Multi-agent learning

Evolutionary game theory

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Wednesday 15th May, 2019

Plan for today

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1. Symmetric games in normal form.

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Definition. A game is symmetric when players have equal actions and payoffs:

$$u_i(a_1,\ldots,a_i,\ldots,a_j,\ldots,a_n) = u_j(a_1,\ldots,a_j,\ldots,a_i,\ldots,a_n).$$

for all i and j.

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So a 2-player game $G = (A, B)$ is symmetric iff $m = n$ and $B = A^T$.

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  Two asymmetric equilibria and one symmetric equilibrium $(1/3, 1/3)$.

Hawk vs. Dove

Evolutionary game theory

Evolutionary game theory: the idea

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- The average fitness is: $\bar{f} = \sum_{i=1}^5 p_i f_i = p^T Ap$.

Evolutionarily stable strategies (ESSs)

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Definition. Let A be a symmetric game. A state $p \in \Delta^m$ is evolutionarily stable if for every $q \in \Delta^m$, $q \neq p$, there is an $0 < \epsilon_q < 1$ such that for all $\epsilon < \epsilon_q$:

$$p^T A[(1-\epsilon)p + \epsilon q] \ > \ q^T A[(1-\epsilon)p + \epsilon q].$$

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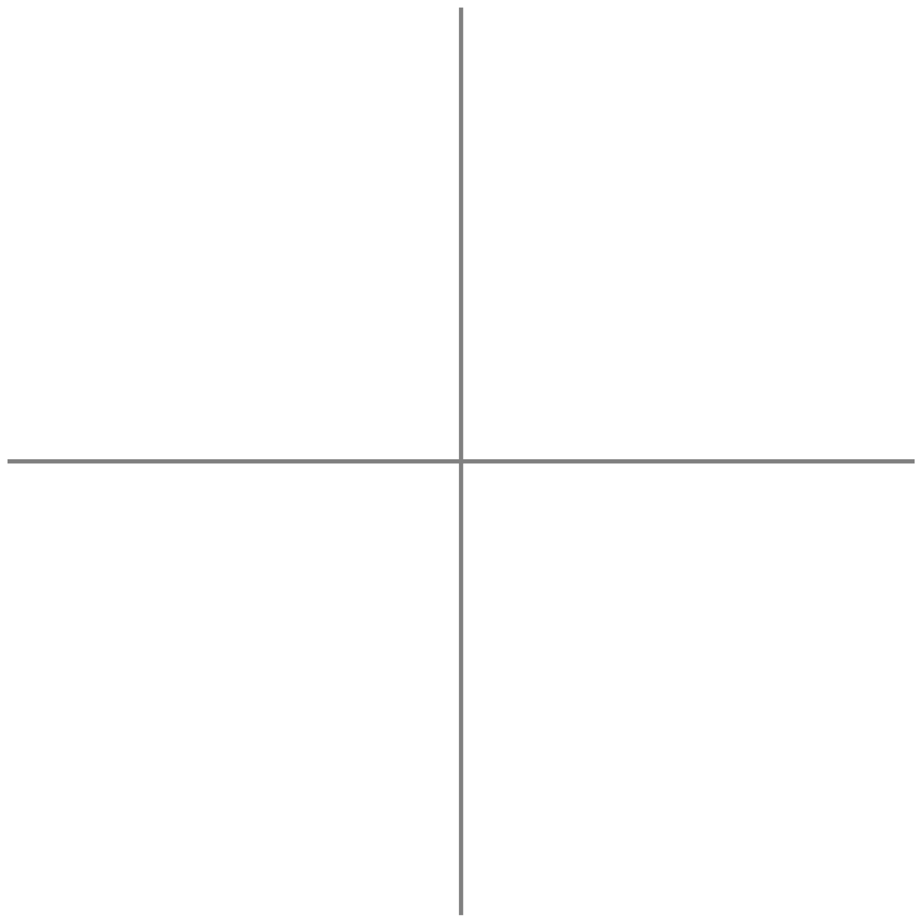

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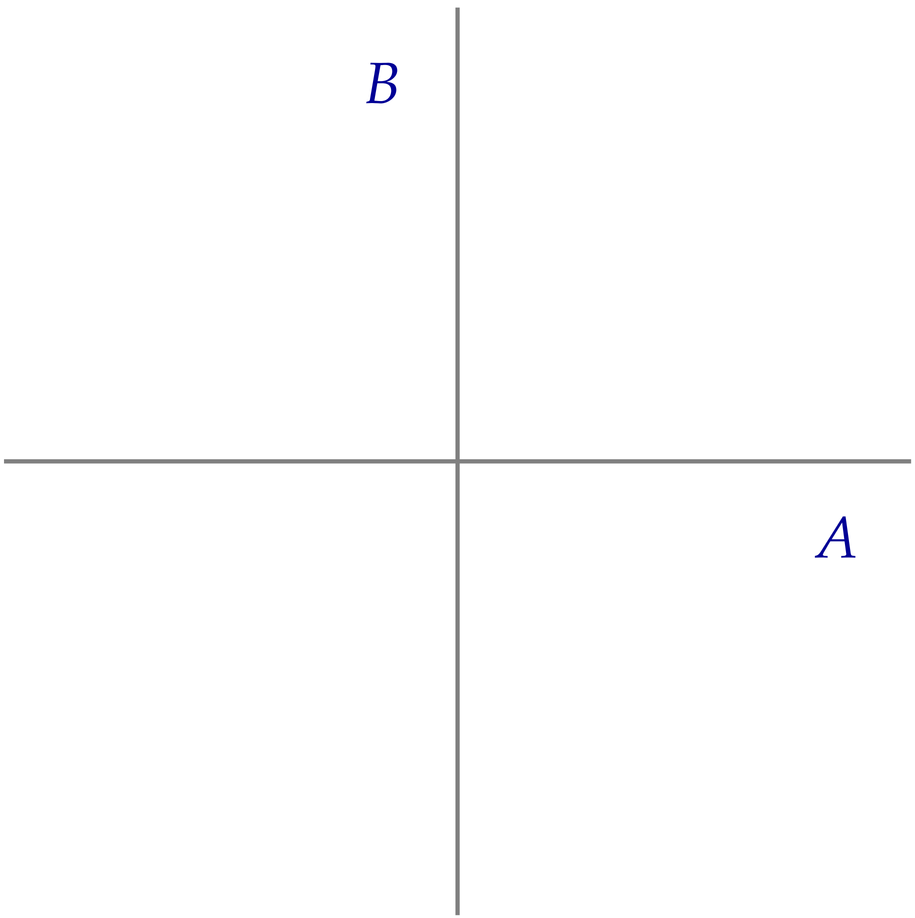

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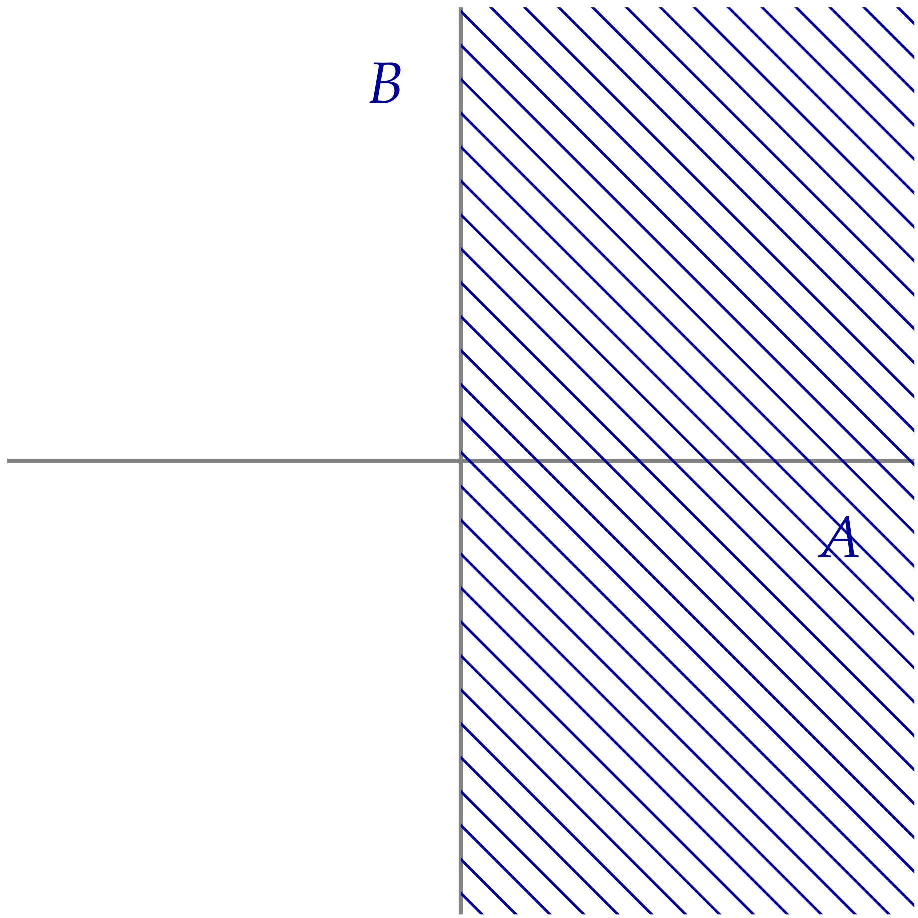

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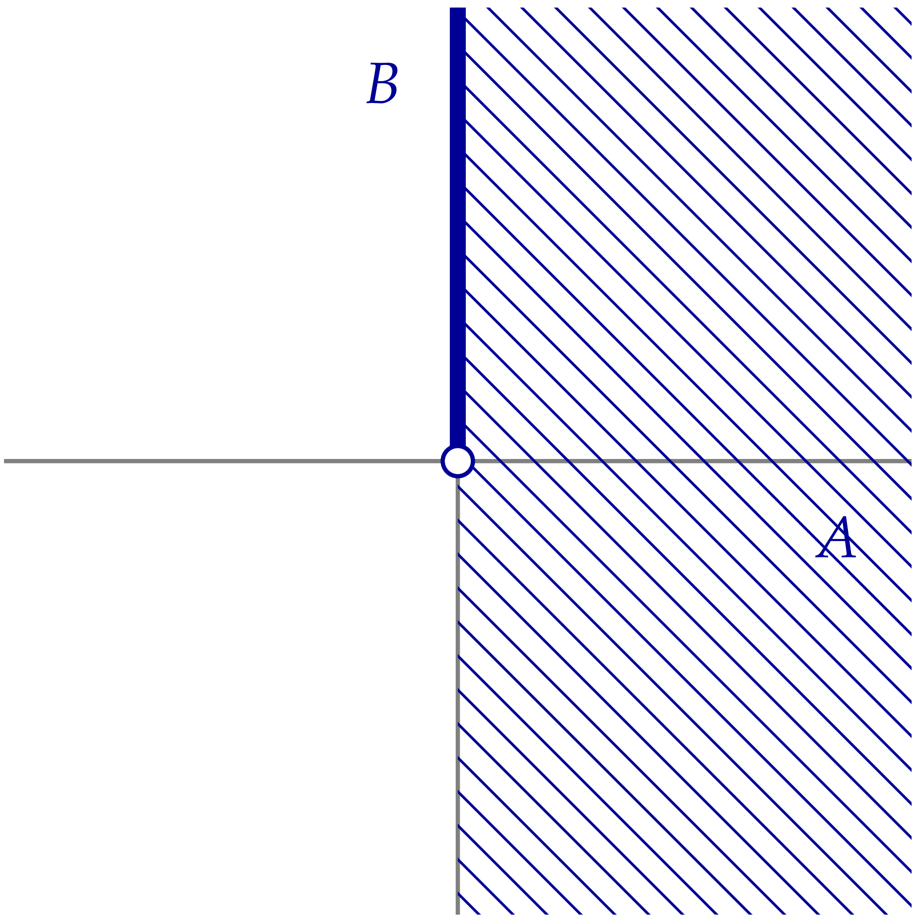

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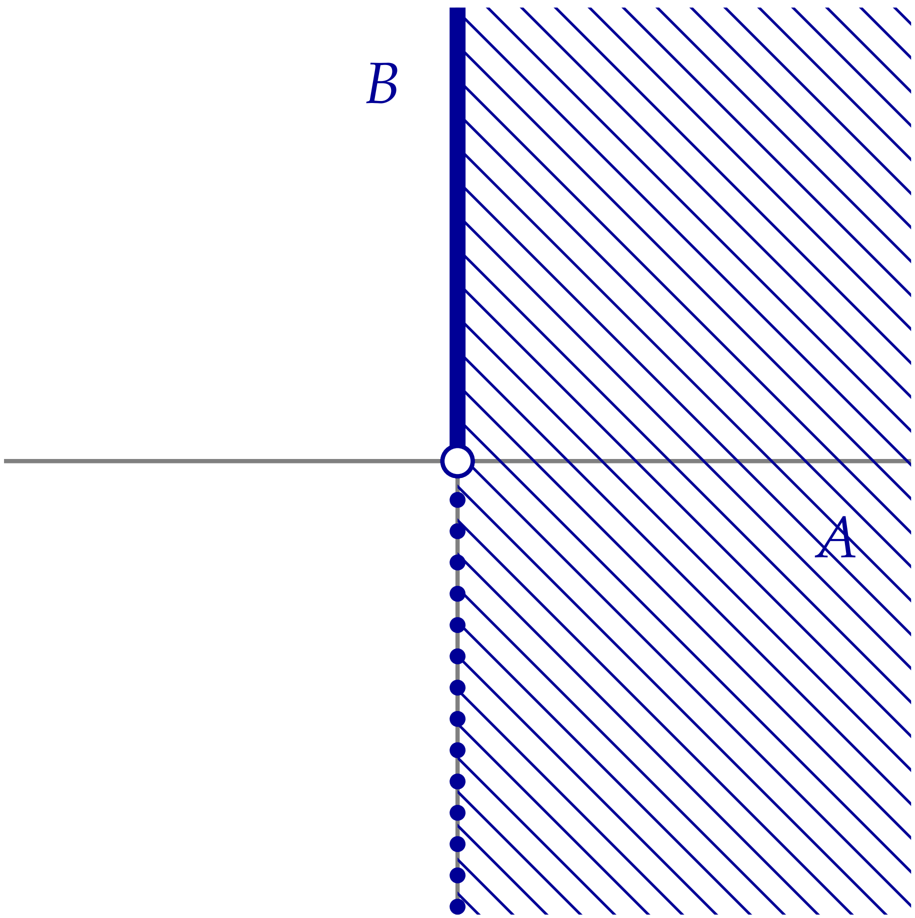

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A lemma

Lemma. Let A an B be real numbers. The following statements are equivalent:

1. For small $\epsilon > 0$:

$$(1-\epsilon)A + \epsilon B > 0.$$

2. $A > 0$, or $A = 0$ and $B > 0$.
3. $A \geq 0$, and $A = 0 \Rightarrow B > 0$.

Proof. (2$\Leftrightarrow$3) is proven through logical manipulation, or by shading areas in an A-B axis system.

Now for (1$\Leftrightarrow$2).

Evidently:

$$\lim_{\epsilon \to 0}(1-\epsilon)A + \epsilon B = A.$$

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Proof. Suppose for every $q \in \Delta^m$, $q \neq p$, there is an $0 < \epsilon_q < 1$ such that for all $\epsilon < \epsilon_q$:

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Finding ESS

Example. Hawk-dove game:

	H	D
H	0,0	3,1
D	1,3	2,2

Finding ESS

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$$p^T A q = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = -2y + 2\frac{1}{2},$$

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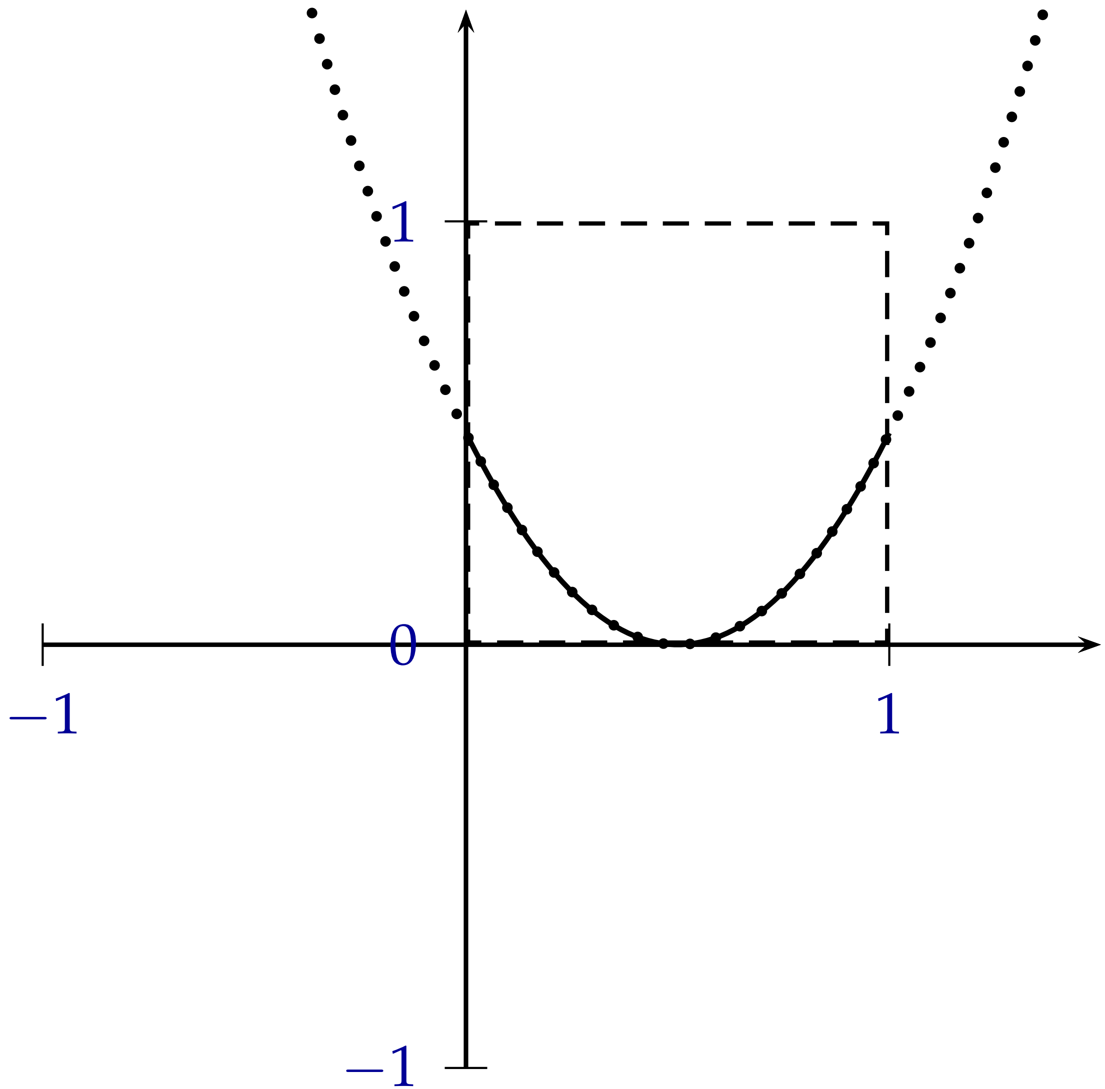
To verify that $p^T Aq > q^T Aq$

