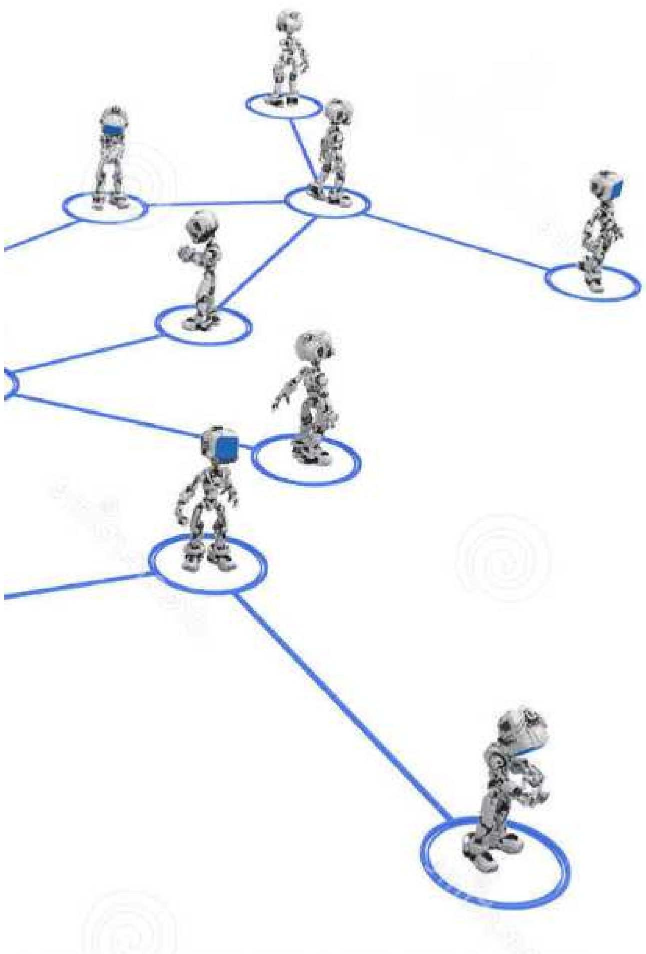

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Exam preparation

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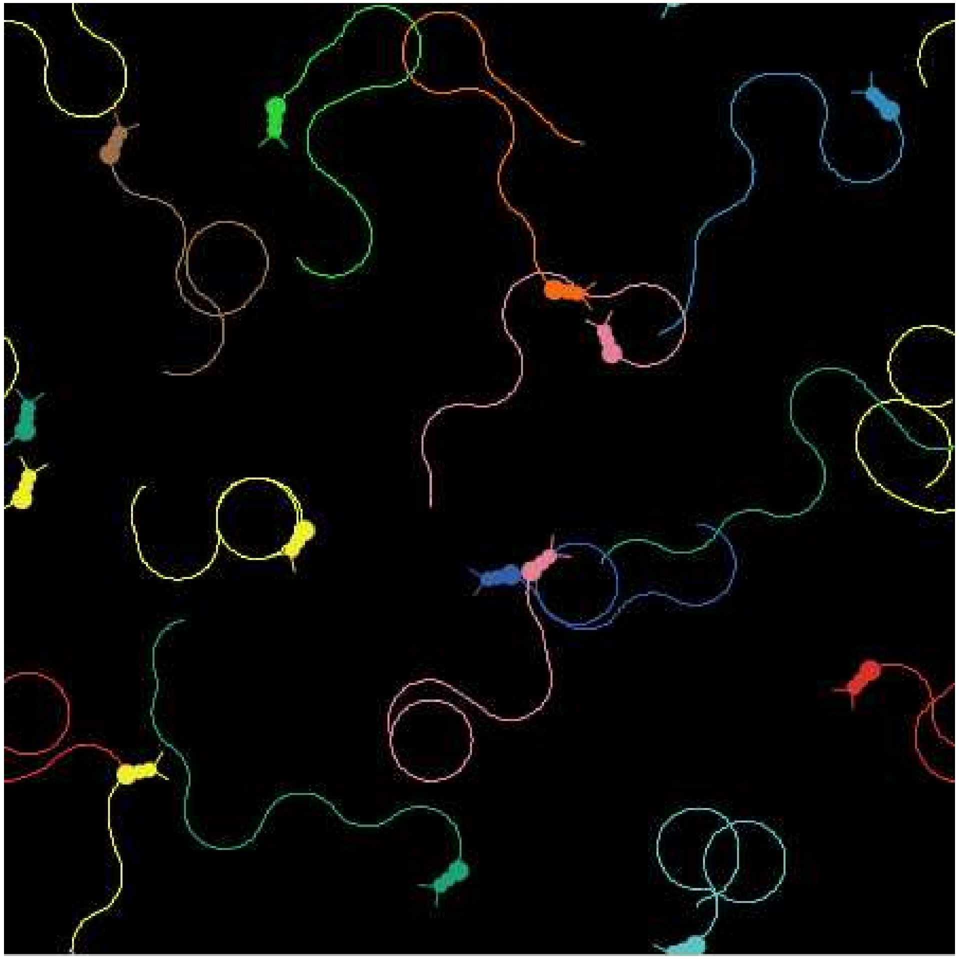

