Multi-agent learning

Hypothesis Testing

Gerard Vreeswijk, Intelligent Systems Group, Computer Science Department, Faculty of Sciences, Utrecht University, The Netherlands.

Wednesday 12th June, 2019

Plan for today

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Basic concepts

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2. A simple hypothesis testing algorithm that leads to ϵ-Nash $1 - \epsilon$ of the time.

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1. Comparison with Bayesian play

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2. Learning to predict with bounded recall.

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1. Comparison with Bayesian play
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3. Tightening learning parameters (= simulated annealing).

Part I:
Motivation

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- Rather than maintaining a "docile" opponent model, why not adhere to a hypothesis of opponent play for as long as possible, until it really is no longer tenable due to strong counter-evidence.

  A hypothesis $\approx$ an opponent model formed on the basis of a large and (therefore hopefully) representative sample of stimuli.

Tim Salmon's experiment

Tim Salmon's experiment (2004)

		Player B			
		a	b	c	d
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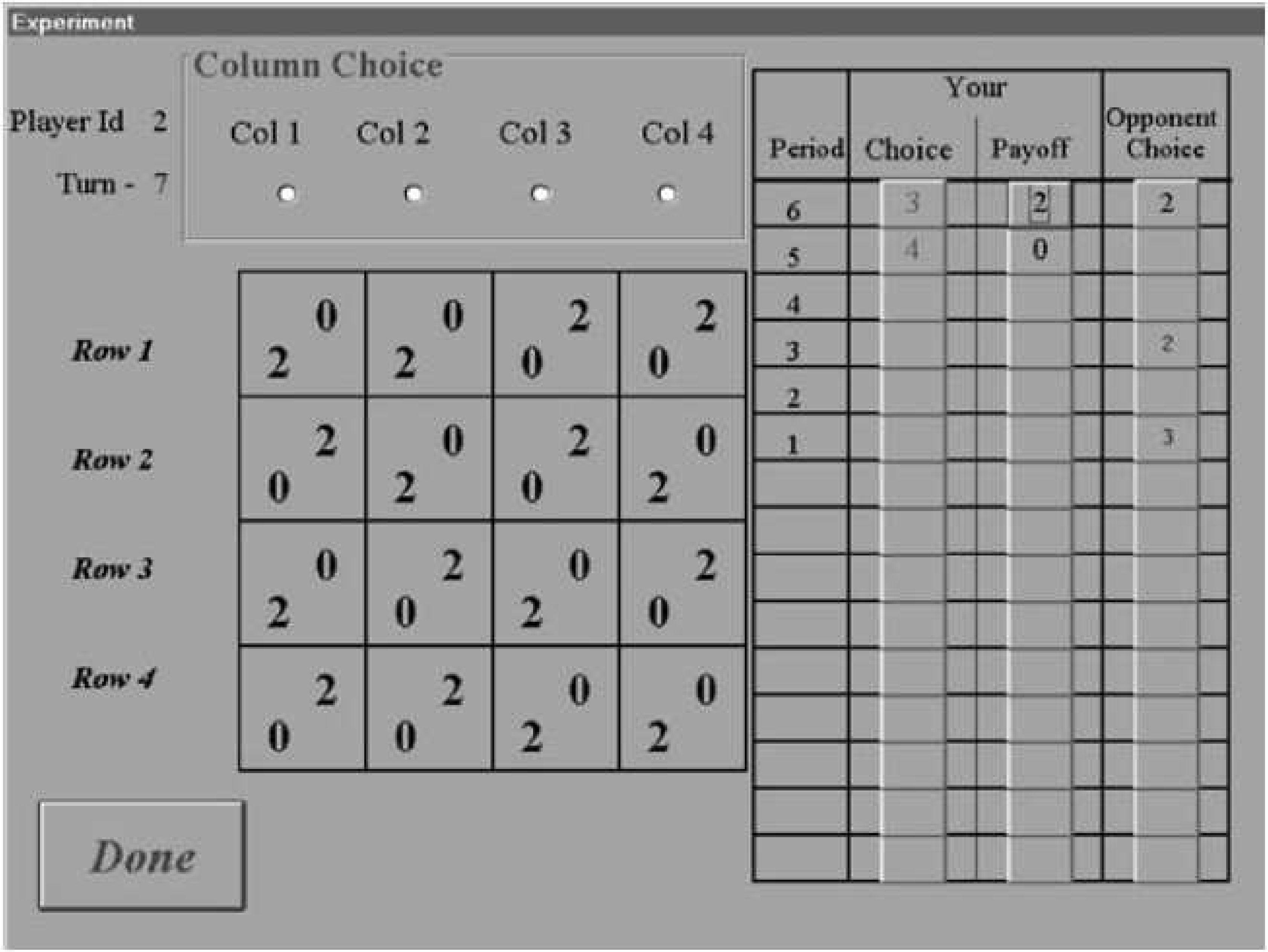
T. Salmon (2004). "Evidence for Learning to Learn Behavior in Normal Form Games," in: *Theory and Decision*, Vol. **56**, No. 4, pp. 367-404.

Tim Salmon's experiment (user interface)



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Round 41: By now I had realized that it was only playing 1 and 4 because they're was no column I could pick that would cover both of them, so it was just a matter of guessing how long it would stay on one or the other. Even though it kept losing by choosing row 1 I thought it might keep trying figuring I would eventually not want to test my luck any more.

Part II:
Hypothesis testing

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   - The next model is chosen randomly all of the time.

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Power. (Dutch: onderscheidend vermogen). The probability β that a Type II error (false negative) will not be made.

