Multi-agent learning

The replicator dynamic

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  - Derivation of the CRE from the DRE.
  - Properties of the replicator dynamic, connection with Nash equilibria.

Symmetric games in normal form

Ways to denote the payoff matrix of a symmetric game

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Hawk vs. Dove

Symmetric normal-form games

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Definition. A game is symmetric when players have equal actions and payoffs:

$$u_i(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) = u_j(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_n).$$

for all i and j.

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So a 2-player game $G=(A,B)$ is symmetric iff $m=n$ and $B=A^T$.

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  Two asymmetric equilibria and one symmetric equilibrium $(1/3, 1/3)$.

Evolutionary game theory

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The replicator equation

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- So $p_i \propto q_i$ and $p_1 + \cdots + p_n = 1$.

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  The fitness vector, f, can now be computed as follows:

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- Species 2 and 3 have fitness 2.4 and 2.3, respectively.

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The continuous replicator equation has an extremely intuitive reading:

$$\dot{p}_i(t) = p_i(t)[\, f_i(t) - \bar{f}(t) \,],$$

where $\dot{p}_i(t)$ is shorthand for the change of p_i in time:

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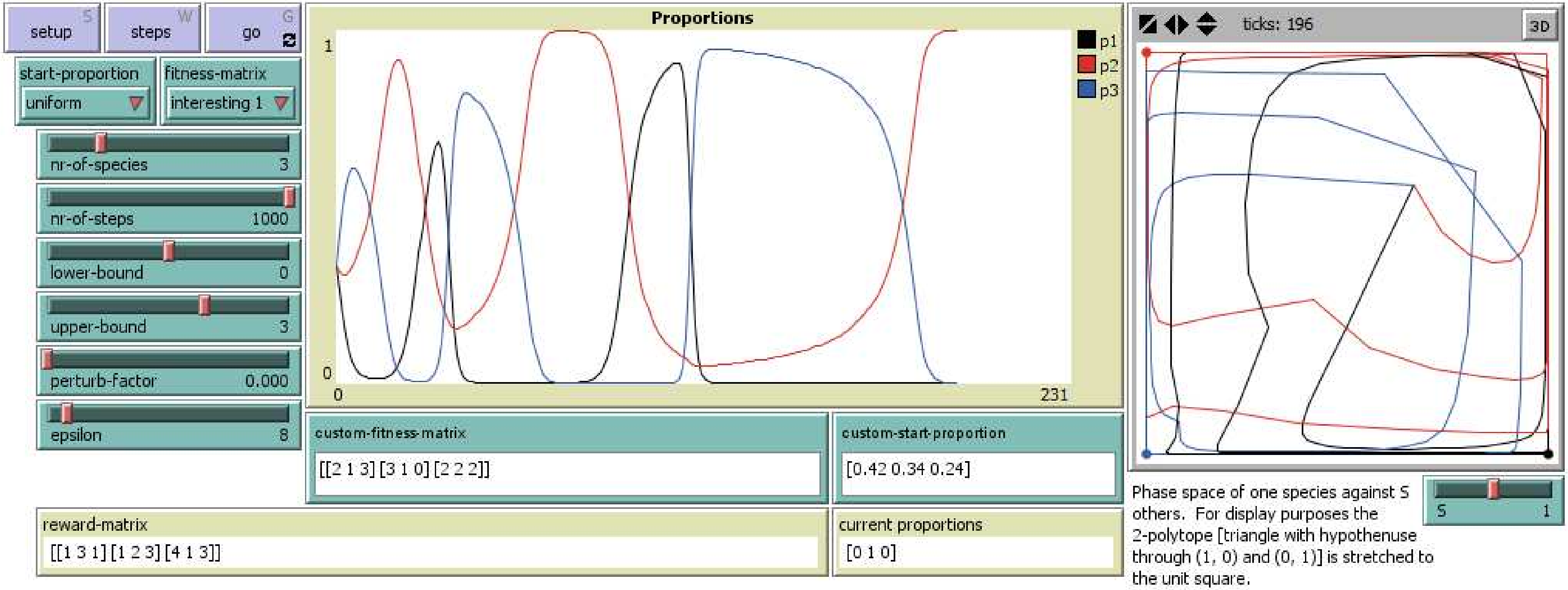
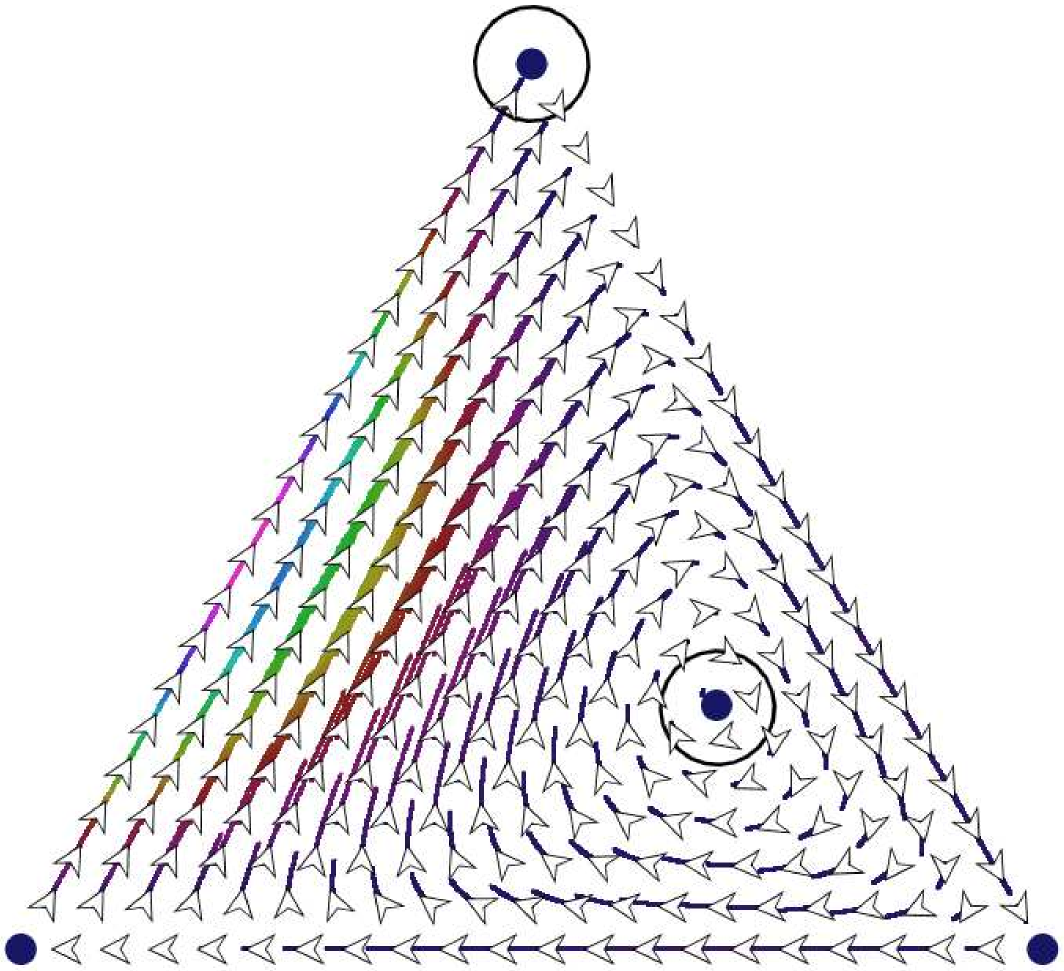
Answer. $\dot{p}_5(t) = p_5(t)[f_5(t) - \bar{f}(t)] = 0.2(6 - 4) = 0.4.$ □