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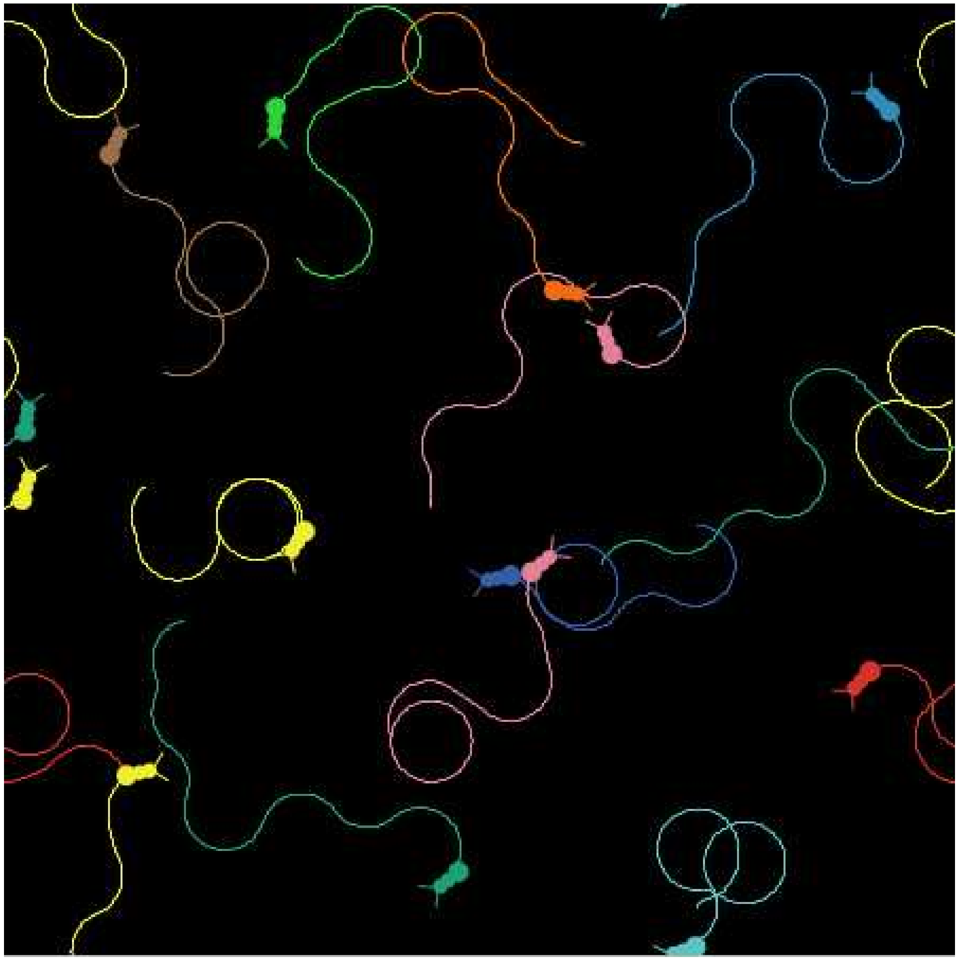

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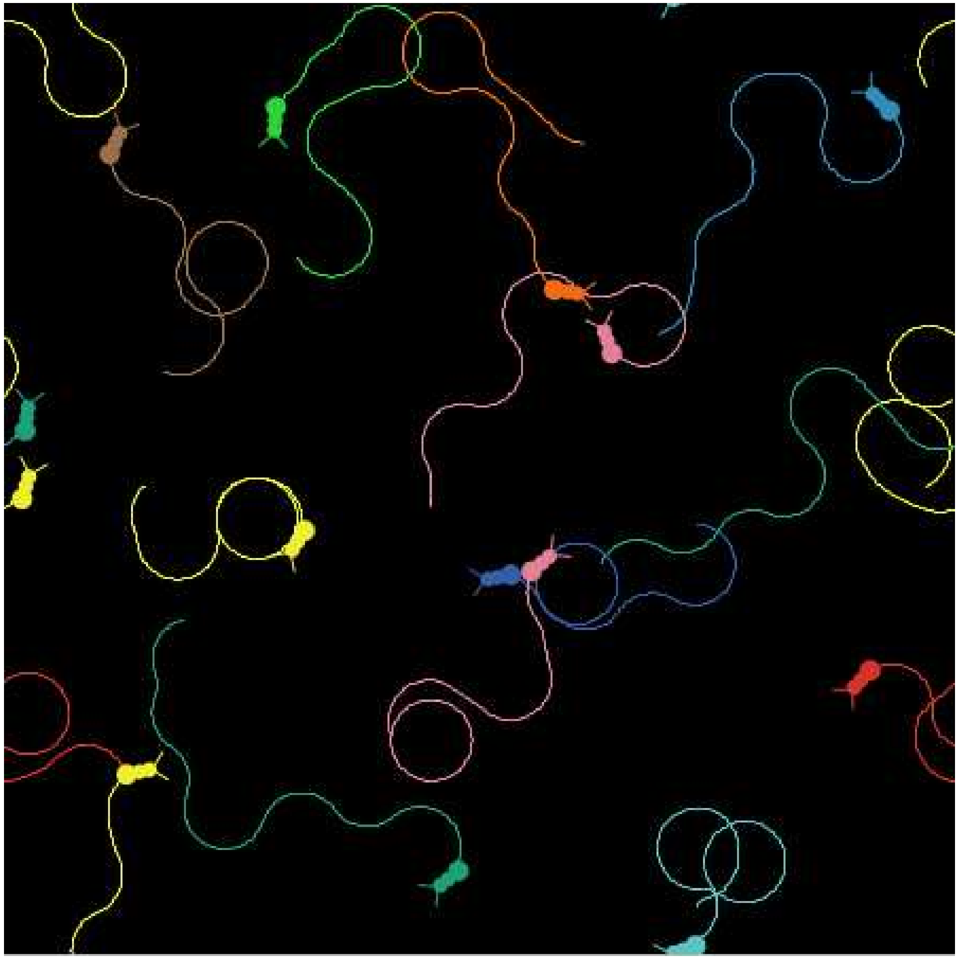