Equivalent statements:



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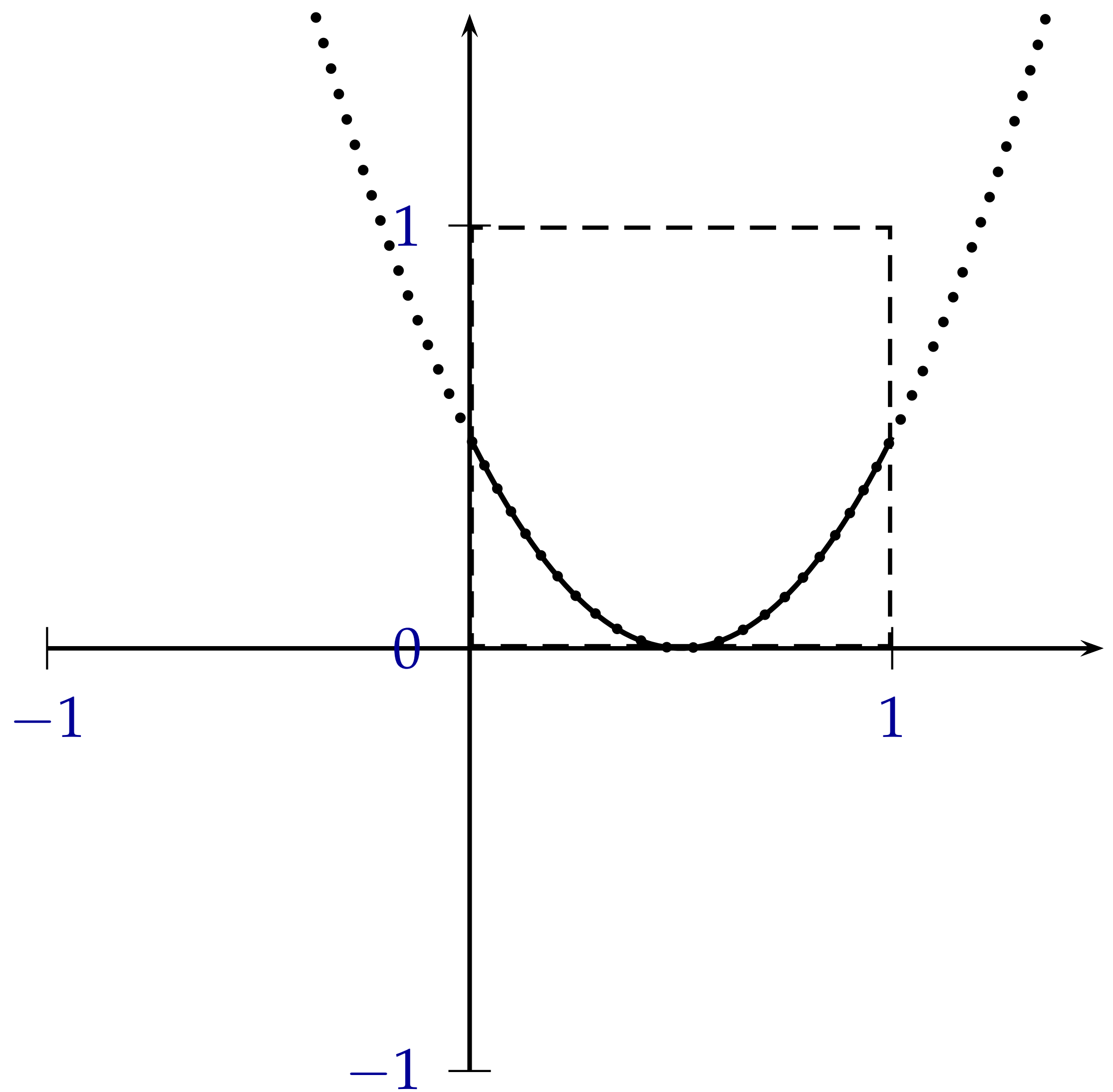

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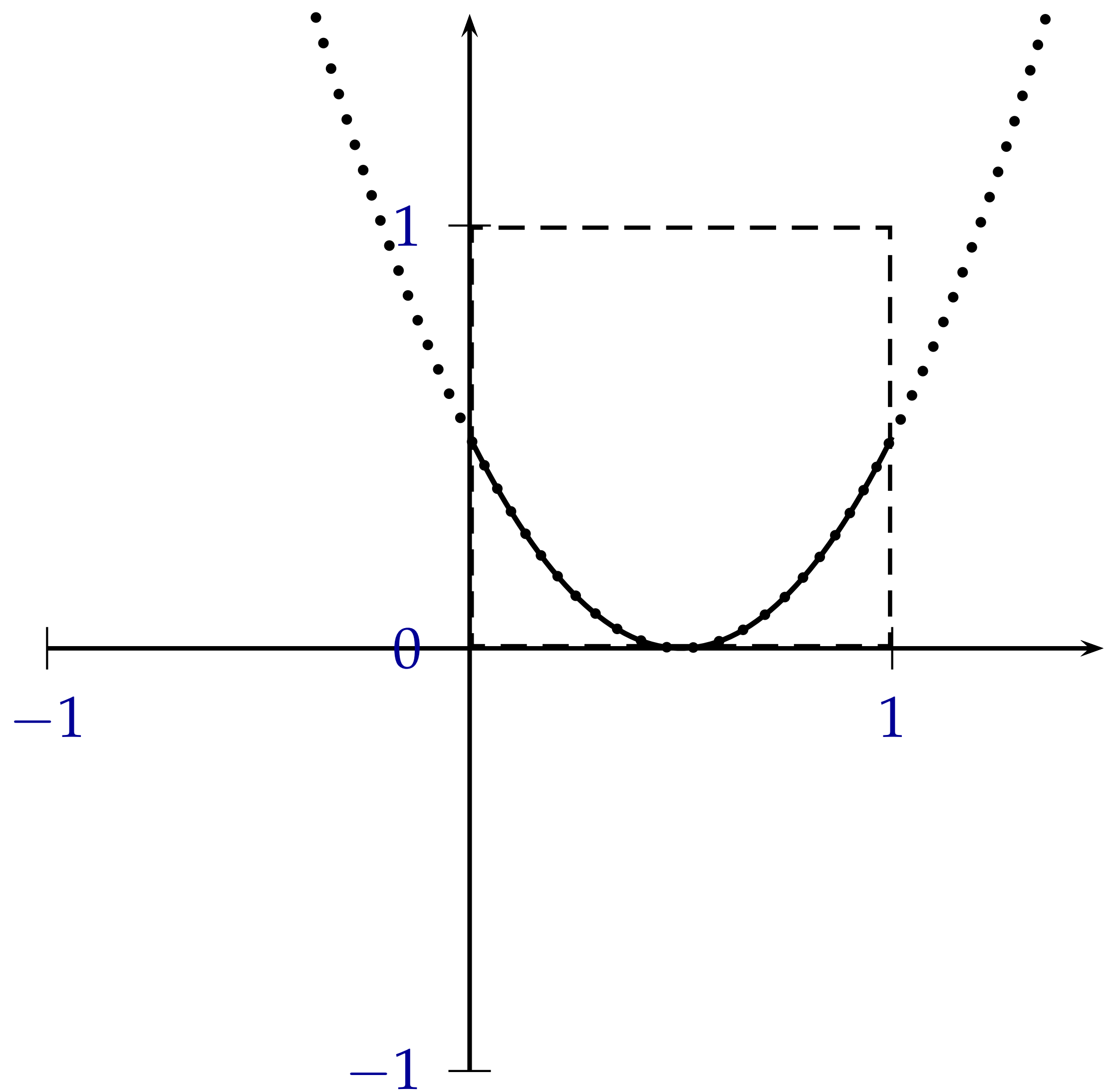

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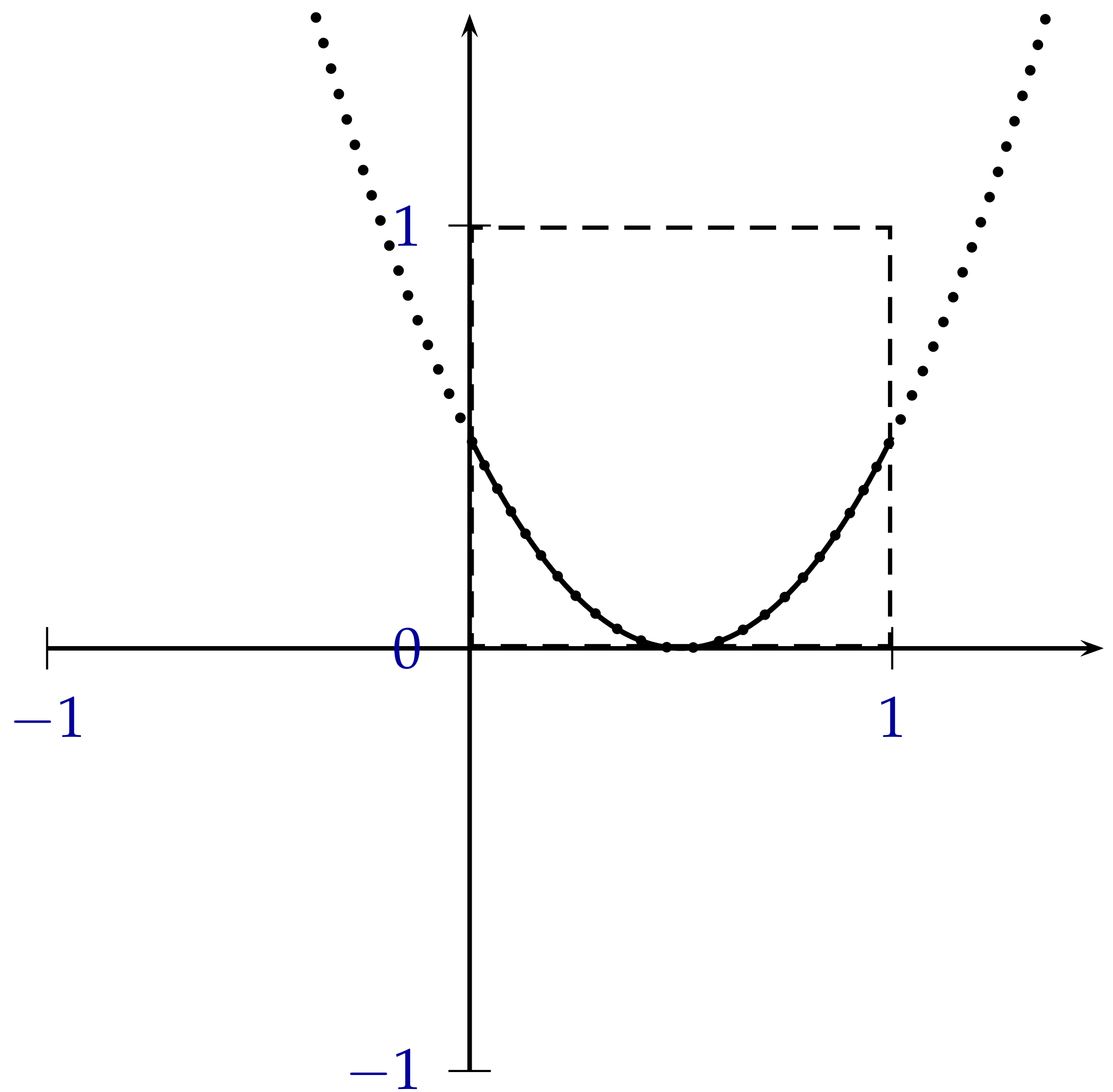

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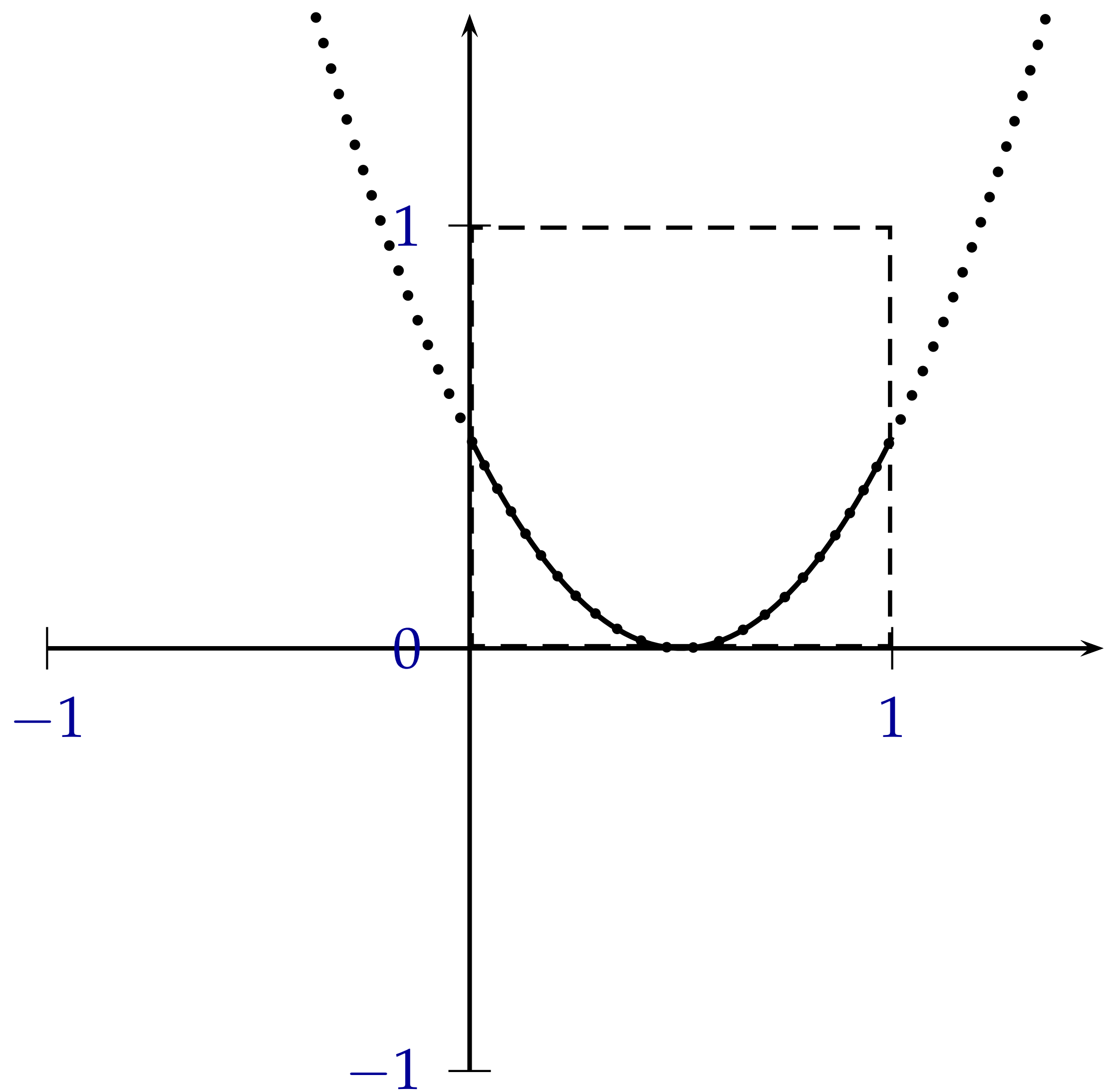

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None of the implications can be reversed, i.e., all set inclusions are strict.

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- For $\alpha = 0$, we have symmetric NE for all p.

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So if $\alpha p = 0$ then (p, p) is a symmetric NE.

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Claim: $\Delta^{\text{NE}<} \subset \Delta^{\text{ESS}} \subset \Delta^{\text{NSS}} \subset \Delta^{\text{NE}}$, i.e., all inclusions are strict.

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- Similarly, $\alpha \in \{-1, 1\}$ makes the last two inclusions strict. $\square$

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Question to contemplate: is it OK that ESSs may not exist?

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Theorem. If $p, q \in \Delta^m$ such that p is an ESS, $q \neq p$, and $\text{support}(q) \subseteq \text{support}(p)$, then $q \notin \Delta^{\text{NE}}$.

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1. The set Δ^{ESS} is finite and possibly empty.

