Multi-agent learning

Gradient ascent

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Gradient ascent: idea

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- Every opponent is identified with a (possibly mixed) strategy.

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  - In fictitious play, players learn projected opponent strategies, and play a best response to it.

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- Every opponent is identified with a (possibly mixed) strategy.
- Players can observe the (possibly mixed) strategy of their opponent.
- After observation, every player changes its strategy a tiny bit in the right direction.
- Comparison with fictitious play.
  - Like in fictitious play, opponents are modelled through a mixed strategy.
  - In fictitious play, players learn projected opponent strategies, and play a best response to it.
  - In gradient ascent, players do not project a mixed strategy, and do not play a best response.

Plan for today

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   - Convergence of IGA.
4. **IGA-WoLF**: **Win or Learn Fast**. Bowling and Veloso (2001, 2002).
   - Convergence of IGA-WoLF + analysis of the proof of convergence.

Part 1:
Payoffs of general 2x2 games
in normal form

Two-player, two-action, general sum games

In its most general form, a two-player, two-action game in normal form with real-valued payoffs can be represented by

$$M = \begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} r_{11}, c_{11} & r_{12}, c_{12} \\ r_{21}, c_{21} & r_{22}, c_{22} \end{pmatrix} \end{array}$$

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Row plays mixed $(\alpha, 1-\alpha)$. Column plays mixed $(\beta, 1-\beta)$.

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Expected payoff:

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Row plays mixed $(\alpha, 1-\alpha)$. Column plays mixed $(\beta, 1-\beta)$.

Expected payoff:

$$u_1(\alpha, \beta) = \alpha[\beta r_{11} + (1-\beta) r_{12}] + (1-\alpha)[\beta r_{21} + (1-\beta) r_{22}]$$

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$$\begin{aligned} u_1(\alpha, \beta) &= \alpha[\beta r_{11} + (1-\beta) r_{12}] + (1-\alpha)[\beta r_{21} + (1-\beta) r_{22}] \\ &= u\,\alpha\beta + \alpha(r_{12} - r_{22}) + \beta(r_{21} - r_{22}) + r_{22}. \end{aligned}$$

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where $u = (r_{11} - r_{12}) - (r_{21} - r_{22})$ and $u' = (c_{11} - c_{21}) - (c_{12} - c_{22})$.

Gradient of expected payoff

Gradient:

$$\frac{\partial u_1(\alpha,\beta)}{\partial\alpha} = \beta u \ + (r_{12} - r_{22})$$
$$\frac{\partial u_2(\alpha,\beta)}{\partial\beta} = \alpha u' + (c_{21} - c_{22})$$

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$$\frac{\partial u_1(\alpha,\beta)}{\partial \alpha} = \beta u \; + (r_{12} - r_{22})$$
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As an affine map:

$$\begin{bmatrix} \partial u_1 / \partial \alpha \\ \partial u_2 / \partial \beta \end{bmatrix} = \begin{bmatrix} 0 & u \\ u' & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} r_{12} - r_{22} \\ c_{21} - c_{22} \end{bmatrix} = U \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + C$$

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Stationary point:

$$(\alpha^*, \beta^*) = \left(\frac{c_{22} - c_{21}}{u'}, \frac{r_{22} - r_{12}}{u}\right)$$

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Remarks:

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Remarks:

- There is at most one stationary point.

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- If a stationary point exists, it may lie outside $[0,1]^2$.

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$$(\alpha^*, \beta^*) = (\frac{c_{22} - c_{21}}{u'}, \frac{r_{22} - r_{12}}{u})$$

Remarks:

- There is at most one stationary point.
- If a stationary point exists, it may lie outside $[0,1]^2$.
- If there is a stationary point inside $[0,1]^2$, it is a weak (i.e., non-strict) Nash equilibrium.

Example: payoffs in Stag Hunt (r=4, t=3, s=1, p=3)

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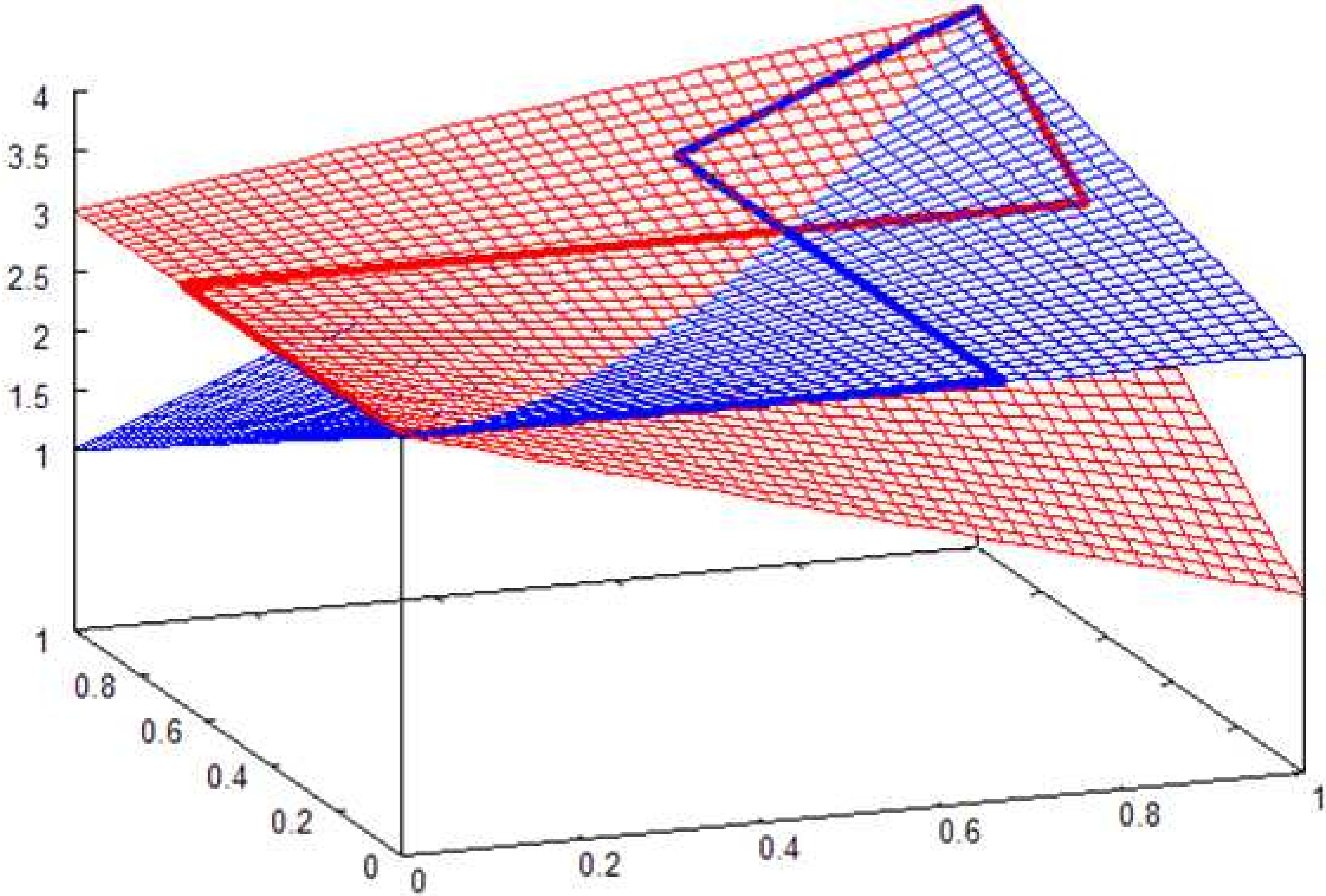

Player 1 may only move "back – front"; Player 2 may only move "left – right".

Part 2: IGA

Gradient ascent

Affine differential map:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{t+1} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_t + \eta \begin{bmatrix} \partial u_1 / \partial \alpha \\ \partial u_2 / \partial \beta \end{bmatrix}_t$$

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- Because $\alpha, \beta \in [0, 1]$, the dynamics must be confined to $[0, 1]^2$.

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- Because $\alpha, \beta \in [0, 1]$, the dynamics must be confined to $[0, 1]^2$.
- Suppose the state (α, β) is on the boundary of the probability space $[0, 1]^2$, and the gradient vector points outwards.

  Intuition: one of the players has an incentive to improve, but cannot improve further.

Gradient ascent

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  Intuition: one of the players has an incentive to improve, but cannot improve further.
- To maintain dynamics within $[0, 1]^2$, the gradient is projected back on to $[0, 1]^2$.

  Intuition: if one of the players has an incentive to improve, but *cannot* improve, then he *will not* improve.

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  Intuition: one of the players has an incentive to improve, but cannot improve further.
- To maintain dynamics within $[0, 1]^2$, the gradient is projected back on to $[0, 1]^2$.

  Intuition: if one of the players has an incentive to improve, but *cannot* improve, then he *will not* improve.
- If nonzero, the projected gradient is parallel to the (closest) boundary of $[0, 1]^2$.

