Multi-agent learning

Repeated games

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Repeated games: motivation

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5. Therefore, familiarity with the basic concepts and results from the theory of repeated games is essential to understand MAL.

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* H. Peters (2008): *Game Theory: A Multi-Leveled Approach*. Springer, ISBN: 978-3-540-69290-4. Ch. 8: Repeated games.

Part I:
Nash equilibria

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in normal form games

Part I:
Nash equilibria
in normal form games
that are repeated

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Nash equilibria in playing the PD twice

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- The following diagram (hopefully) shows that playing the PD two times in succession does not yield an essentially new NE.

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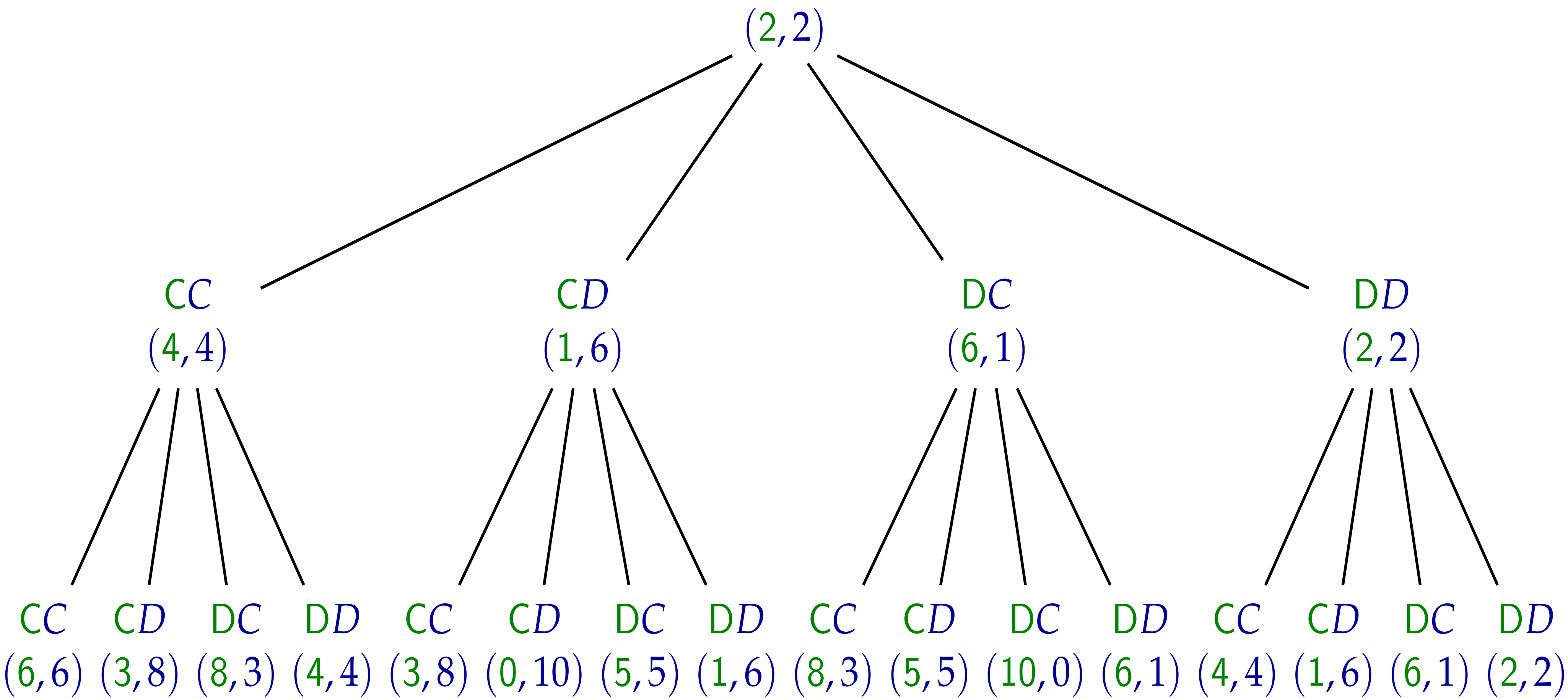