2. If an ESS is fully mixed, it is unique.

The replicator equation

Idea of the replicator equation

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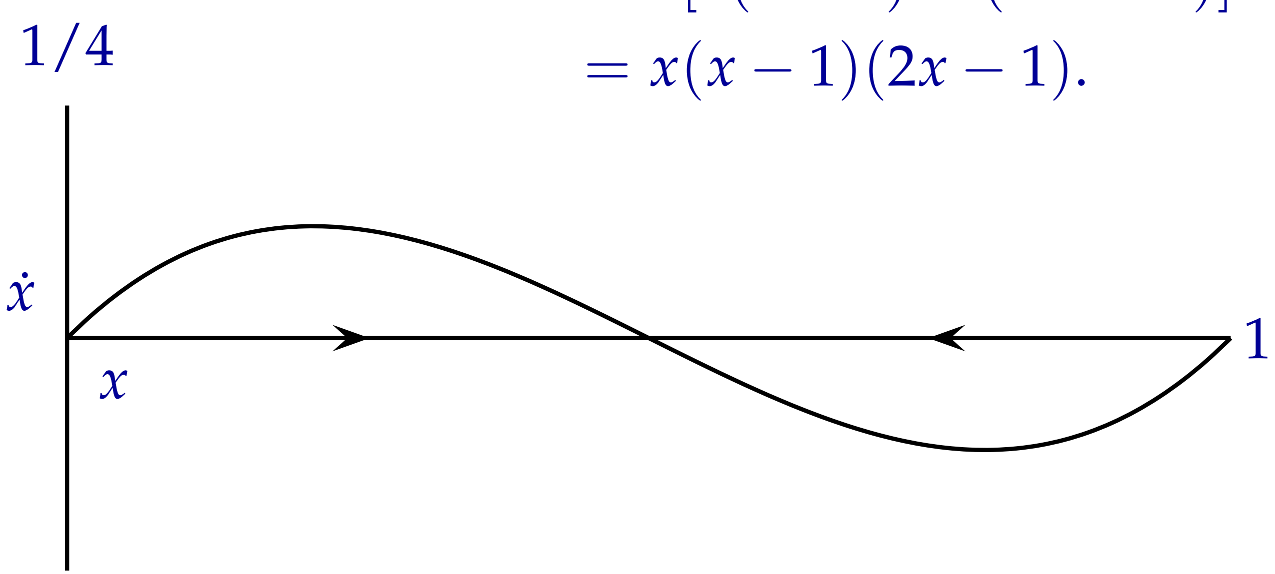

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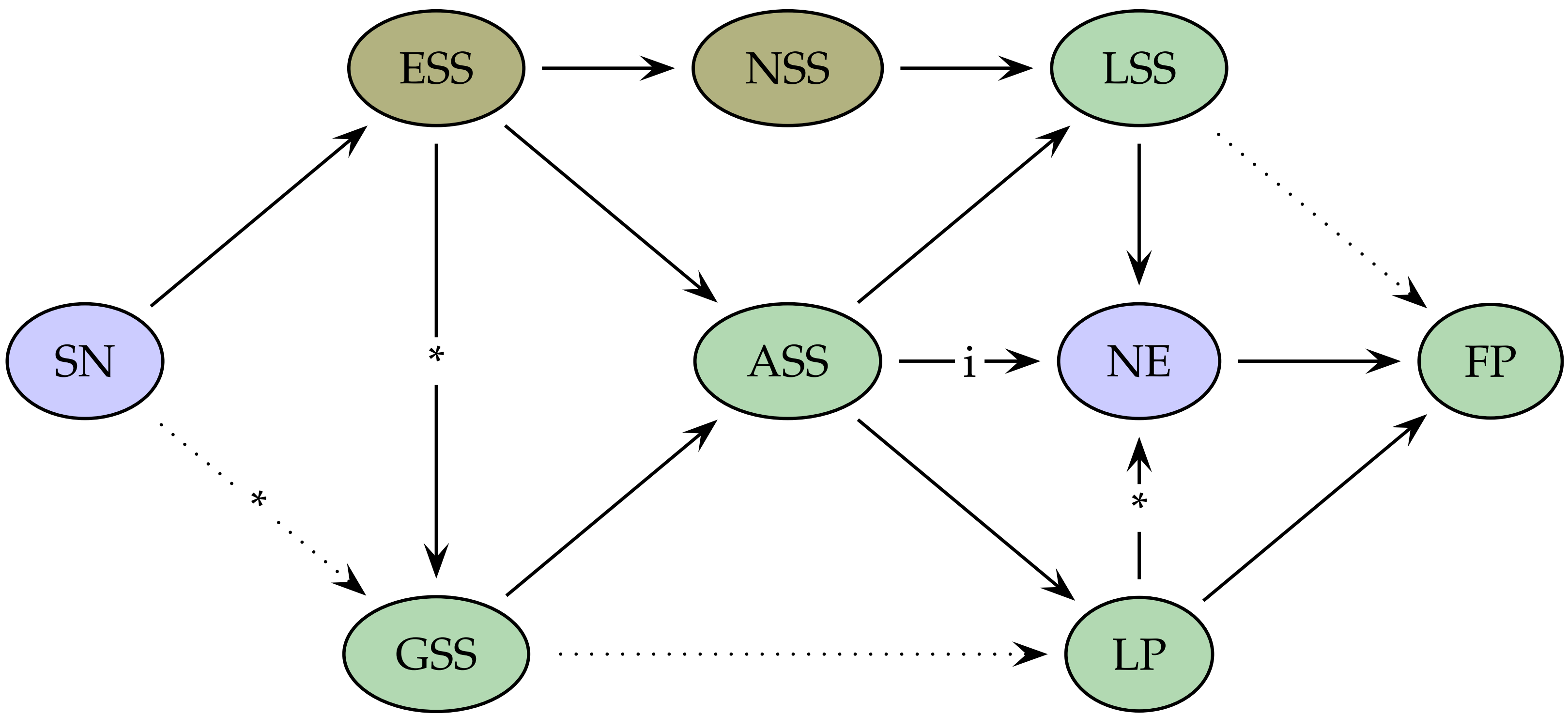
    These are further discussed in, e.g.,

    * H. Peters (2008): *Game Theory: A Multi-Leveled Approach.* Springer, ISBN: 978-3-540-69290-4. Ch. 15: Evolutionary Games.

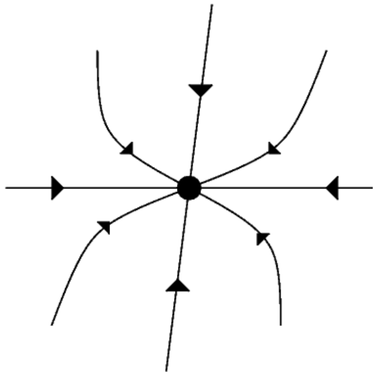
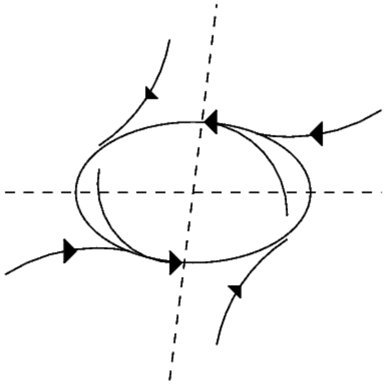
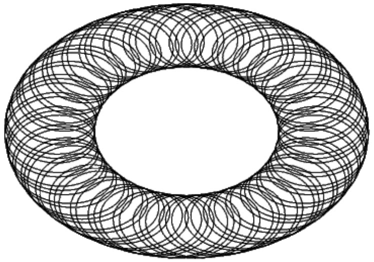
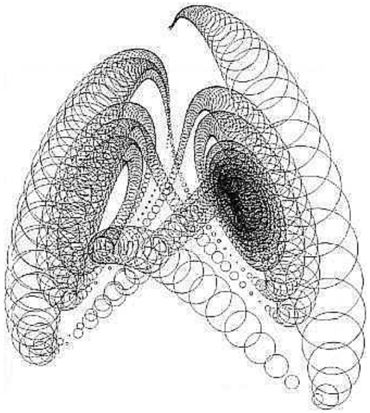
Implications



SN = strict Nash, ESS - evol'y stable strategy, GSS = glob'y stable state, ASS = asymp'y stable state, NSS = neutrally stable strategy, LP = limit point, LSS = Lyapunov stable state, NE = Nash eq., FP = fixed point, * = only if fully mixed, i = isolated Nash eq. Dotted lines are indirect implications.

The dynamics of the replicator equation

Possible dynamics

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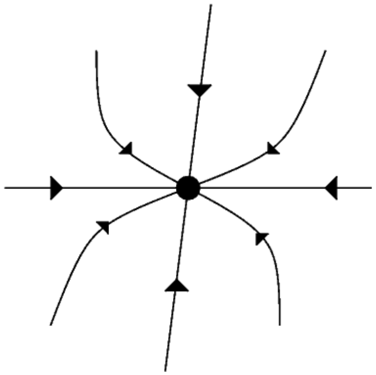
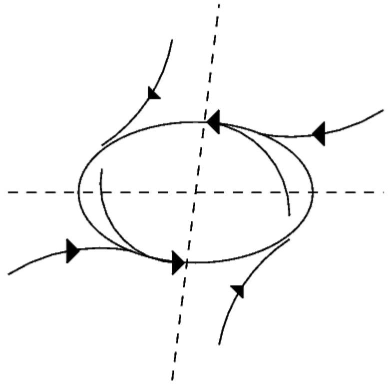
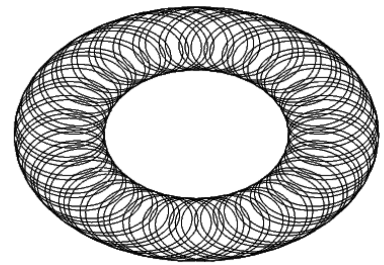
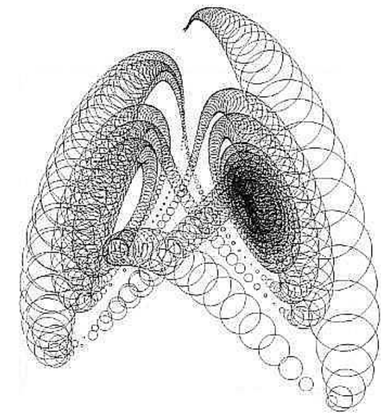
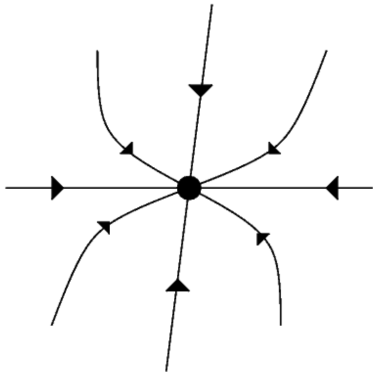
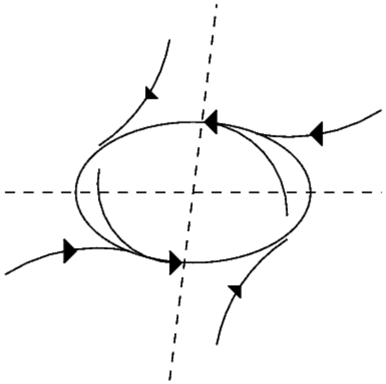
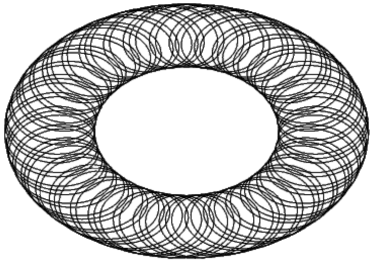
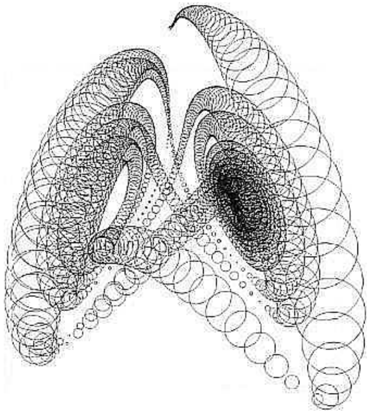
   

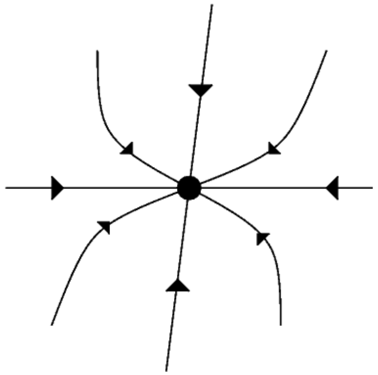
1. **Stability**.

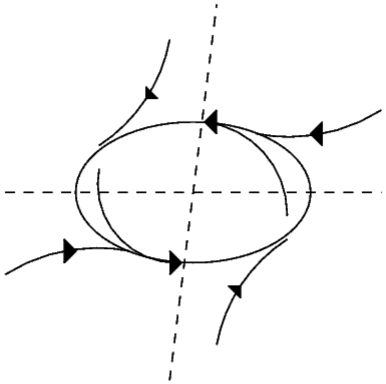
Possible dynamics

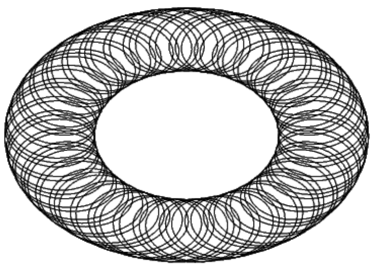
   

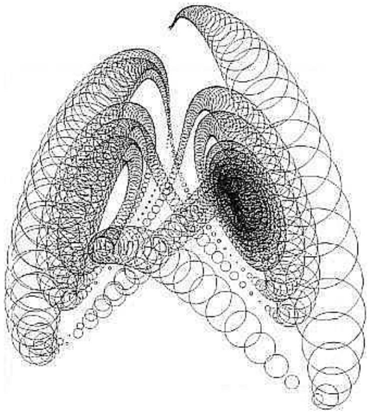
1. **Stability**. Convergence to a fixed proportion.

Possible dynamics





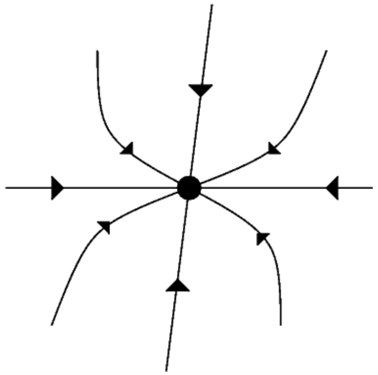




1. **Stability**. Convergence to a fixed proportion.

2. **Periodicity**.

Possible dynamics



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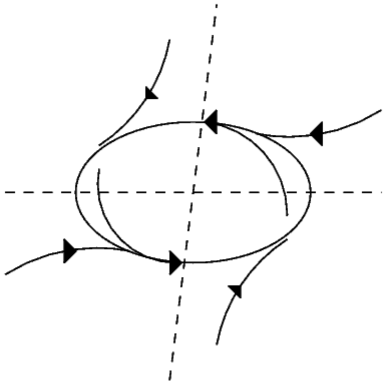
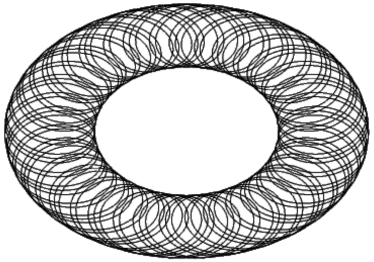
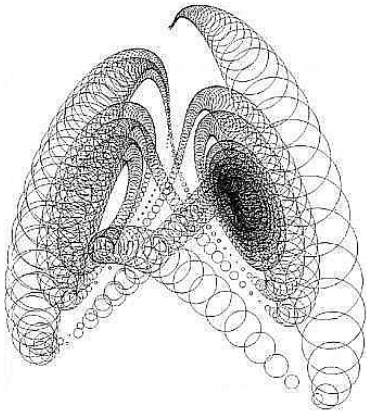


2. **Periodicity**. Convergence to a cycle of proportions





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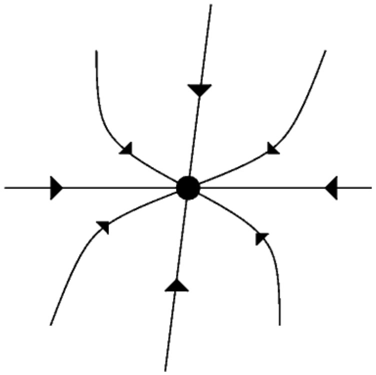
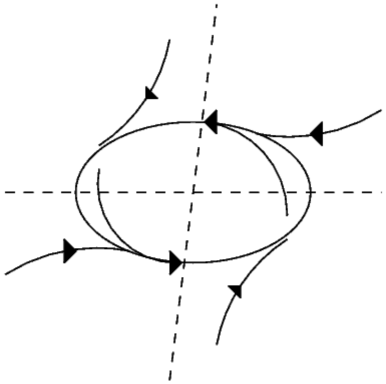
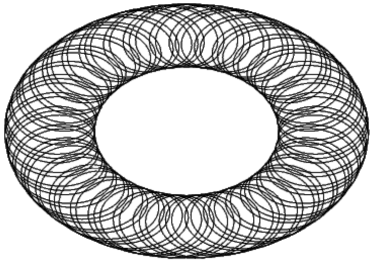
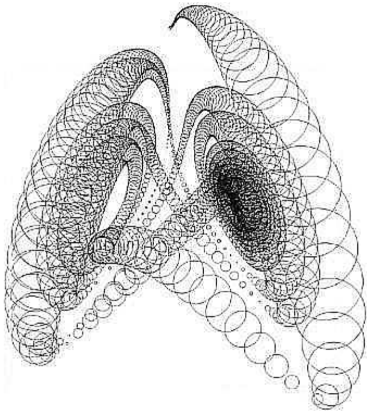


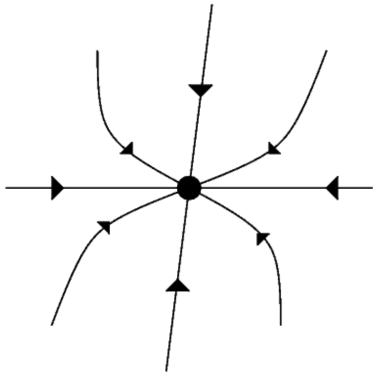






1. **Stability**. Convergence to a fixed proportion.
2. **Periodicity**. Convergence to a cycle of proportions, with some period $n \geq 2$.

Possible dynamics



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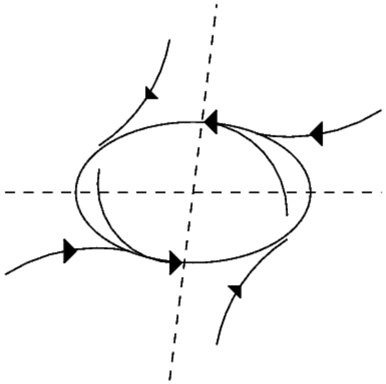

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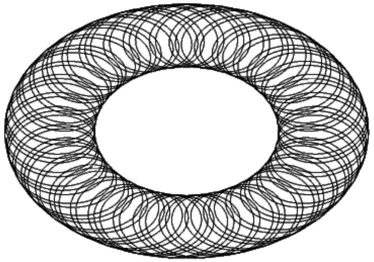
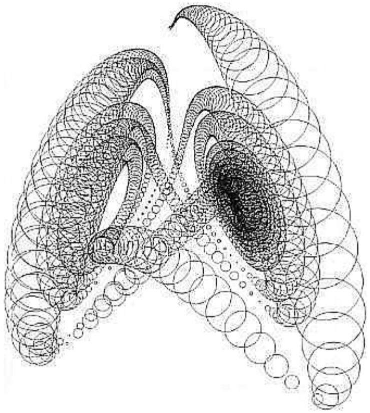


3. **Semi-periodicity**.