Infinitesimal Gradient Ascent : IGA (Singh *et al.*, 2000)

Affine differential map:

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Theorem (Singh, Kearns and Mansour, 2000) *If players follow IGA, where* $\eta \to 0$*, their average payoffs will converge to the (expected) payoffs of a NE. If their strategies converge, they will converge to that same NE.*

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1. If U is invertible, and its eigenvalue λ (solution of $Ux = \lambda x \Leftrightarrow$ solution of $\mathrm{Det}[U - \lambda I] = 0$) is real, $\exists$ stationary point

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1. If U is invertible, and its eigenvalue λ (solution of $Ux = \lambda x \Leftrightarrow$ solution of $\mathrm{Det}[U - \lambda I] = 0$) is real, $\exists$ stationary point, w.i. is a saddle point.
2. If U is invertible, and its eigenvalue λ is imaginary, there is a stationary point

Infinitesimal Gradient Ascent : IGA (Singh *et al.*, 2000)

Affine differential map:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{t+1} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{t} + \eta \left(\begin{bmatrix} 0 & u \\ u' & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} r_{12} - r_{22} \\ c_{21} - c_{22} \end{bmatrix} \right)$$

Theorem (Singh, Kearns and Mansour, 2000) *If players follow IGA, where* $\eta \to 0$, *their average payoffs will converge to the (expected) payoffs of a NE. If their strategies converge, they will converge to that same NE.*
The proof is based on a qualitative result in the the theory of differential equations, which says that the behaviour of an affine differential map is determined by the multiplicative matrix U:

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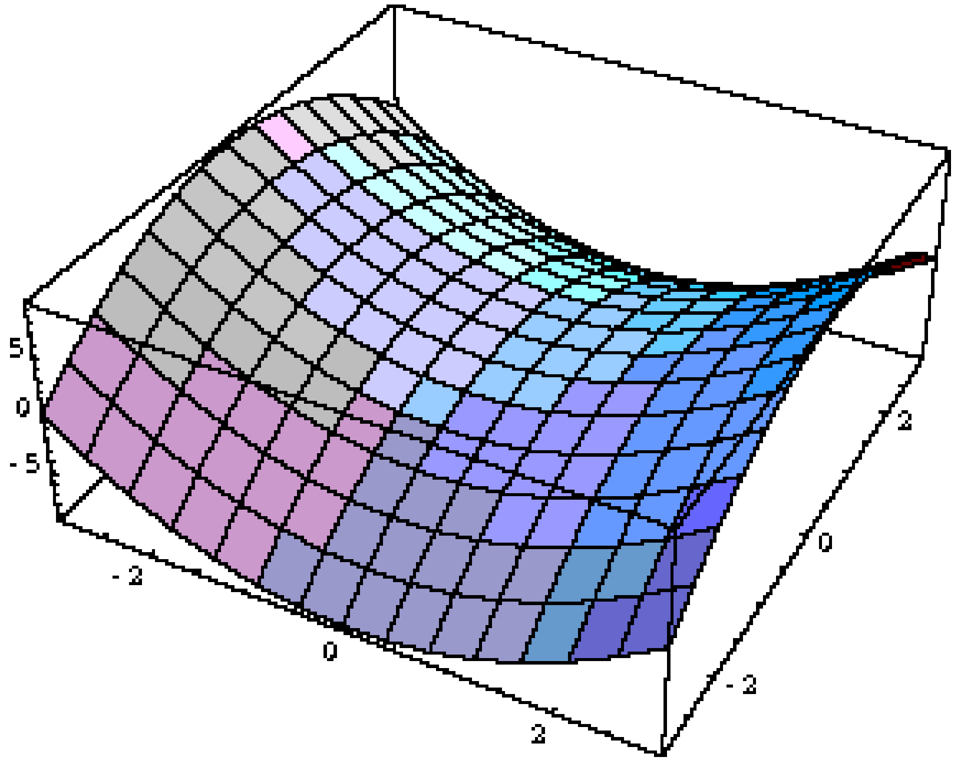
2. If U is invertible, and its eigenvalue λ is imaginary, there is a stationary point, which, in particular, is a centric point.
3. If U is not invertible (iff $u = 0$ or $u' = 0$), there is no stationary point.

Saddle point



Gradient ascent: Coordination game

Gradient ascent: Coordination game

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{array}{c} \text{T} \\ \text{B} \end{array} & \begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 1,1 \end{pmatrix} \end{array}$$

Gradient ascent: Coordination game

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 1,1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} 2\cdot \beta - 1 \\ 2\cdot \alpha - 1 \end{bmatrix}$$

Gradient ascent: Coordination game

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 1,1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} 2\cdot \beta - 1 \\ 2\cdot \alpha - 1 \end{bmatrix}$$

- Stationary at $(1/2, 1/2)$.

Gradient ascent: Coordination game

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{array}{c} \text{T} \\ \text{B} \end{array} & \begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 1,1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} 2 \cdot \beta - 1 \\ 2 \cdot \alpha - 1 \end{bmatrix}$$

- Stationary at $(1/2, 1/2)$.
- Matrix

$$U = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

has real eigenvalues: $\lambda^2 - 4 = 0$.

Gradient ascent: Coordination game

- Symmetric, but not zero sum:

$$\begin{array}{c} \\ \mathrm{T} \\ \mathrm{B} \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 1,1 \end{pmatrix} \end{array}$$

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- Matrix

$$U = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

has real eigenvalues: $\lambda^2 - 4 = 0$.
Saddle point inside $[0,1]^2$.

Gradient ascent: Coordination game

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 1,1 \end{pmatrix} \end{array}$$

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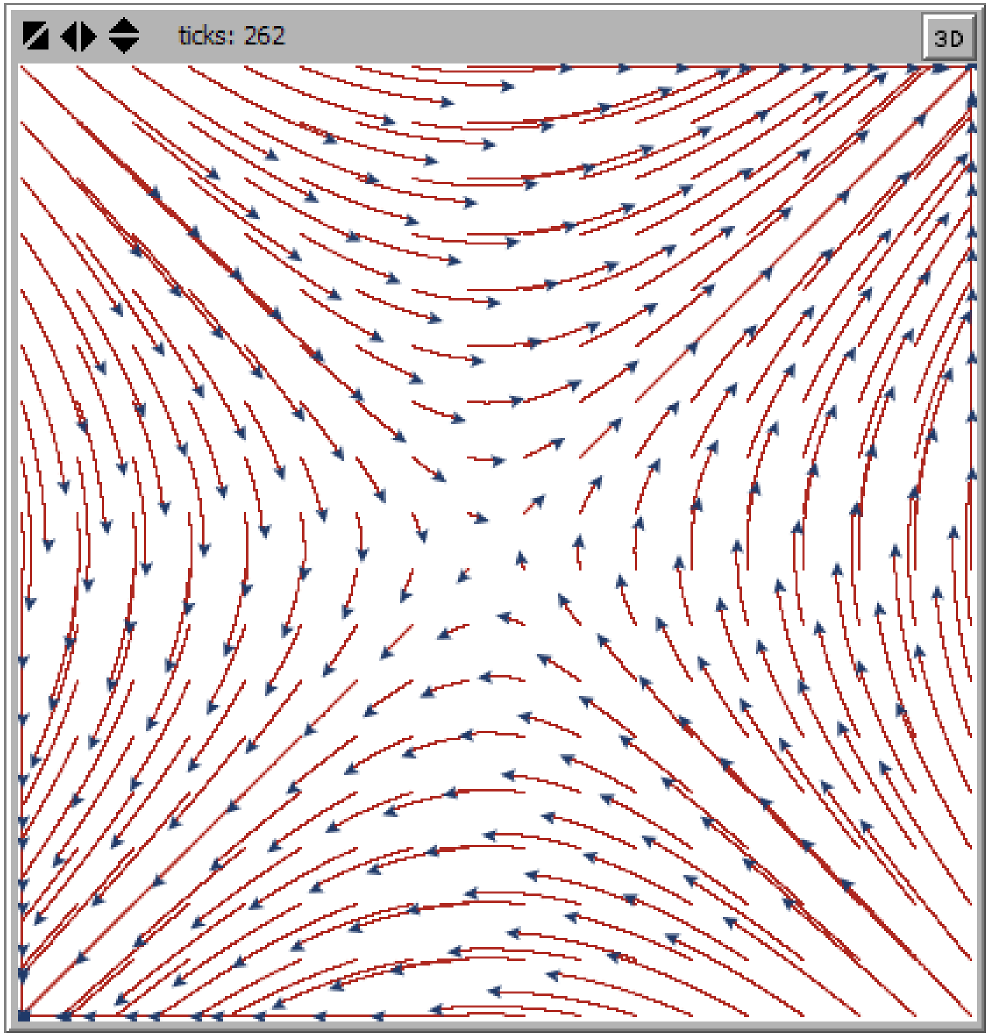

Gradient ascent: Prisoners' Dilemma

Gradient ascent: Prisoners' Dilemma

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 3,3 & 0,5 \\ 5,0 & 1,1 \end{pmatrix} \end{array}$$

Gradient ascent: Prisoners' Dilemma

- Symmetric, but not zero sum:

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{pmatrix} 3,3 & 0,5 \\ 5,0 & 1,1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} -1\cdot \beta - 1 \\ -1\cdot \alpha - 1 \end{bmatrix}$$

Gradient ascent: Prisoners' Dilemma

- Symmetric, but not zero sum:

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} \quad L & \quad R \end{array} \\ \begin{pmatrix} 3,3 & 0,5 \\ 5,0 & 1,1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} -1 \cdot \beta - 1 \\ -1 \cdot \alpha - 1 \end{bmatrix}$$

- Stationary at $(-1, -1)$.

