Multi-agent learning

Uncoupled learning and NE

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The computational complexity of finding Nash equilibria

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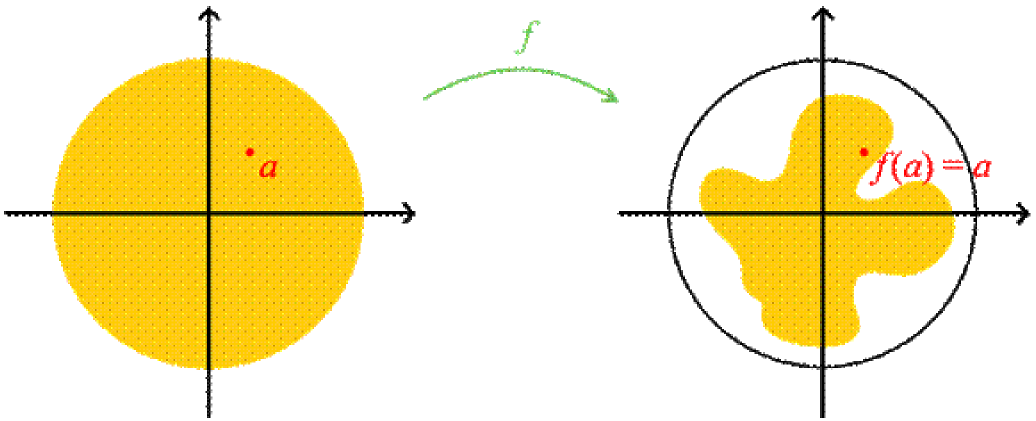

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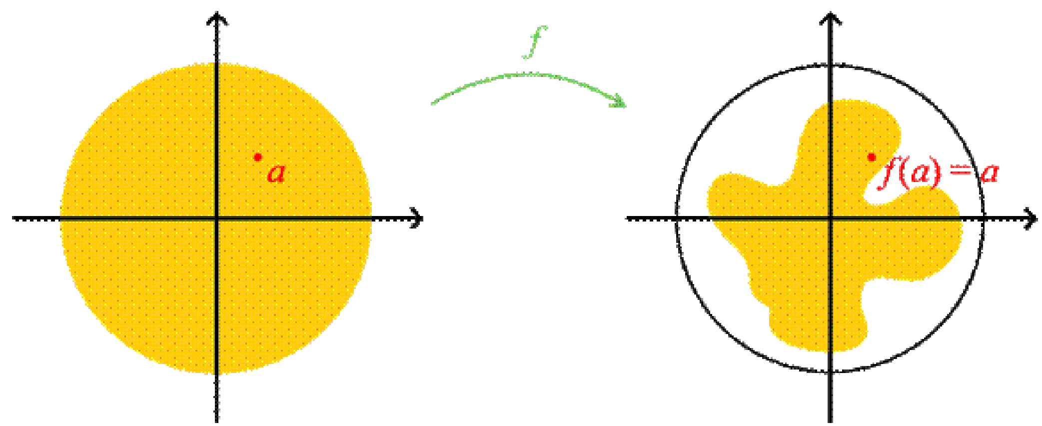

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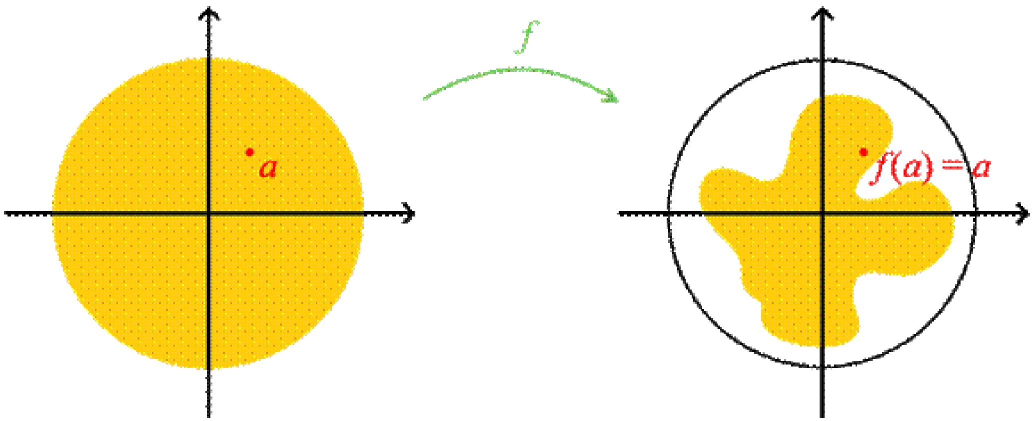

The proof of Brouwer's theorem is non-constructive. So is the proof of Nash's theorem!

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– C.H. Papadimitriou (2007): "The complexity of finding Nash equilibria" in: Algorithmic Game Theory.
– C. Daskalakis *et al.* (2009): "The complexity of computing a Nash equilibrium" in: SIAM Journal on Computing.

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Conitzer and Sandholm (2008) give simple proofs by reducing (3-) satisfiability problems to such questions.

Conitzer, Vincent, and Tuomas Sandholm. "New complexity results about Nash equilibria." Games and Economic Behaviour 63.2 (2008): 621-641.

Gambit: software to find Nash equilibria

Gambit is a library of game theory software and tools.

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 $ ls -l
total 42760
-rwxr-xr-x 1 Gerard None 3179712 May 15 2017 gambit-enummixed
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Finding Nash equilibria through multi-agent learning

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Completely uncoupled. Also no knowledge of opponent's actions and can't observe them.