The dynamics of the replicator equation



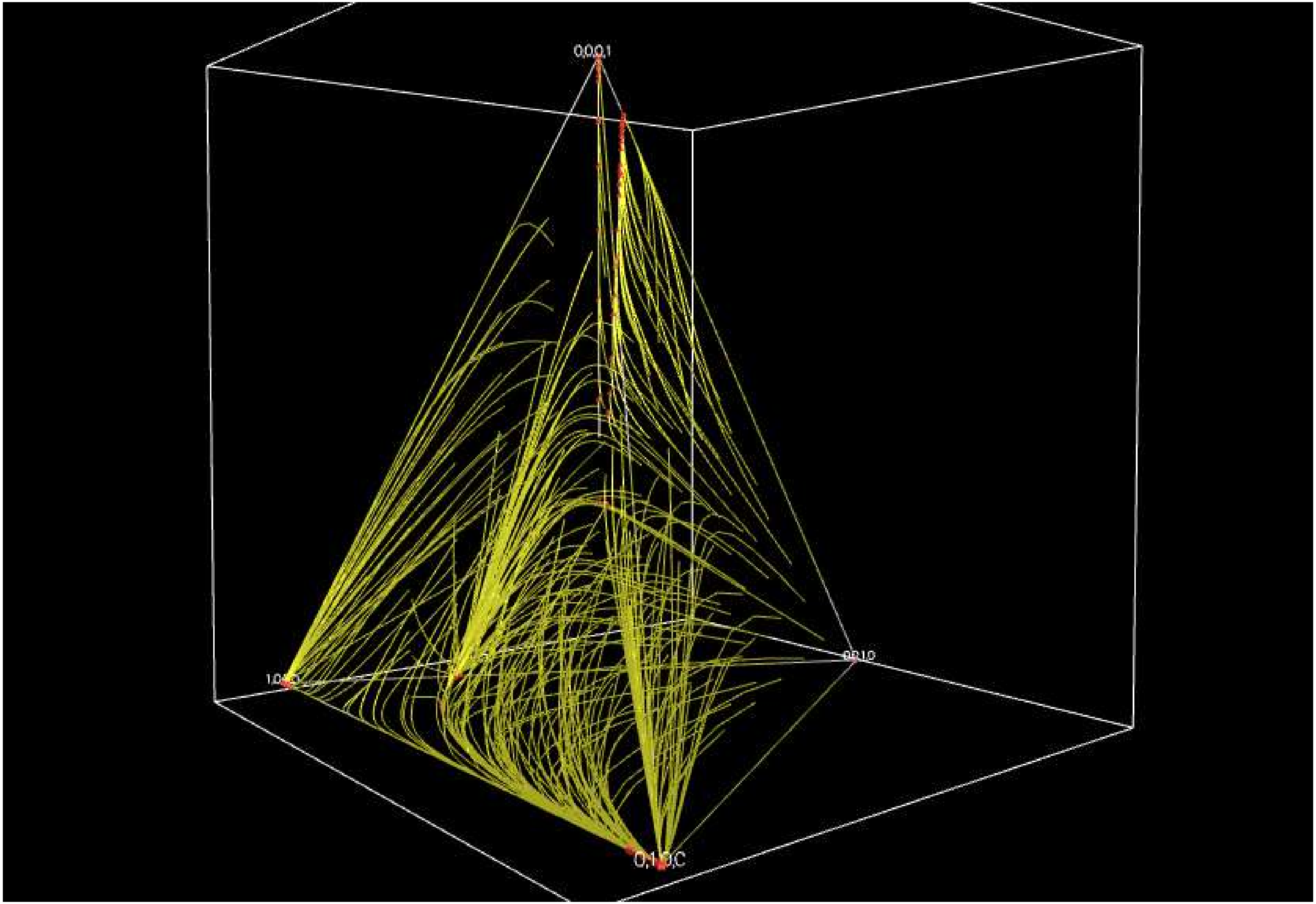
Relative score matrix $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix}$, start proportions $p = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$.