Gradient ascent: Prisoners' Dilemma

- Symmetric, but not zero sum:

$$\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 3,3 & 0,5 \\ 5,0 & 1,1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} -1 \cdot \beta - 1 \\ -1 \cdot \alpha - 1 \end{bmatrix}$$

- Stationary at $(-1, -1)$.
- Matrix

$$U = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

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Gradient ascent: Prisoners' Dilemma

- Symmetric, but not zero sum:

$$
\begin{array}{c} \\ T \\ B \end{array}
\begin{array}{c}
\begin{array}{cc} \;\; L & \;\;\; R \end{array} \\
\begin{pmatrix} 3,3 & 0,5 \\ 5,0 & 1,1 \end{pmatrix}
\end{array}
$$

- Gradient:

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- Stationary at $(-1, -1)$.
- Matrix

$$
U = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}
$$

has real eigenvalues: $\lambda^2 - 1 = 0$.
Saddle point outside $[0, 1]^2$.

Gradient ascent: Prisoners' Dilemma

- Symmetric, but not zero sum:

$$\begin{array}{c} \\ \mathrm{T} \\ \mathrm{B} \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \left(\begin{array}{cc} 3,3 & 0,5 \\ 5,0 & 1,1 \end{array} \right) \end{array}$$

- Gradient:

$$\left[\begin{array}{c} -1\cdot\beta - 1 \\ -1\cdot\alpha - 1 \end{array} \right]$$

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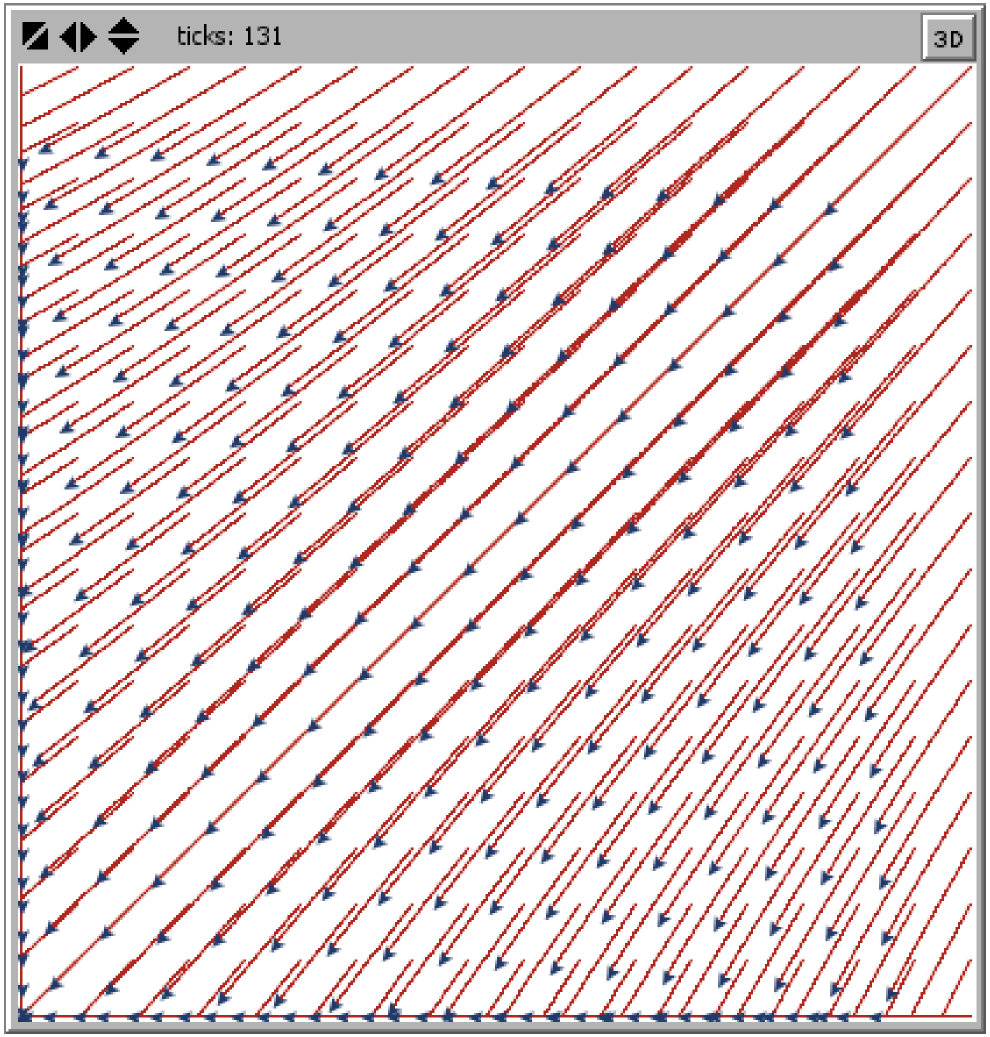
Saddle point outside $[0, 1]^2$.



Gradient ascent: Stag hunt

Gradient ascent: Stag hunt

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 5,5 & 0,3 \\ 3,0 & 2,2 \end{pmatrix} \end{array}$$

Gradient ascent: Stag hunt

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 5,5 & 0,3 \\ 3,0 & 2,2 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} 4 \cdot \beta - 2 \\ 4 \cdot \alpha - 2 \end{bmatrix}$$

Gradient ascent: Stag hunt

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 5,5 & 0,3 \\ 3,0 & 2,2 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} 4\cdot \beta - 2 \\ 4\cdot \alpha - 2 \end{bmatrix}$$

- Stationary at $(1/2, 1/2)$.

Gradient ascent: Stag hunt

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \left(\begin{array}{cc} 5,5 & 0,3 \\ 3,0 & 2,2 \end{array} \right) \end{array}$$

- Gradient:

$$\left[\begin{array}{c} 4 \cdot \beta - 2 \\ 4 \cdot \alpha - 2 \end{array} \right]$$

- Stationary at $(1/2, 1/2)$.
- Matrix

$$U = \left[\begin{array}{cc} 0 & 4 \\ 4 & 0 \end{array} \right]$$

has real eigenvalues:
$\lambda^2 - 16 = 0$.

Gradient ascent: Stag hunt

- Symmetric, but not zero sum:

$$\begin{array}{c} \\ \mathrm{T} \\ \mathrm{B} \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{pmatrix} 5,5 & 0,3 \\ 3,0 & 2,2 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} 4 \cdot \beta - 2 \\ 4 \cdot \alpha - 2 \end{bmatrix}$$

- Stationary at $(1/2, 1/2)$.

- Matrix

$$U = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$$

has real eigenvalues: $\lambda^2 - 16 = 0$. Saddle point inside $[0, 1]^2$.

Gradient ascent: Stag hunt

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 5,5 & 0,3 \\ 3,0 & 2,2 \end{pmatrix} \end{array}$$