RESTRICTIONS ON PAYOFFS

Generic game. Payoffs can be perturbed without introducing new equilibria or removing existing ones.

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Limit frequency. 1: all of the time, a.s; $1 - \epsilon$ of the time, a.s.

Overview of learning NE in MAL (column "Result")

	Conditions	Uncoupled	Restrictions	Equilibrium	Limit freq.	Result nr.	Result	Who	When	
1.	1-recall	yes		pure	1	Th. 1	$-$	Hart *et al.*	2006	
2.	1-recall, 2p	yes	generic	pure	1	Pr. 2	$+$	Hart *et al.*	2006	
3.	2-recall	yes		pure	1	Th. 3	$+$	Hart *et al.*	2006	
4.	fin. recall	yes		mix-ϵ	$1-\epsilon$ (m)	Pr. 4	$+$	Hart *et al.*	2006	
5.	fin. recall	yes		mix-ϵ	$1-\epsilon$ (j)	Th. 5	$+$	Hart *et al.*	2006	
6.	fin. recall	yes		mix-ϵ	$1-\epsilon$ (b)	Th. 6	$-$	Hart *et al.*	2006	
7.	fin. mem	yes		mix-ϵ	$1-\epsilon$ (b)	Th. 7	$+$	Hart *et al.*	2006	
8.	fin. mem	yes		mix-ϵ	1		$+$	slides Bab'ko	2013 –	Th. X of Hart *et al.*?
9.	fin. mem	radically		pure	1	Tb. 1	$-$	Babichenko	2012	
10.	fin. mem	radically	generic	pure	1	Tb. 3	$-$	Babichenko	2012	
11.	fin. mem	radically		pure	$1-\epsilon$	Tb. 5	$-$	Babichenko	2012	
12.	fin. mem	radically	generic	pure	$1-\epsilon$	Tb. 7	$+$	H.P. Young	2009 –	k. a. trial and error learning
13.	fin. mem	radically		mix-ϵ	1	Tb. 2	$-$	Babichenko	2012	
14.	fin. mem	radically	generic	mix-ϵ	1	Tb. 4	$+$	Babichenko	2012	
15.	fin. mem	radically		mix-ϵ	$1-\epsilon$	Tb. 6	$+$	Babichenko	2012	
16.	fin. mem	radically	generic	mix-ϵ	$1-\epsilon$	Tb. 8	$+$	Germano *et al.*	2007 –	regret testing (Foster *et al.*, 2006)

(m) marginal frequencies; (j) joint frequencies; (b) behaviour frequencies

Work of S. Hart and A. Mas-Colell

"Stochastic uncoupled dynamics and Nash equilibrium" in: *Games and Economic Behavior*, **57** (2006), pp 286-303.

The possibility of learning pure equilibria

Naive algorithm to learn pure equilibria

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IDEA

- Let Row and Col play pure strategies x and y, and let A be the payoff matrix.
- The scalar $x^T Ay$ equals Row's expected payoff for x.
- The vector Ay equals Row's expected payoffs per action.
- If $x^T Ay < \max\{Ay\}$, then x wasn't optimal.

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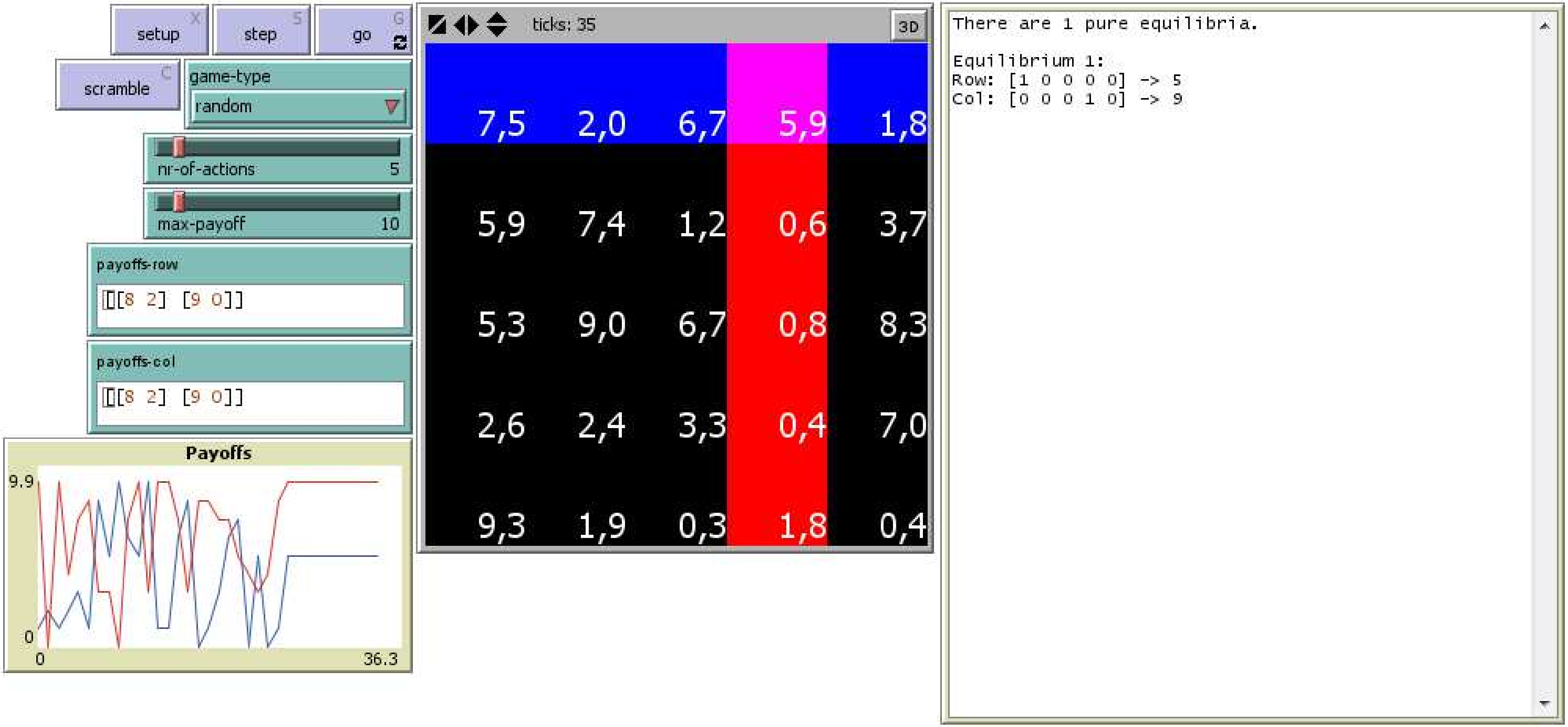
ALGORITHM