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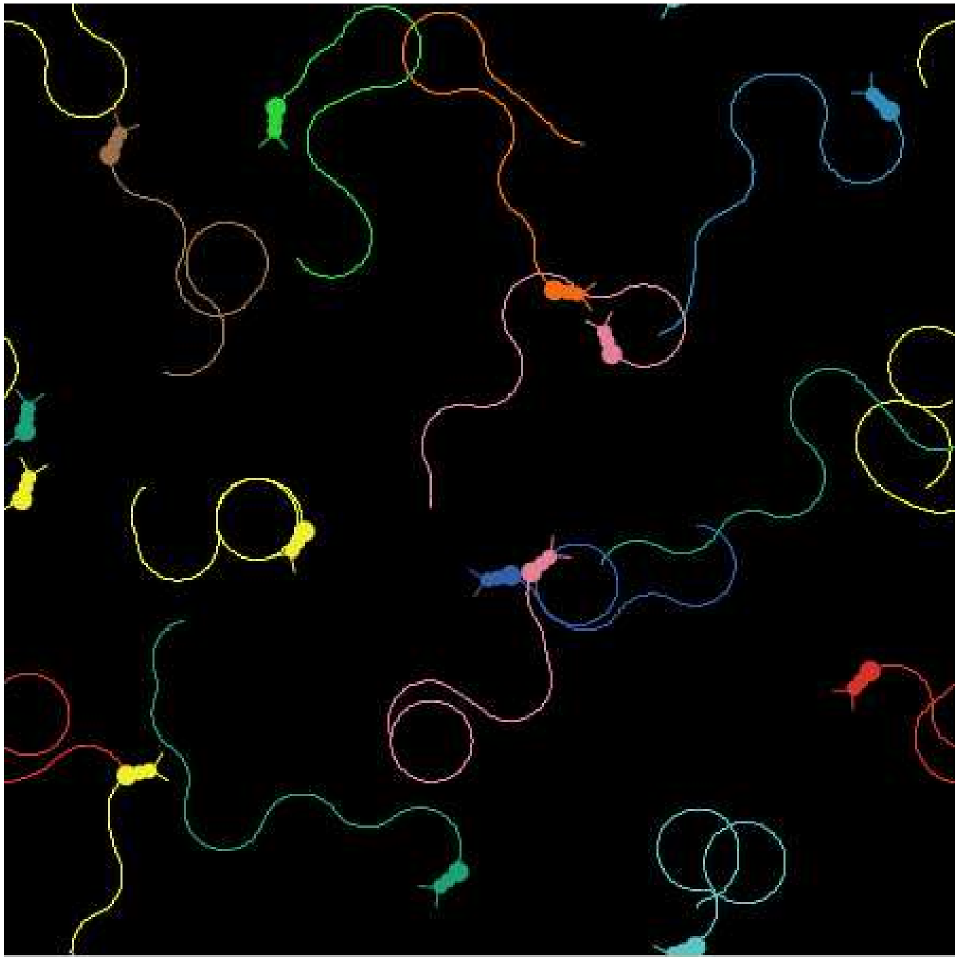