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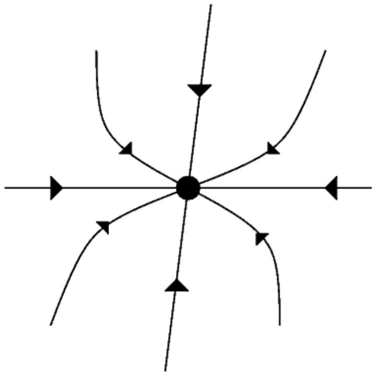

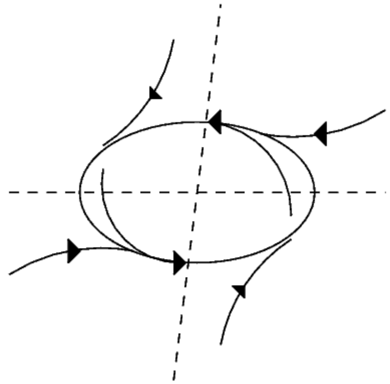

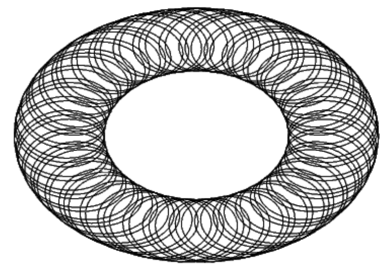

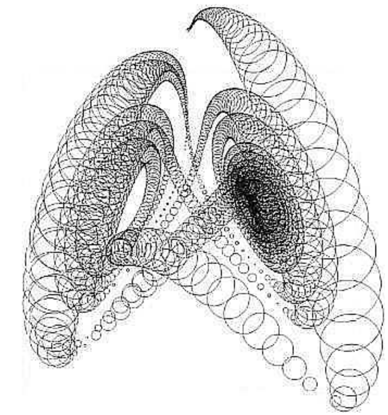

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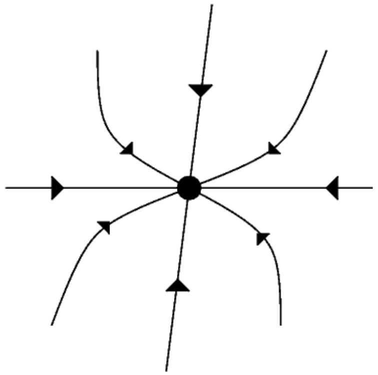

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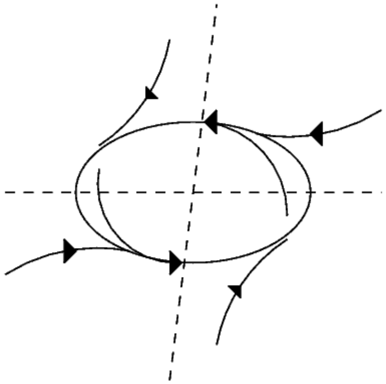

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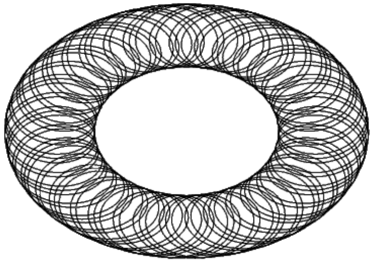

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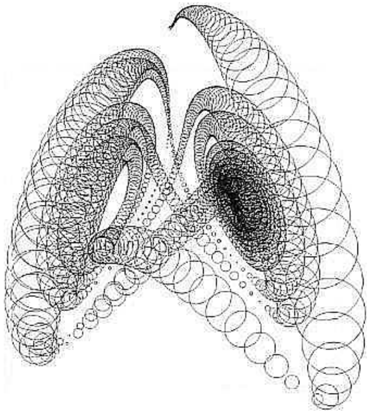
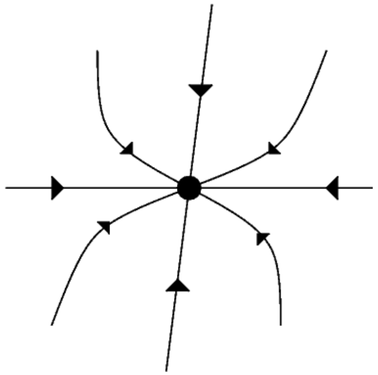


4. **Chaos**.

Possible dynamics



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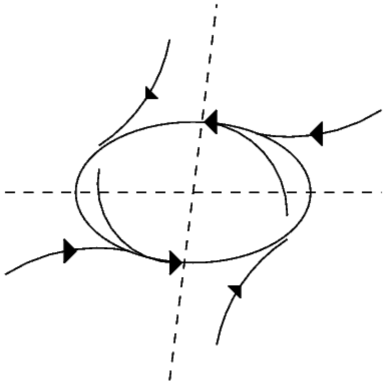

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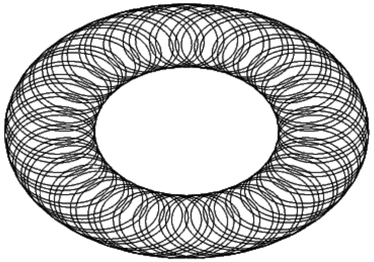

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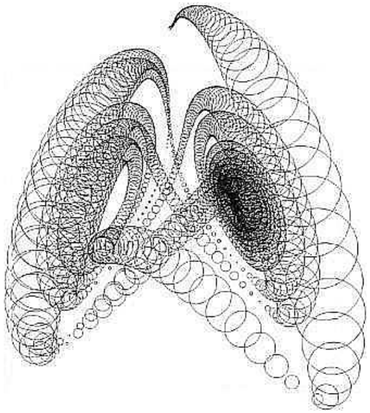
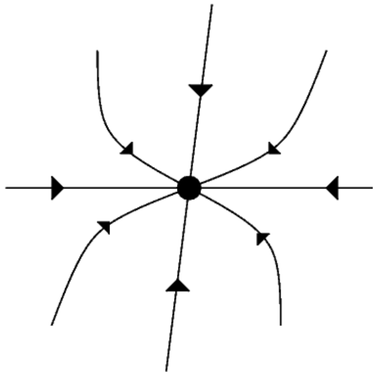


4. **Chaos**. Like 1, 2 and 3 deterministic, but otherwise different.

Predictable only by execution.

Possible dynamics



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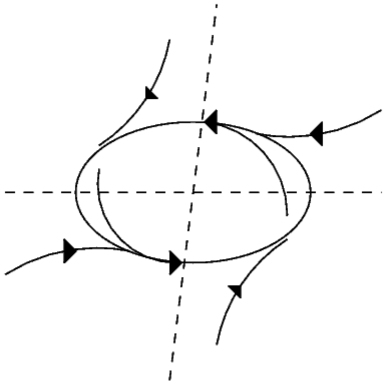

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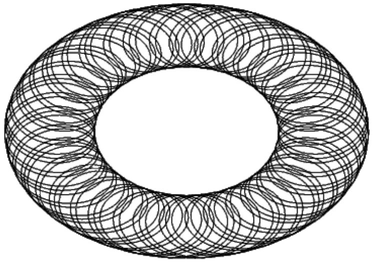

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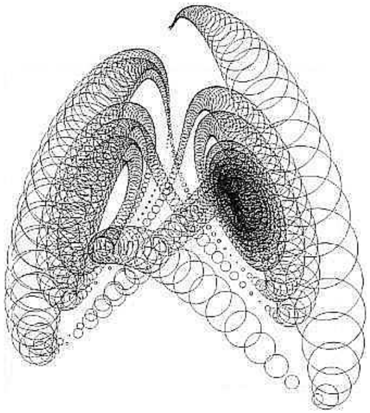

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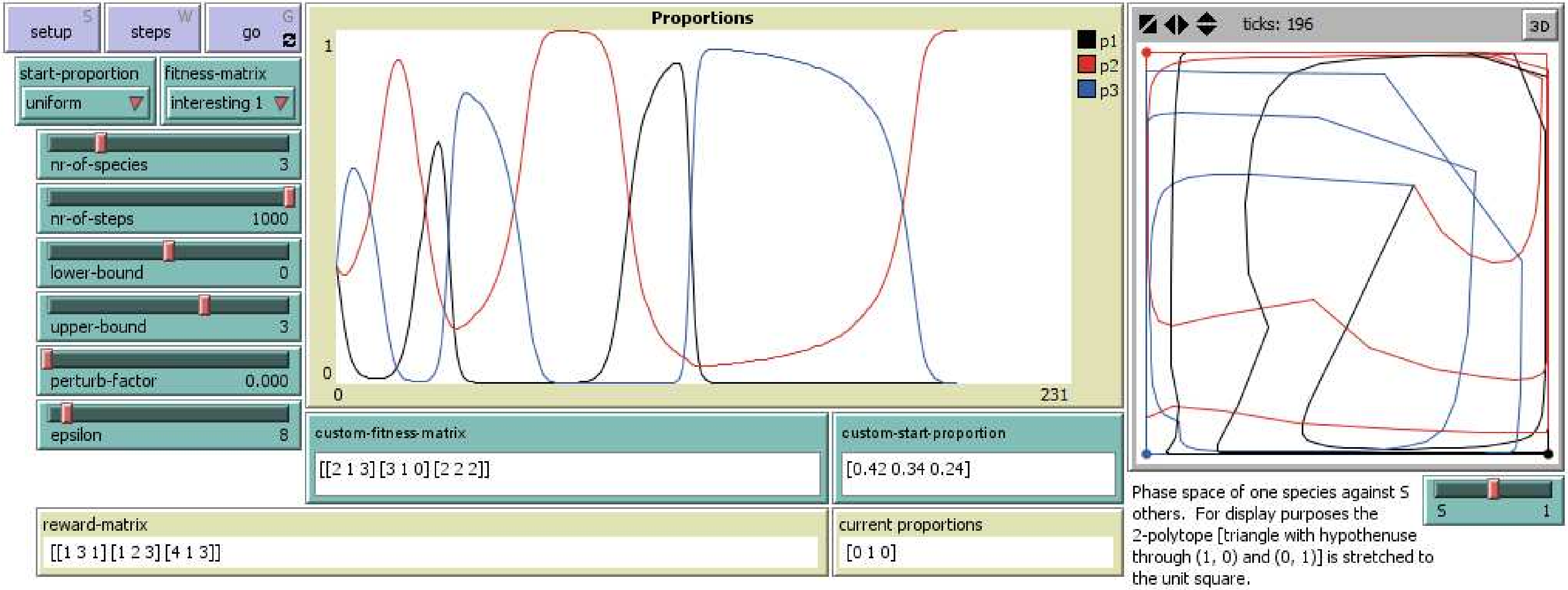
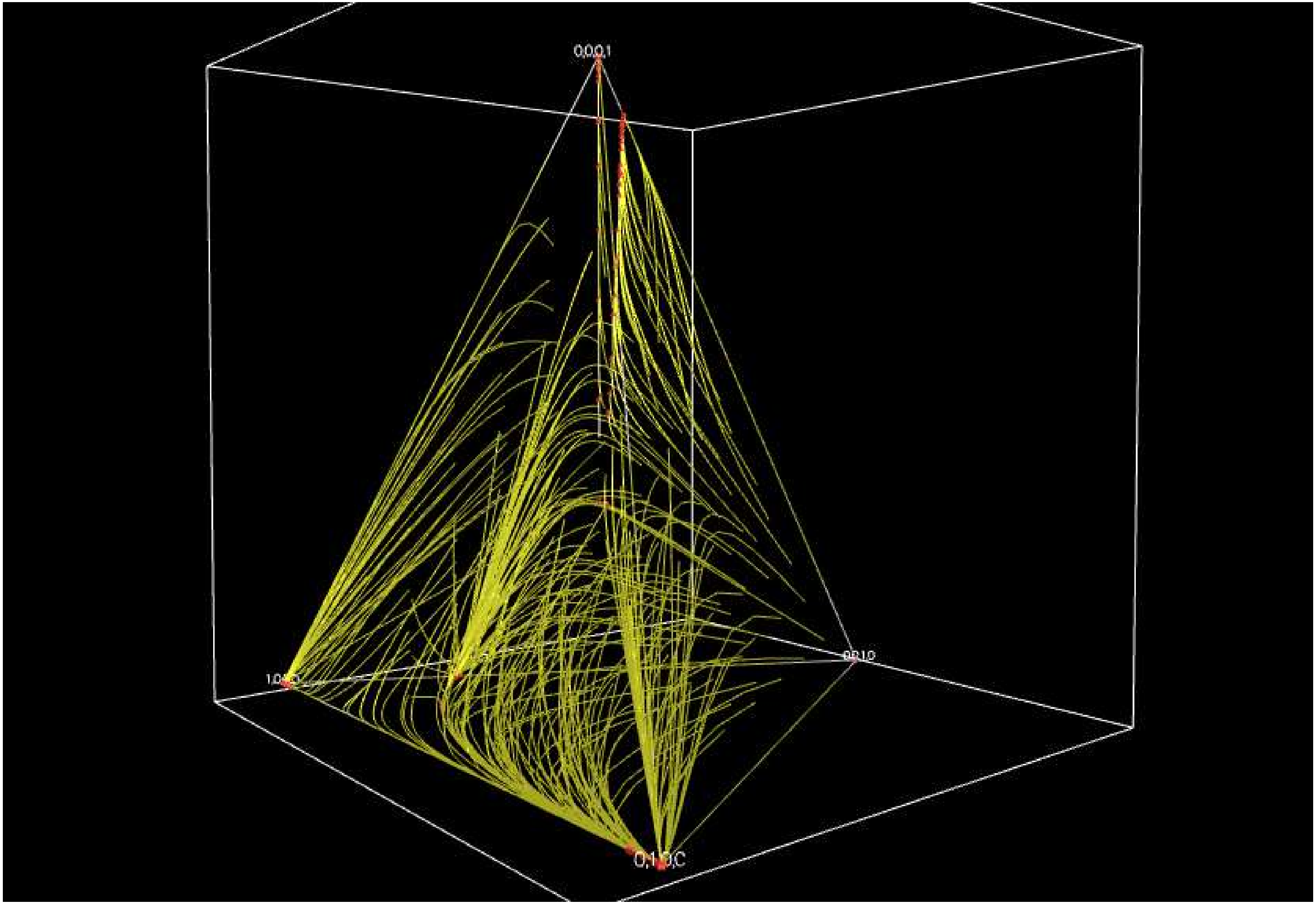
Images from: *The Computional Beaty of Nature*, W.G. Flake (1998).

The dynamics of the replicator equation



Relative score matrix $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix}$, initial proportions $p = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$.

The dynamics of the replicator equation



Symmetric 2x2 games

	L	R
T	R, R	S, T
B	T, S	P, P

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- R, S, T, and P represent payoffs named *reward*, *sucker*, *temptation* and *punishment*, respectively.

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- So all interesting symmetric 2×2 games can be generated by

$$(S, T) \in [-\epsilon, 1+\epsilon] \times [-\epsilon, 1+\epsilon], \;\; R = 1, \; P = 0, \; \epsilon > 0.$$

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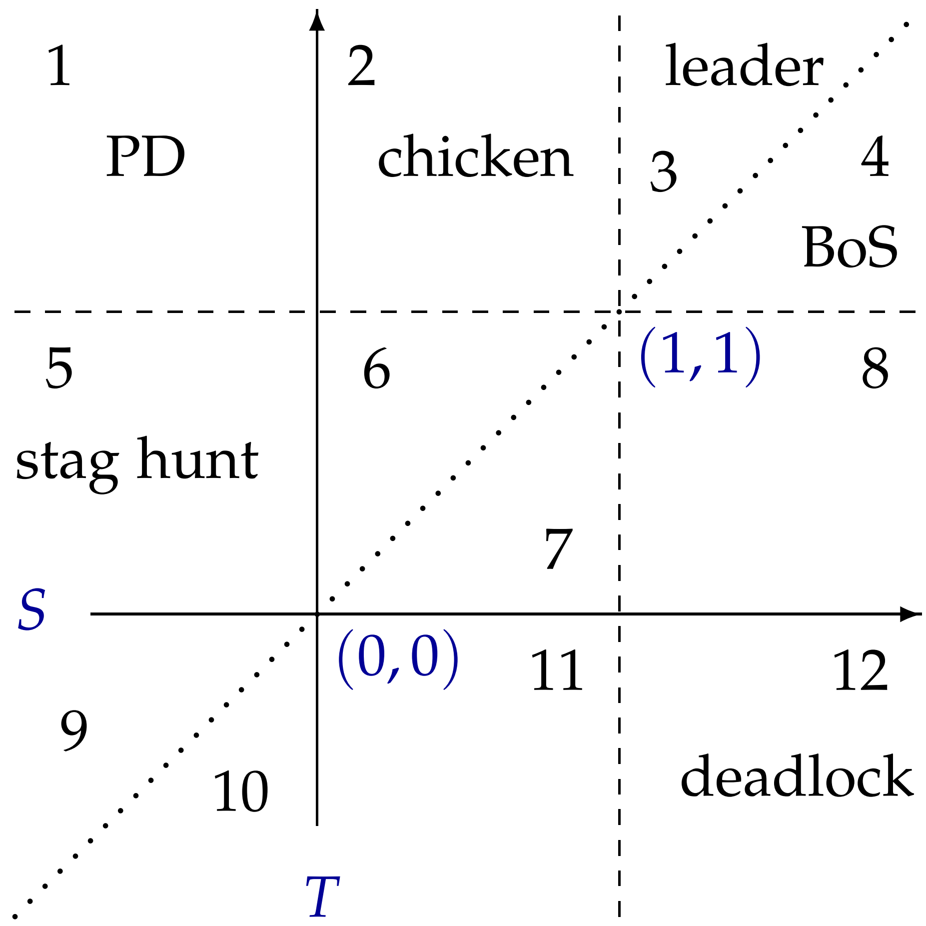
- For simplicity, take $\epsilon = 1$ so that (S, T) is in $[-1, 2] \times [-1, 2]$.

The (S,T) plane



Partition of the (S, T) plane which displays various symmetric 2×2 games.

A discrete replicator equation

A discrete replicator equation

- Of the existing discrete replicator equations, two of them are well known. The simplest of these two is

$$x^i_{t+1} = x^i_t + x^i_t(f^i_t - \bar{f}_t), \tag{1}$$

where x^i represents the ratio (proportion) of species i, f^i represents the fitness of species i, and $\bar{f}$ represents the average fitness.

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- Eq. (1) can perhaps best be understood by assuming that the growth of x_i in one time unit is entirely determined by the difference in fitness and average fitness:

$$\frac{x^i_{t+1} - x^i_t}{1} = x^i_t(f^i_t - \bar{f}_t). \tag{2}$$

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  In this way, Eq. (2) very much resembles the continuous replicator equation $\dot{x}^i = x^i_t(f^i_t - \bar{f}_t)$.

A discrete replicator equation

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- Suppose the RSTP-game is played, x_t represents the proportion with which the first action is played, $1 - x_t$ represents the proportion with which the second action is played, and the payoffs are plugged into the discrete replicator equation (1).

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- We obtain

$$f_t^1 = x_t R + (1 - x_t)S, \quad f_t^2 = x_t T + (1 - x_t)P, \quad \bar{f}_t = x_t f_t^1 + (1 - x_t) f_t^2,$$

$$x_{t+1} = x_t + x_t(f_t^1 - \bar{f}_t) \tag{3}$$

$$= x_t + x_t(1 - x_t)\left[S - P + (P - S + R - T)x_t\right]. \tag{4}$$

The step from (3) to (4) follows with some algebra.