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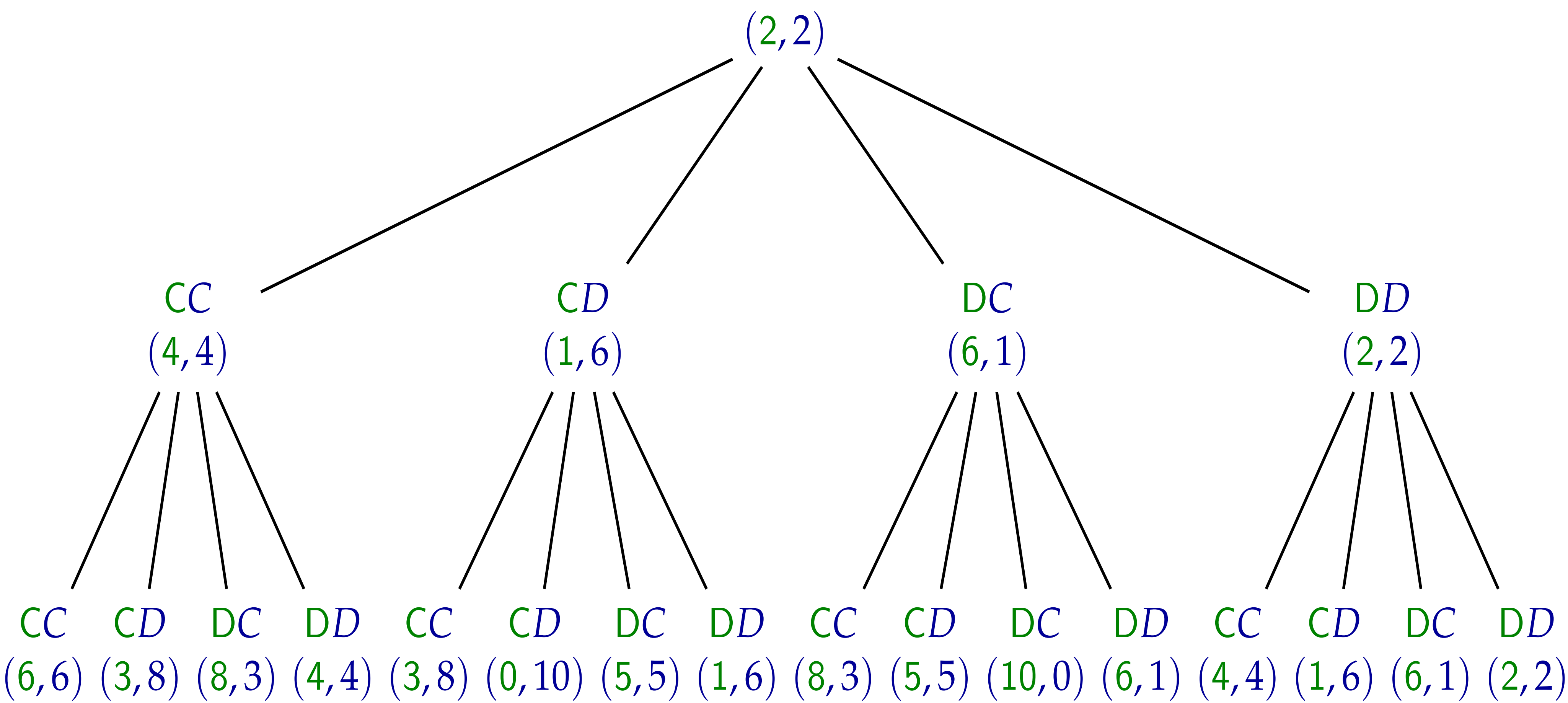

P.S. This is just a payoff tree, not a game in extensive form!

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- Generalise to N repetitions: $(\mathrm{D}D^{N-1}, DD^{N-1})$ still is the only Nash equilibrium in a repeated game where the PD is played N times in succession.

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- Here we discuss one version of "the" Folk Theorem.

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  Example: (for the prisoner's dilemma):

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Example on next page.

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$$\begin{aligned}\text{Expected payoff}_1(s) &= \sum_{t=0}^{\infty} \left[\left(\frac{1}{2}\right)^t [0.8(0.7 \cdot 3 + 0.3 \cdot 0) + 0.2(0.7 \cdot 5 + 0.3 \cdot 1)] \right] \\ &= \frac{1}{1-1/2} [\ldots] \approx \frac{1}{1-1/2} 2.44 = 2 \times 2.44 = 4.88.\end{aligned}$$

Subgame perfect equilibria of $G^*(\delta)$: D^*

Definition. A strategy profile s for $G^*(\delta)$ is a subgame-perfect Nash equilibrium if (1) it is a Nash equilibrium of this repeated game, and (2) for every subgame (i.e., tail game) of this repeated game, s restricted to that subgame is a Nash equilibrium as well.

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Consider the strategy of iterated defection D^*: "always defect, no matter what".[1]

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[1] A notation like D^* or (worse) D^∞ is suggestive. Mathematically it makes no sense, but intuitively it does.

Part III:
Trigger strategies

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Therefore, if $\delta > 1/2$ every player forfeits payoff by deviating from T. $\square$

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An analysis of this situation and a proof of this claim can be found in (Peters, 2008), pp. 104-105.*

*H. Peters (2008): *Game Theory: A Multi-Leveled Approach*. Springer, ISBN: 978-3-540-69290-4.

Generalisation of trigger strategies

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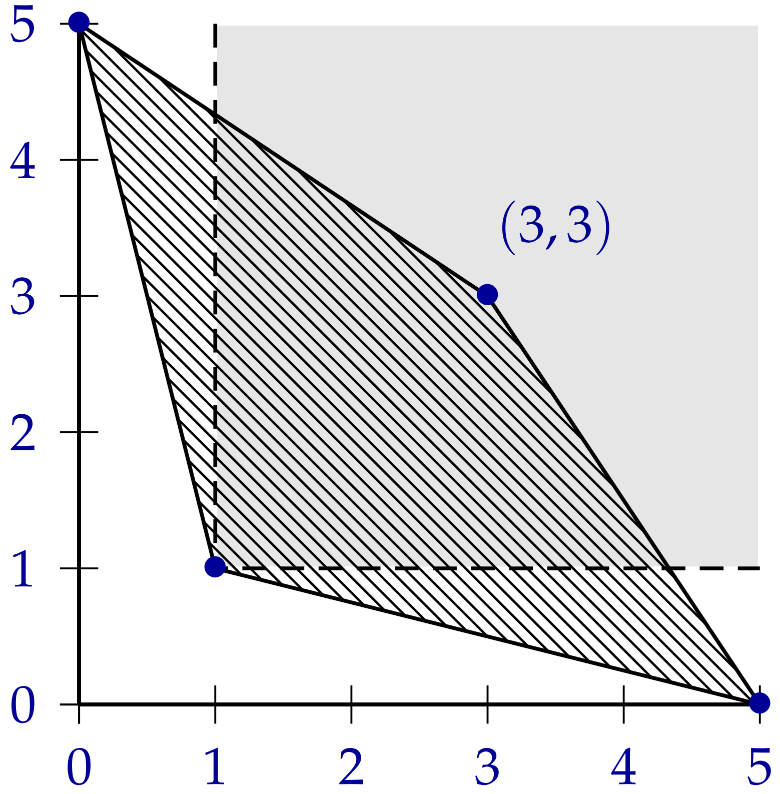
- For δ high enough, these strategies again form a SGP NE.