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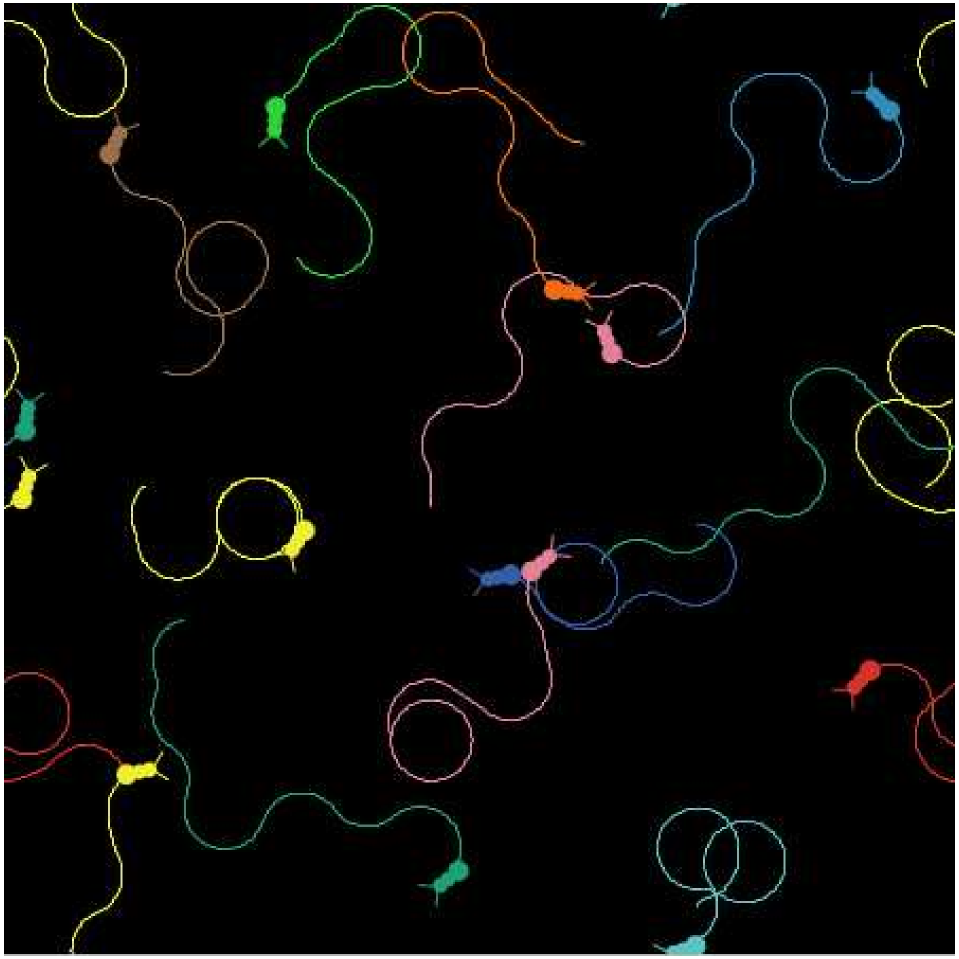

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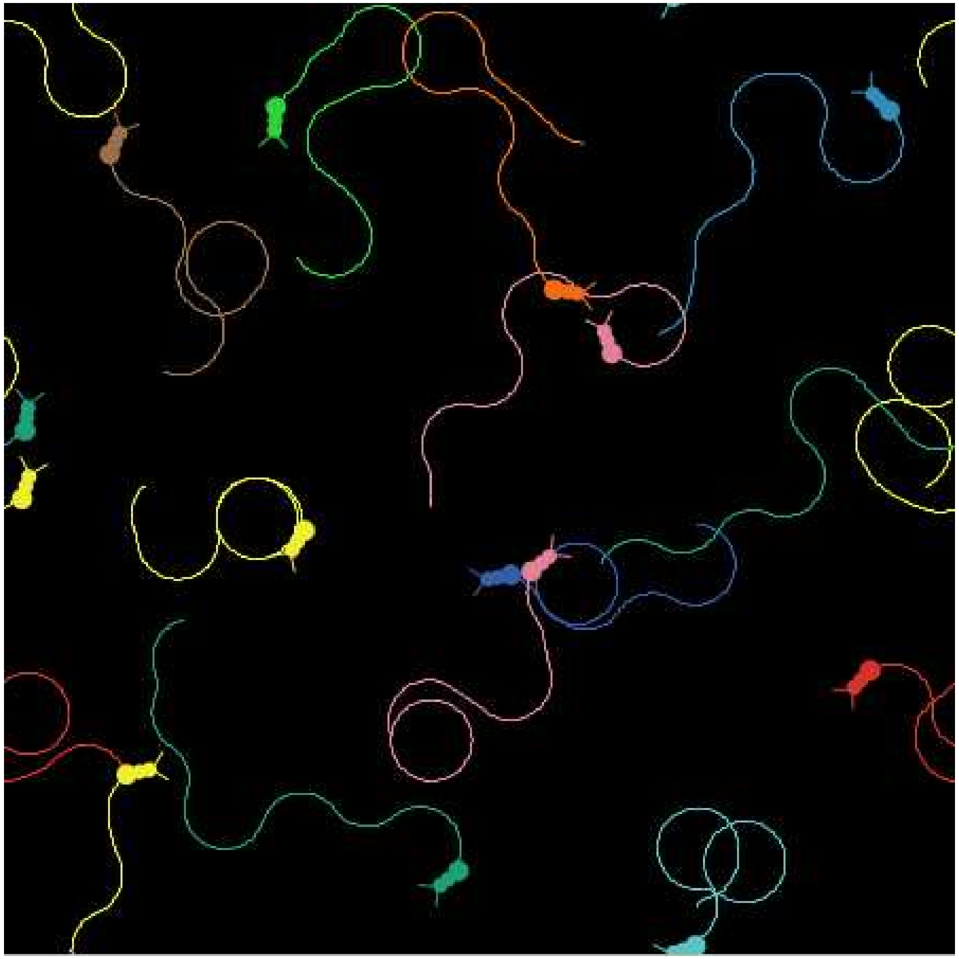