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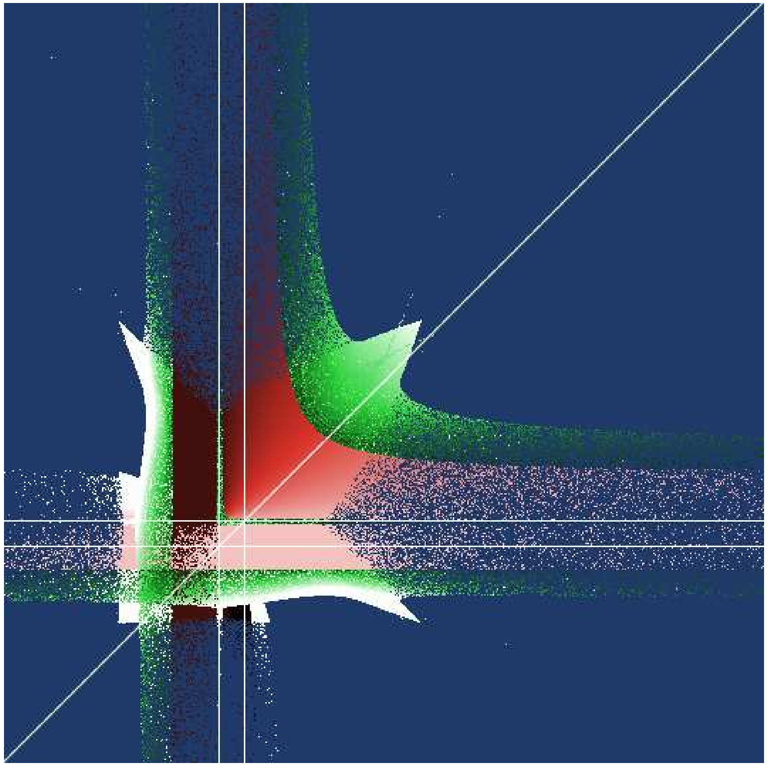
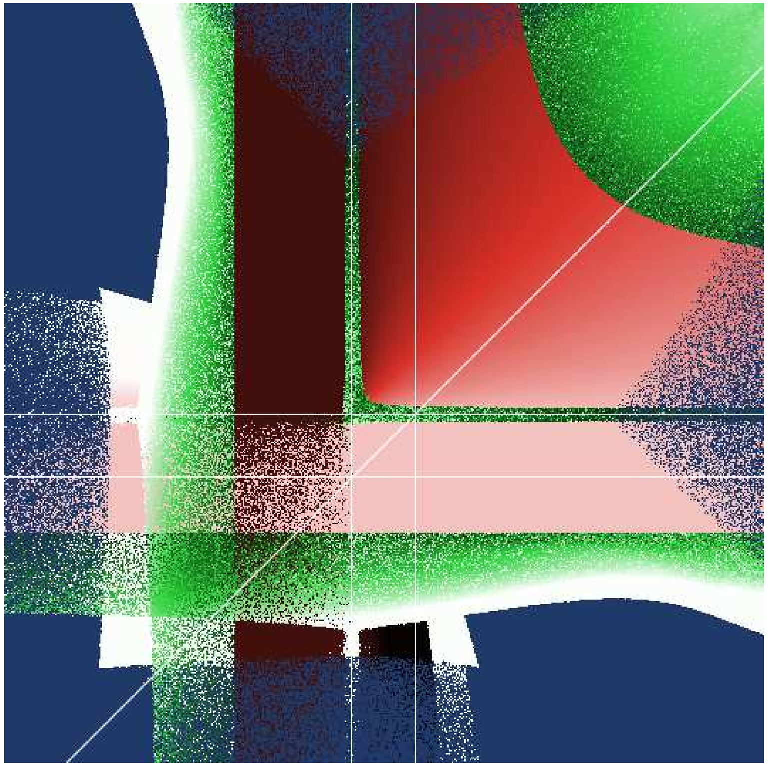
The step from (3) to (4) follows with some algebra.

- If (4) with $R = 1$ and $P = 0$ is iterated 150 times on the (S, T) square $[-8.5, 21.5] \times [-8.5, 21.5]$ with random start values picked from $[0, 1]$ and the first 50 iterations are thrown away (the so-called *transient phase*), we obtain [next page]

Convergence / divergence of the discrete replicator



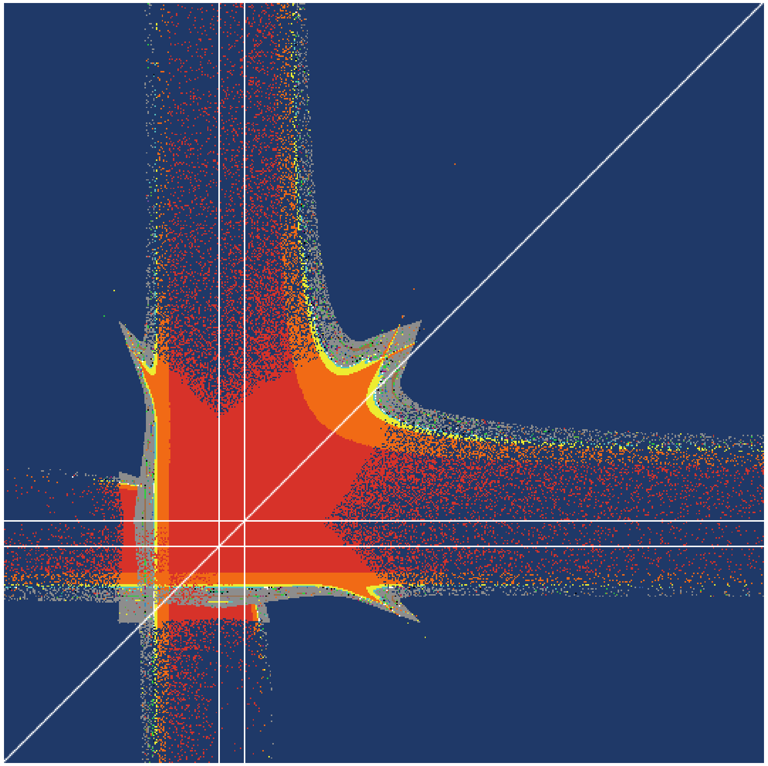
$[-8.5, 21.5] \times [-8.5, 21.5]$



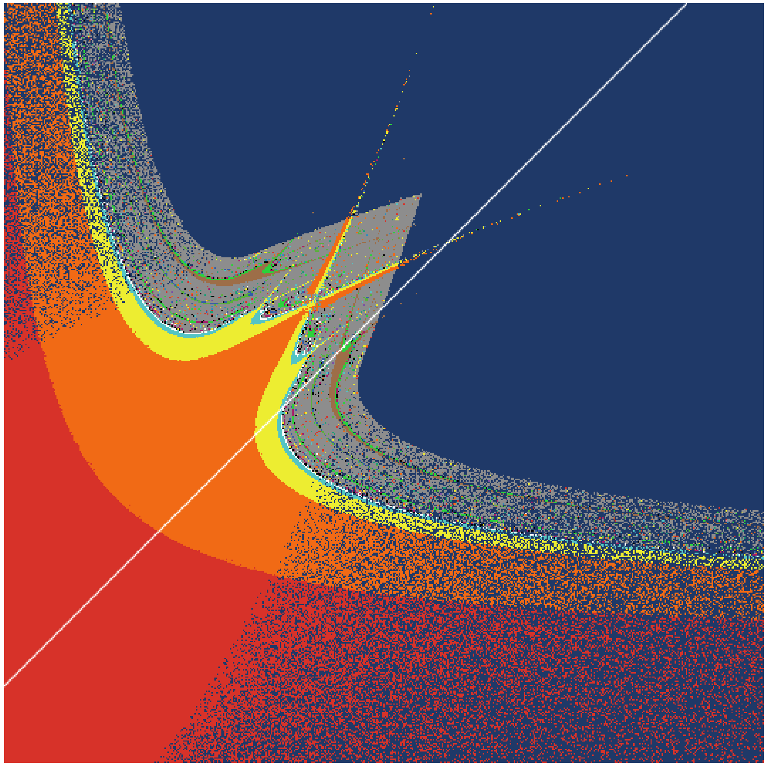
$[-5.5, 6.5] \times [-4.5, 7.5]$

End values (if any) in the (S, T)-plane. Red: fixed point. Dark red: low value; light red: high value. Green: divergent but bounded. Dark green: small amplitude; light green: large amplitude. Blue: divergent.

Bifurcation plot of the discrete replicator



$[-8.5, 21.5] \times [-8.5, 21.5]$



$[2.5, 12.5] \times [1.5, 11.5]$

Periods (if any) in the (S, T)-plane. Grey: 0 (chaotic); red: period 1 (fixed point); orange: period 2; brown: period 3; yellow: 4; green: 5; lime: 6; turquoise: 7; cyan: 8; sky: 9; blue: 10; violet: 11; magenta: 12; pink: 13; white: 16; black: greater than 13 but $\neq$ 16.

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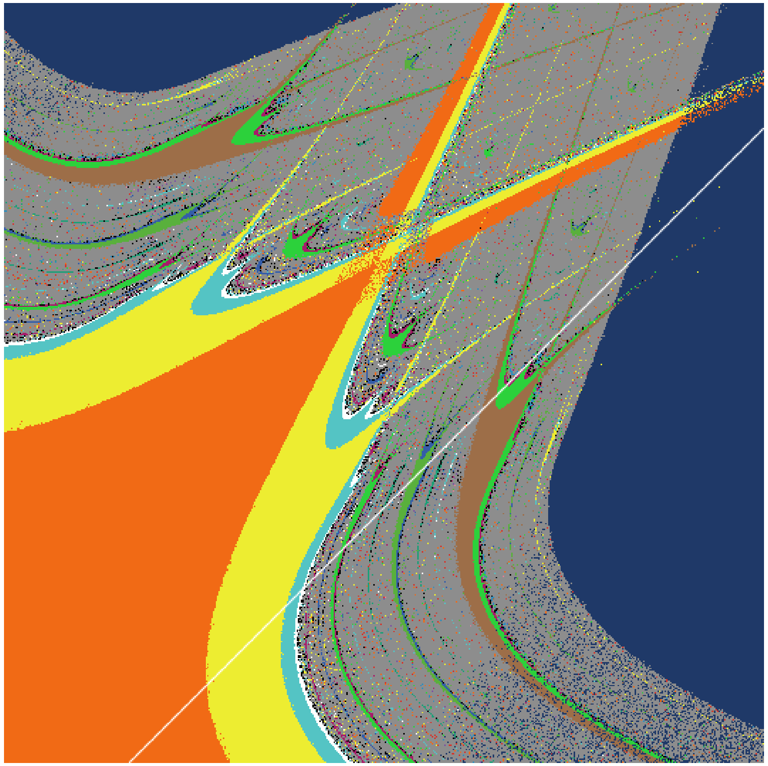
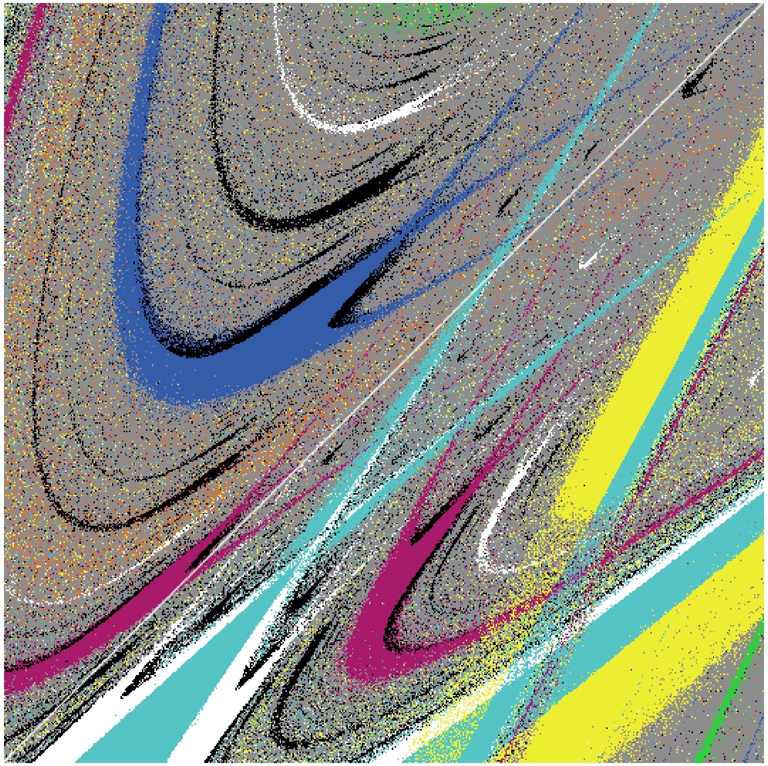


$[5,8] \times [5.5,8.5]$



$[6.4,6.6] \times [6.9,7.1]$

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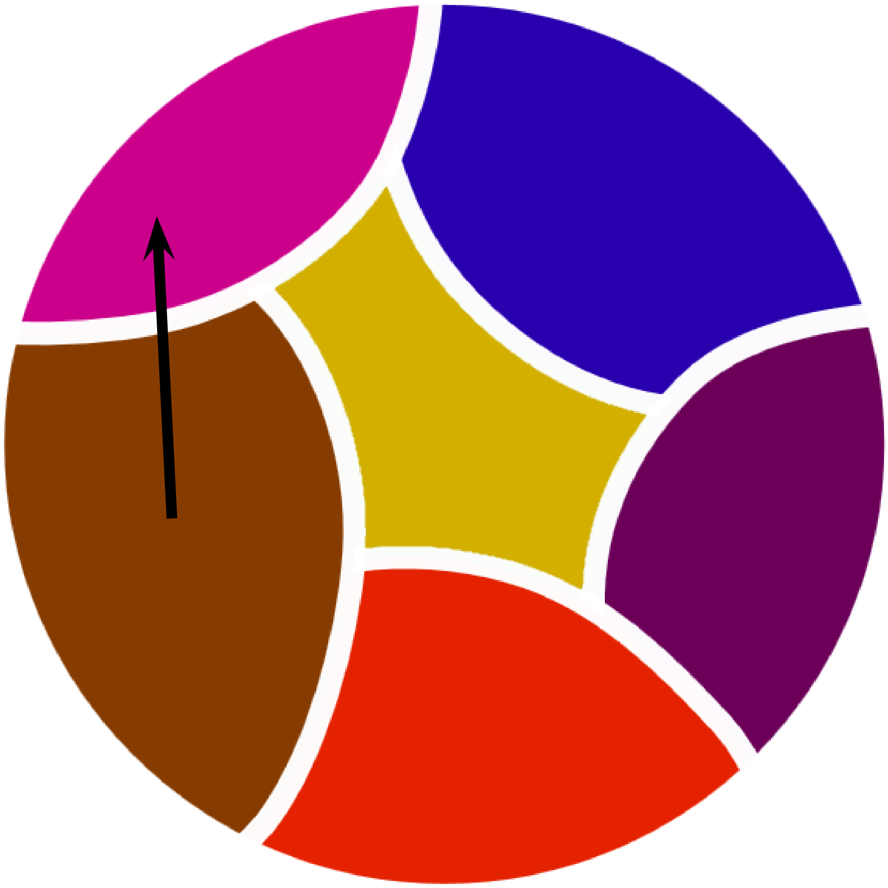
Replicator dynamic for two competing populations

One population

One population



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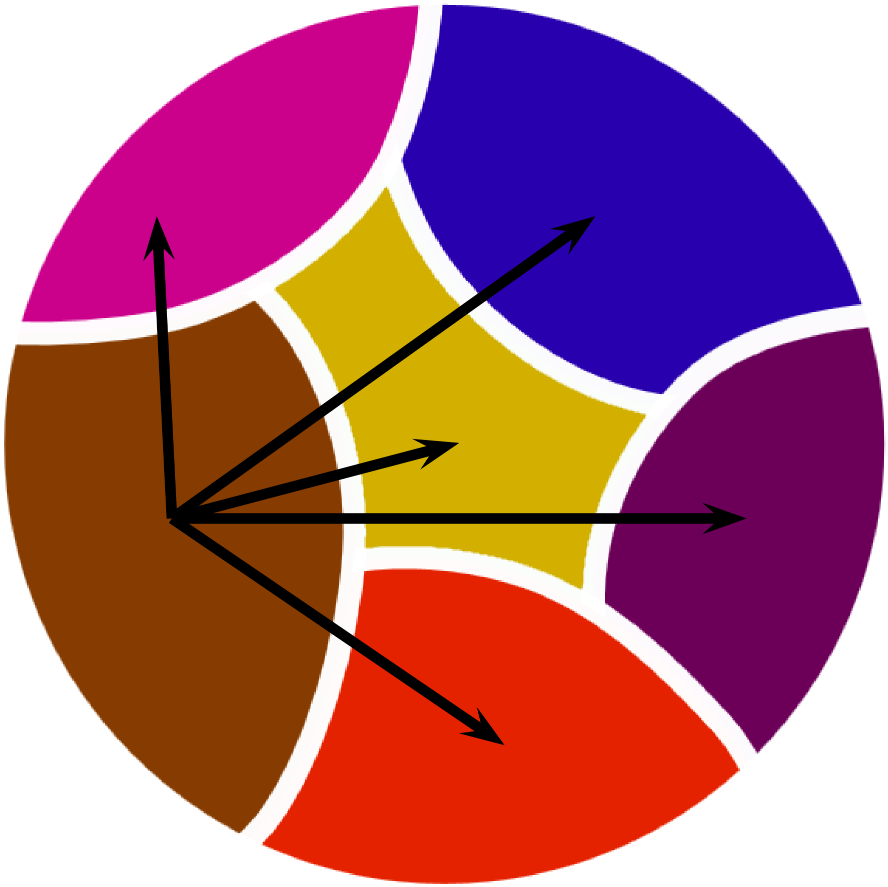