[2]Meaning $\alpha_i \geq 0$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$.

Folk theorem for SGP NE in a repeated PD



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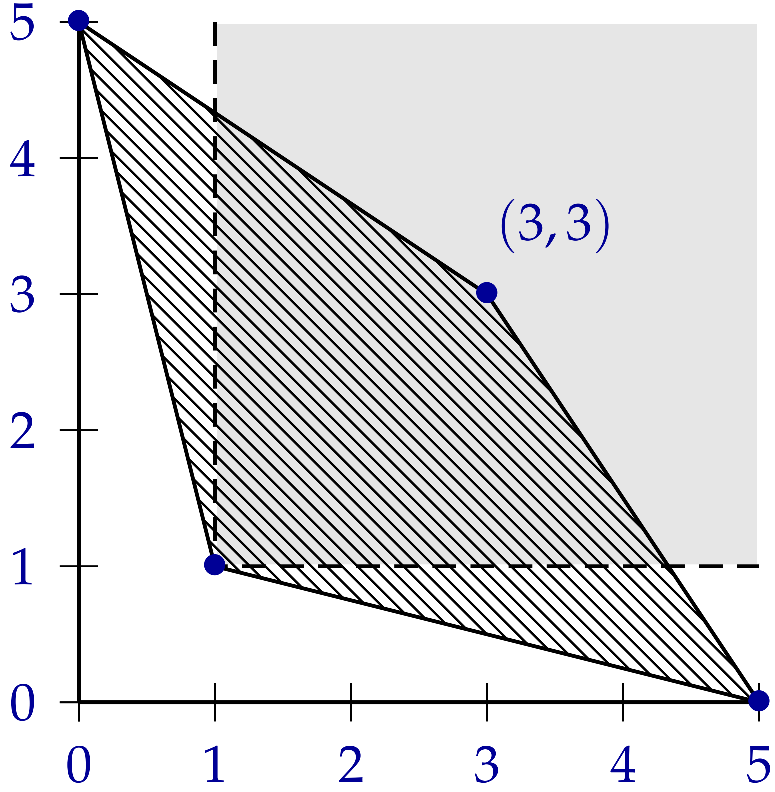

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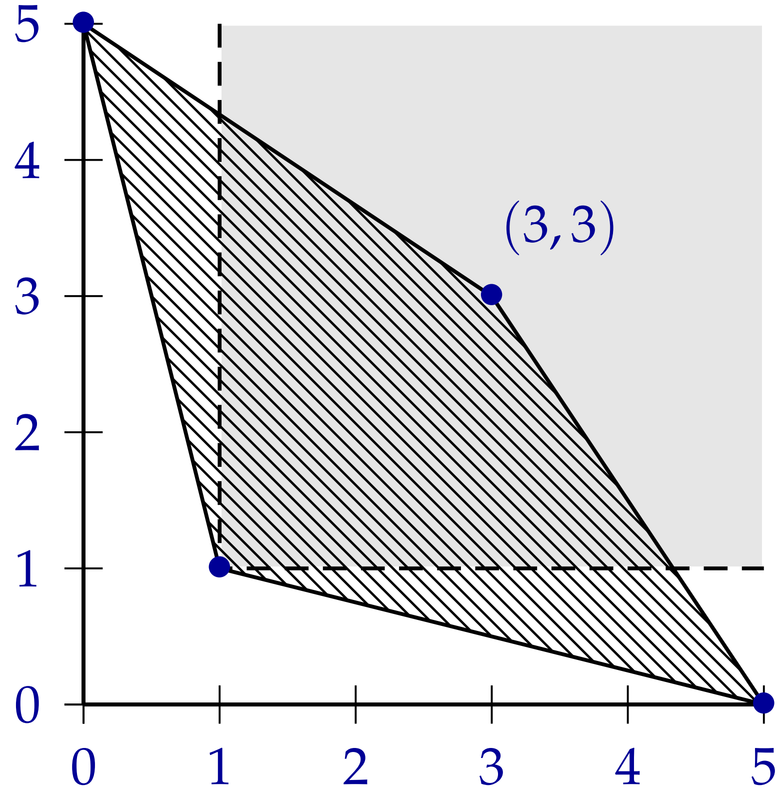

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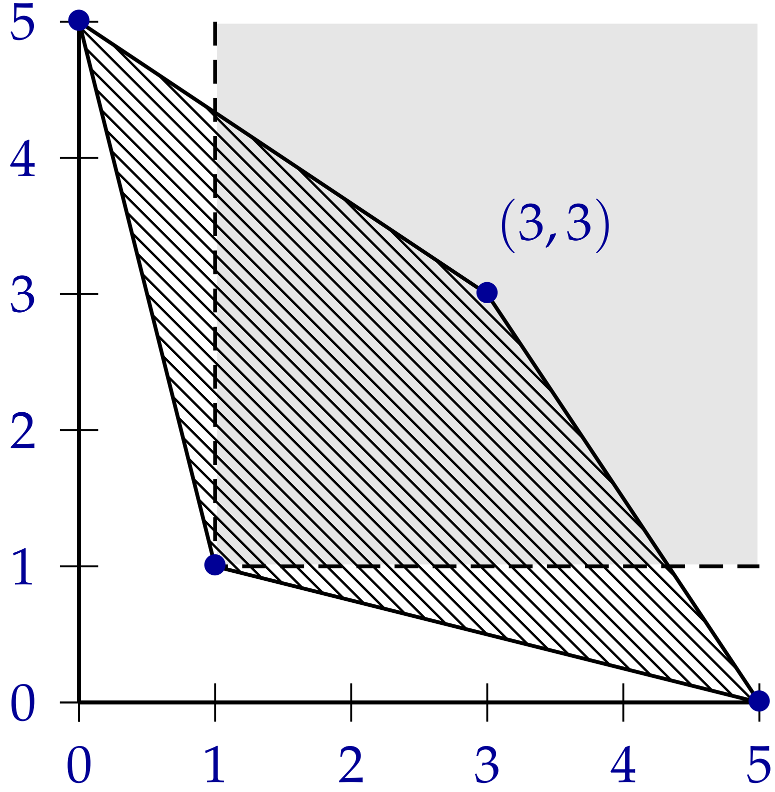