- Repeat: randomise current action if it wasn't a best reply.

Naive algorithm seems to work . . .



Learning pure equilibria with recall one (entry #1)

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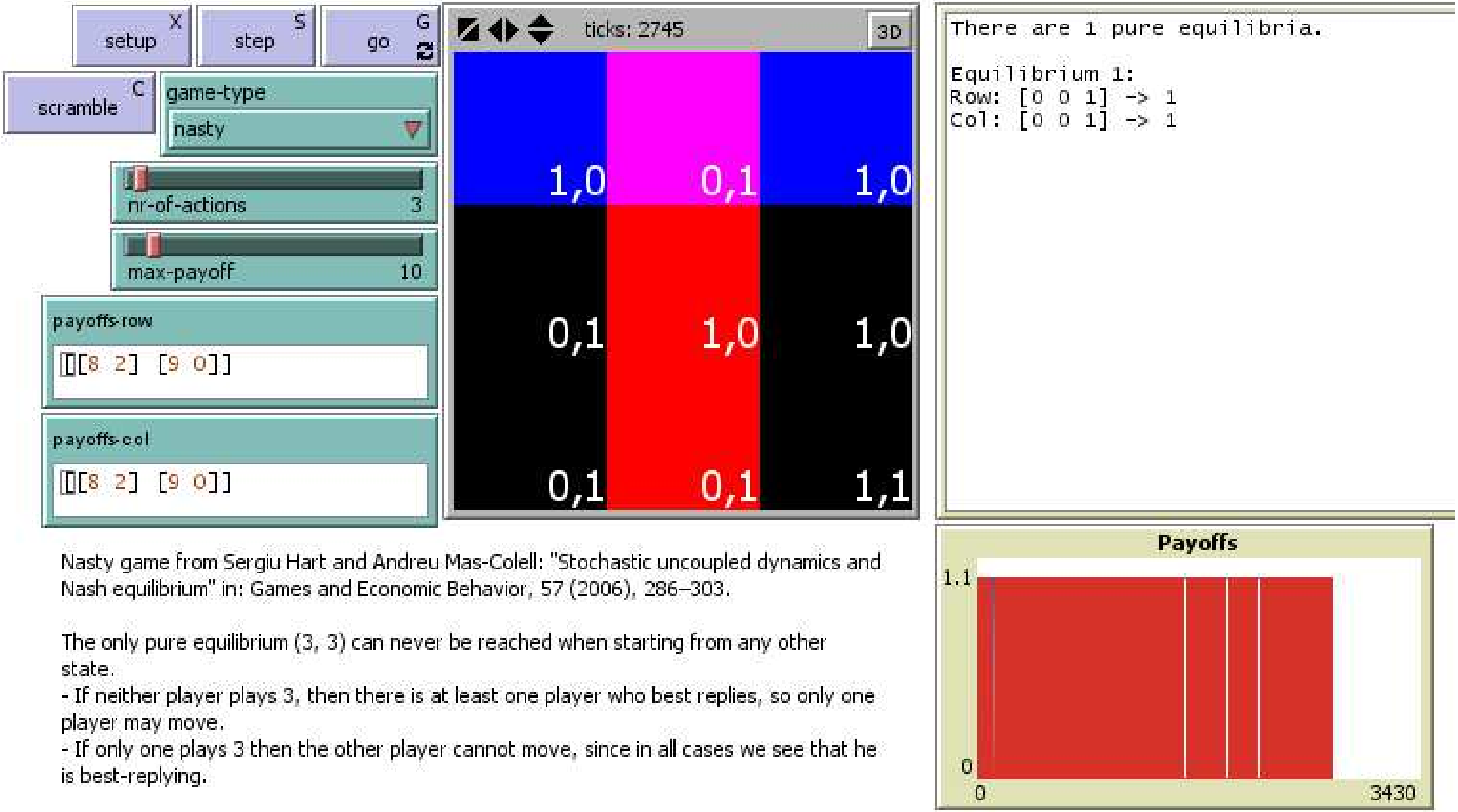
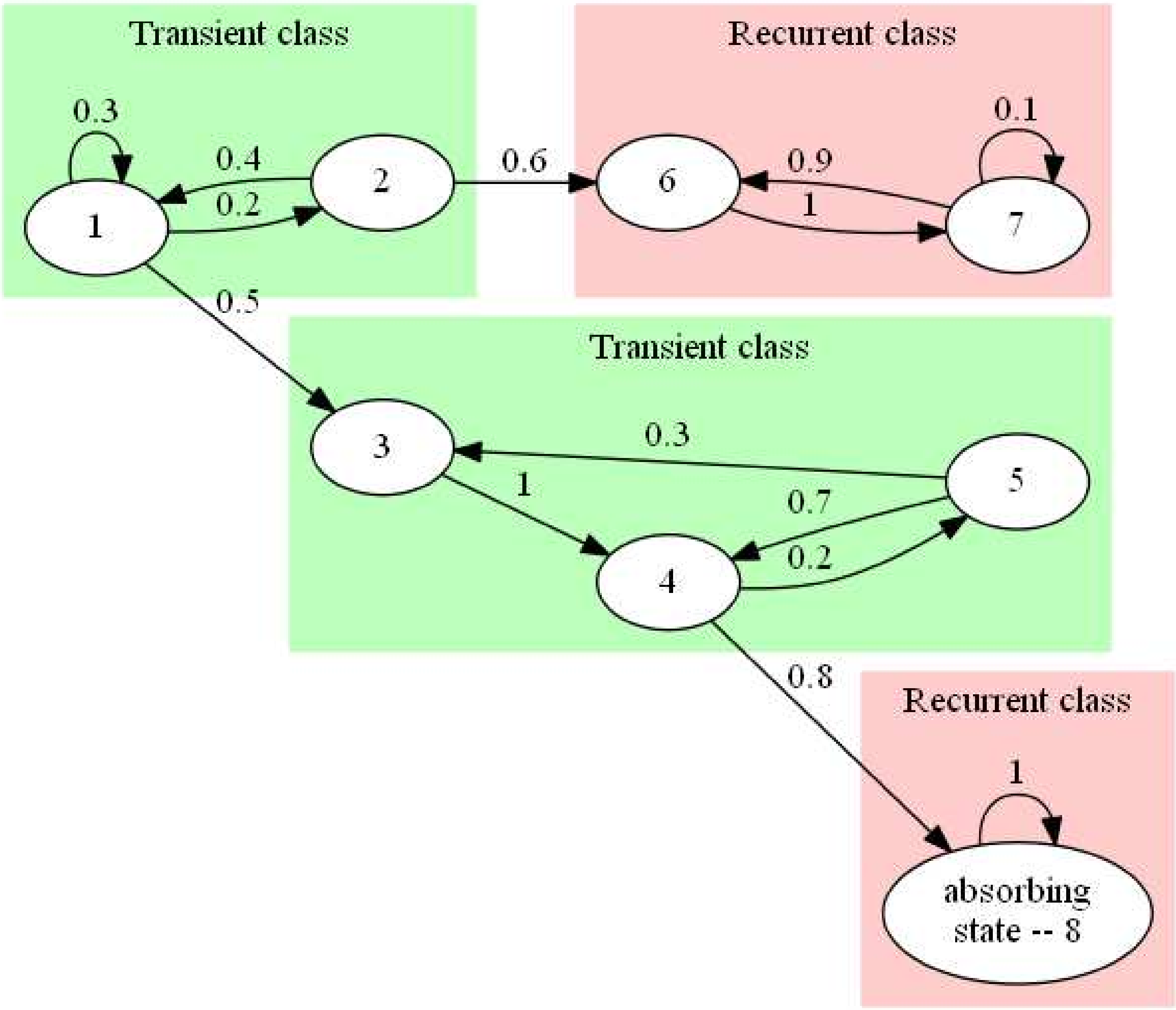
Impossibility of learning equilibria with recall one