Phase space of the replicator on the previous page



Circled rest points indicate Nash equilibria of the score-matrix, interpreted as the payoff matrix of a symmetric game in normal form.

A replicator dynamic in a higher dimension



Rest point, stable point, asymptotically stable point

The continuous replicator equation:

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is a system of differential equations. We have

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State p is a Nash equilibrium:

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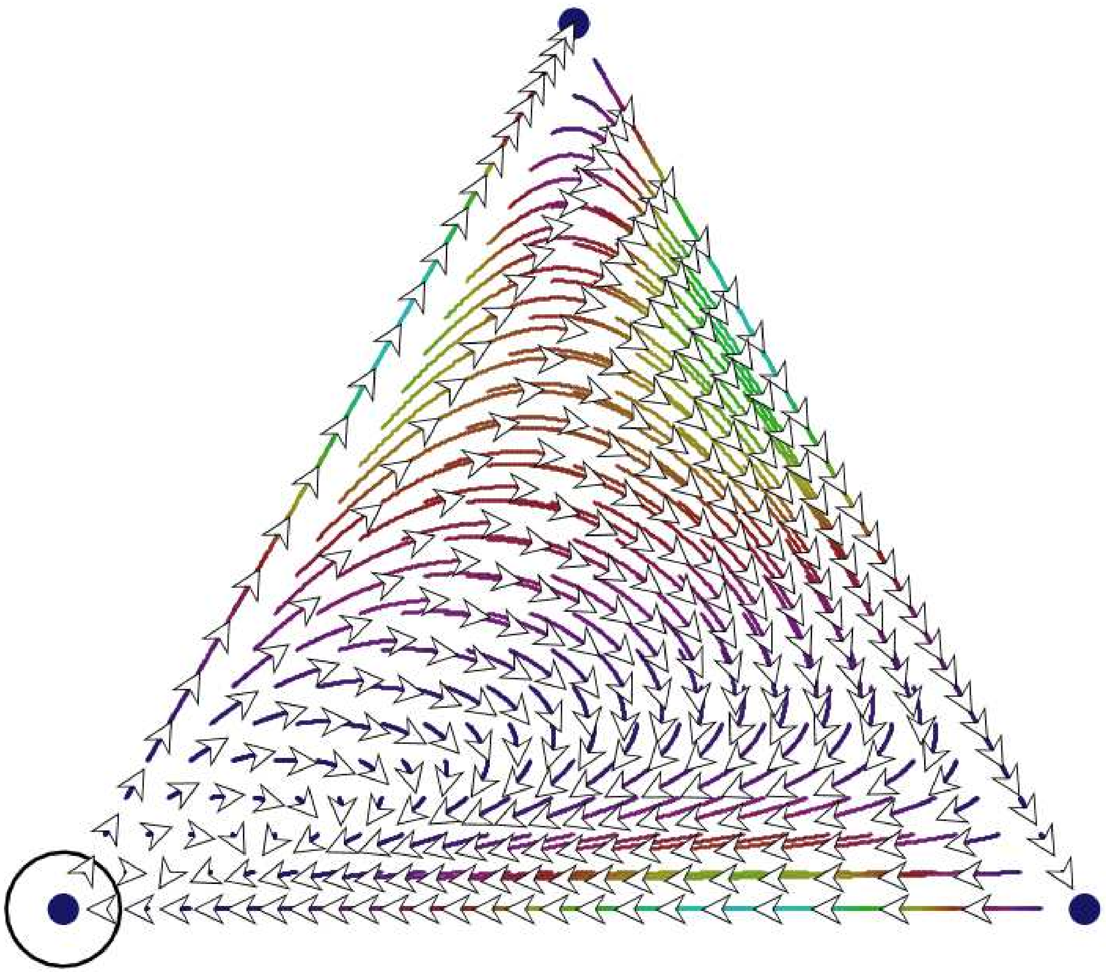
Not all Nash equilibria are Lyapunov stable



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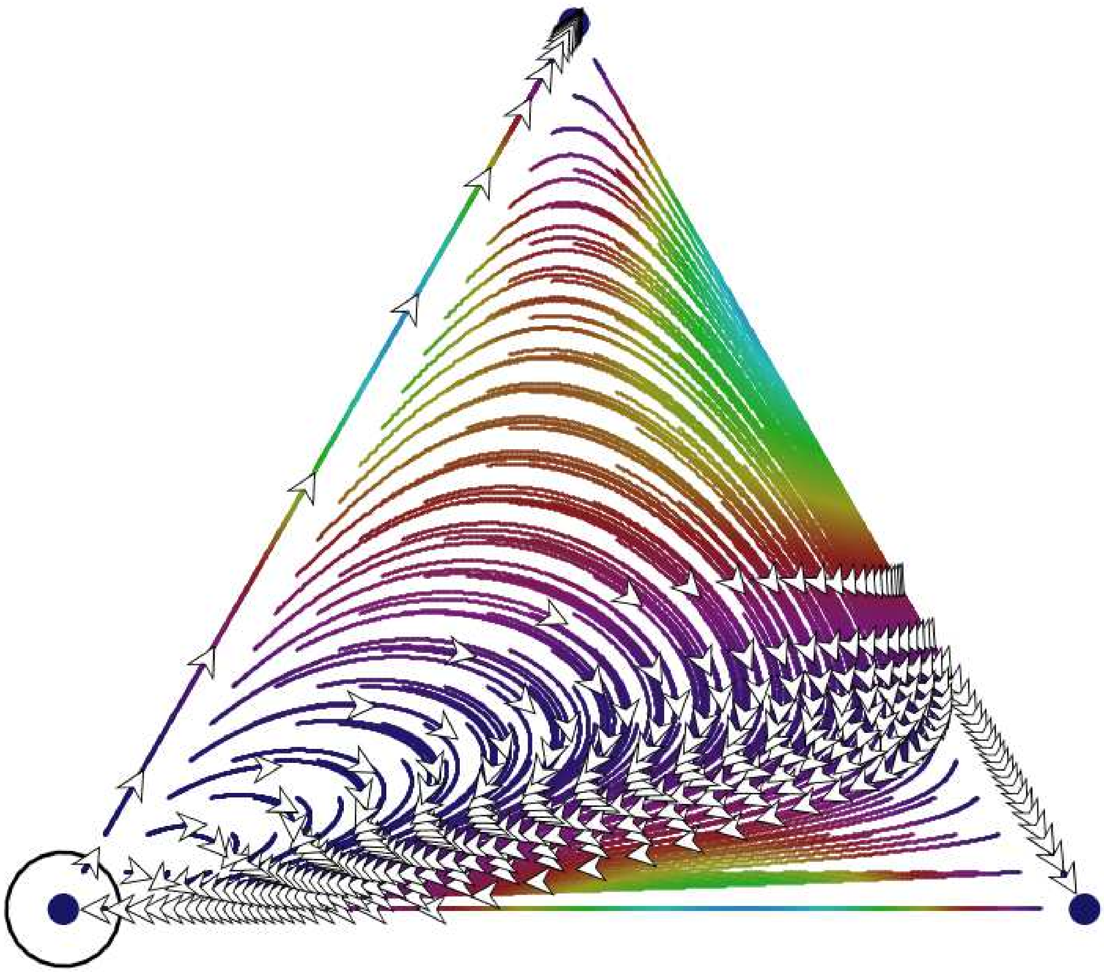