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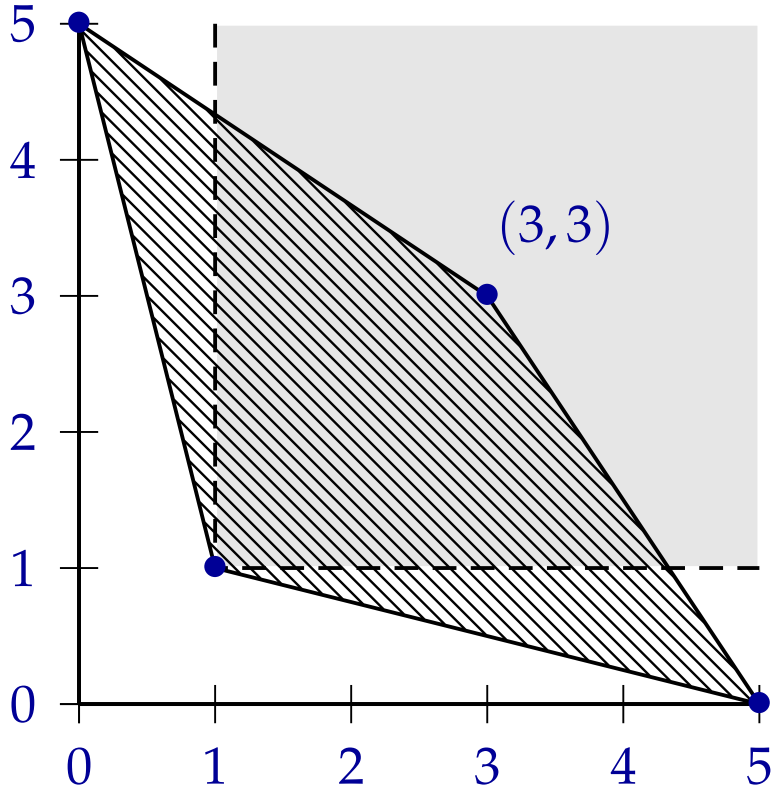

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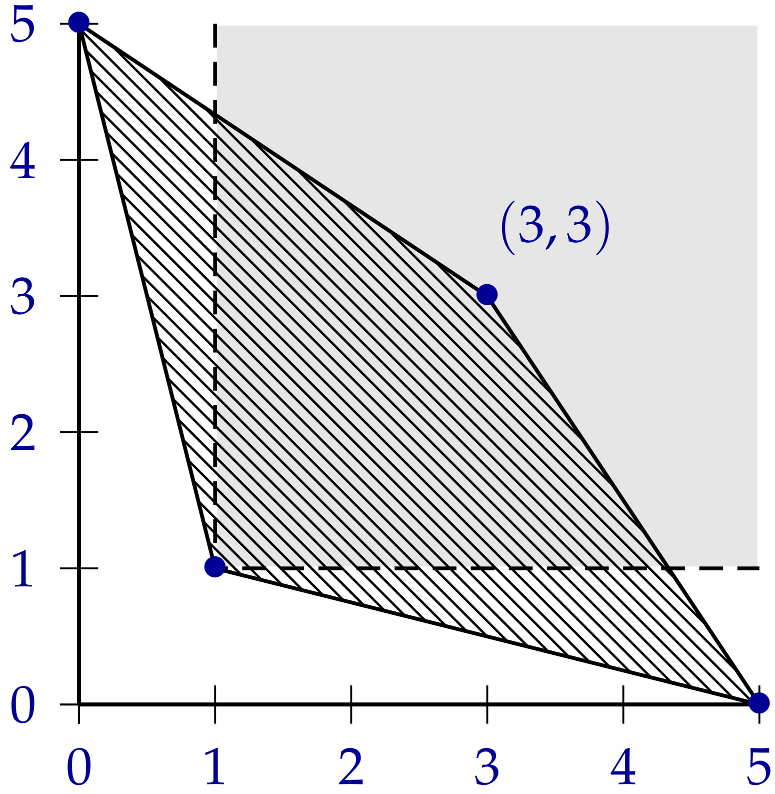

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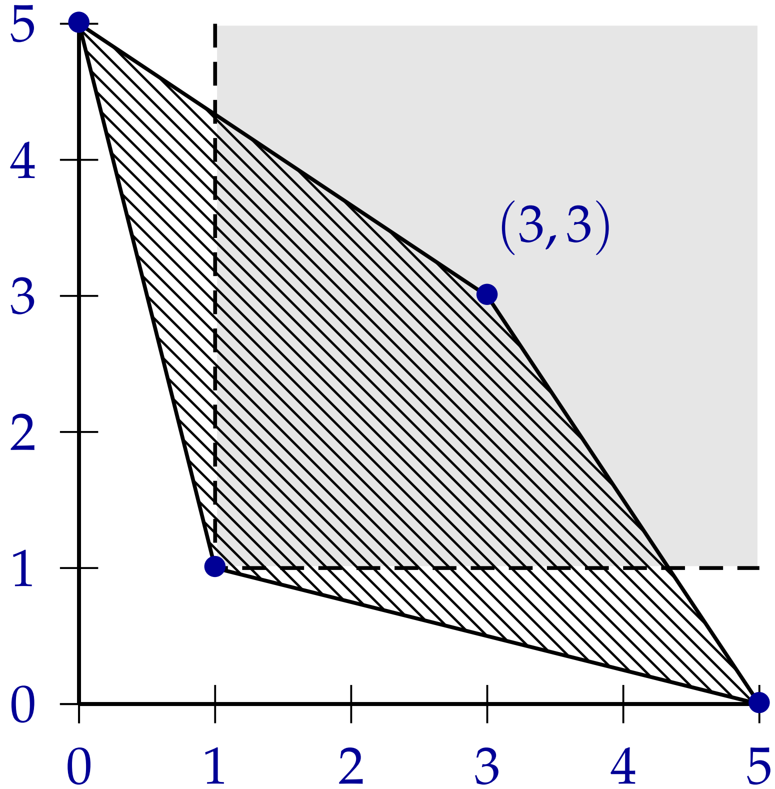