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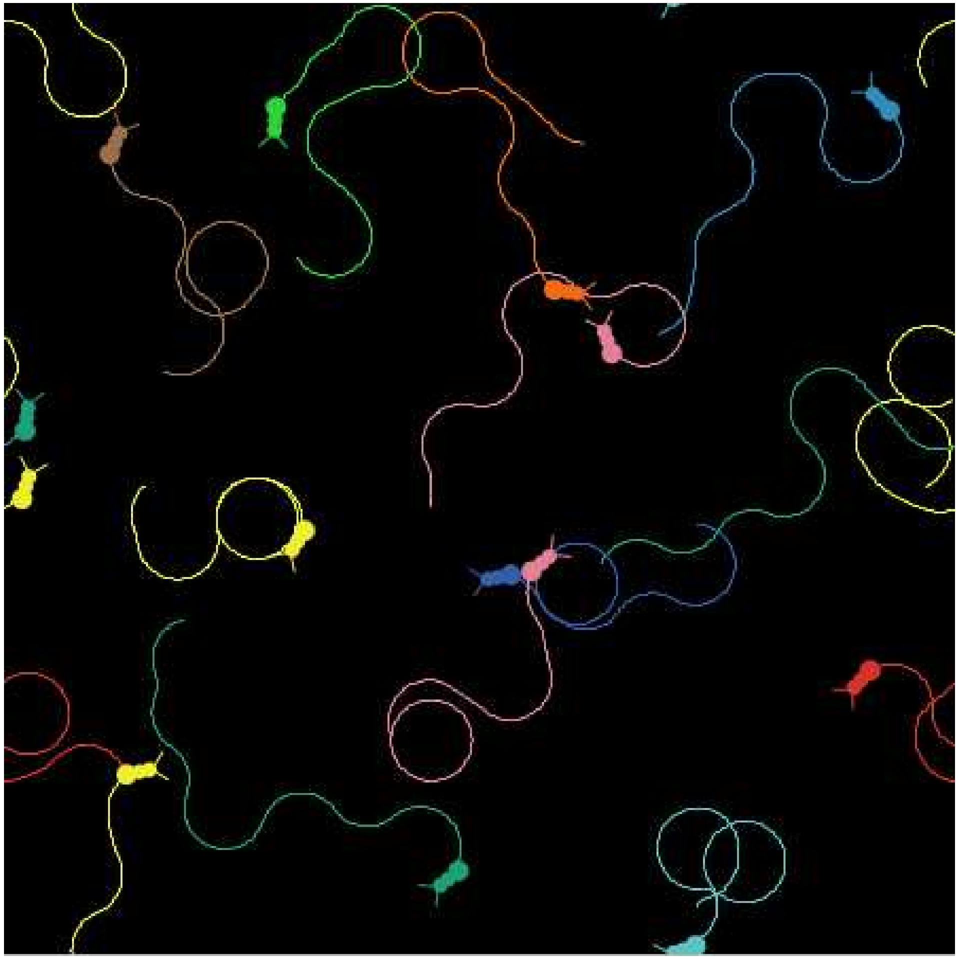

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- They have, depending on the value of a global constant a, one of the following objectives
  - $a = +1$: to pursue other bugs.
  - $a = -1$: to evade other bugs.
- Bug Hunt is a simple instance of a so-called n-type pursuit ($a = 1$) or evasion ($a = -1$) game.

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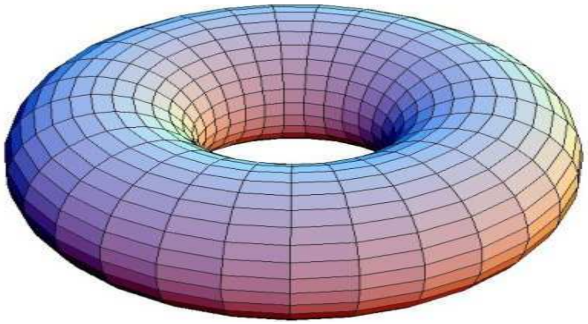
- The centroid of k points on a torus is defined as in a 2D plane, but we will have to take into account that the neighborhood and direction on a torus are defined differently.