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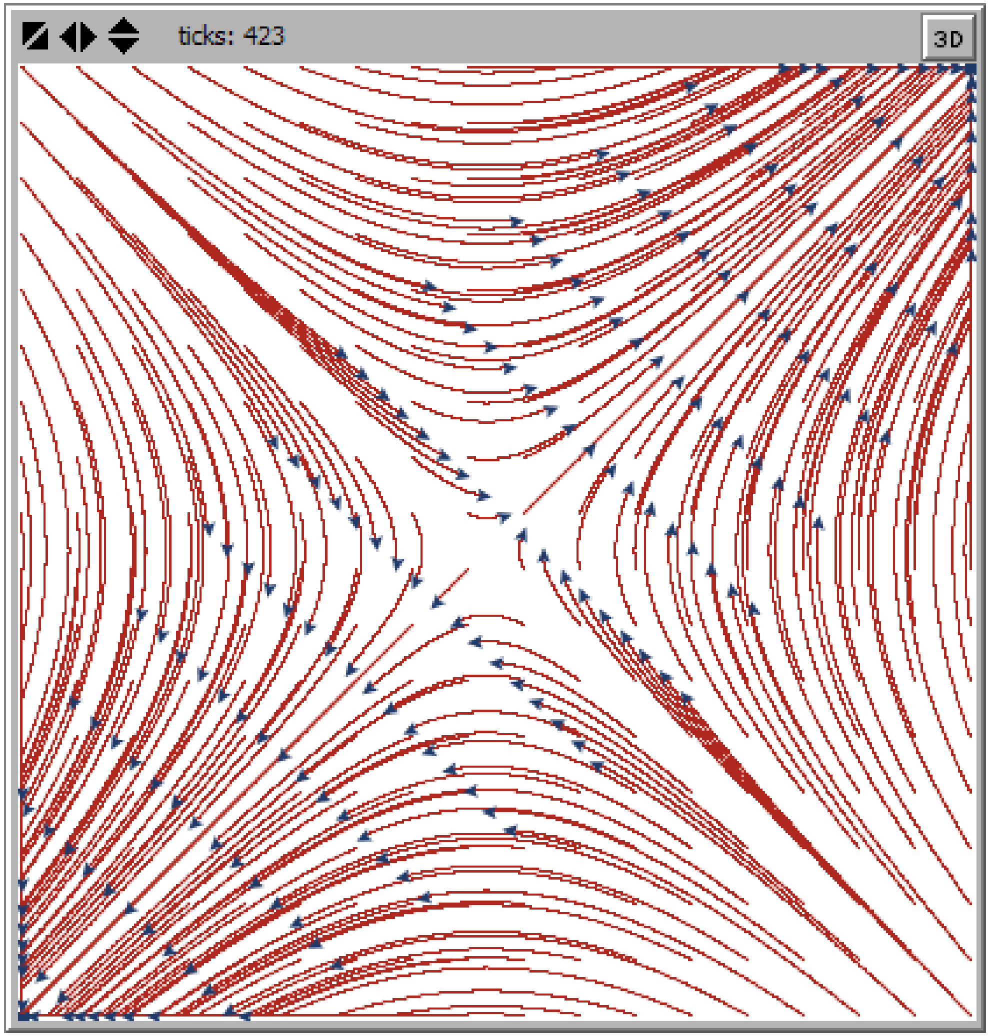


Gradient ascent: Game of Chicken

Gradient ascent: Game of Chicken

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 0,0 & -1,1 \\ 1,-1 & -3,-3 \end{pmatrix} \end{array}$$

Gradient ascent: Game of Chicken

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{array}{c} \text{T} \\ \text{B} \end{array} & \begin{pmatrix} 0,0 & -1,1 \\ 1,-1 & -3,-3 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} -3\cdot \beta + 2 \\ -3\cdot \alpha + 2 \end{bmatrix}$$

Gradient ascent: Game of Chicken

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 0,0 & -1,1 \\ 1,-1 & -3,-3 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} -3\cdot\beta+2 \\ -3\cdot\alpha+2 \end{bmatrix}$$

- Stationary at $(2/3, 2/3)$.

Gradient ascent: Game of Chicken

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 0,0 & -1,1 \\ 1,-1 & -3,-3 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} -3\cdot\beta + 2 \\ -3\cdot\alpha + 2 \end{bmatrix}$$

- Stationary at $(2/3, 2/3)$.
- Matrix

$$U = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$$

has real eigenvalues: $\lambda^2 - 9 = 0$.

Gradient ascent: Game of Chicken

- Symmetric, but not zero sum:

$$\begin{array}{c c} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 0,0 & -1,1 \\ 1,-1 & -3,-3 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} -3\cdot\beta + 2 \\ -3\cdot\alpha + 2 \end{bmatrix}$$

- Stationary at $(2/3, 2/3)$.
- Matrix

$$U = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$$

has real eigenvalues: $\lambda^2 - 9 = 0$.
Saddle point inside $[0,1]^2$.

Gradient ascent: Game of Chicken

- Symmetric, but not zero sum:

$$\begin{array}{c c} & \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{array}{c} \text{T} \\ \text{B} \end{array} & \begin{pmatrix} 0,0 & -1,1 \\ 1,-1 & -3,-3 \end{pmatrix} \end{array}$$

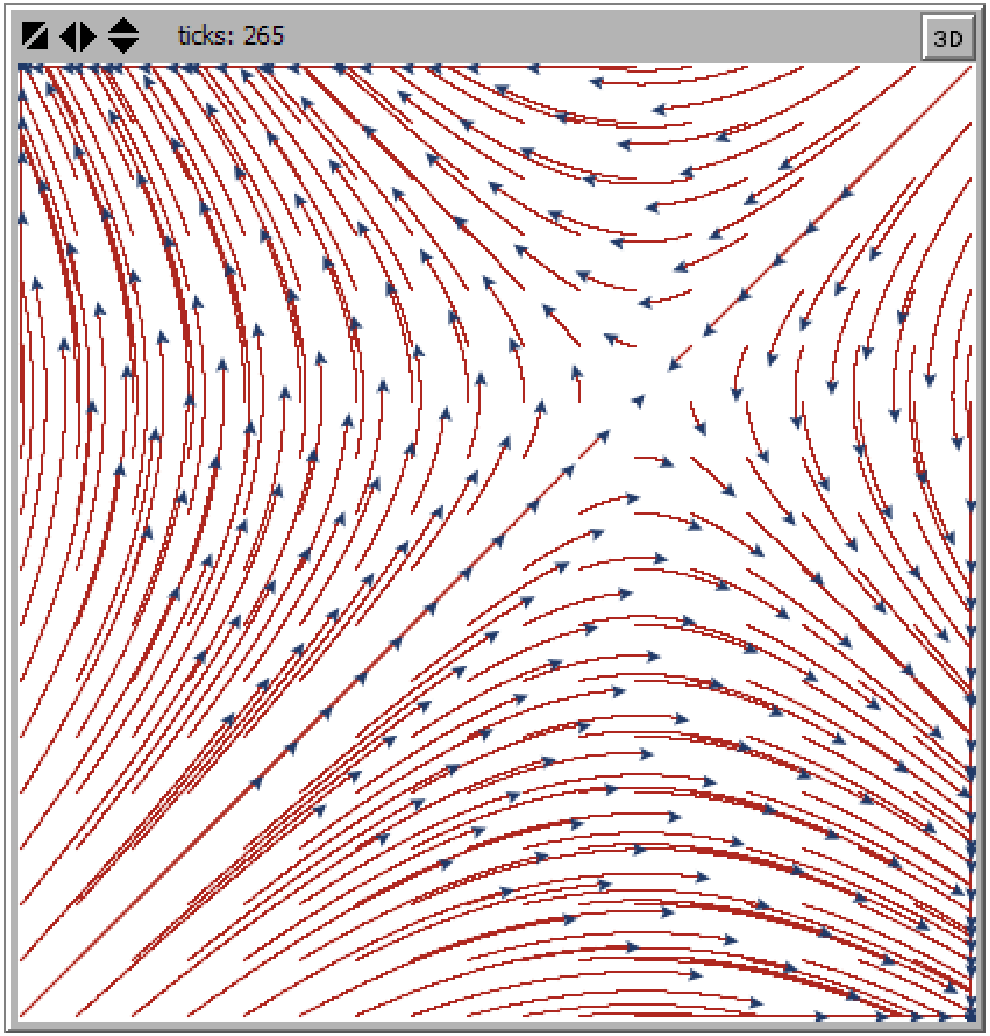
- Gradient:

$$\begin{bmatrix} -3 \cdot \beta + 2 \\ -3 \cdot \alpha + 2 \end{bmatrix}$$

- Stationary at $(2/3, 2/3)$.
- Matrix

$$U = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$$

has real eigenvalues: $\lambda^2 - 9 = 0$.
Saddle point inside $[0,1]^2$.



Gradient ascent: Battle of the Sexes

Gradient ascent: Battle of the Sexes

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{array}{c} \text{T} \\ \text{B} \end{array} & \begin{pmatrix} 0,0 & 2,3 \\ 3,2 & 1,1 \end{pmatrix} \end{array}$$

Gradient ascent: Battle of the Sexes

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{array}{c} \text{T} \\ \text{B} \end{array} & \begin{pmatrix} 0,0 & 2,3 \\ 3,2 & 1,1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} -4 \cdot \beta + 1 \\ -4 \cdot \alpha + 1 \end{bmatrix}$$

Gradient ascent: Battle of the Sexes

- Symmetric, but not zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 0,0 & 2,3 \\ 3,2 & 1,1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} -4\cdot \beta + 1 \\ -4\cdot \alpha + 1 \end{bmatrix}$$

- Stationary at $(1/4, 1/4)$.

Gradient ascent: Battle of the Sexes

- Symmetric, but not zero sum:

$$\begin{array}{c} \\ \mathrm{T} \\ \mathrm{B} \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{pmatrix} 0,0 & 2,3 \\ 3,2 & 1,1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} -4\cdot\beta + 1 \\ -4\cdot\alpha + 1 \end{bmatrix}$$

- Stationary at $(1/4, 1/4)$.
- Matrix

$$U = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

has real eigenvalues:
$\lambda^2 - 16 = 0$.