1. **Feasible payoffs** (striped): payoff combos that can be obtained by jointly repeating patterns of actions (more accurately: patterns of action profiles).

2. **Enforceable payoffs** (shaded): everyone resides above punishment minmax.

For every payoff pair (x, y) in (1) $\cap$ (2), there is a $\delta(x, y) \in (0, 1)$, such that for all $\delta \geq \delta(x, y)$ the payoff (x, y) can be obtained as the limiting average in a subgame perfect equilibrium of $G^*(\delta)$.

Part IV:
non-SGP Nash equilibria

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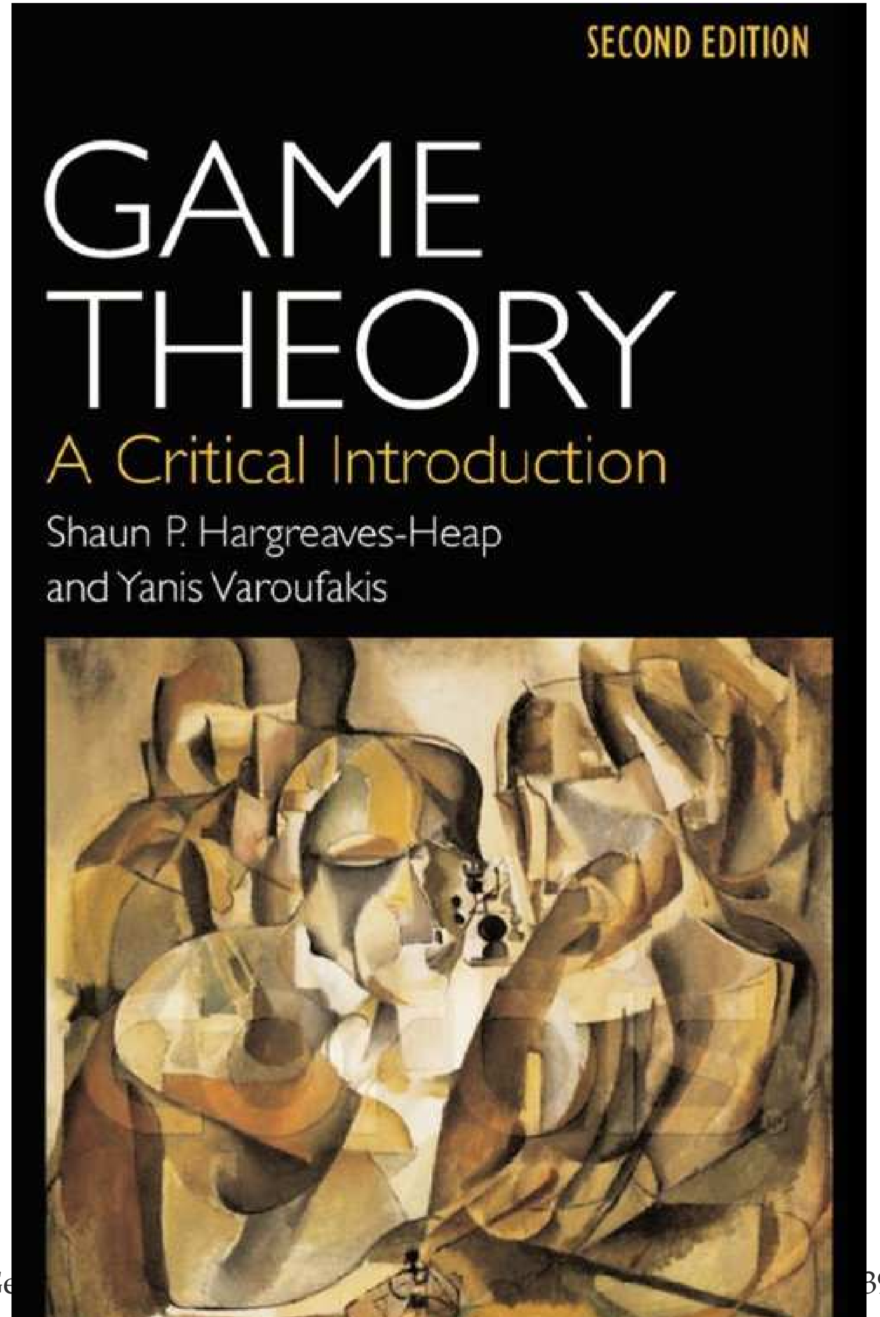
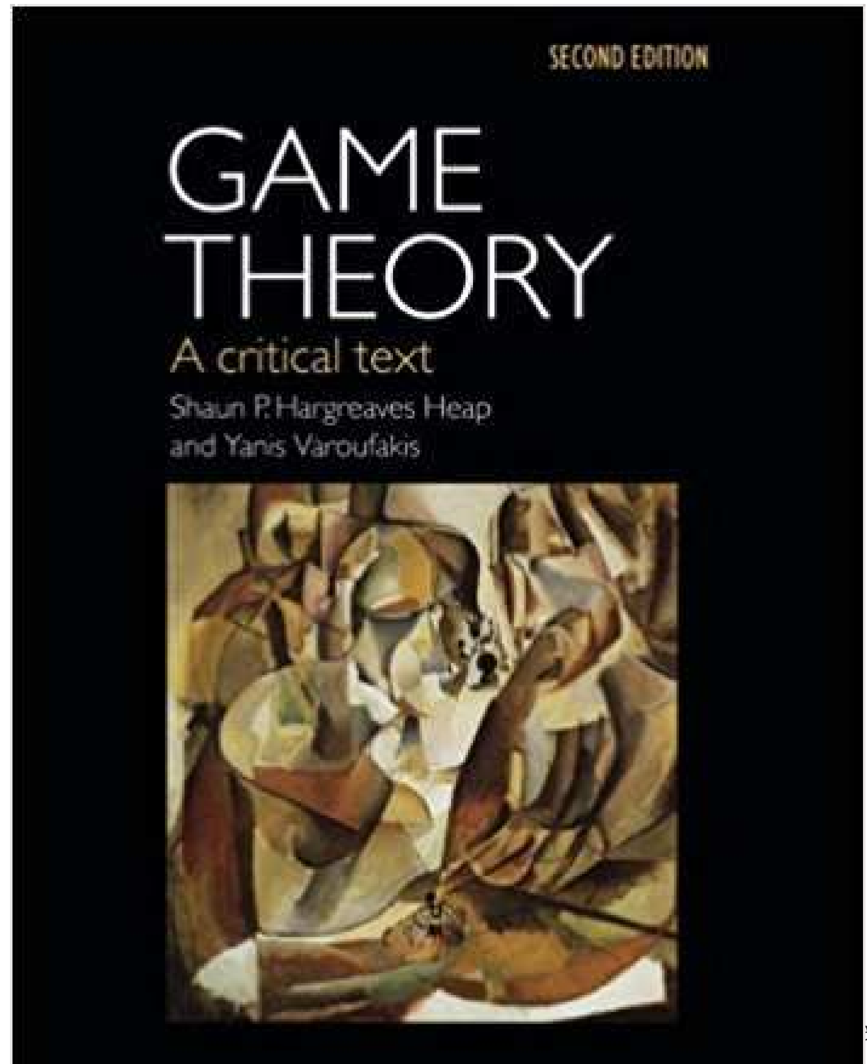
- Without the requirement of subgame perfection, deviations can be punished more severely: the equilibrium does not have to induce SGPs.
- However, non-SGPs implies threats that are not credible.