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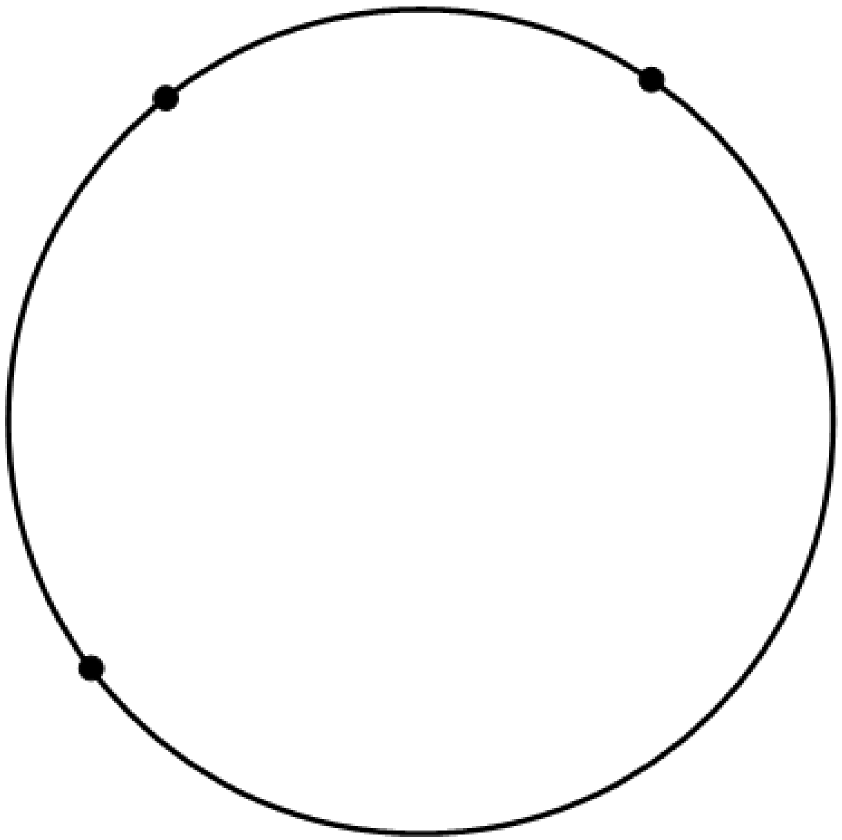
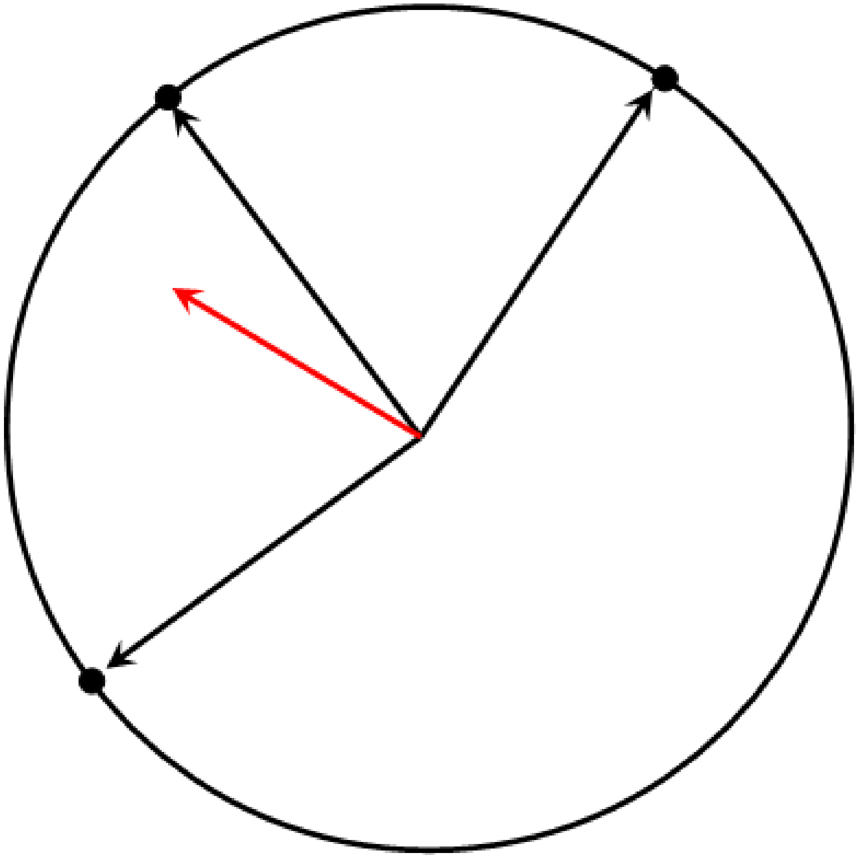


The centroid of points on a torus is computed in the same way as the centroid of points on a circle is computed.

Let us consider the latter.