Markov chain = a probabilistic process among states



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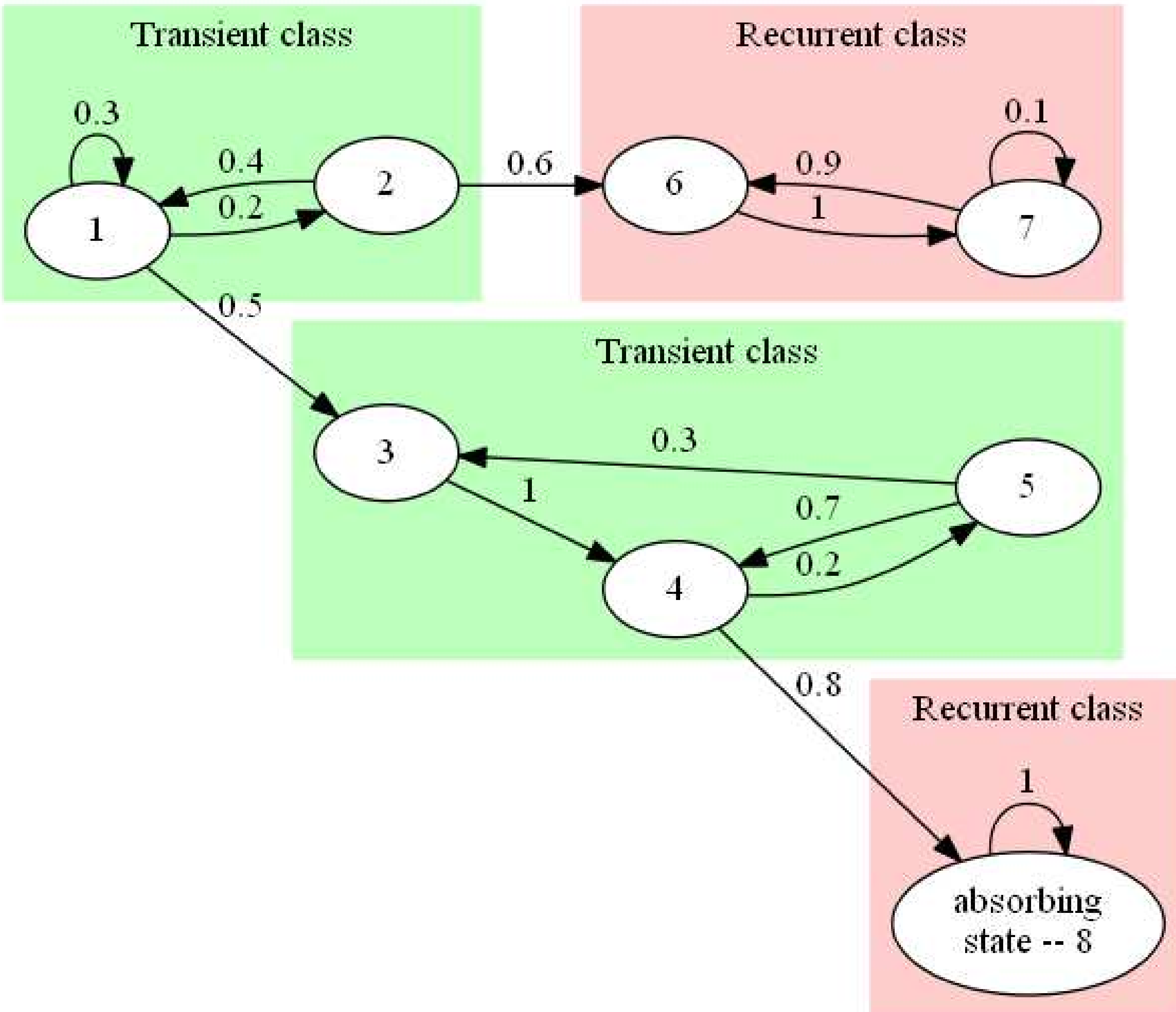

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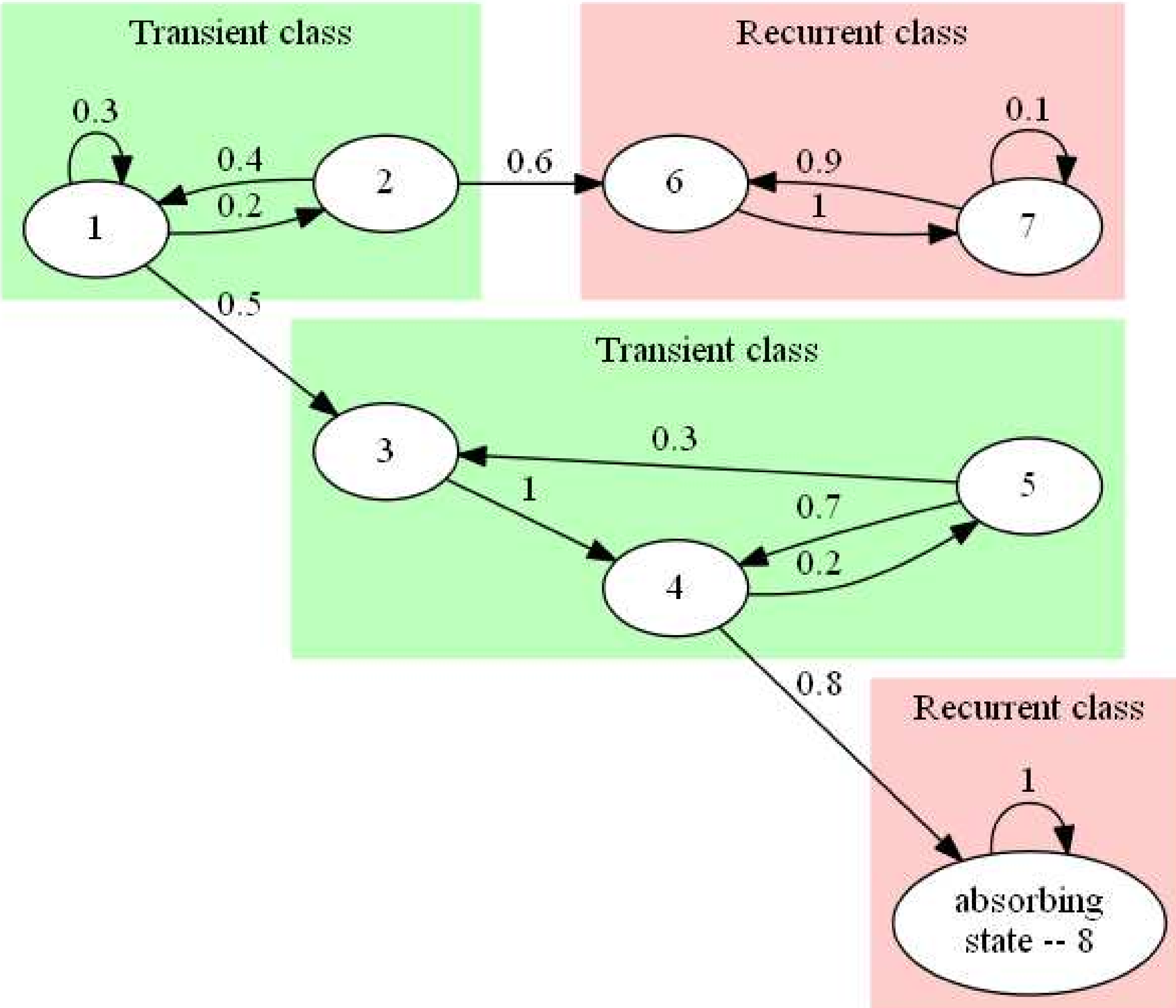