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The discrete replicator equation

The discrete step equation

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$$q_i(t+1) =_{Def} q_i(t)[1+\beta+f_i(t)],$$

where 1 is the reproduction factor, β is the birth and death rate, and $f_i(t)$ indicates the percentage that is added / subtracted due to fitness.

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- The absolute growth of species i is

$$\begin{aligned} \Delta q_i(t) &= q_i(t+1) - q_i(t) \\ &= q_i(t)[1 + \beta + f_i(t)] - q_i(t) = q_i(t)[\beta + f_i(t)]. \end{aligned}$$

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- The DRE follows from the discrete step equation:

$$q_i(t+1) =_{Def} q_i(t)[1+\beta+f_i(t)].$$

Derivation of the DRE from the DSE

$$
\begin{aligned}
p_i(t+1) &= \frac{q_i(t+1)}{\sum_{j=1}^n q_j(t+1)} = \frac{q_i(t)[1+\beta+f_i(t)]}{\sum_{j=1}^n q_j(t)[1+\beta+f_j(t)]} \\
&= \frac{\frac{1}{q(t)} q_i(t)[1+\beta+f_i(t)]}{\frac{1}{q(t)} \sum_{j=1}^n q_j(t)[1+\beta+f_j(t)]} \\
&= \frac{p_i(t)[1+\beta+f_i(t)]}{\sum_{j=1}^n p_j(t)[1+\beta+f_j(t)]} \\
&= \frac{p_i(t)[1+\beta+f_i(t)]}{\sum_{j=1}^n p_j(t) + \beta \sum_{j=1}^n p_j(t) + \sum_{j=1}^n p_j(t) f_j(t)]} \\
&= \frac{p_i(t)[1+\beta+f_i(t)]}{1+\beta+\bar{f}(t)} \\
&= p_i(t) \frac{1+\beta+f_i(t)}{1+\beta+\bar{f}(t)}.
\end{aligned}
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Properties of the DRE

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Claim. If a species is present, is was present and will be present forever. Same for absent.

Proof. Just look at the discrete replicator equation:

$$p_i(t+1) = p_i(t)\frac{1+\beta+f_i(t)}{1+\beta+\bar{f}(t)}$$

and recall that $1+\beta+f_i(t) > 0$ for all t and i, hence $1+\beta+\bar{f}(t) > 0$ for all t. So all the p_i are always multiplied by a positive number.

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and recall that $1+\beta+f_i(t) > 0$ for all t and i, hence $1+\beta+\bar{f}(t) > 0$ for all t. So all the p_i are always multiplied by a positive number.

■ If p_i was 0 it remains 0.

Properties of the DRE

Claim. If a species is present, is was present and will be present forever. Same for absent.

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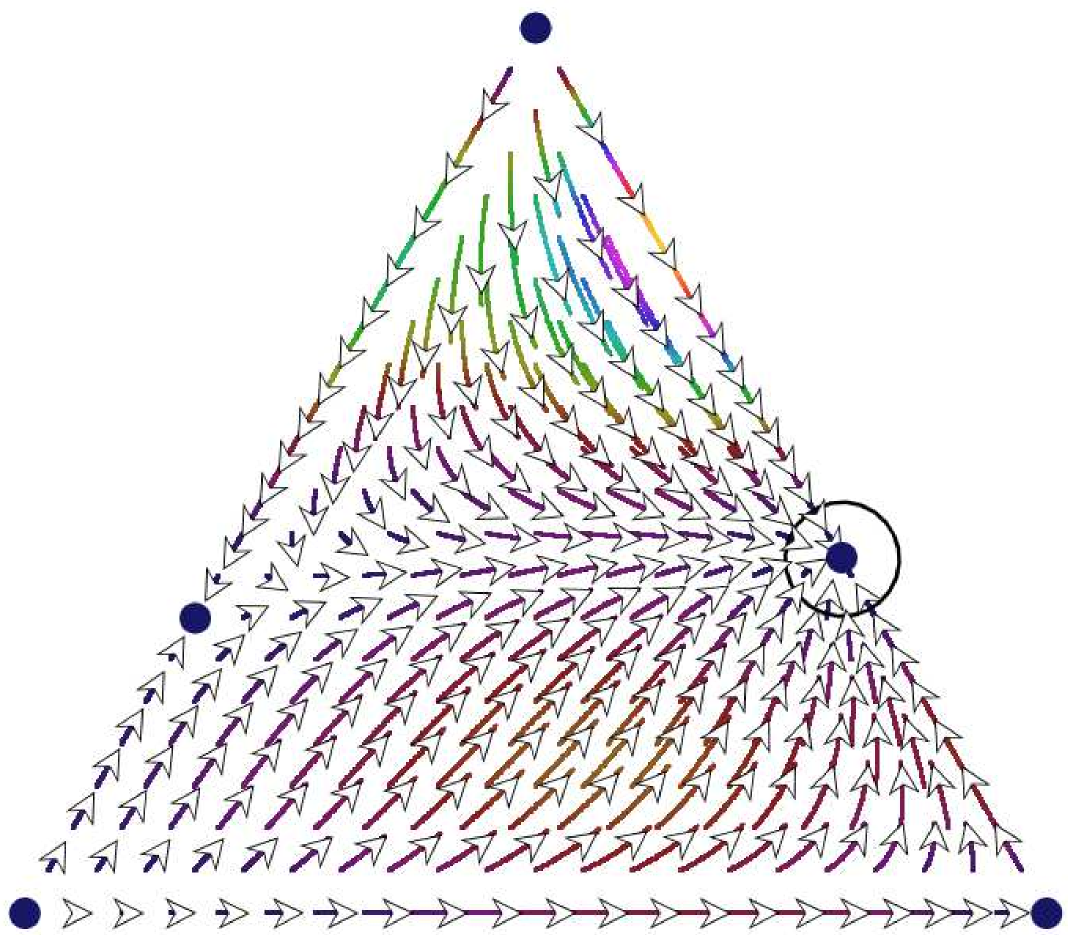
- If p_i was 0 it remains 0.
- If p_i was positive it remains positive.