Part III:
Naive hypothesis testing

Naive hypothesis testing: concepts and parameters

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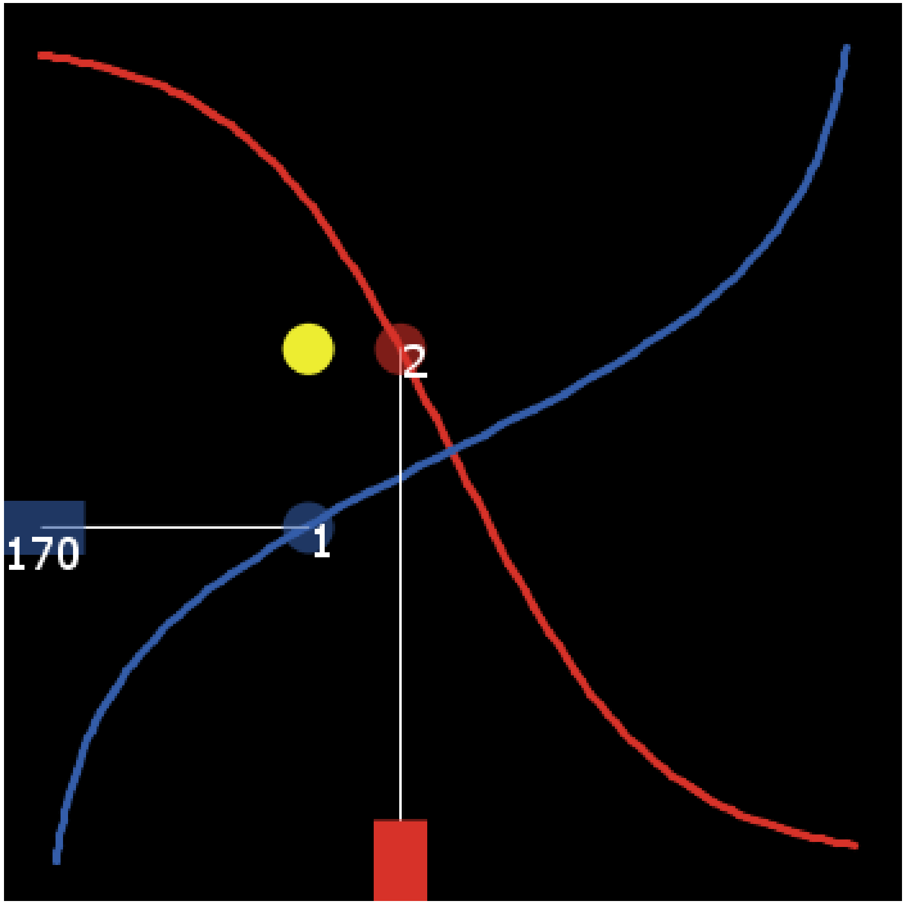
   Else, the hypothesis is rejected, and a new hypothesis is generated. With probability $1 - \rho$ the new hypothesis is taken equal to ϕ^{-it}. Else, it is a random element of Δ_{-i}.

Demo: the matching pennies game



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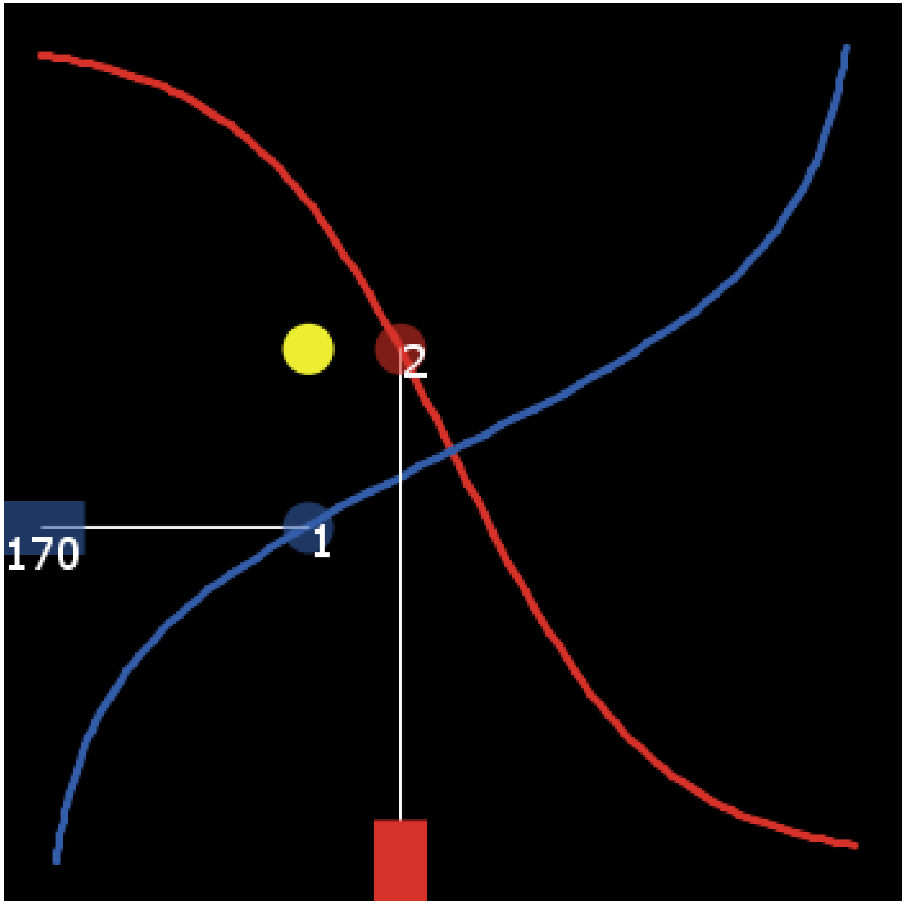

- Player 1: blue, Player 2: red.