Game Theory: A Critical [what?]





Example: A repeated game with a non-SGP NE

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This expression is positive only if $\delta \geq 0.54$. (Solve 3rd-degree equation.)

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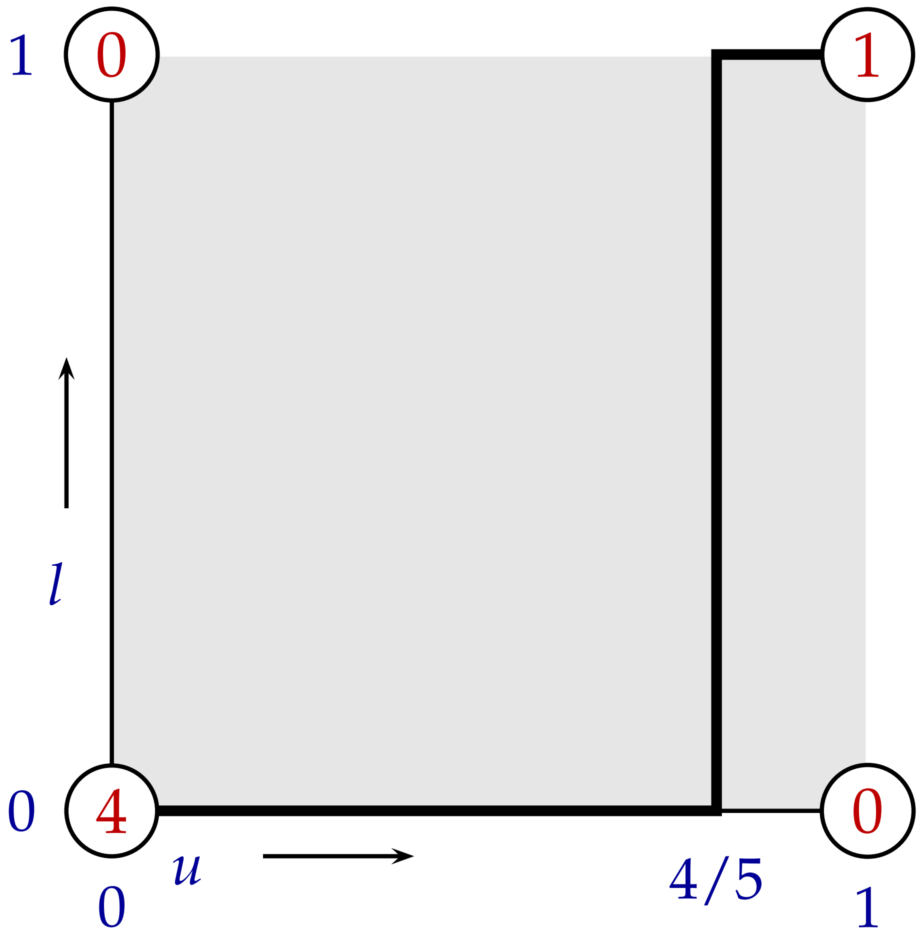