If a species is absent, it will be absent forever



Phase space of a replicator. Notice that corners, edges, and the interior map into themselves. This is always the case.

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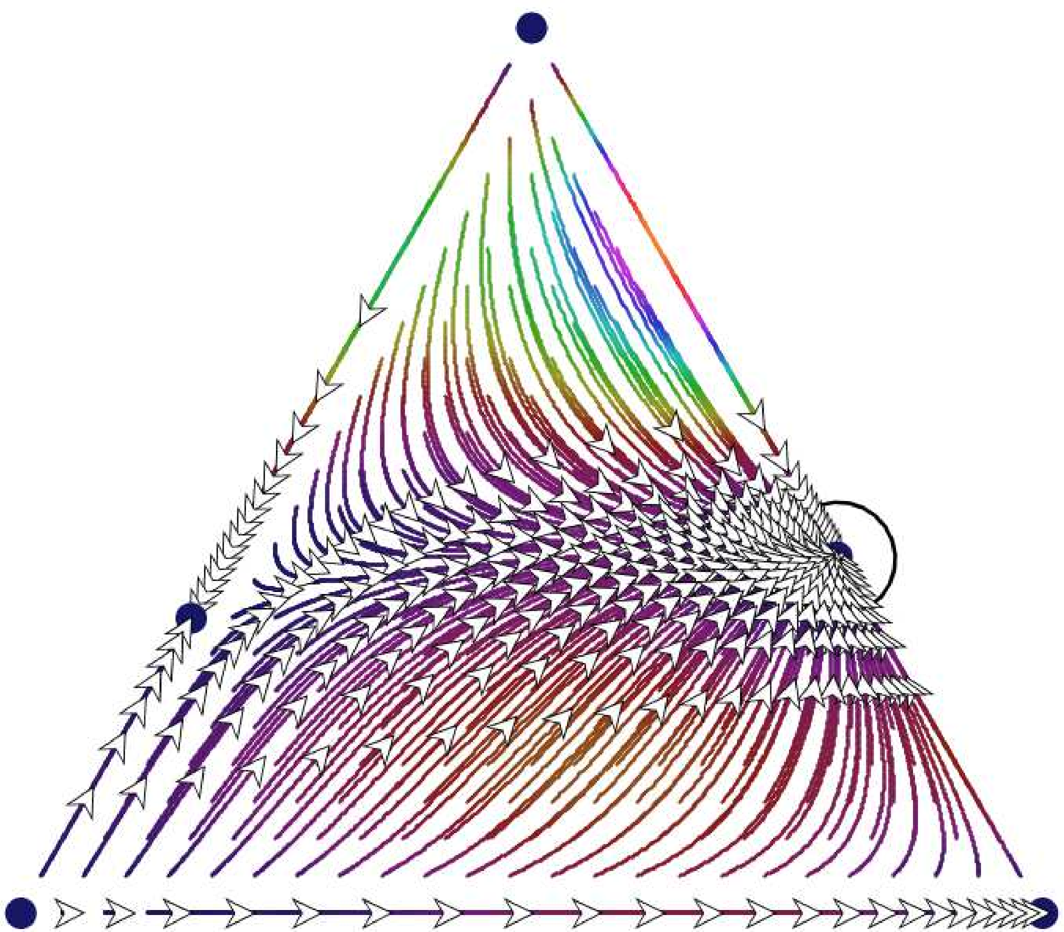