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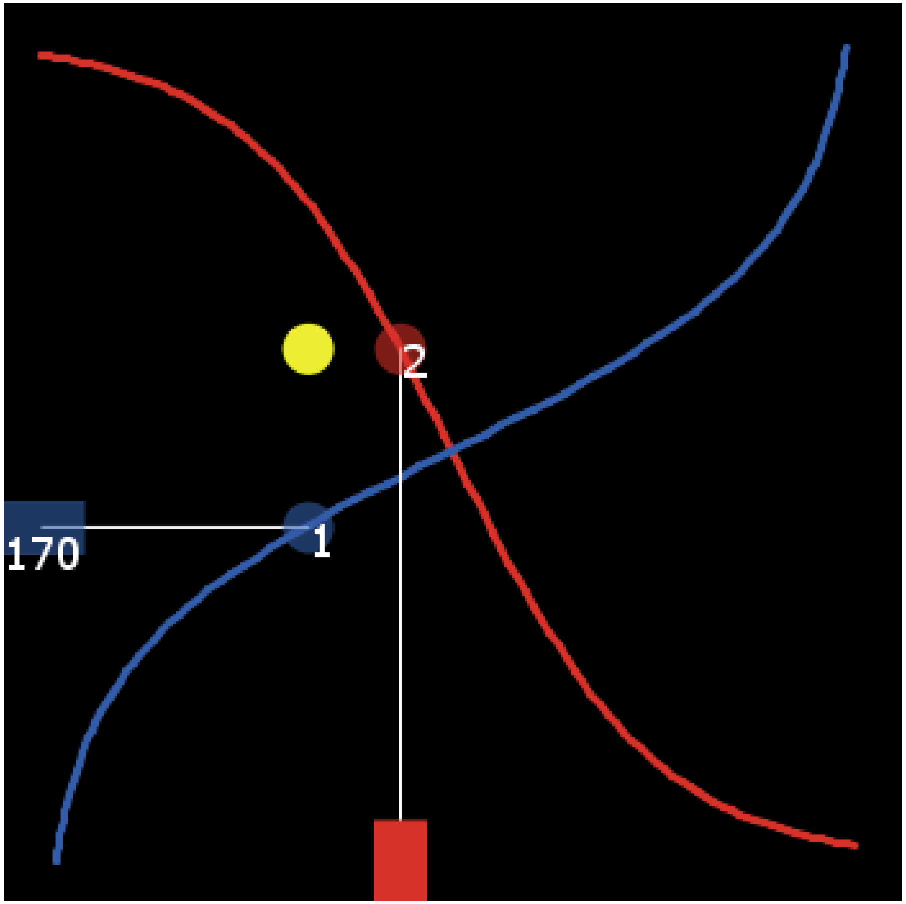

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- — Horizontal axis **rectangle**: Player 2's model of Player 1's

actions = empirical frequency of Player 1.

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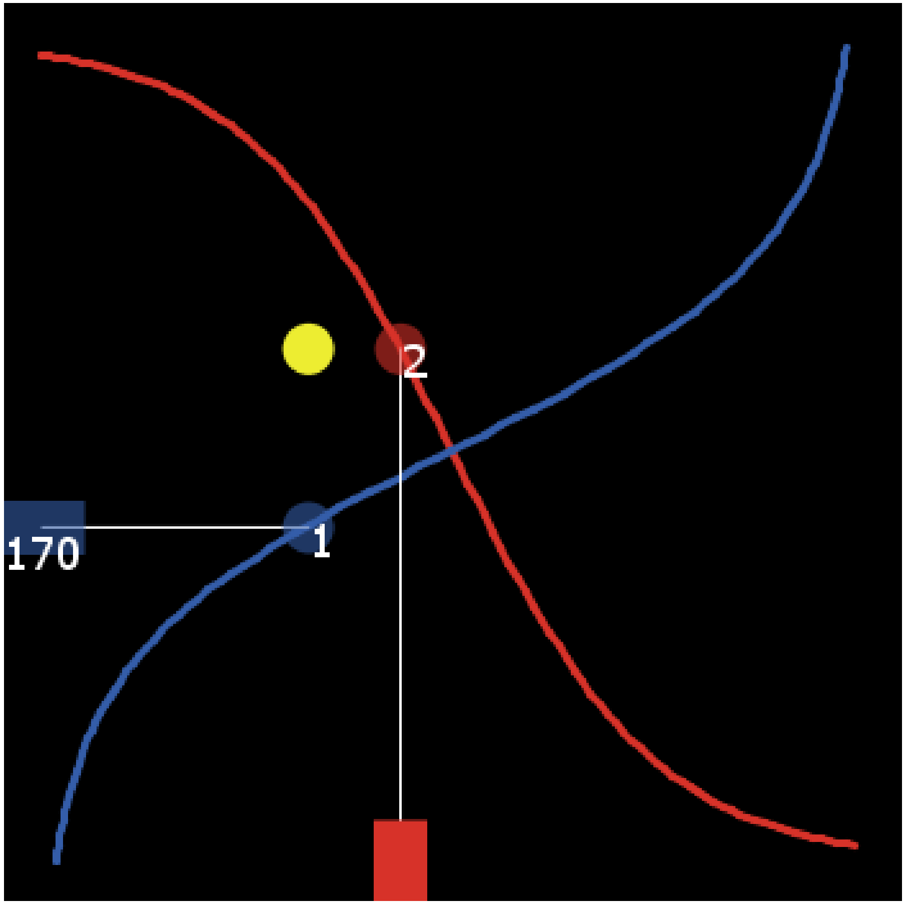