Gradient ascent: Battle of the Sexes

- Symmetric, but not zero sum:

$$\begin{array}{c c} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 0,0 & 2,3 \\ 3,2 & 1,1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} -4\cdot\beta + 1 \\ -4\cdot\alpha + 1 \end{bmatrix}$$

- Stationary at $(1/4, 1/4)$.
- Matrix

$$U = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

has real eigenvalues:
$\lambda^2 - 16 = 0$. Saddle point
inside $[0,1]^2$.

Gradient ascent: Battle of the Sexes

- Symmetric, but not zero sum:

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \left(\begin{array}{cc} 0,0 & 2,3 \\ 3,2 & 1,1 \end{array} \right) \end{array}$$

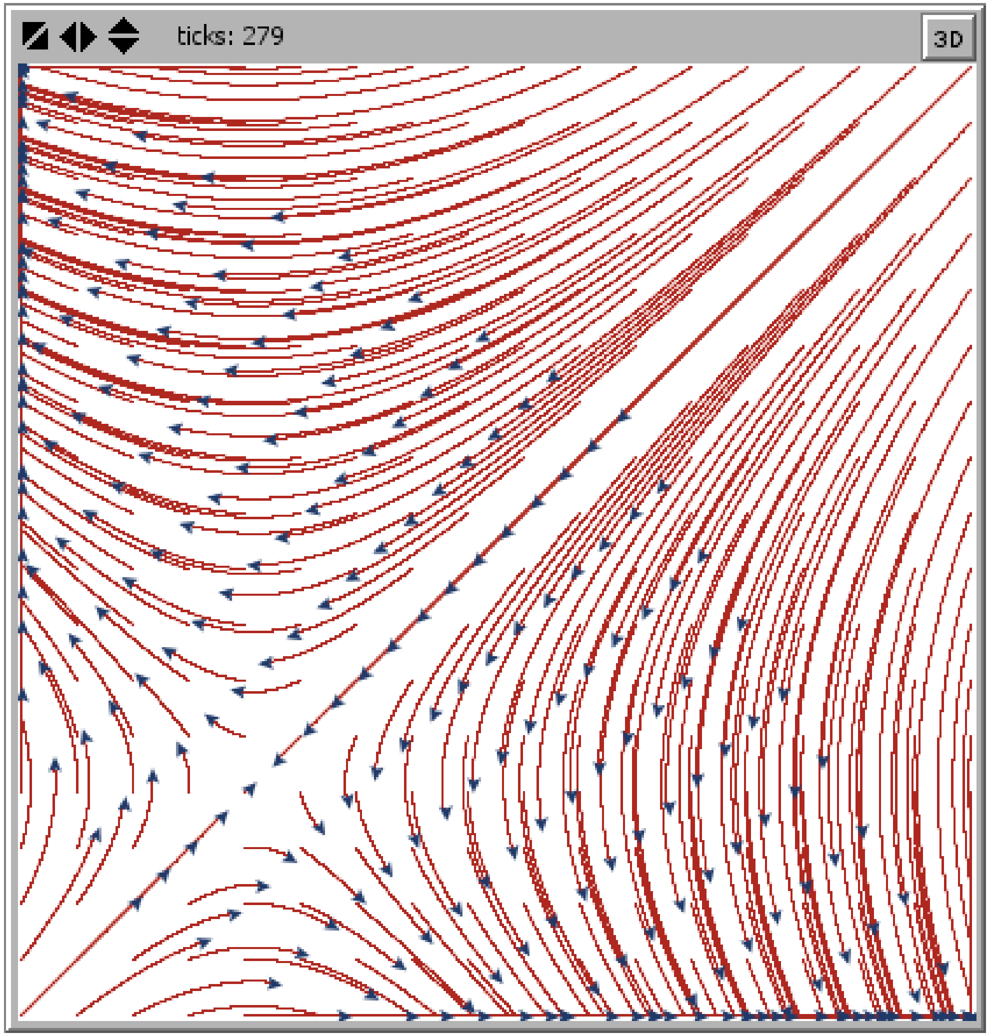
- Gradient:

$$\left[\begin{array}{c} -4\cdot \beta + 1 \\ -4\cdot \alpha + 1 \end{array} \right]$$

- Stationary at $(1/4, 1/4)$.
- Matrix

$$U = \left[\begin{array}{rr} 0 & -4 \\ -4 & 0 \end{array} \right]$$

has real eigenvalues: $\lambda^2 - 16 = 0$. Saddle point inside $[0,1]^2$.



Gradient ascent: Matching pennies

Gradient ascent: Matching pennies

- Symmetric, zero sum:

$$
\begin{array}{cc}
 & \begin{array}{cc} \text{L} & \text{R} \end{array} \\
\begin{array}{c} \text{T} \\ \text{B} \end{array} & \begin{pmatrix} 1,-1 & -1,1 \\ -1,1 & 1,-1 \end{pmatrix}
\end{array}
$$

Gradient ascent: Matching pennies

- Symmetric, zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 1,-1 & -1,1 \\ -1,1 & 1,-1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} 4\cdot\beta - 2 \\ -4\cdot\alpha + 2 \end{bmatrix}$$

Gradient ascent: Matching pennies

- Symmetric, zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} 1,-1 & -1,1 \\ -1,1 & 1,-1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} 4\cdot \beta - 2 \\ -4\cdot \alpha + 2 \end{bmatrix}$$

- Stationary at $(1/2, 1/2)$.

Gradient ascent: Matching pennies

- Symmetric, zero sum:

$$
\begin{array}{cc}
 & \begin{array}{cc} \text{L} & \text{R} \end{array} \\
\begin{array}{c} \text{T} \\ \text{B} \end{array} & \begin{pmatrix} 1,-1 & -1,1 \\ -1,1 & 1,-1 \end{pmatrix}
\end{array}
$$

- Gradient:

$$
\begin{bmatrix} 4\cdot\beta - 2 \\ -4\cdot\alpha + 2 \end{bmatrix}
$$

- Stationary at $(1/2, 1/2)$.
- Matrix

$$
U = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}
$$

has imaginary eigenvalues:
$\lambda^2 + 16 = 0$.

Gradient ascent: Matching pennies

- Symmetric, zero sum:

$$\begin{array}{c|cc} & \text{L} & \text{R} \\ \hline \text{T} & 1,-1 & -1,1 \\ \text{B} & -1,1 & 1,-1 \end{array}$$

- Gradient:

$$\begin{bmatrix} 4\cdot\beta - 2 \\ -4\cdot\alpha + 2 \end{bmatrix}$$

- Stationary at $(1/2, 1/2)$.
- Matrix

$$U = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

has imaginary eigenvalues: $\lambda^2 + 16 = 0$. Centric point inside $[0, 1]^2$.

Gradient ascent: Matching pennies

- Symmetric, zero sum:

$$\begin{array}{c} \\ \mathrm{T} \\ \mathrm{B} \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{pmatrix} 1,-1 & -1,1 \\ -1,1 & 1,-1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} 4\cdot \beta - 2 \\ -4\cdot \alpha + 2 \end{bmatrix}$$

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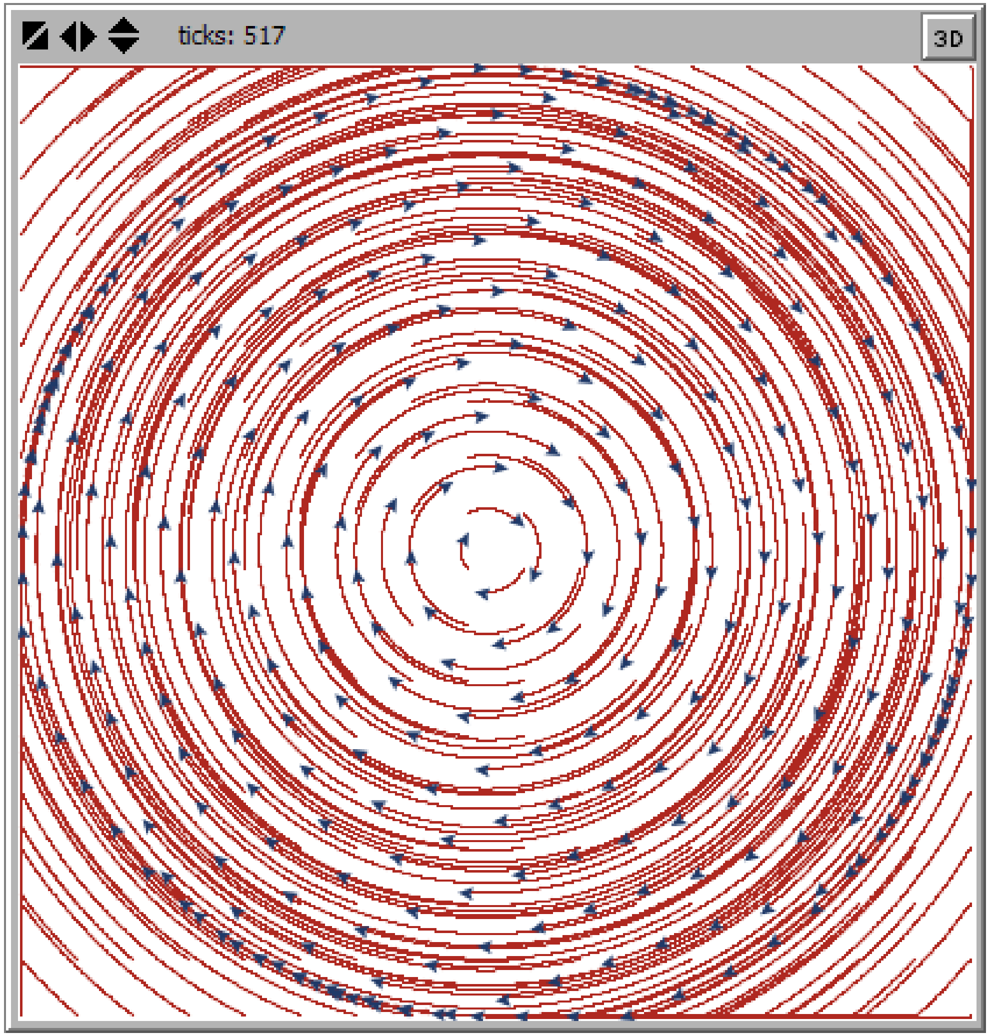
$$U = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

has imaginary eigenvalues: $\lambda^2 + 16 = 0$. Centric point inside $[0, 1]^2$.