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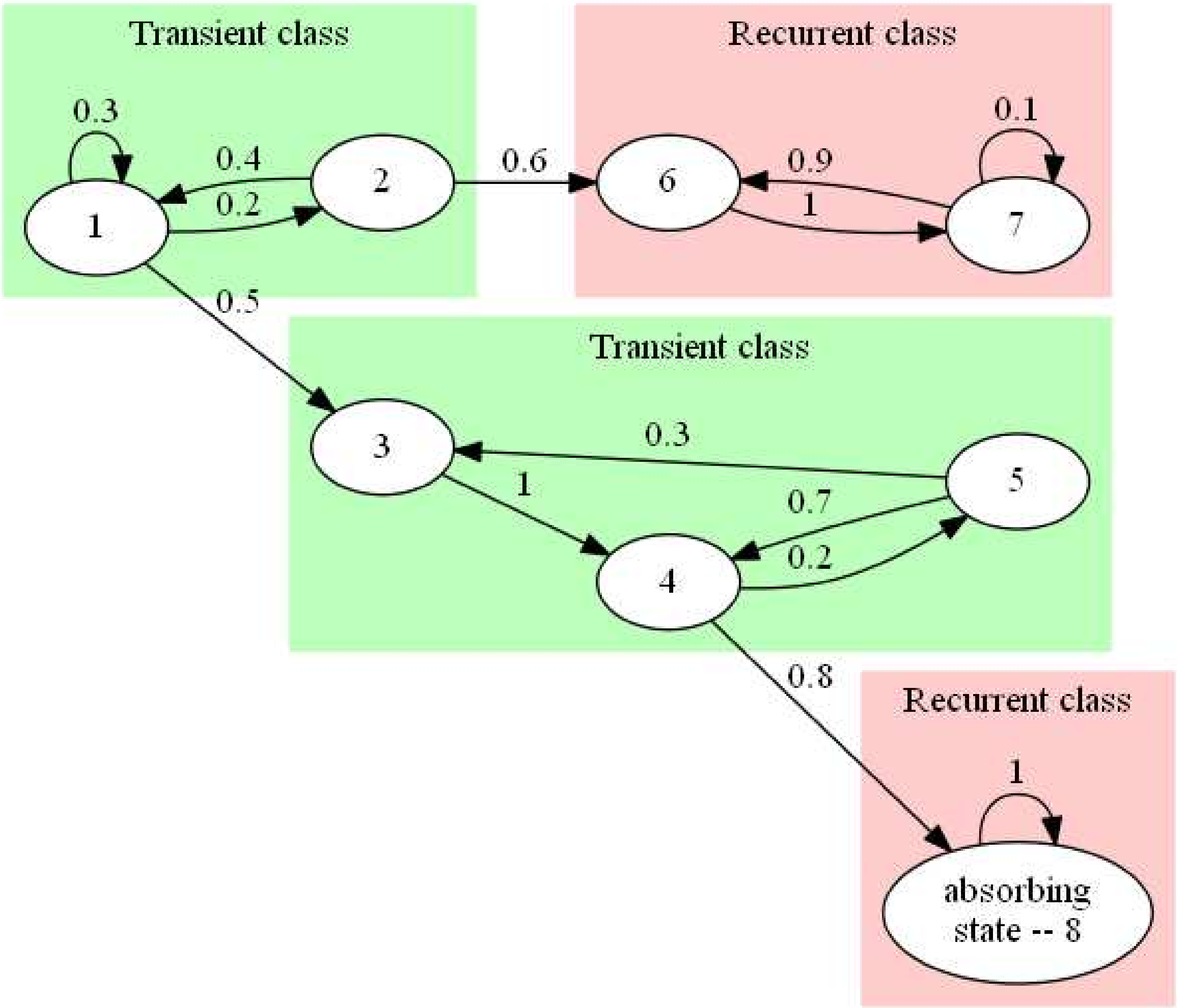

- Numbers represent probabili-ties.
- Probabilities of outgoing edges should add up to one.
- A process always ends up in one of the recurrent classes.

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- It can be proven that generic games are non-degenerate.
- With non-degenerate games, it follows that pure strategies have unique (hence pure) best responses.

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$$(a_1, a_2) \rightsquigarrow_{\exists} (a_1, a_2^*) \rightsquigarrow_{\exists \text{ 1 randomizes if the previous was not a NE}} (a_1^*, a_2^*).$$

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1. $S_1 =_{Def} \{ (a\ , a) \in A^2 \mid a \text{ is Nash} \}$.
2. $S_2 =_{Def} \{ (a', a) \in A^2 \mid a \text{ is Nash} \}$, $a \neq a'$.
3. $S_3 =_{Def} \{ (a', a) \in A^2 \mid a \text{ is not Nash} \}$, $a \neq a'$.
4. $S_4 =_{Def} \{ (a\ , a) \in A^2 \mid a \text{ is not Nash} \}$.