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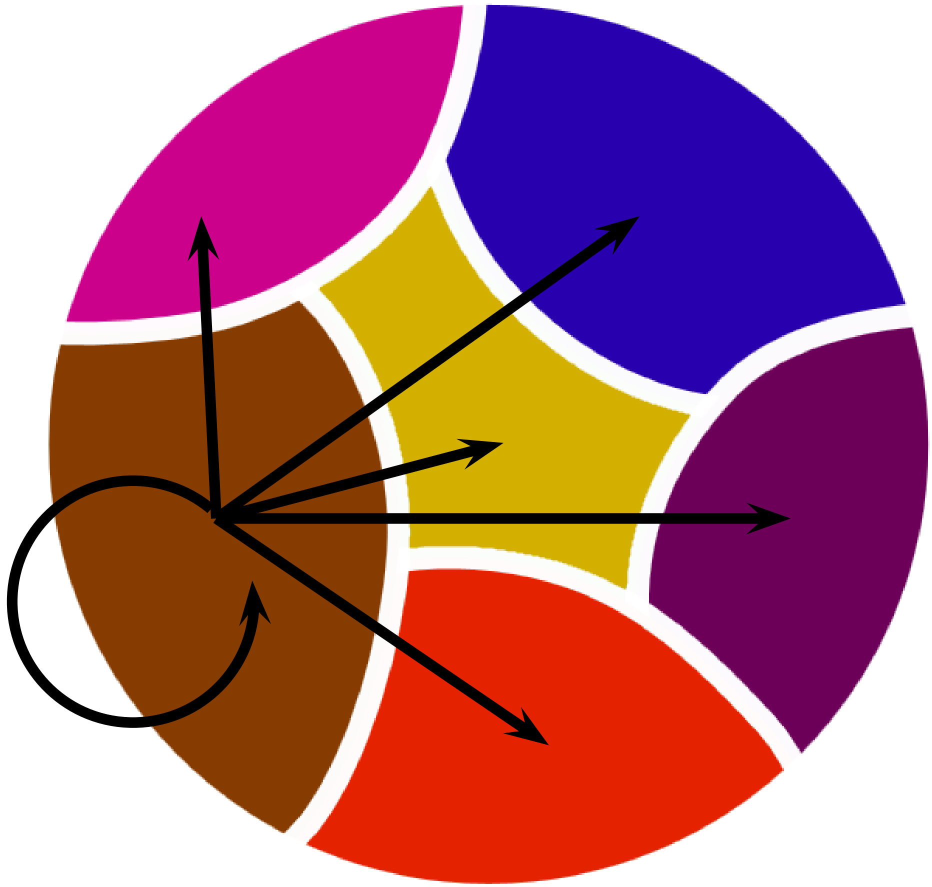

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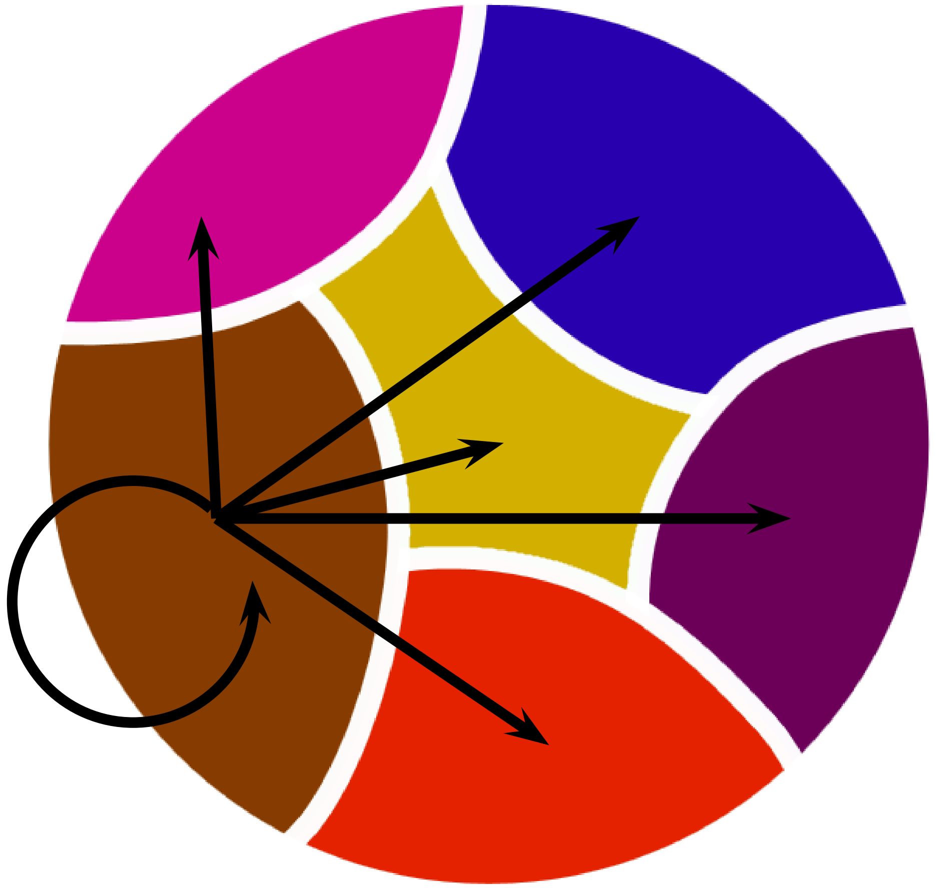
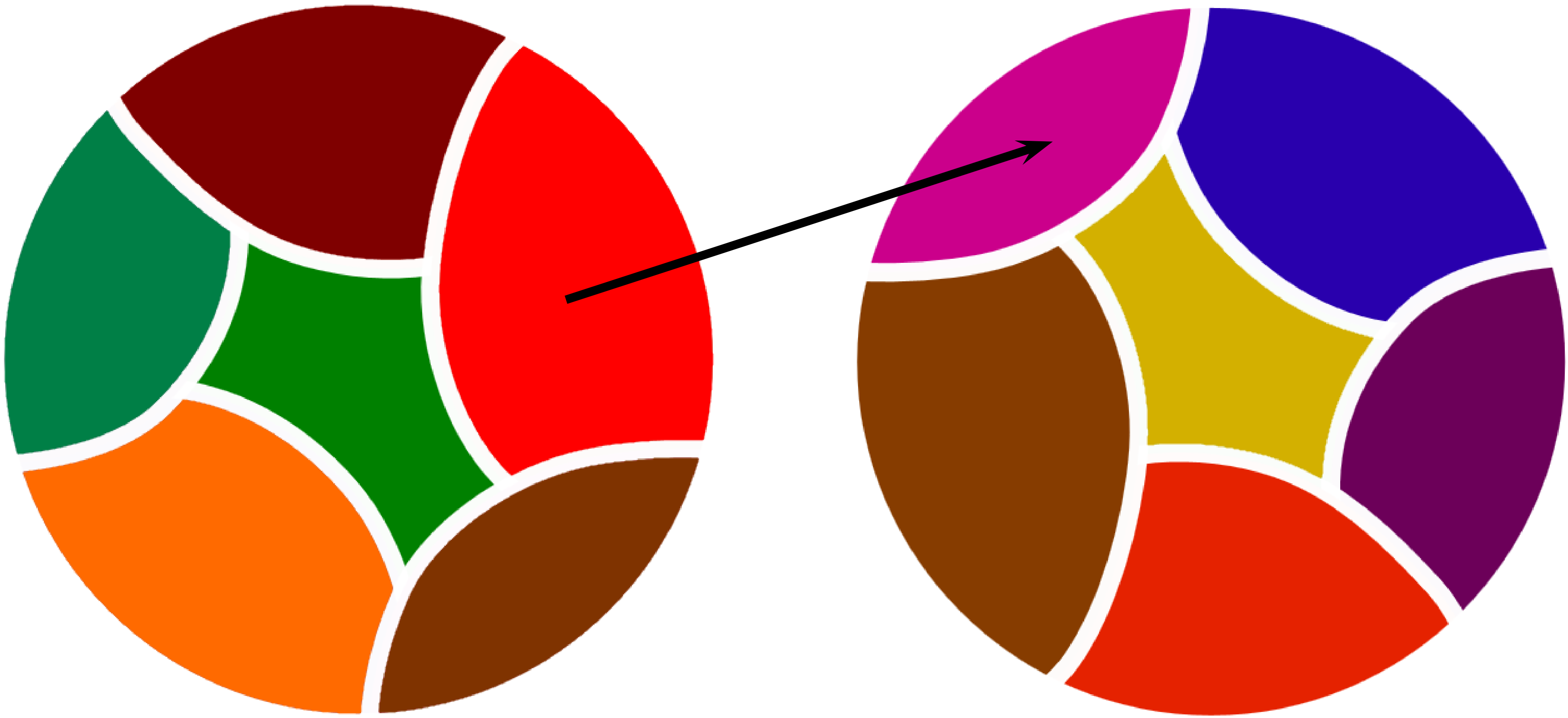


One population: intra-populational interaction.

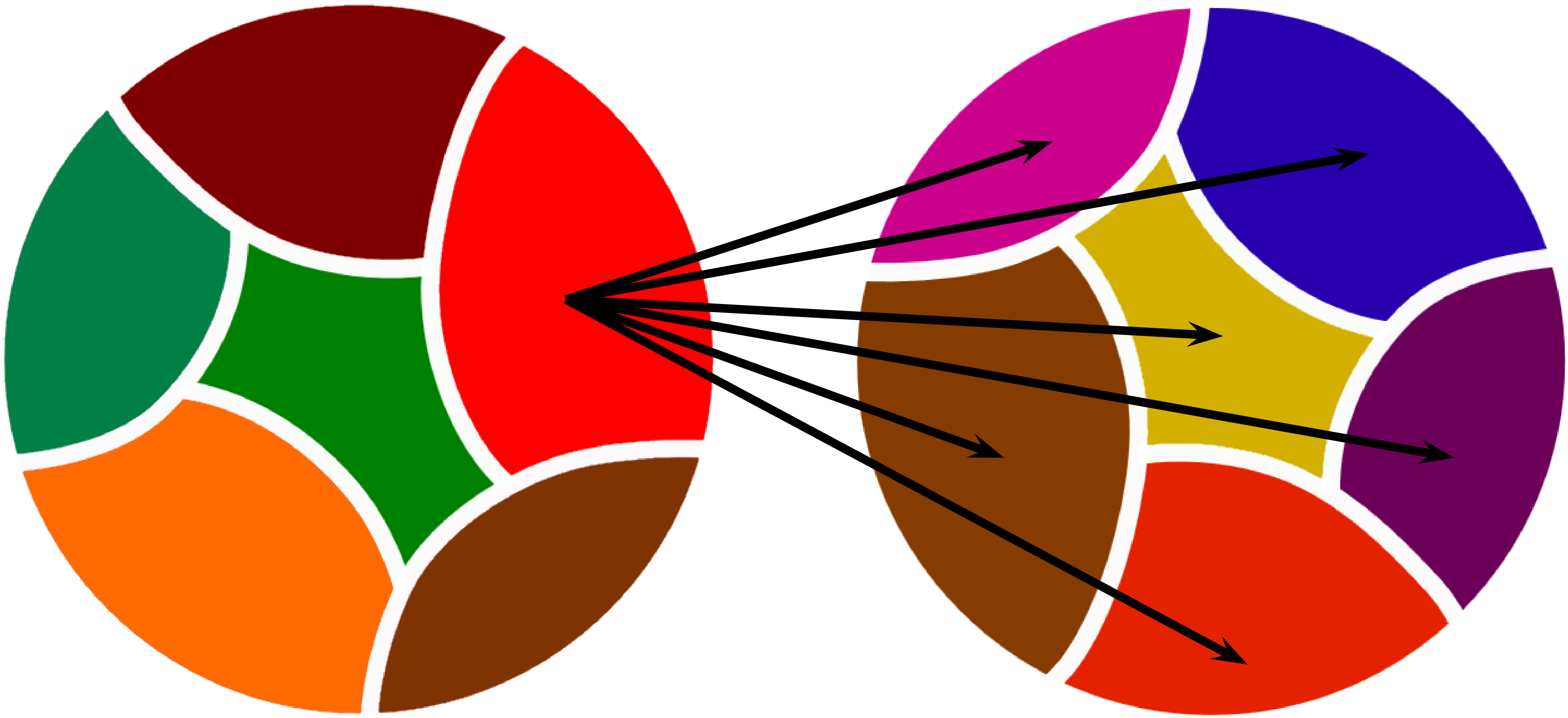
Two competing populations

Two competing populations

Two competing populations



Two competing populations



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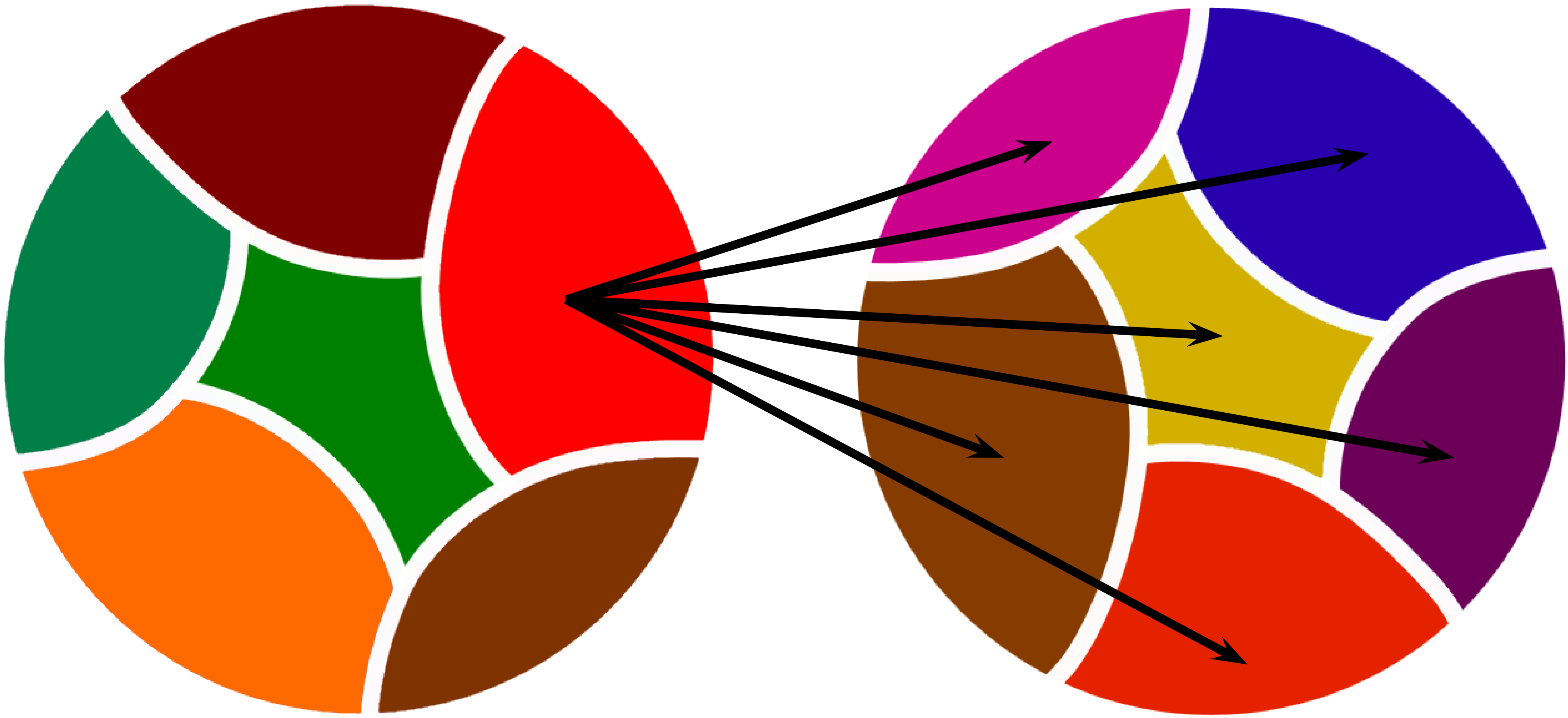

Two populations: inter-populational interaction.

Replicator dynamic for two competing populations

Consider interaction between two competing populations, "the row players" and "the column players":

	L	R
T	0, 0	2, 2
B	1, 5	1, 5

Replicator dynamic for two competing populations

Consider interaction between two competing populations, “the row players” and “the column players”:

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- Assume proportion of row plays T is x; prop. of column plays L is y.
- Expected payoff for T-players :

$$f_T = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 2-2y \\ 1 \end{pmatrix} = 2-2y.$$

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- Expected payoff for the row population: $\bar{f} = x(2-2y) + (1-x)1$.

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- Similarly, the expected payoff for B-players is 1.
- Expected payoff for the row population: $\bar{f} = x(2-2y) + (1-x)1$.
- Replicator equation for the row players: $\dot{x} = x[f_T - \bar{f}] = x[(2-2y) - (x(2-2y)+(1-x)1)] = x(1-x)(1-2y)$.

Replicator dynamic for two competing populations

Similarly for the share y of column players that play L. We obtain a system of differential equations:

$$\begin{cases} \dot{x} = x(x-1)(2y-1) \\ \dot{y} = 2xy(y-1) \end{cases}$$

- Determine (x, y) where $\dot{x} = 0$ (blue).
- Determine (x, y) where $\dot{y} = 0$ (red).
- Determine dynamics elsewhere for x (red) and y (blue).