Gradient ascent: other game with centric

Gradient ascent: other game with centric

- Symmetric, zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{array}{c} \text{T} \\ \text{B} \end{array} & \begin{pmatrix} -2,2 & 1,1 \\ 3,-3 & -2,1 \end{pmatrix} \end{array}$$

Gradient ascent: other game with centric

- Symmetric, zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} -2,2 & 1,1 \\ 3,-3 & -2,1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} -8\cdot \beta + 3 \\ 5\cdot \alpha - 4 \end{bmatrix}$$

Gradient ascent: other game with centric

- Symmetric, zero sum:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{L} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{B} \end{array} & \begin{pmatrix} -2,2 & 1,1 \\ 3,-3 & -2,1 \end{pmatrix} \end{array}$$

- Gradient:

$$\begin{bmatrix} -8\cdot \beta + 3 \\ 5\cdot \alpha - 4 \end{bmatrix}$$

- Stationary at $(4/5, 3/8)$.

Gradient ascent: other game with centric

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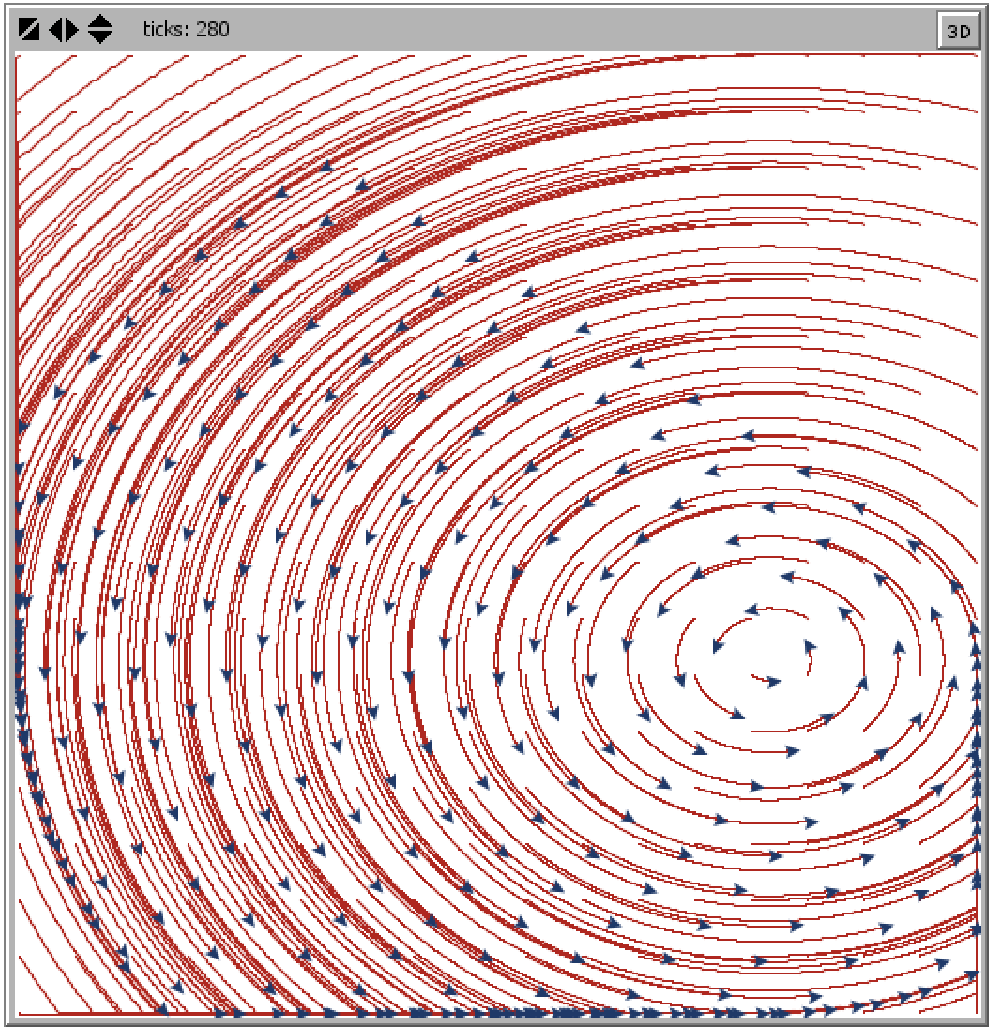

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Part 3: IGA-WoLF

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To this end, IGA-WoLF uses a variable learning rate:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{t+1} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{t} + \eta \begin{bmatrix} l_t^1 \cdot \partial u_1 / \partial \alpha \\ l_t^2 \cdot \partial u_2 / \partial \beta \end{bmatrix}_{t}$$

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where α^e is a row strategy belonging to some NE, chosen by the row player. Similarly for β^e and column player. So (α^e, β^e) need not be Nash!

Case 2d: revolution around Nash equilibrium

Lemma 1. *With fixed l^1 and l^2, the trajectory of the strategy pair (α, β) is an elliptic orbit around (α^*, β^*) with axes*

$$\begin{bmatrix} 0 \\ \sqrt{l^2|u|/l^1|u'|} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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- Bowling *et al.* do not prove this result but refer to Sing *et al.*, who, on their turn refer to a work on differential equations by Reinhard (1987).

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Simplifying and using $\beta u - (r_{12} - r_{22}) = \partial u_1 / \partial \alpha$ yields

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Corollary. *The learning rate is constant throughout any one quadrant.*

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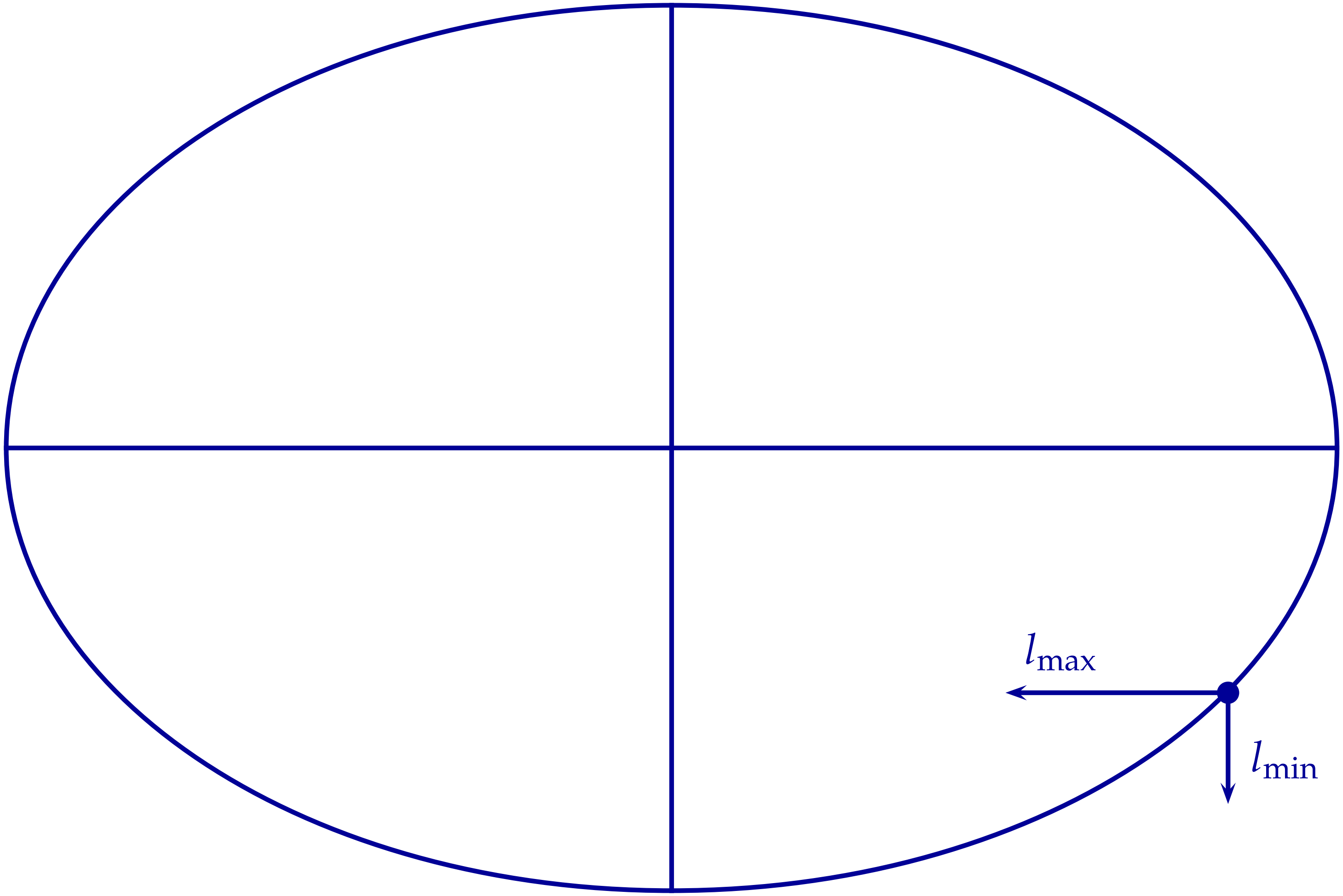
(First a suggestive picture, then the rest of the proof.)