Claim: all states in S_1 are absorbing, all other states are transient:
$S_2 \rightsquigarrow_{\exists} S_1$, $S_3 \rightsquigarrow_{\exists} S_2$, $S_4 \rightsquigarrow_{\exists} S_2 \cup S_3$. $\square$

The possibility of learning mixed equilibria

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Check: $\Phi_t[a] \in [0,1]$, $\Phi_t[a^i] \in [0,1]$, joint $\in \Delta(A)$, marginal $\in \Delta(A^i)$.

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Proof. Let $\epsilon > 0$ be given. Since u^i is linear hence continuous, there exists an integer K such that

$$\|x^i - y^i\| \leq \frac{1}{K} \text{ for all } i \quad \Rightarrow \quad |u^i(x) - u^i(y)| \leq \frac{\epsilon}{2} \text{ for all } i \tag{1}$$

for all mixed strategies x^i, y^i of player i.[1]

[1] The authors use the max-norm: $\|x^i - y^i\| =_{Def} \max_k\{|x^i_k - y^i_k|\}$.

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Theorem 5. For every $\epsilon > 0$ there exists an R and an uncoupled, R-recall, family of response rules that guarantees, in every game, the a.s. convergence of the empirical joint distributions of play to Nash ϵ-equilibria.

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Theorem 6. For every small enough $\epsilon > 0$, there are no uncoupled, finite recall response rules that guarantee, in every game, the almost sure convergence of the behaviour probabilities to ϵ-equilibria of the stage game.

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- y is ϵ-Nash in U', so $g_1(\bar{a})(a_1) > \rho$.

Contradiction. $\square$

Work of D. Foster and H. Peyton Young on completely uncoupled learning

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Young, H. Peyton. "Learning by trial and error." *Games and economic behaviour* **65**(2) (2009), pp. 626-643.

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Foster, Dean P., and H. Peyton Young. “Regret testing: Learning to play Nash equilibrium without knowing you have an opponent” in: *Theoretical Economics* **1**(3) (2006), pp. 341-367.

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Germano, F. and Lugosi, G.. "Global Nash convergence of Foster and Young's regret testing" in *Games and Economic Behaviour*, **60**(1), pp. 135-154.

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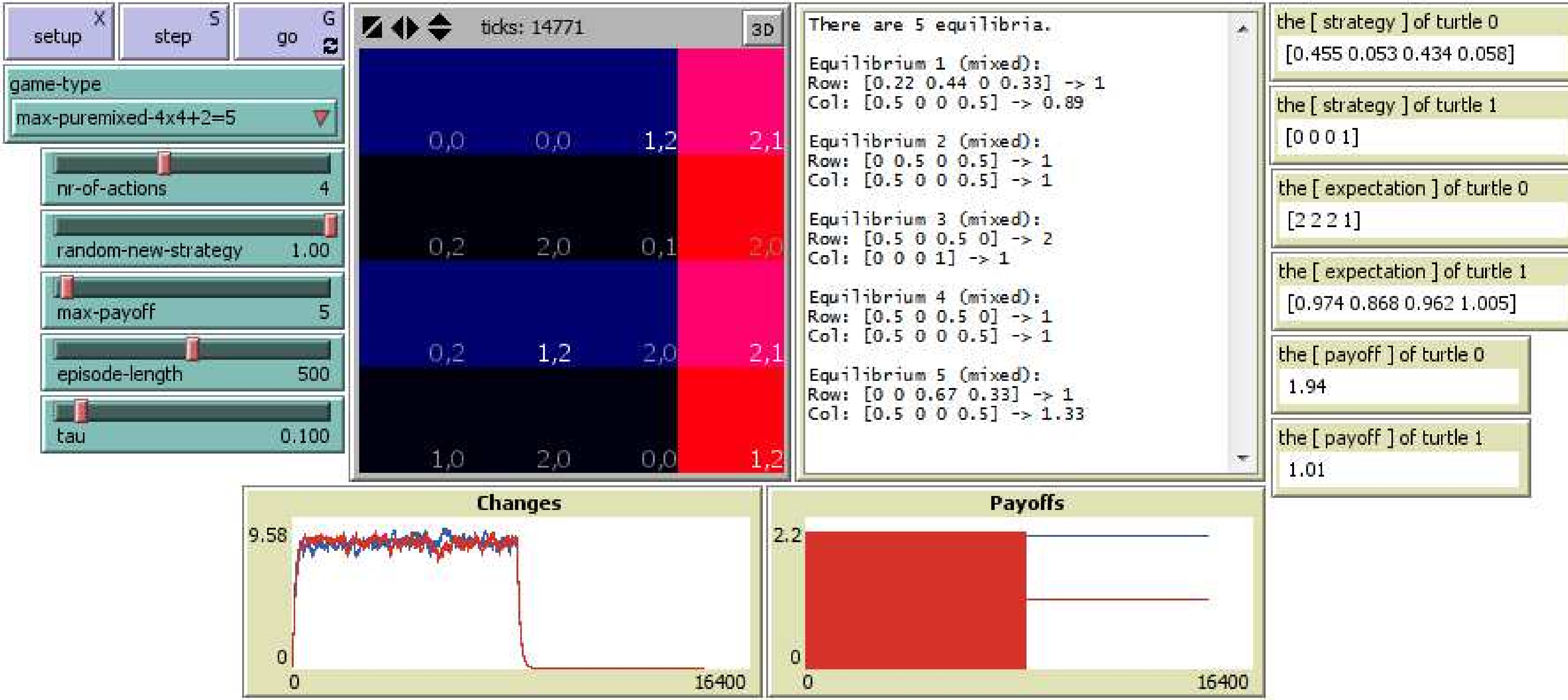
2. If, at the end of sampling, the revenue plus τ does not exceed the yield of every individual action when experimenting, adopt a new random x.