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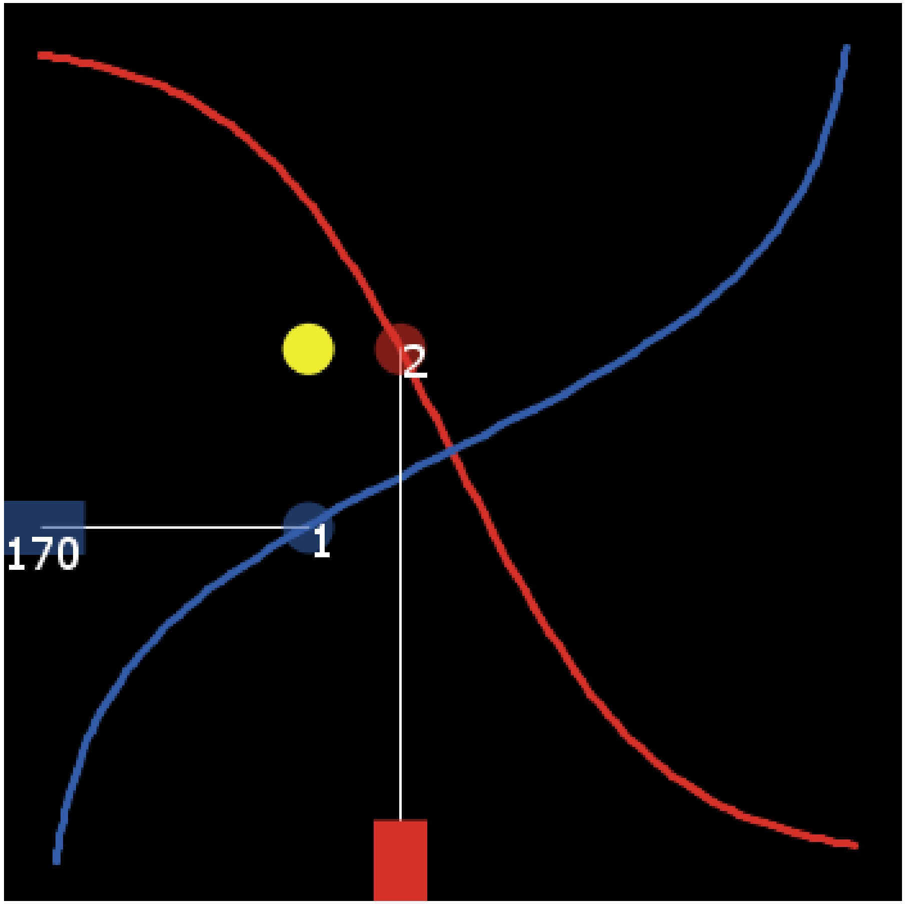

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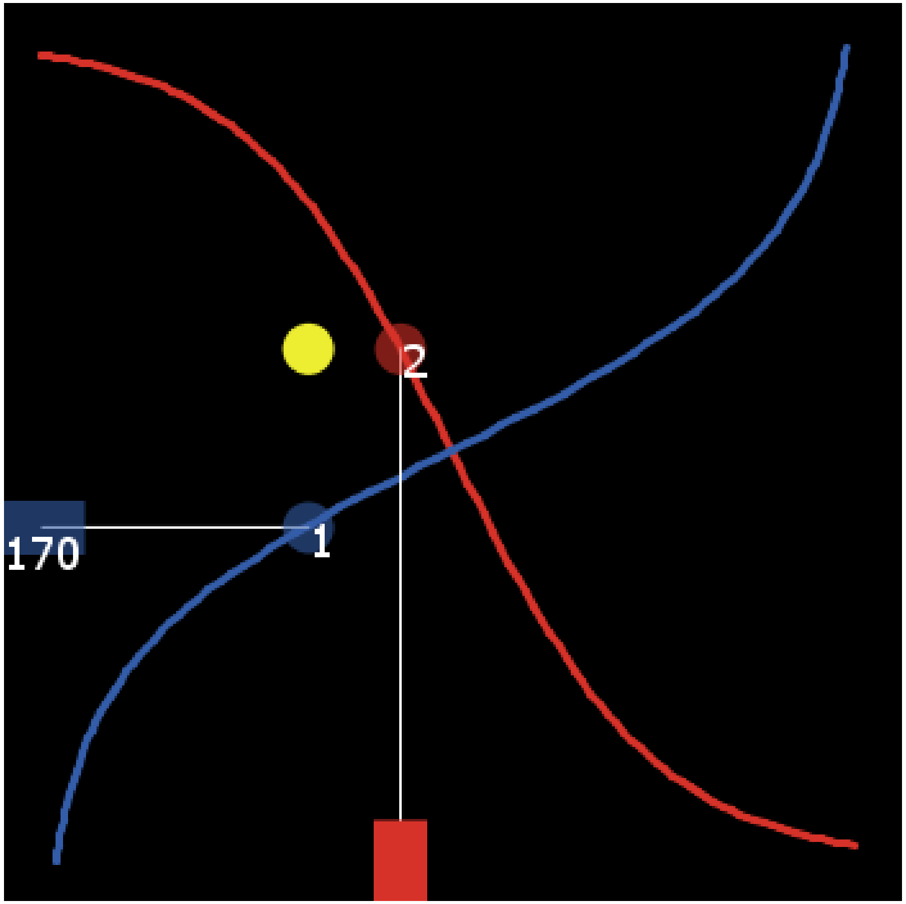

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- — Red **ball**: Player 2's smoothed reply to what it observes.