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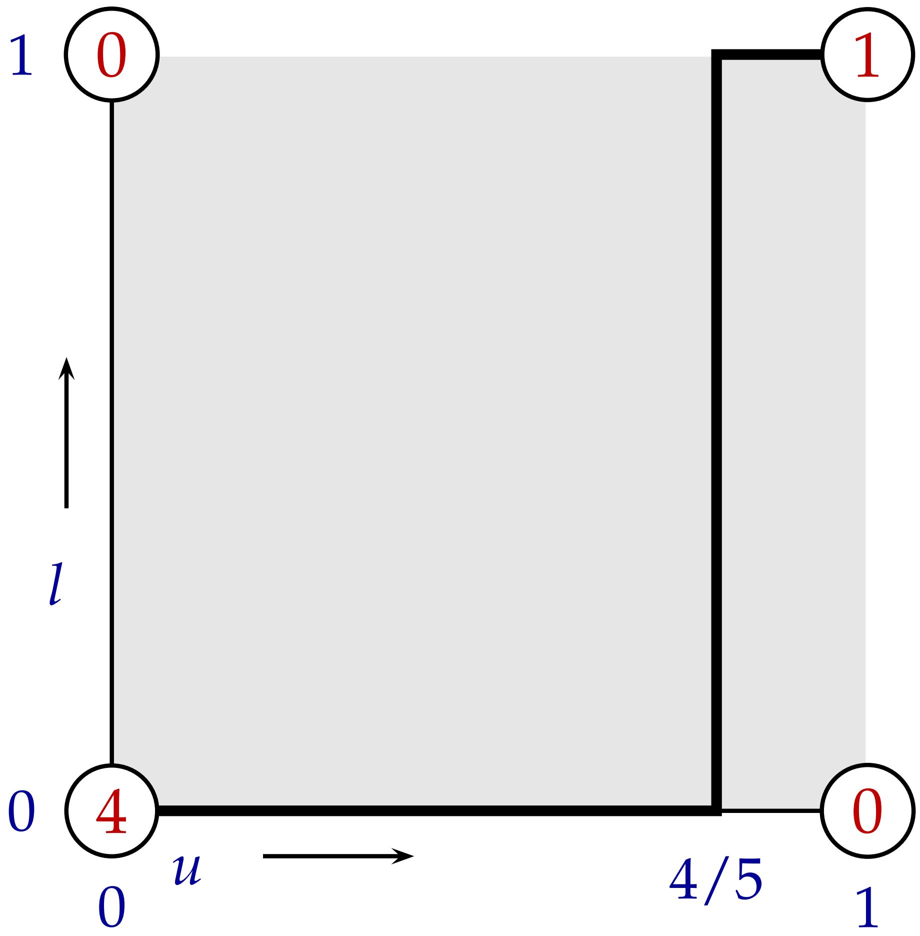

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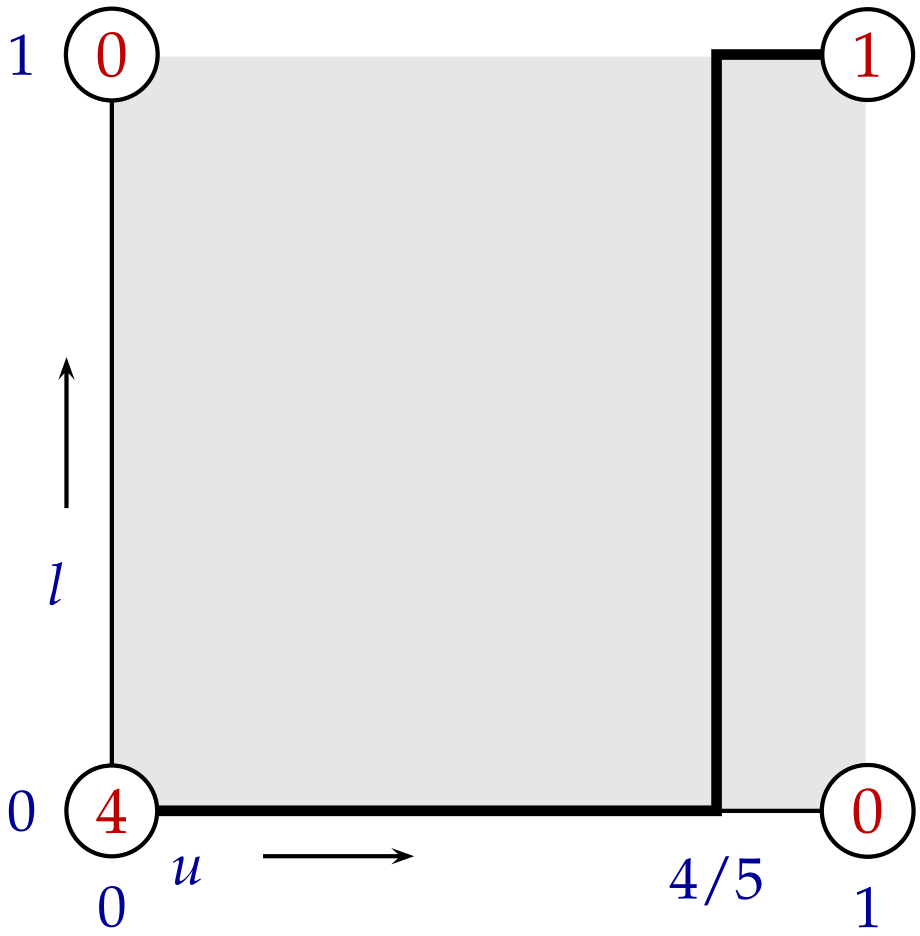