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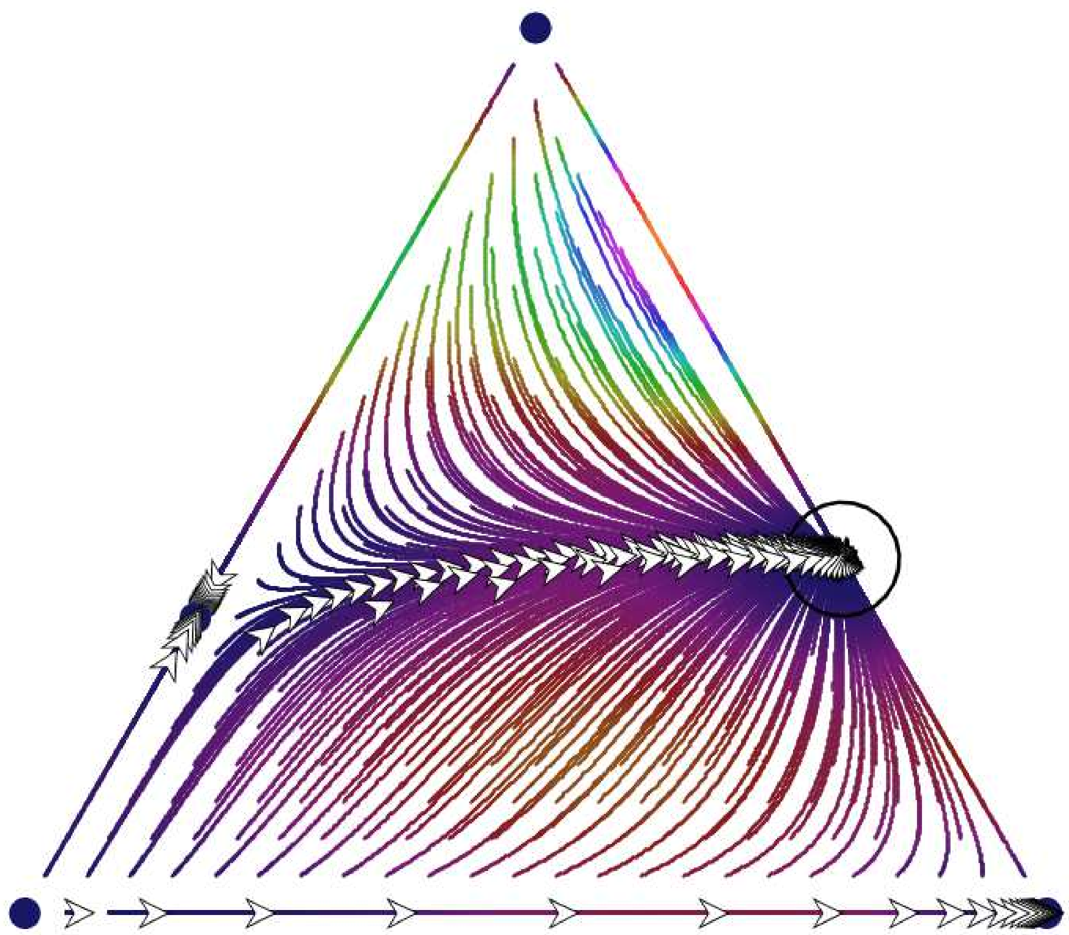

Phase space of a replicator. Notice that corners, edges, and the interior map into themselves. This is always the case.

Properties of the DRE

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Claim. Species i grows if and only if $p_i(t) > 0$ and it has above average fitness.

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\begin{aligned}
& p_i(t+1) > p_i(t) \\
\Leftrightarrow \quad & p_i(t)\frac{1+\beta+f_i(t)}{1+\beta+\bar{f}(t)} > p_i(t) \\
\Leftrightarrow \quad & 1+\beta+f_i(t) > 1+\beta+\bar{f}(t), \qquad p_i(t) > 0 \\
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Question. What if β is large?

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Question. What if β is large? **Answer**. If β is large then the differences in growth among species is smaller, and the dynamics is slower ("bluer").

The continuous replicator equation

Derivation of the CRE from the DRE

Discrete step equation:

$$q_i(t+1) = q_i(t)[1 + \beta + f_i(t)]$$

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- The idea is the following:

  Per small step $t = \delta$ the largest part $1 - \delta$ of species i remains unchanged, while a smaller part δ of species i does change:

$$q_i(t+\delta) = (1-\delta)\ \underbrace{q_i(t)}_{\text{remains}}\ + \delta \underbrace{q_i(t)(1 + \beta + f_i(t))}_{\text{changes}}.$$

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- What if $\delta = 0$? What if $\delta = 1$?

Derivation of the CRE from the DRE

We have:

$$\frac{q_i(t+\delta) - q_i(t)}{\delta} = \frac{(1-\delta)q_i(t) + \delta q_i(t)(1+\beta+f_i(t)) - q_i(t)}{\delta}$$
$$= \dots$$
$$= q_i(\beta + f_i(t)).$$

So:

$$\dot{q}_i = \frac{dq_i(t)}{dt}$$
$$= \lim_{\delta \to 0} \frac{q_i(t+\delta) - q_i(t)}{\delta}$$
$$= \lim_{\delta \to 0} q_i(\beta + f_i(t))$$
$$= q_i(\beta + f_i(t)).$$

Derivation of the CRE from the DRE

$$
\begin{aligned}
p_i(t+\delta) &= \frac{q_i(t+\delta)}{\sum_{j=1}^n q_j(t+\delta)} \\
&= \frac{(1-\delta)q_i(t) + \delta q_i(t)(1+\beta+f_i(t))}{\sum_{j=1}^n \left[(1-\delta)q_j(t) + \delta q_j(t)(1+\beta+f_j(t))\right]} && (/q(t)) \\
&= \frac{(1-\delta)p_i(t) + \delta p_i(t)(1+\beta+f_i(t))}{\sum_{j=1}^n \left[(1-\delta)p_j(t) + \delta p_j(t)(1+\beta+f_j(t))\right]} && \text{(yields proportions)} \\
&= \frac{p_i(t)[1+\delta(\beta+f_i(t))]}{\sum_{j=1}^n p_j(t)[1+\delta(\beta+f_j(t))]} && (\textstyle\sum p_i(t) = 1) \\
&= p_i(t)\frac{1+\delta\left((\beta+f_i(t)\right)}{1+\delta\left(\beta+\bar{f}(t)\right)}.
\end{aligned}
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\end{aligned}
$$

- What if $\delta = 0$? What if $\delta = 1$?