1

$y = 1/2$

0 x 1

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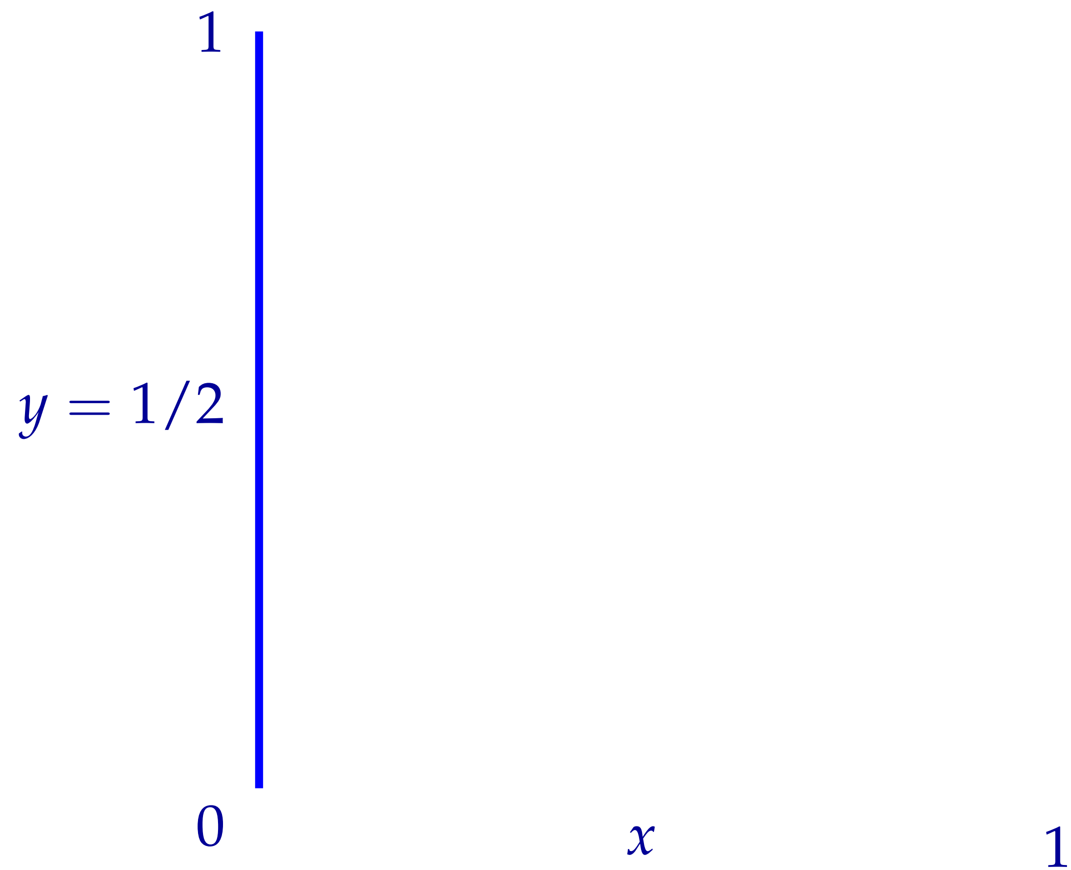

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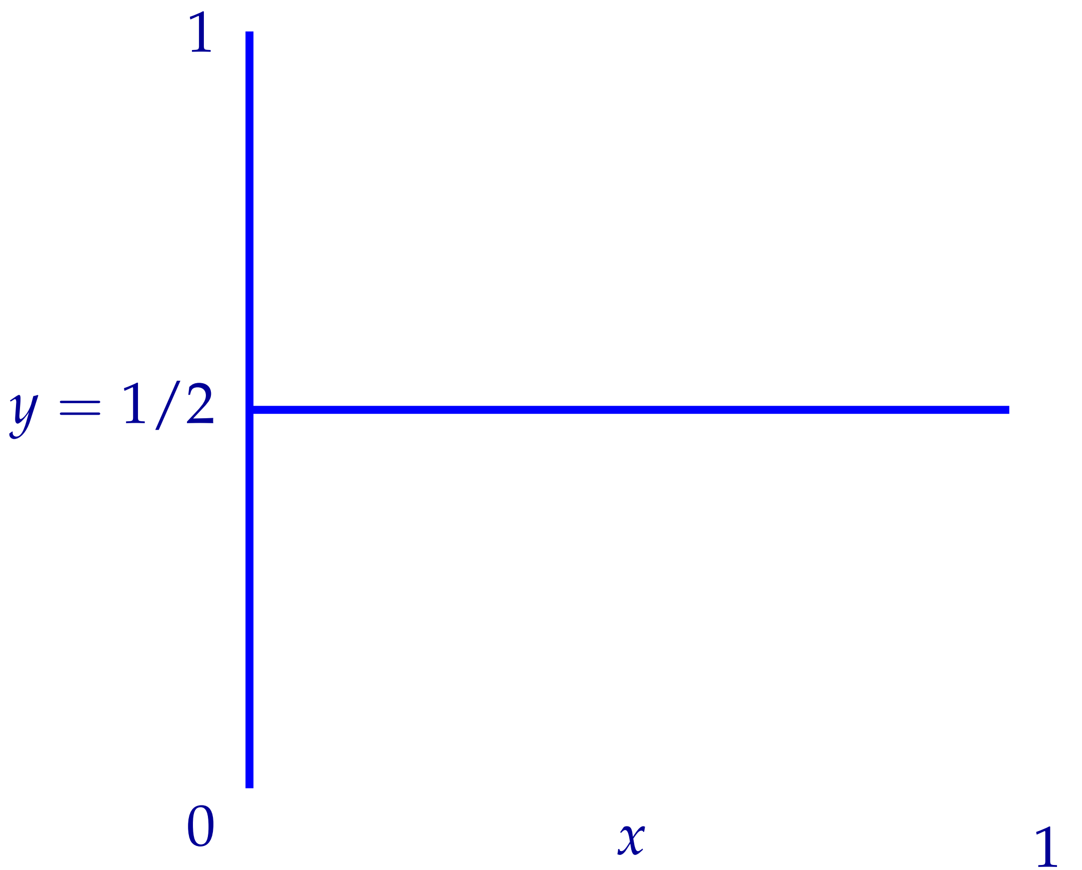

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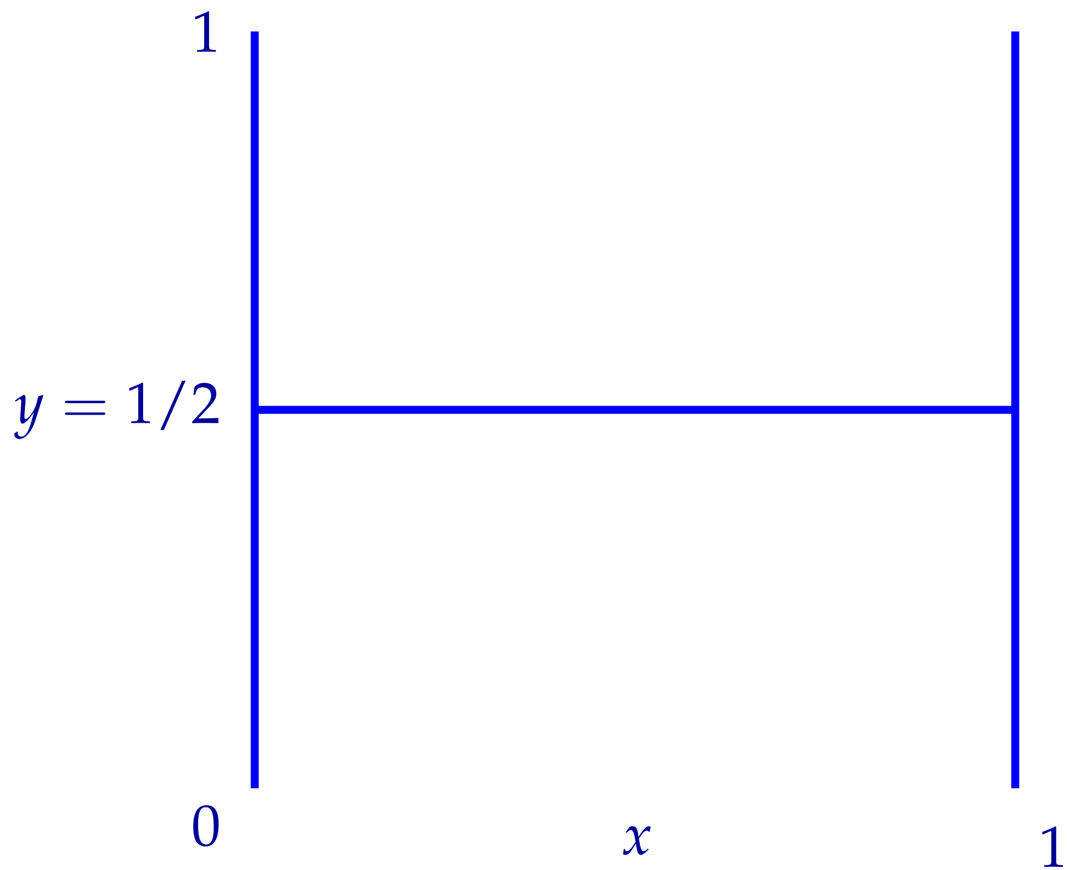

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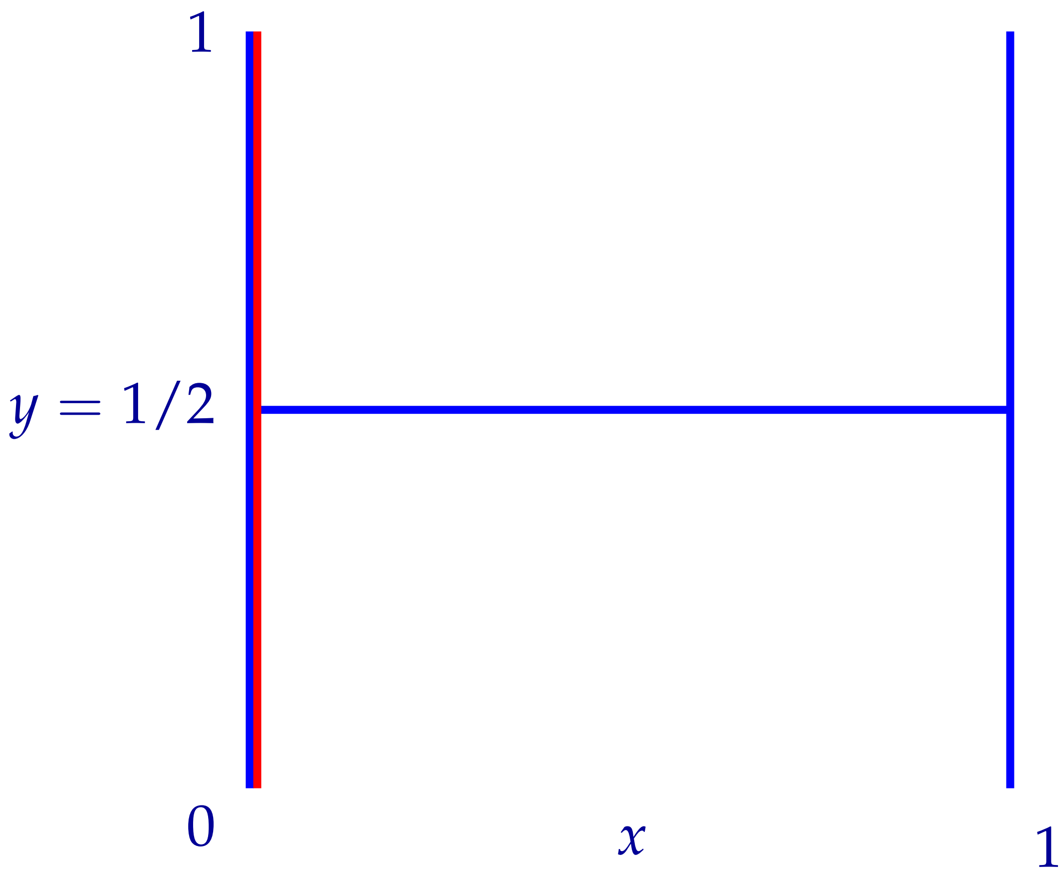

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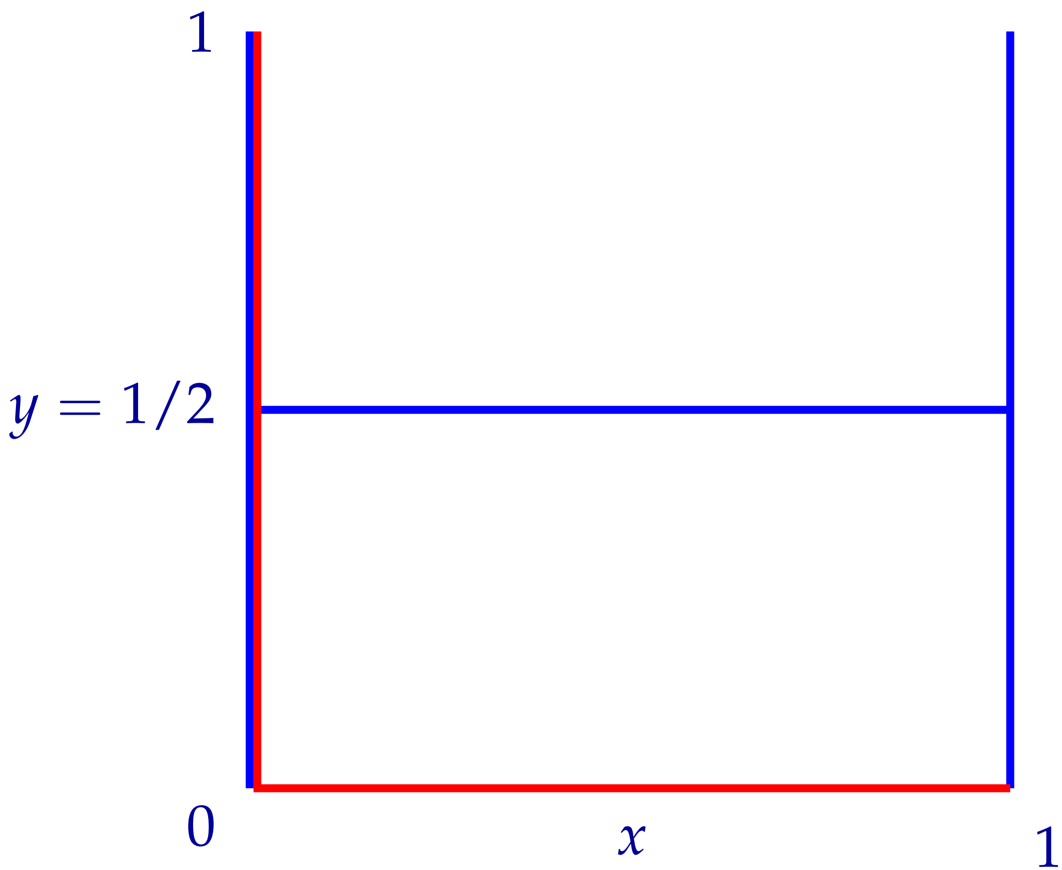

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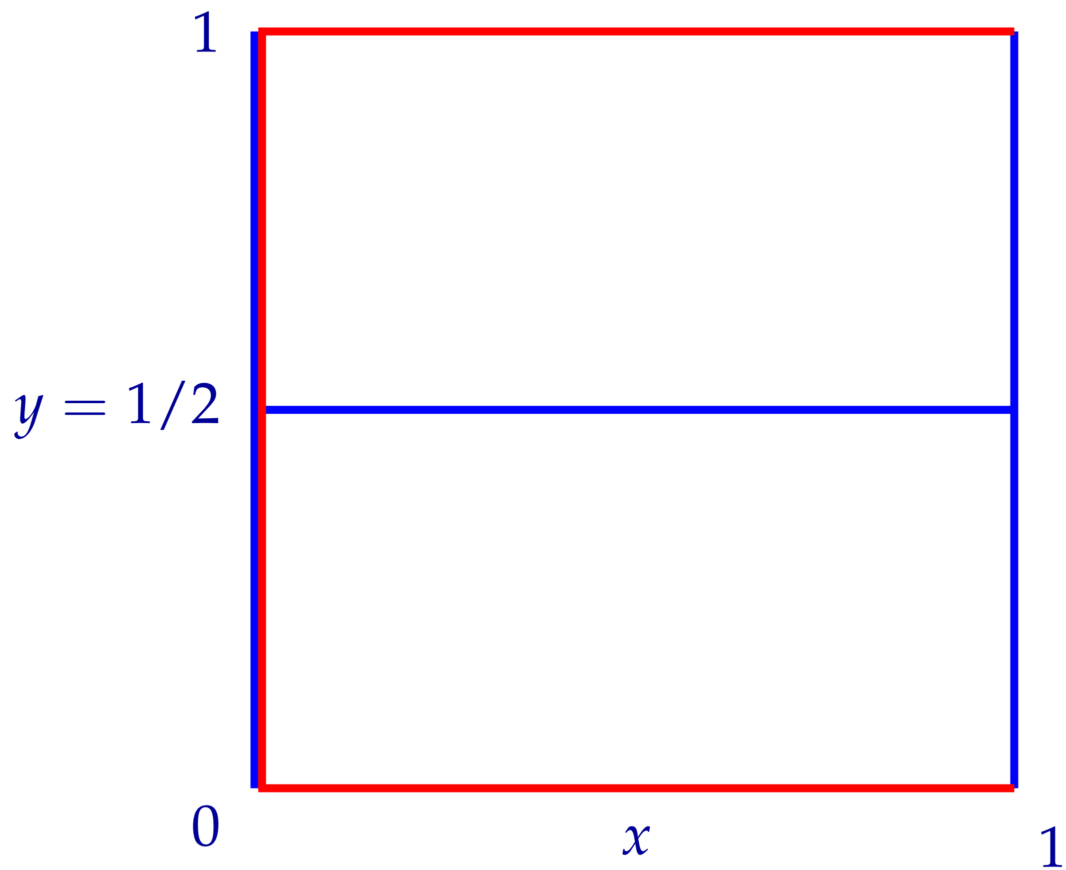

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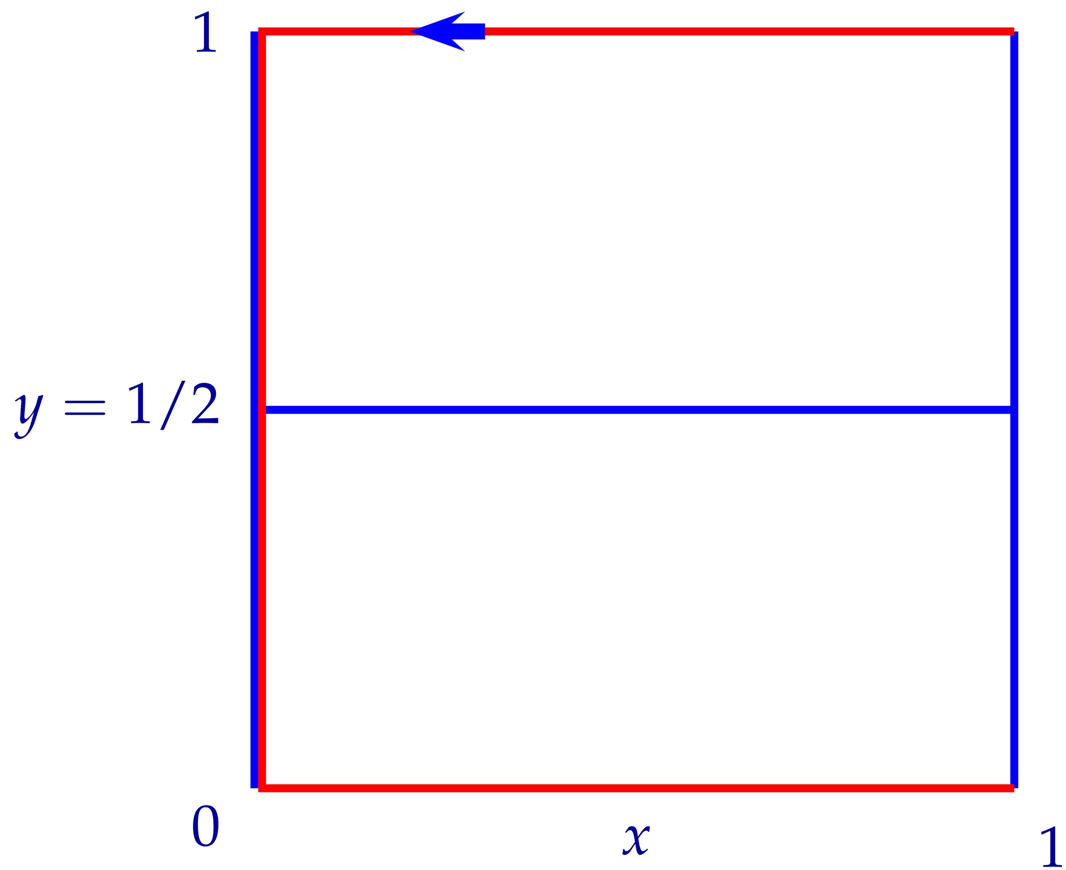