Example: Pursuit / evasion game



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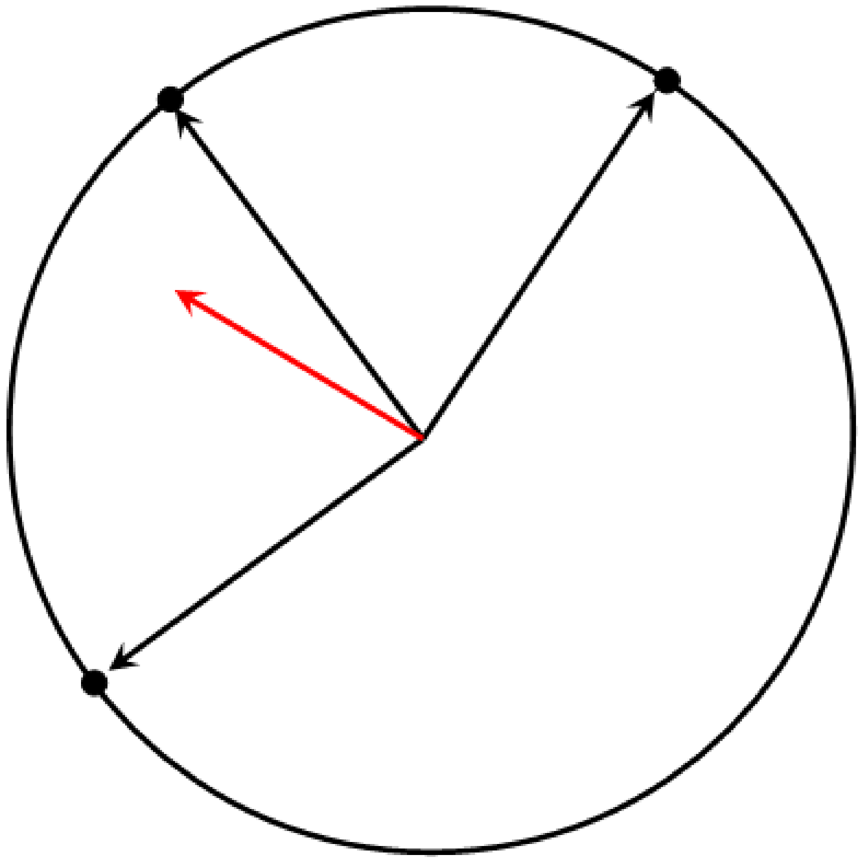

The average of a number of directions is computed by representing degrees as vectors, and then add the vectors.

Then map the sum of the vectors back to the circle. The certainty factor of this average is the length of the sum vector.

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  - The plane (Unbounded 2D). The absence of boundaries would enable simple evading behaviour. (Just flee into open space.)

Example: Pursuit / evasion game

Literature on n-person differential games:

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Warning. Work on DG’s is highly analytical (rather than philosophical, conceptual, or empirical).

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- If col has any sense he/she/it will pick up the signal and play R.

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- Is there a learning rule that ensures coordination without an external co-ordinator?

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Models of MAL: descriptive models and prescriptive models

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  - Empirical frequencies end up in periodic or chaotic dynamics.

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- No learning algorithm performs optimal against all possible opponents.

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- At the end of each week q_1 and q_2 are made public, and both firms adapt their production to the new situation

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- Let a be fixed.
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  - The production costs per unit is defined as c, with $0 < c < a$.
  - The profit for Firm 1 per unit is therefore

$$\text{Profit}_1(q_1) = \begin{cases} q_1(a - (q_1 + q_2)) - q_1 \cdot c & \text{if } q_1 + q_2 \leq a, \\ -q_1 \cdot c & \text{else.} \end{cases}$$

- At the end of each week q_1 and q_2 are made public, and both firms adapt their production to the new situation (= max. 2nd degree eq.).

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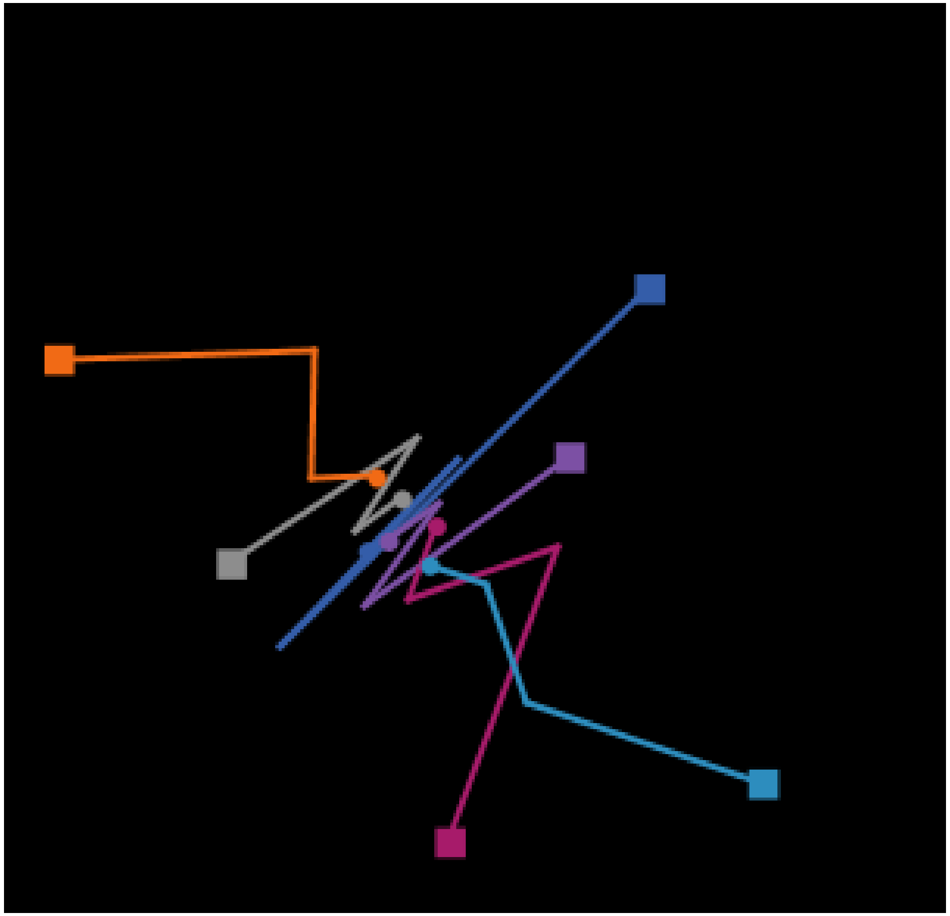