Trajectory in different quadrants (clockwise)



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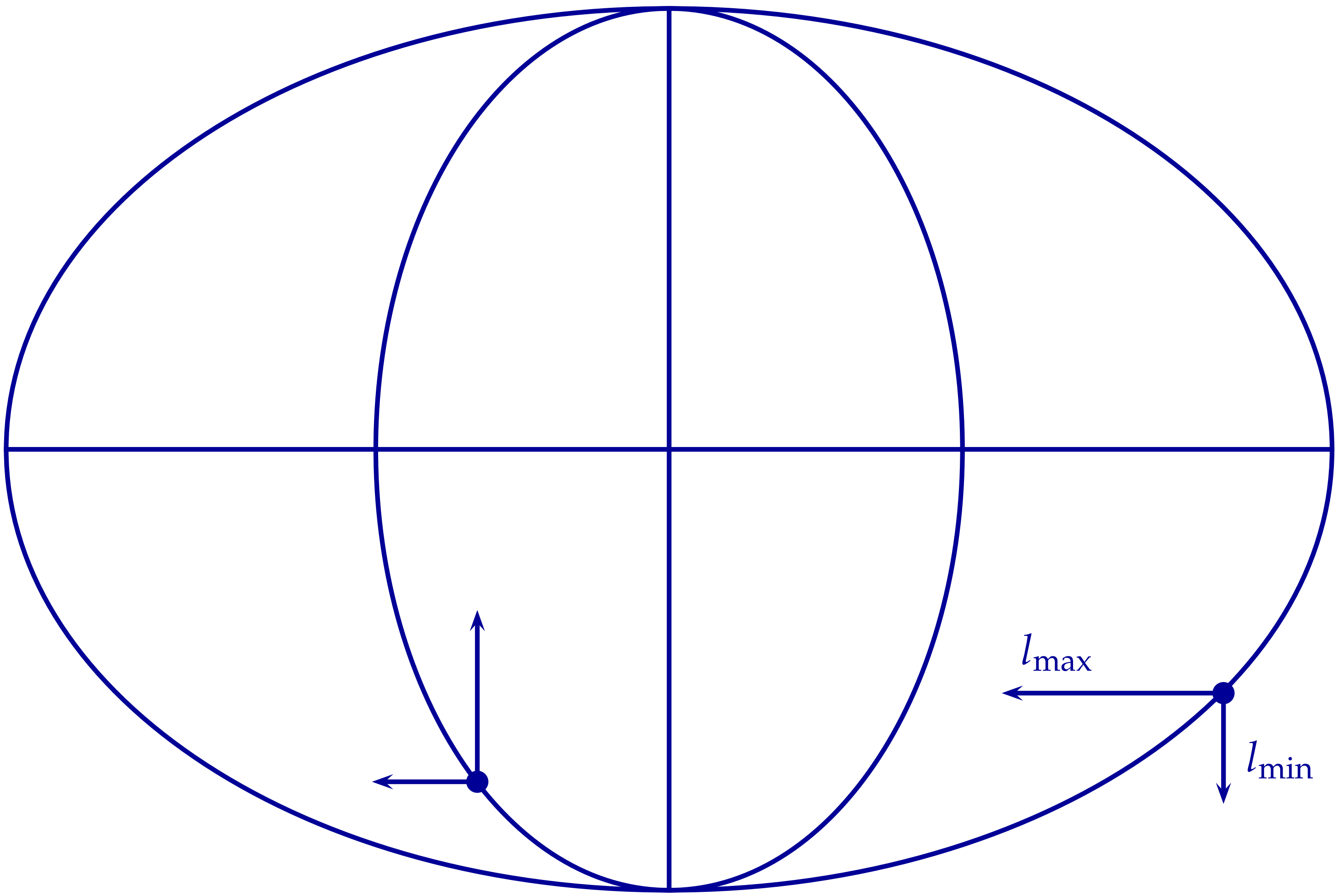