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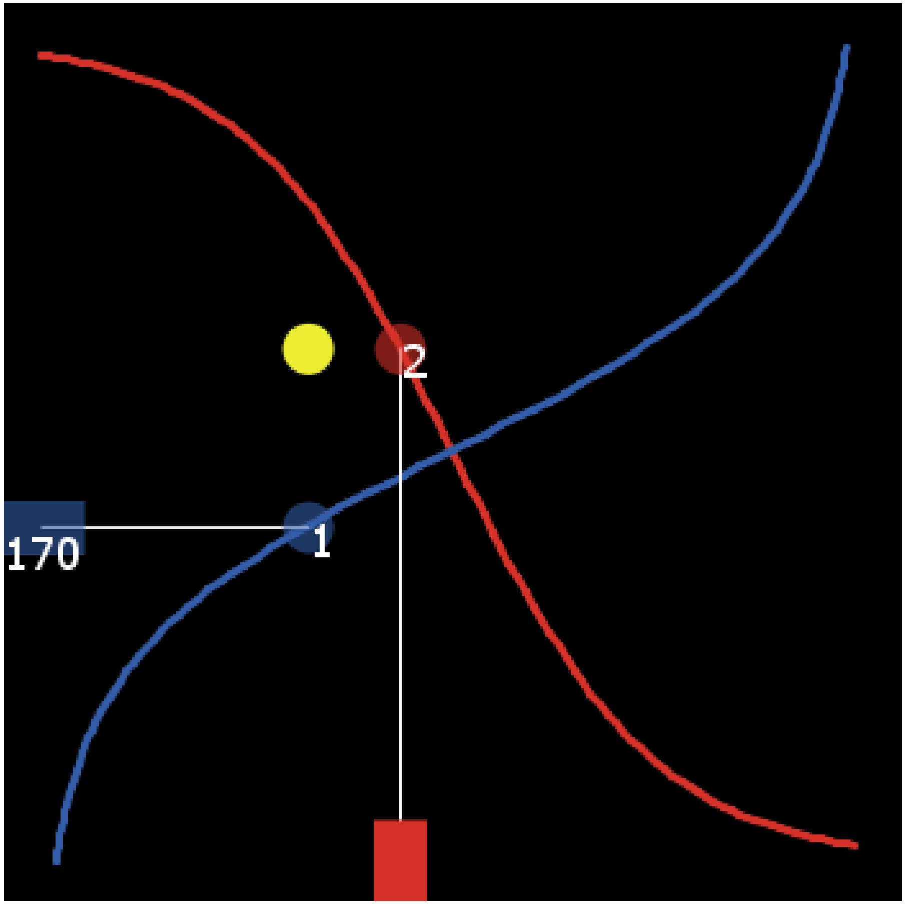

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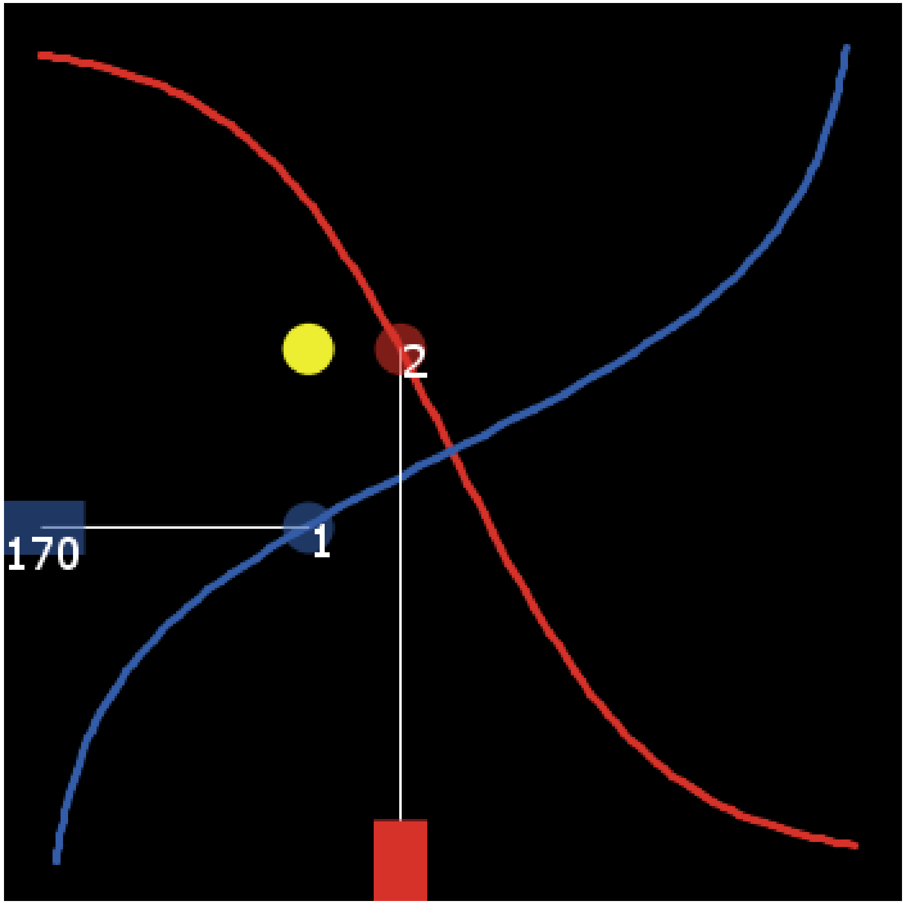