Demo: regret testing



Row player: blue; column player: white. To test the algorithm, a game with mixed equilibria only is chosen. Pareto sub-optimal action profiles are grey. In experiments, the strategy profiles are near the third equilibrium most of the time.

The following slides were not used.

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6. Form a differentiable function from mixed strategy profiles to $[0, \rightarrow)$ that is zero if and only if its argument is a Nash-equilibrium.

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– McKelvey, R. D., A Liapunov function for Nash equilibria, mimeo, California Institute of Technology (1992).
– Judd, Kenneth L. Numerical methods in economics. MIT press, 1998. (Sec. 4.9: Computing Nash equilibrium.)

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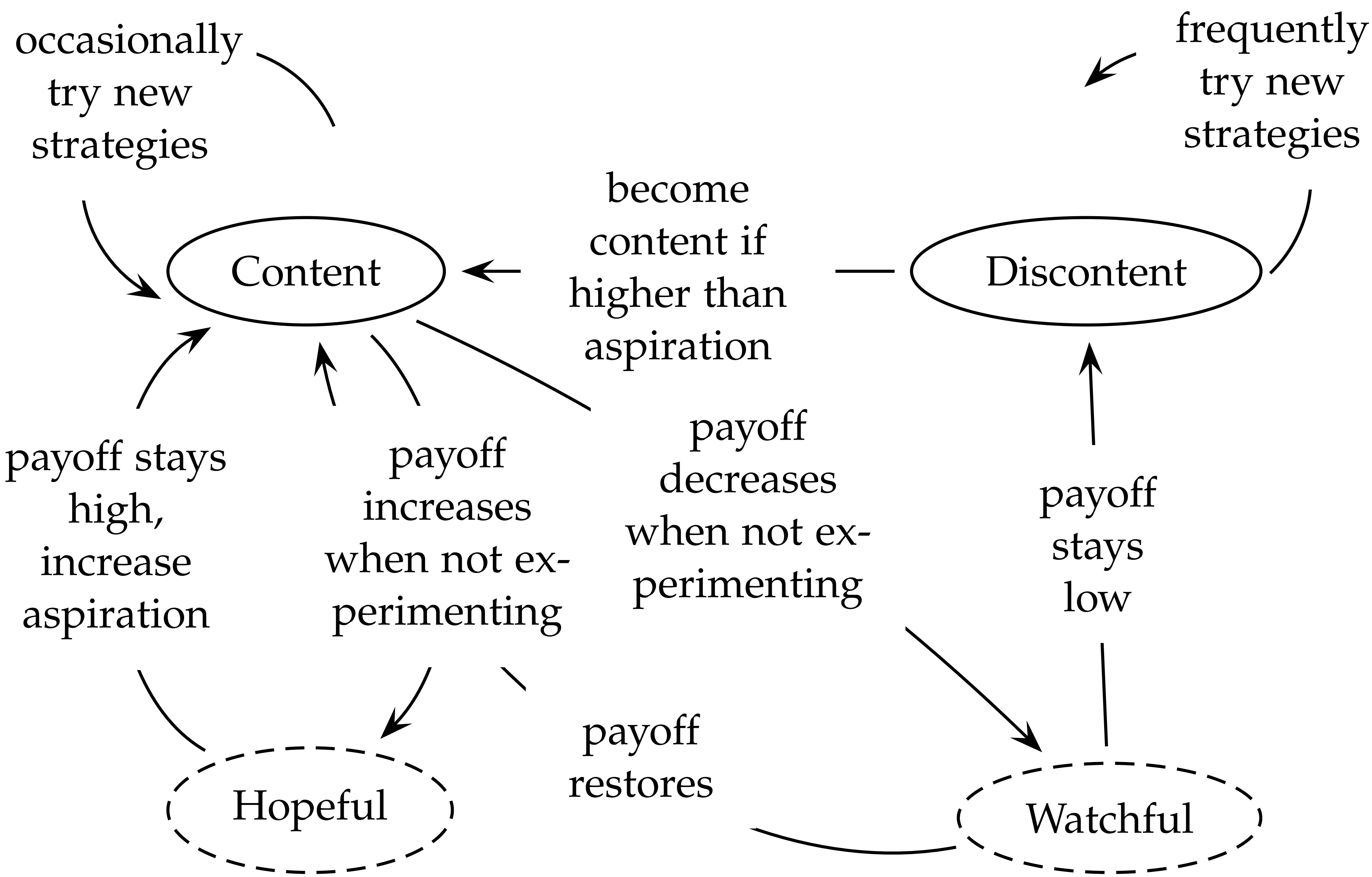
- Strategies can become erroneous due to a change of behaviour by someone else.
- Suppose a learning rule that conditionalises on whether a player experiences an error of the first or second type.
- This rule leads strategies into pure Nash equilibria most of the time in generic games that have pure Nash equilibria.

Idea: four states ("moods")



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- It can be shown that, *when the probability of calming down is sufficiently large relative to the probability of "thrashing around"*, the process is in a pure Nash equilibrium state much more often than in a disequilibrium state.