Fig: traces from 6 different starts.

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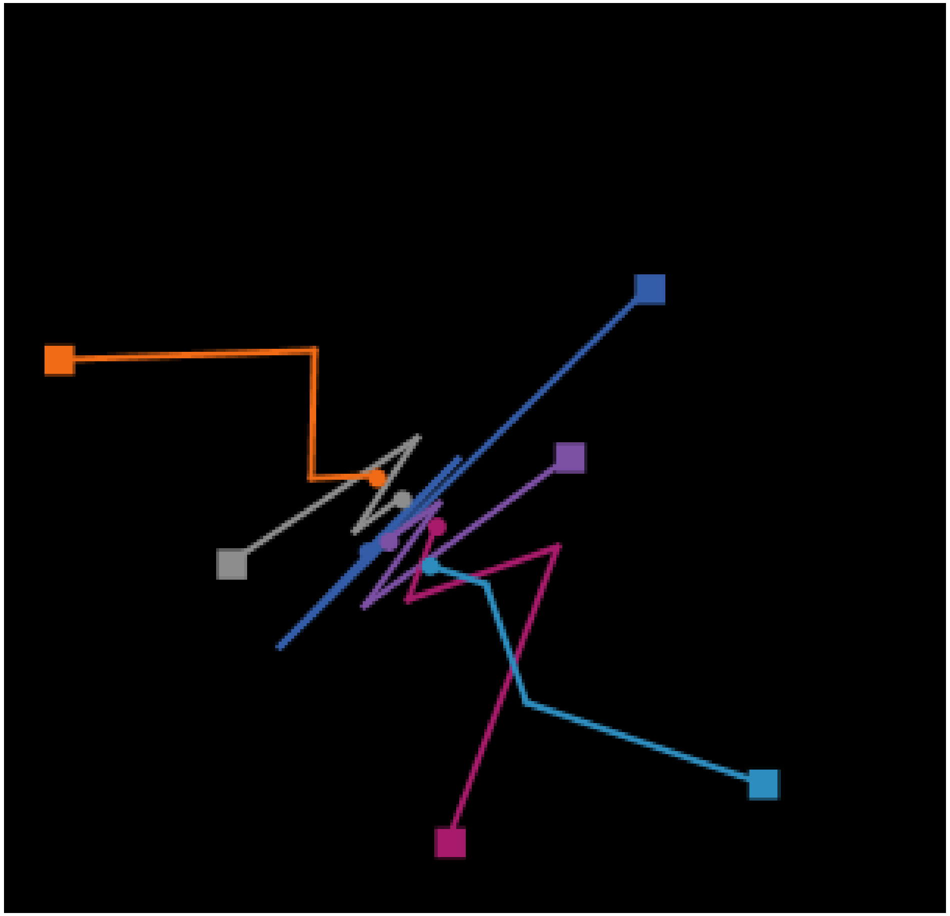
- Dynamic approach. Start with random quantities. Adapt them at the end of every week.
- Static approach. The pair (Q_1, Q_2) is called a Cournot equilibrium iff Q_1 is a best response to Q_2 and *vice versa*.



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- Set partial derivatives to zero. We then have two equations with two unknowns. Solution:

$$\begin{aligned} q_1 &= \begin{cases} (a-c-q_2)/2 & \text{if } q_2 \leq a-c, \\ 0 & \text{else.} \end{cases} \\ q_2 &= \begin{cases} (a-c-q_1)/2 & \text{if } q_1 \leq a-c, \\ 0 & \text{else.} \end{cases} \end{aligned}$$

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- For linear prices and costs $\Rightarrow$ convergence to a unique equilibrium.

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- If $\alpha/\beta \in (4/25, 25/4)$ then trajectory remains bounded.*
- Whenever $0.16 < \alpha/\beta \leq 0.171\dots$ or $5.828\dots \leq \alpha/\beta \leq 6.25$ there is periodicity, semi-periodicity, or chaos.*

Fig: chaotic trajectory.

* Tönu Puu. Chaos in Duopoly Pricing. *Chaos, Solitions & Fractions* **1**(6), pp. 573-581, 1991.