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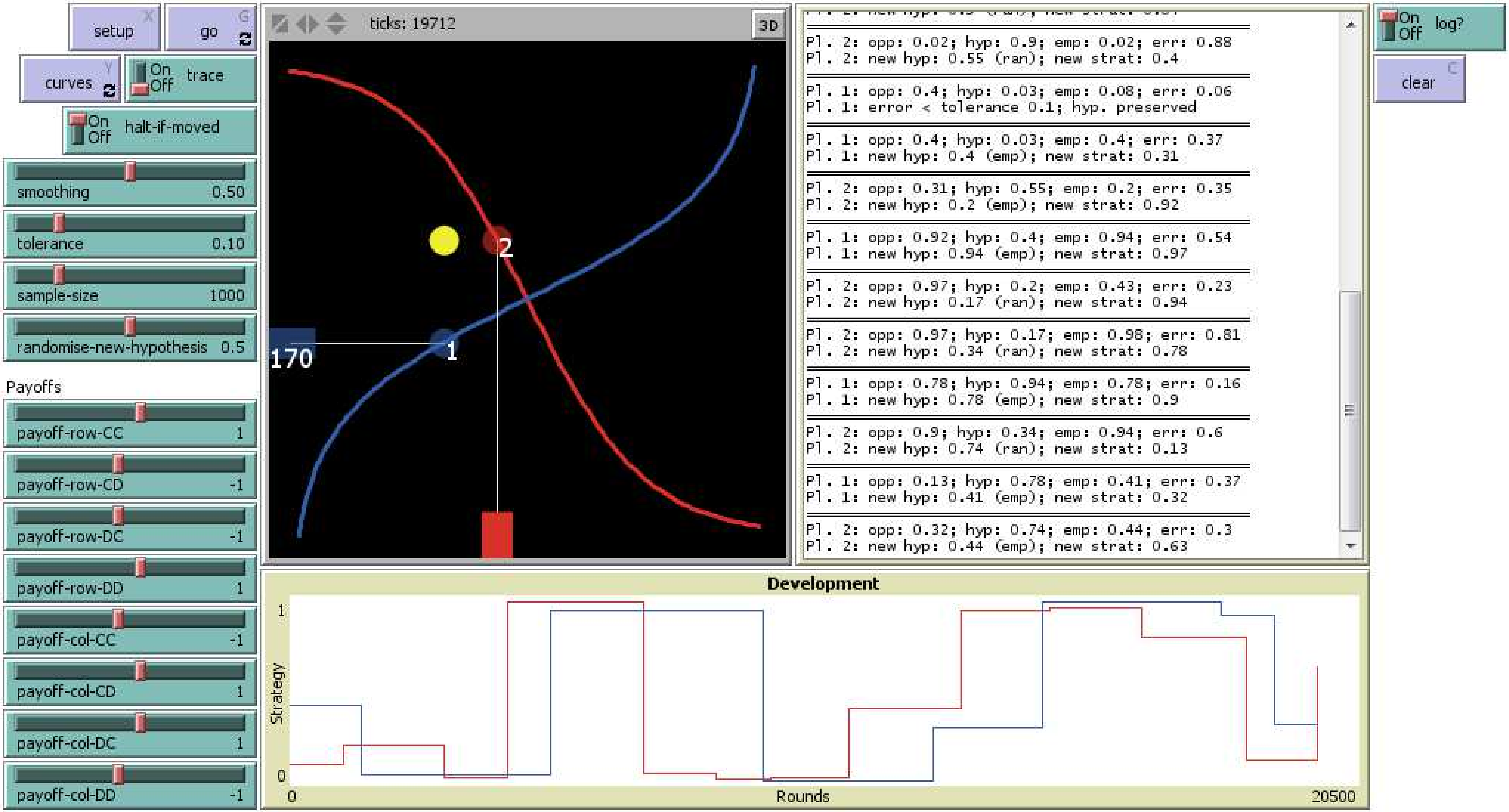
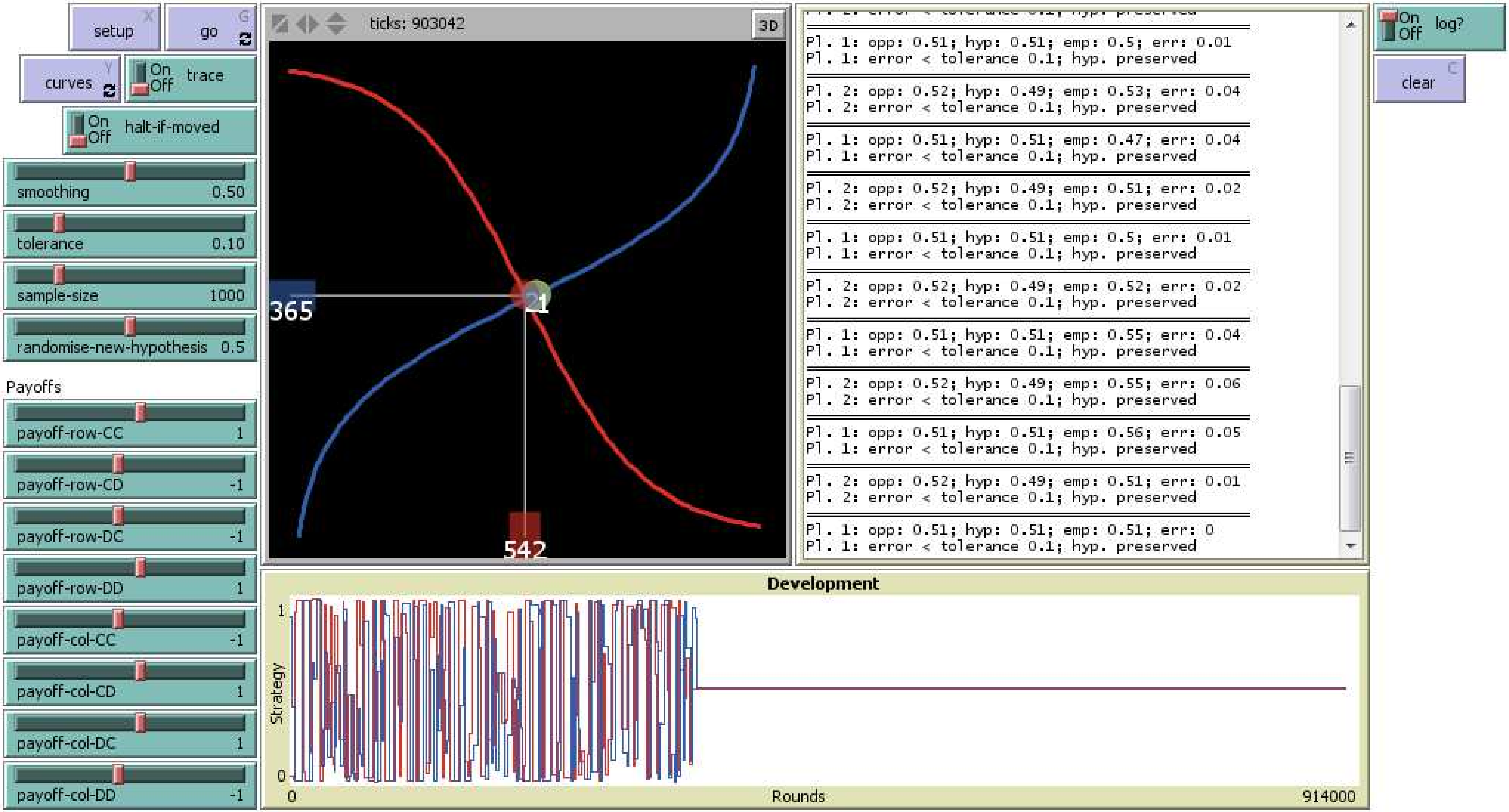
  — Vertical axis **rectangle**: Player 1's model of Player 2's actions = empirical frequency of Player 2. In 170th round of collecting data.
- — Red **ball**: Player 2's smoothed reply to what it observes.
  — Blue **ball**: Player 1's smoothed reply to what it observes.
- Yellow ball: strategy profile.