Derivation of the CRE from the DRE

Now $$\frac{p_i(t+\delta)-p_i(t)}{\delta} = \frac{p_i(t)\frac{1+\delta((\beta+f_i(t))}{1+\delta\left(\beta+\bar{f}(t)\right)} - p_i(t)}{\delta}$$

$$= p_i(t)\frac{\frac{1+\delta((\beta+f_i(t))}{1+\delta\left(\beta+\bar{f}(t)\right)} - 1}{\delta} \quad \text{multiply w. } 1+\delta\left(\beta+\bar{f}(t)\right)$$

$$= p_i(t)\frac{1+\delta\left((\beta+f_i(t)) - \left(1+\delta\left(\beta+\bar{f}(t)\right)\right)\right)}{\delta\left(1+\delta\left(\beta+\bar{f}(t)\right)\right)}$$

$$= p_i(t)\frac{f_i(t)-\bar{f}(t)}{1+\delta(\beta+\bar{f}(t))}.$$

So

$$\dot{p}_i = \lim_{\delta\to 0}\frac{p_i(t+\delta)-p_i(t)}{\delta}$$

$$= \lim_{\delta\to 0} p_i(t)\frac{f_i(t)-\bar{f}(t)}{1+\delta(\beta+\bar{f}(t))} = p_i(t)\frac{f_i(t)-\bar{f}(t)}{1+0\cdot C} = p_i(t)[\ f_i(t)-\bar{f}(t)\ ].$$

Calculating stationary points of the replicator

Finding stationary points of the replicator: example

Consider the replicator with

$$A = \begin{pmatrix} 6 & 1 & 6 \\ 4 & 10 & 1 \\ 8 & 5 & 1 \end{pmatrix} \text{ and } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

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Stationary points (fixed points, rest points):

$$\begin{cases} p \in \Delta_2 \\ x((Ap)_x - p(Ap)) = 0 \\ y((Ap)_y - p(Ap)) = 0 \\ z((Ap)_z - p(Ap)) = 0. \end{cases}$$

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Solve with Maple / Mathematica / SciPy / ...

Finding stationary points of the replicator: example

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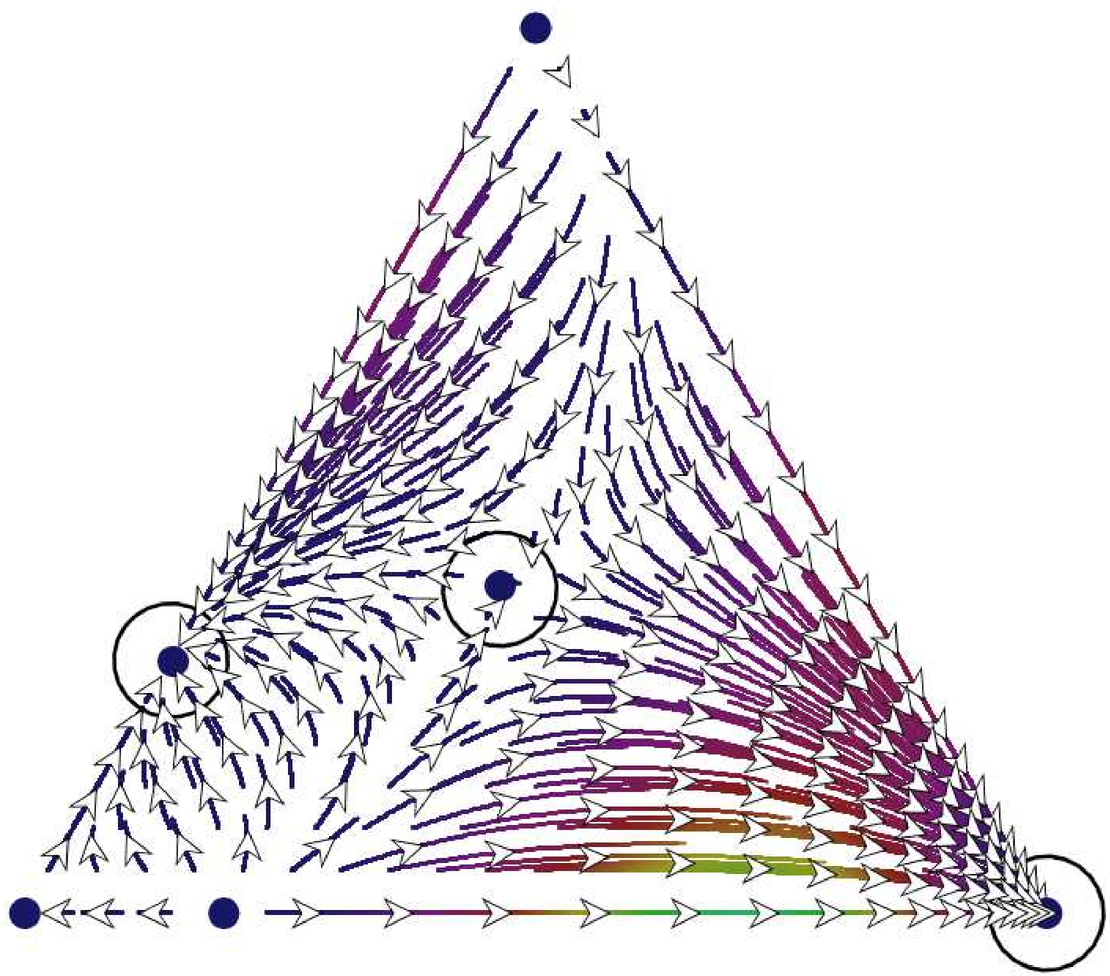
Solve with Maple / Mathematica / SciPy / ... (Nash equilibria are blue):

$$\left\{ (1,0,0), (0,1,0), (0,0,1), \left(\frac{25}{71}, \frac{20}{71}, \frac{26}{71}\right), \left(\frac{5}{7}, 0, \frac{2}{7}\right), \left(\frac{9}{11}, \frac{2}{11}, 0\right) \right\}.$$