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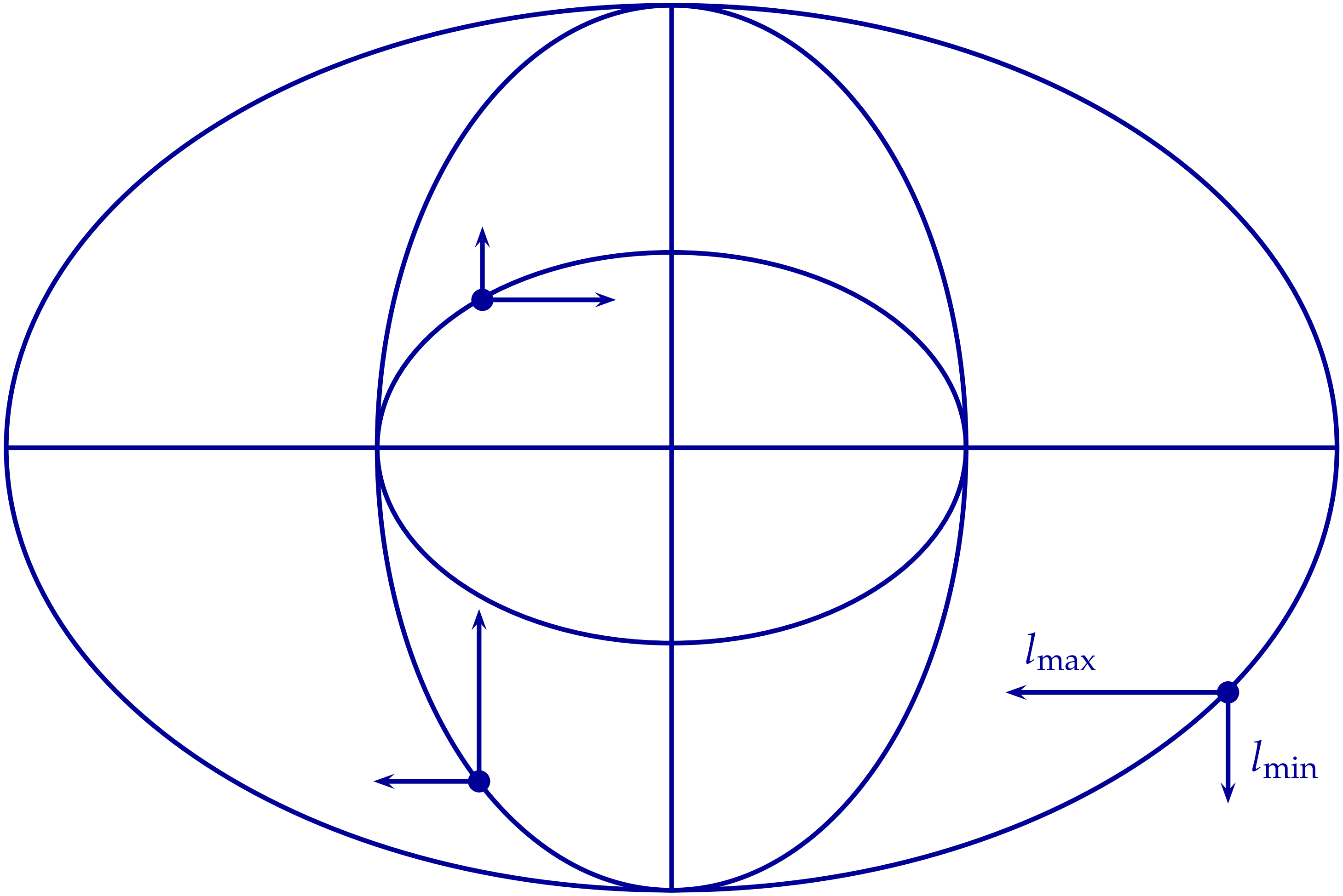
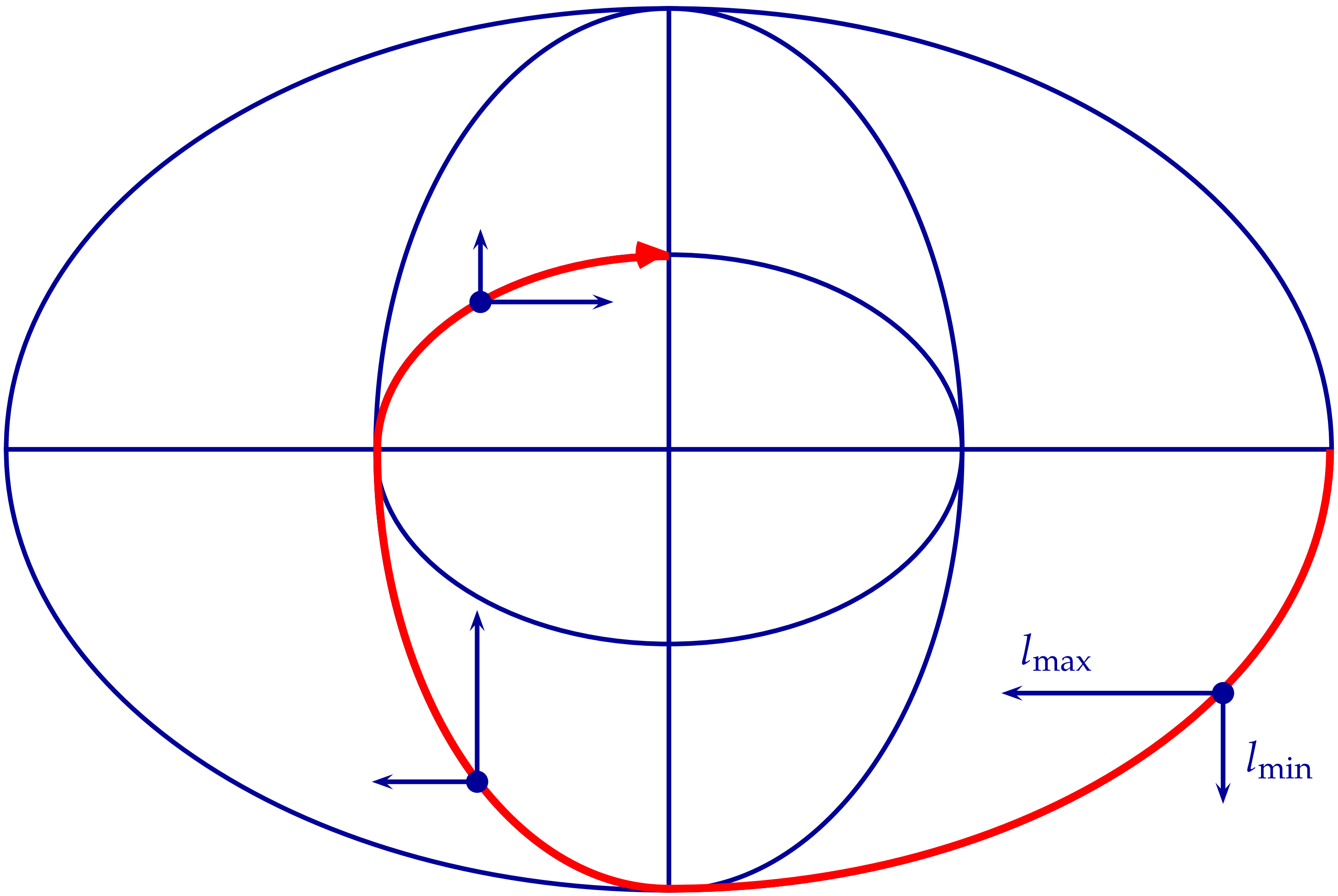


Compound trajectory



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Claim. *The learning parameters* $l_{\min}$ / $l_{\max}$ *alternate in such a way that the ellipse that forms the trajectory in clockwise movement "lies flat" when* (α, β) *is in* $Q1$ *and* $Q3$ *of the ellipse and "stands" otherwise.*

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Clearly, the reasoning is similar when
– the strategy pair (α, β) is in the other two quadrants, or
– when movement is counter-clockwise. $\square$

Part 4: Another solution

Why not utilise Singh *et al.*'s result on emp. frequencies

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- **Theorem** (Singh, Kearns and Mansour, 2000) *If players follow IGA, where* $\eta \to 0$*, then their strategies will converge to a Nash equilibrium. If not, then at least their average payoffs will converge to the expected payoffs of a Nash equilibrium.*

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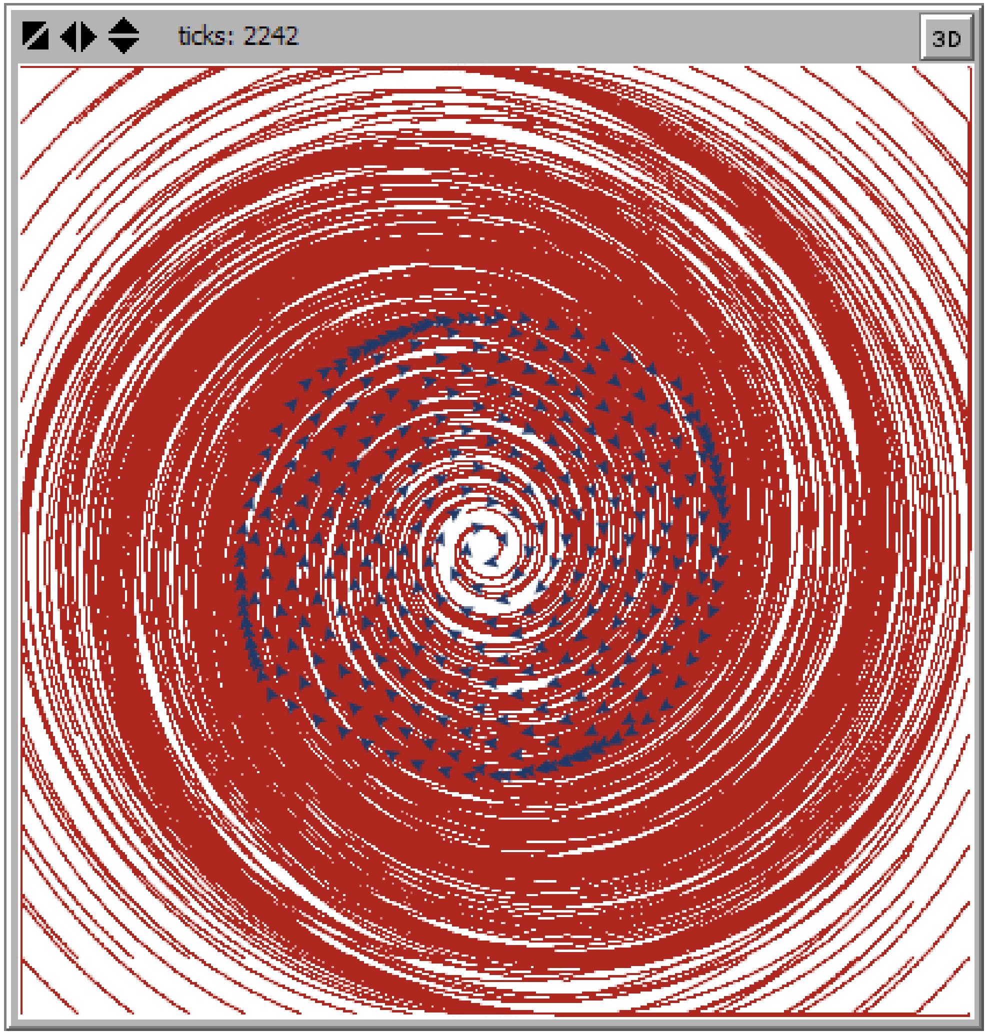

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Bowling and Veloso (2002). "Multiagent Learning Using a Variable Learning Rate". In: *Artificial Intelligence* **136**, pp. 215-250, 2002.

What next?

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- With fictitious play, or gradient ascent, opponents are modelled by a single mixed strategy.

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- With fictitious play, or gradient ascent, opponents are modelled by a single mixed strategy.
- With Bayesian play, opponents are modelled by a probability distribution over all opponent strategies

$$\Delta \left[\Pi_{j \neq i} \, \Delta(X_j)^H \right].$$

  - $\Delta(A)$ denotes the set of all probability distributions over A.
  - B^A denotes the set of all functions from A to B.
  - $\Pi_{j \neq i} A_j$ denotes the Cartesian product of $\{A_j\}_{j \neq i}$. In case of a finite product, this can be written as

$$\Pi_{j \neq i} A_j \; = \; A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n.$$