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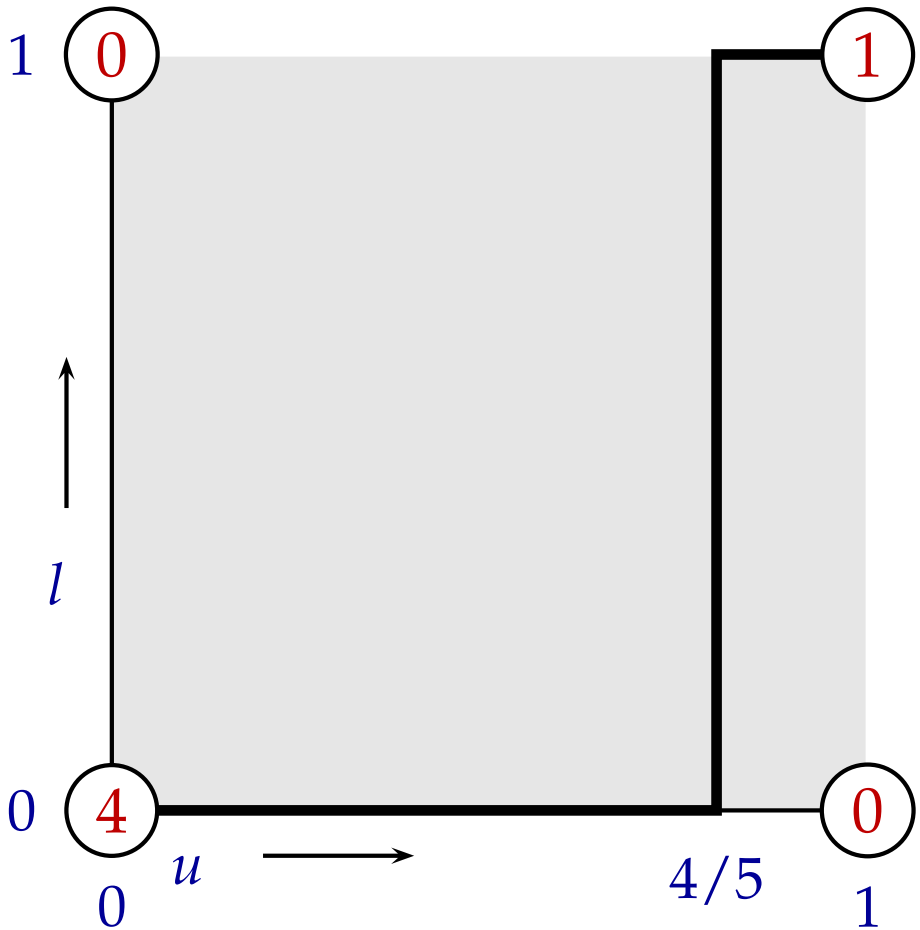

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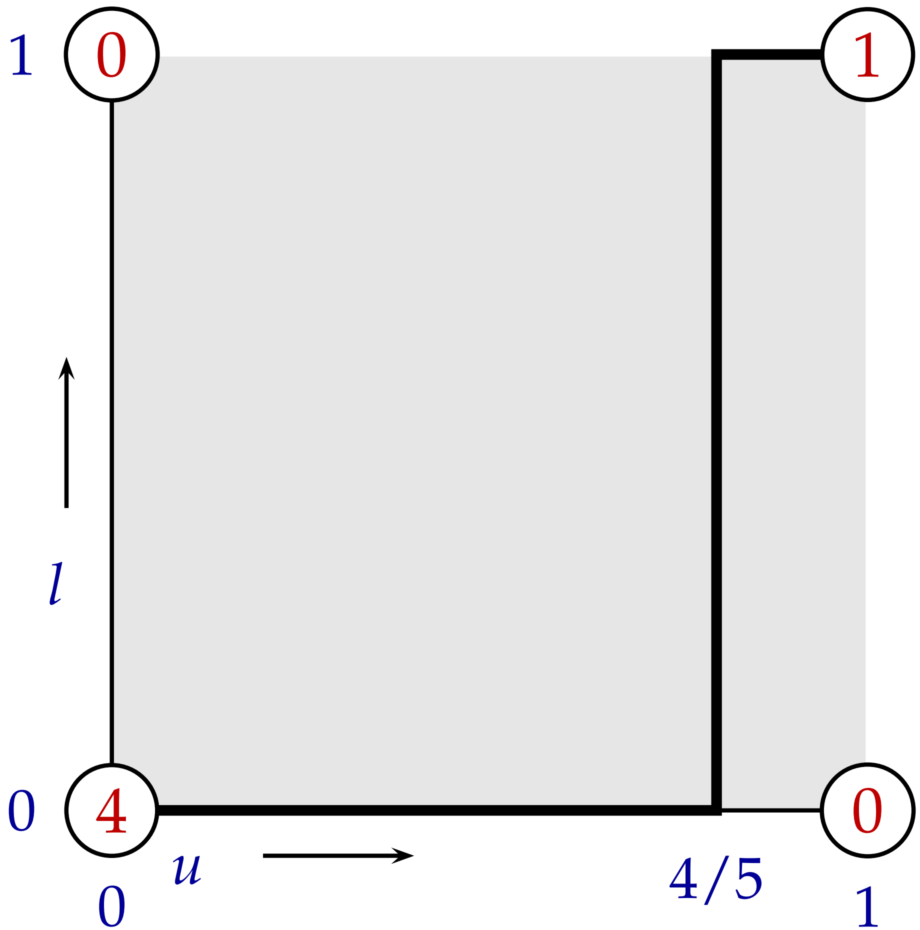

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These calculations are done by hand, and do not easily generalise to higher dimensions.

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- **Gradient Dynamics**. This is to approximate NE of single-shot games (stage games) through gradient ascent (hill-climbing).