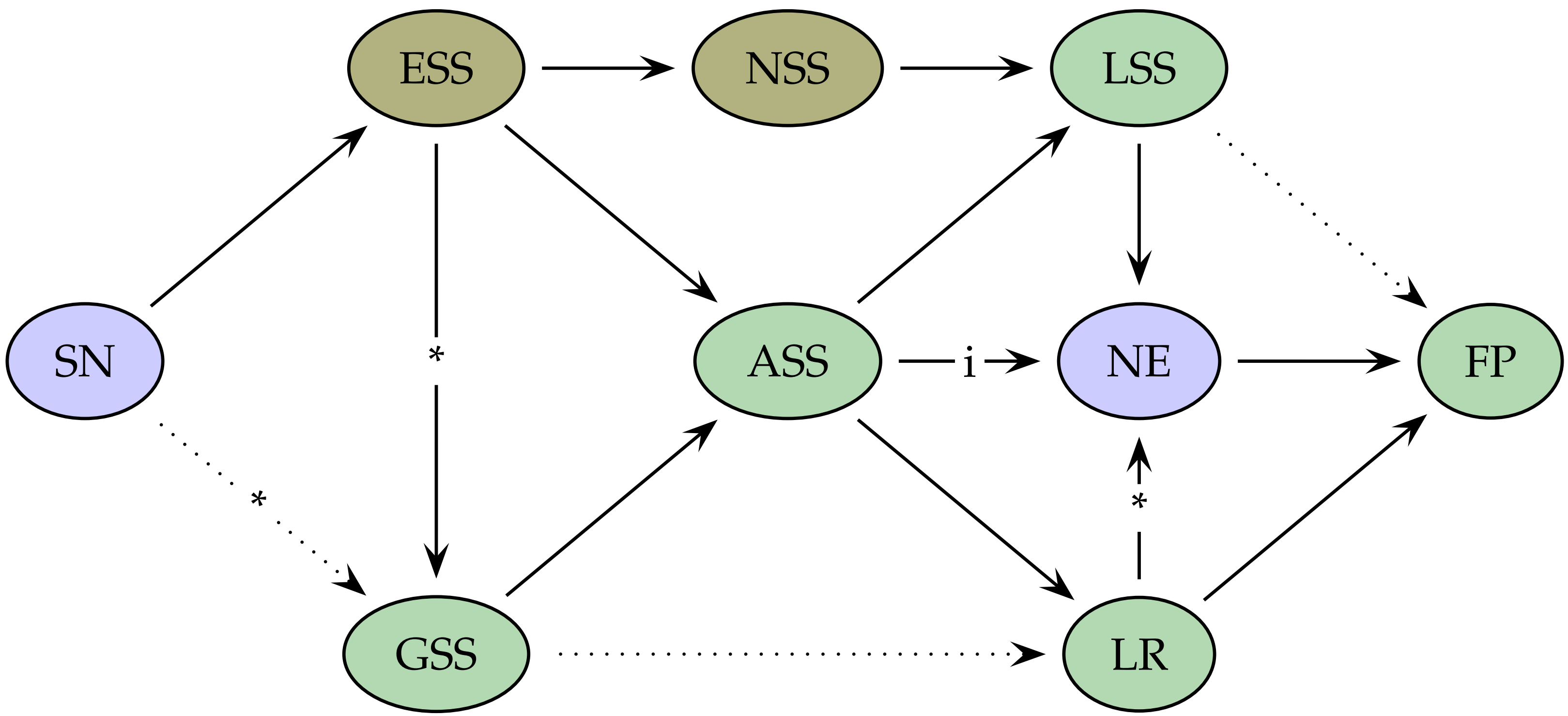
Finding stationary points of the replicator: example



Phase space of the replicator as discussed. Circled rest points indicate Nash equilibria in the corresponding symmetric game.

Summary

Implications



SN = strict Nash, ESS - evol'y stable strategy, GSS = glob'y stable state, ASS = asymp'y stable state, NSS = neutrally stable strategy, LR = limit of replicator, LSS = Lyapunov stable state, FP = fixed point, * = only if fully mixed, i = isolated NE. Dotted: indirect implication.

Blue: game theory; olive: evolutionary game theory; green: the replicator dynamic.

Justifications of the implications

- SN $\Rightarrow$ ESS: cf. slides evolutionary games and, e.g., Th 7.7.12 of Sh&LB.
- ESS $\Rightarrow$ NSS: cf. slides evolutionary games and, e.g., Game Theory Evolving (2nd ed.) by H. Gintis.
- ESS $\Rightarrow$ NE: cf. slides evolutionary games and, e.g., Sh&LB Th 7.7.11.
- ESS $\Rightarrow_*$ GSS: cf., e.g., Th. 12.7 Gintis.
- ESS $\Rightarrow$ ASS: cf., e.g., Th. 7.7.13 Sh&LB, Th. 12.7 Gintis, Sec. 3.5 (begin) of Evol. Game Theory by J.G. Weibull.
- NSS $\Rightarrow$ LSS: cf. Sec. 3.5 Weibull.
- GSS $\Rightarrow$ ASS: by definition of the two concepts.
- ASS $\Rightarrow$ LSS: by definition of the two concepts.
- ASS $\Rightarrow$ LR: by definition of the two concepts.
- ASS $\Rightarrow_i$ NE: Th 7.7.8 Sh&LB, Th. 12.6 Gintis.
- LSS $\Rightarrow$ NE: Th 7.7.6 Sh&LB, 7.2.1(c) Hofbauer & Sigmund.
- LR $\Rightarrow_*$ NE: Th. 7.2.1(b) H&S.
- NE $\Rightarrow$ FP: Th. 7.2.1(a) H&S, Th 7.7.5 Sh&LB, Th. 12.6 Gintis.
- LR $\Rightarrow$ FP: Ch. 6 Weibull.

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