Demo: the matching pennies game



Demo: the matching pennies game (ϵ-equilibrium)



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- Meanwhile your are still sampling.

  Notice different opponent behaviour in $[1, 1746]$ and $[1746, 1812]$.

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Part IV:
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Part V: Generalisation

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[1] Such families of tests would discriminate very sharply between models that are near the truth (nearer than τ) and those that are far from the truth (further than τ).

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$$(x^{t-m+1}, \dots, x^t) \mapsto \left.\underbrace{\begin{pmatrix} p_{1,1} & \dots & p_{k,1} \\ \vdots & \ddots & \vdots \\ p_{1,n} & \dots & p_{k,n} \end{pmatrix}}_{\#\text{actions}}\right\} \#\text{players} - 1$$

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- There is also the actual distribution of play, formed by $a^t = (a_1^t, \dots, a_n^t)$.
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- This rate of distinction is related exponentially to s and τ.

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  Such a model revision process is said to be flexible.

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Informally: more or less any reasonable hypothesis testing strategy 'works' in the sense that learned behaviours are eventually close to Nash equilibrium most of the time.

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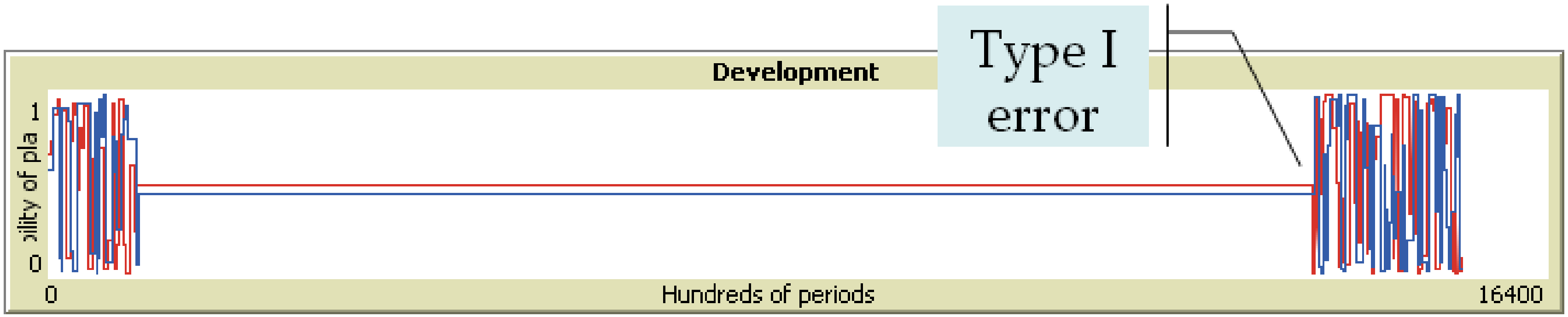

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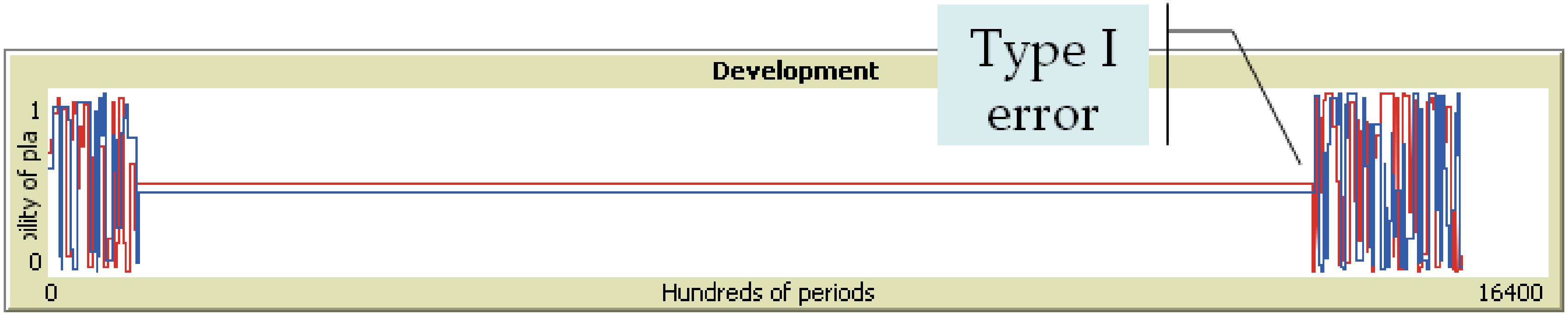

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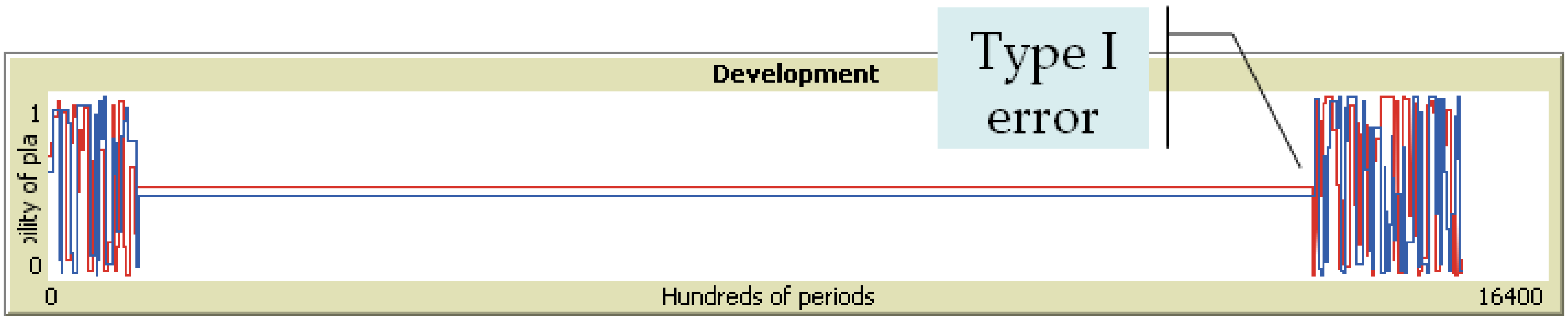

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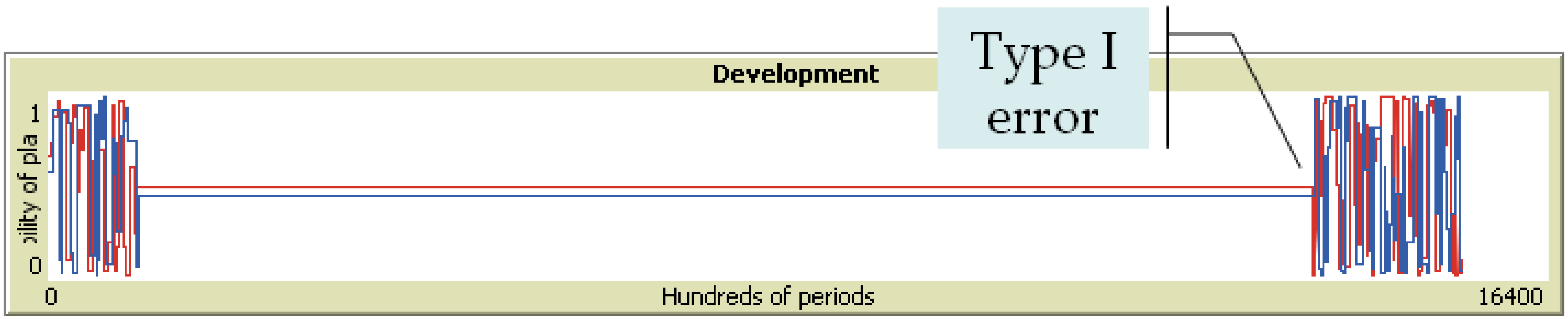

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H. Peyton Young, Dean P. Foster (2003). "Learning, Hypothesis Testing, and Nash Equilibrium," in *Games and Economic Behavior*, Vol. **45**, pp. 73-96.

Part VI:
Observations

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- **Annealing.** Tighten learning parameters slowly $\Rightarrow$ convergence in probability to the set of Nash equilibria.

Part VII:
Extensions

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Theorem (Foster and Young, 2003). Let G be a finite n-person game. If the hypothesis testing parameters are tightened sufficiently slowly, the players' responses converge set-wise[a] in probability[b] to the set of Nash equilibria $\mathcal{N}$ of the repeated game.

[a] Not necessarily pointwise.

[b] For all $\epsilon > 0$, $\lim_{n\to\infty} \Pr\{|X_n - X| > \epsilon\} = 0$.

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