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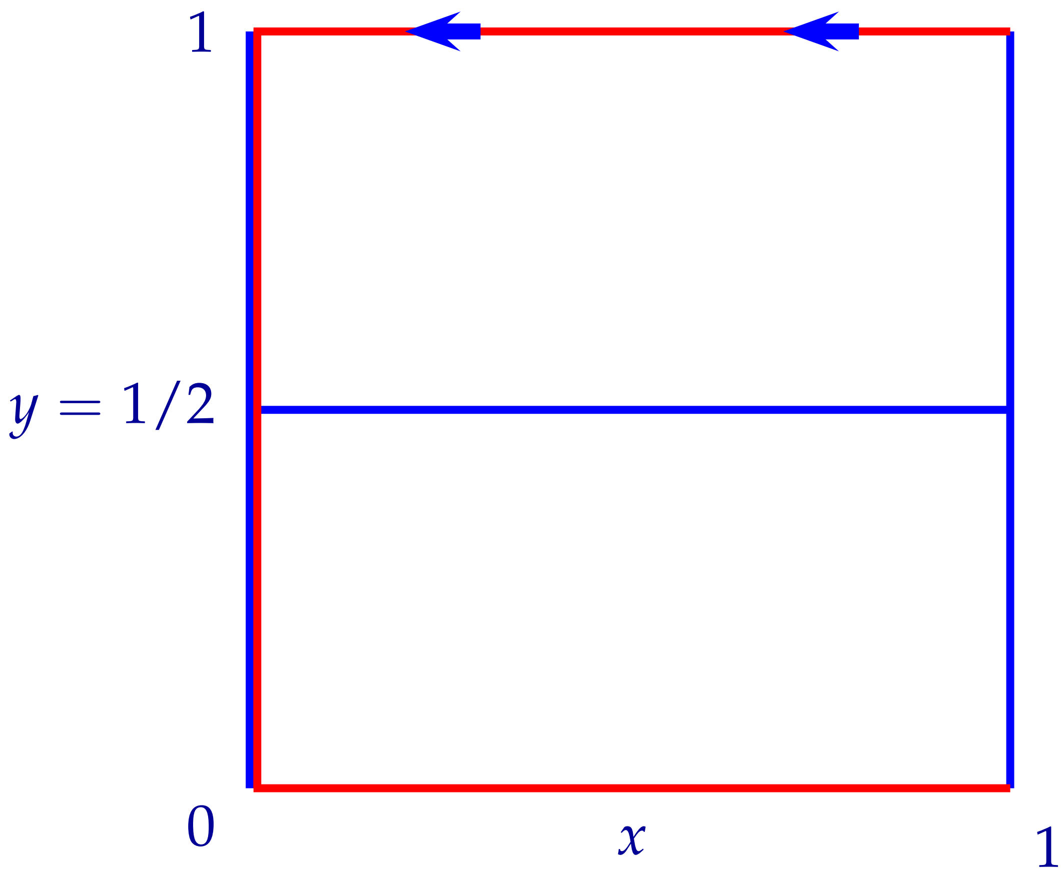

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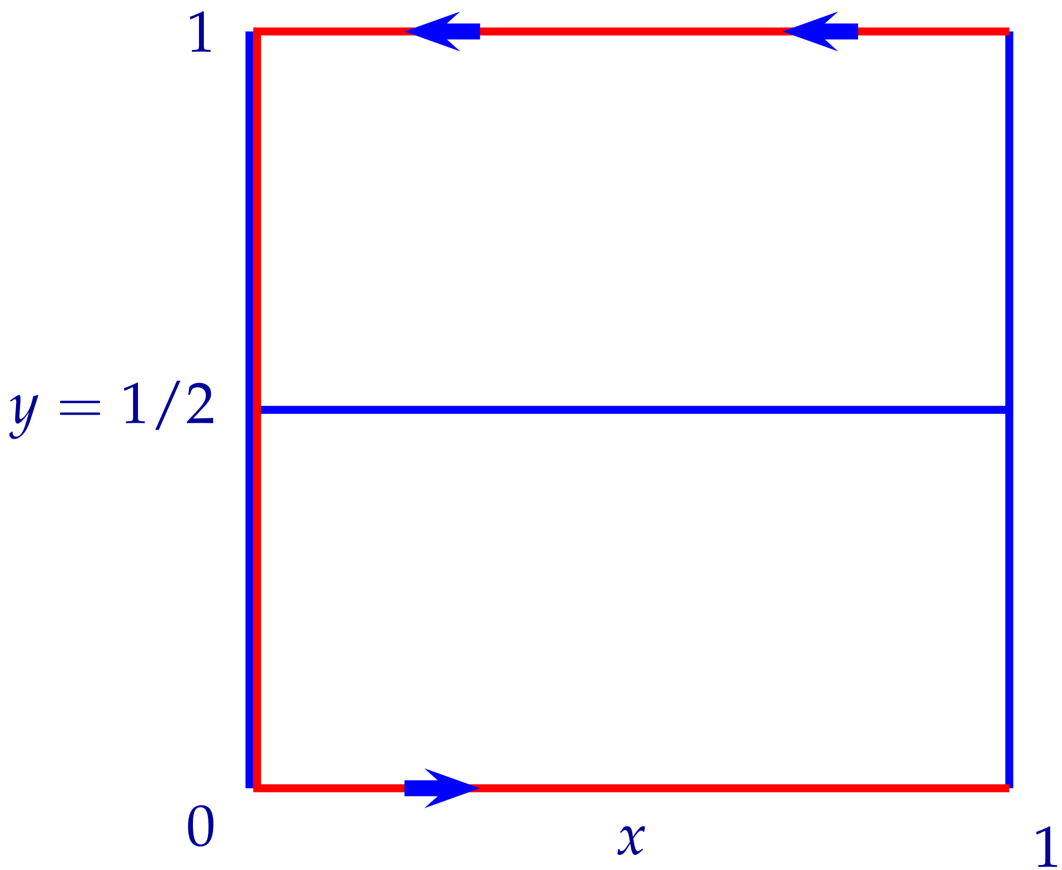

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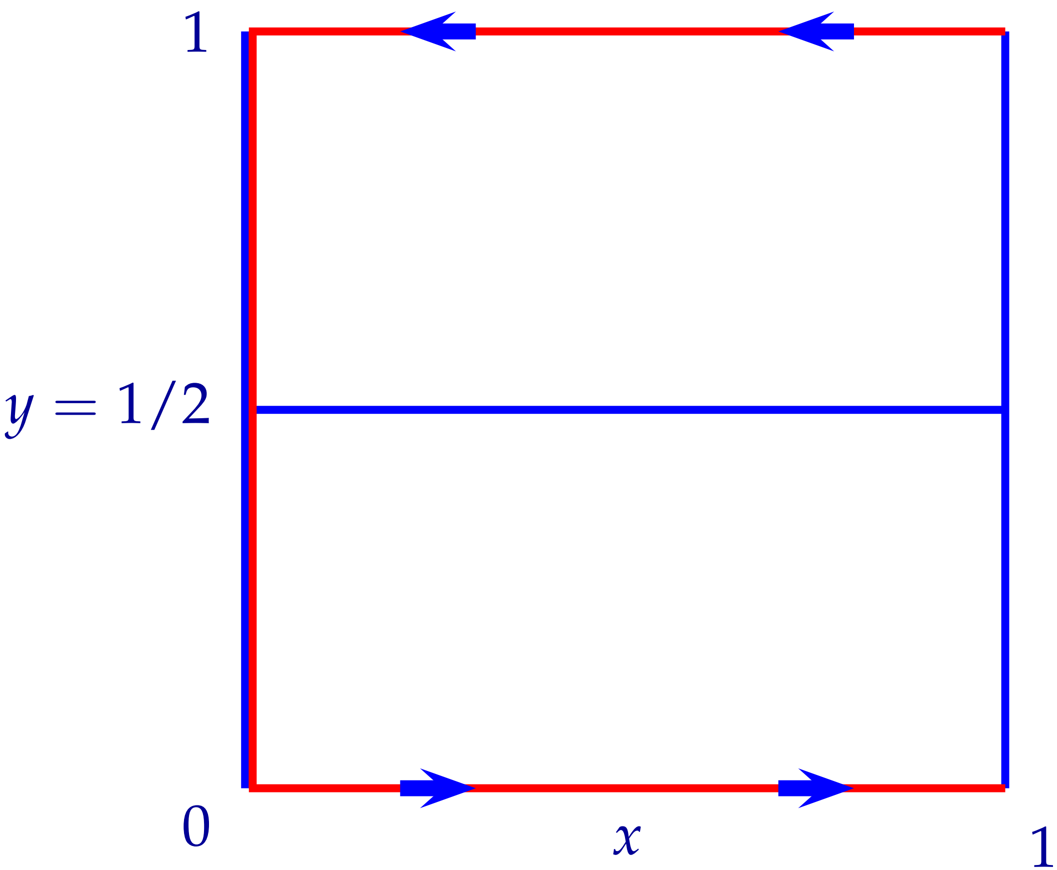

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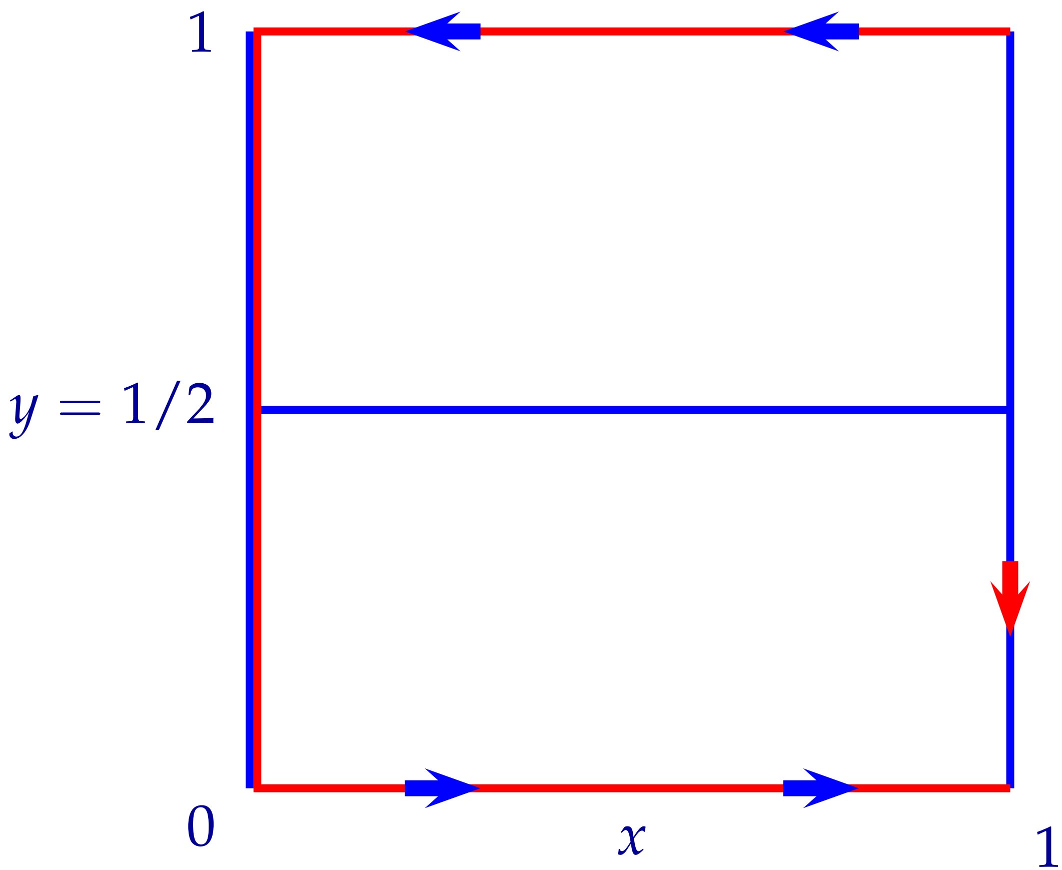

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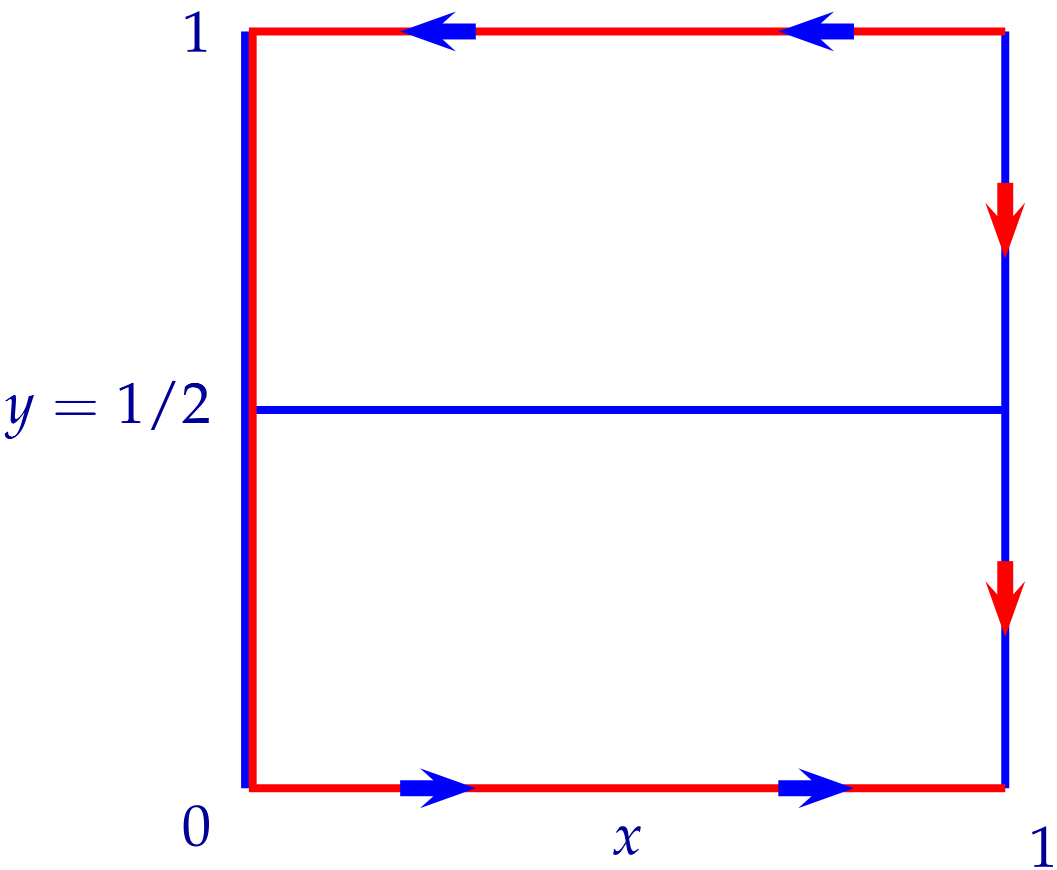

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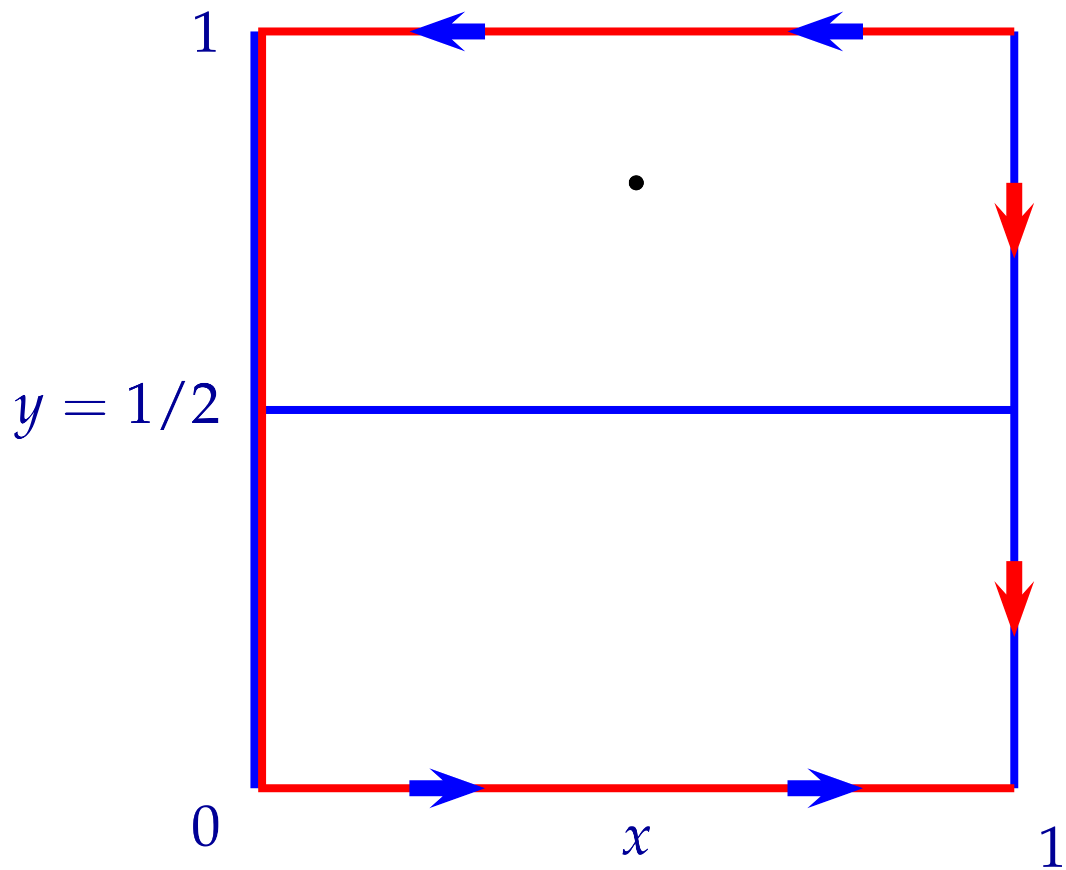

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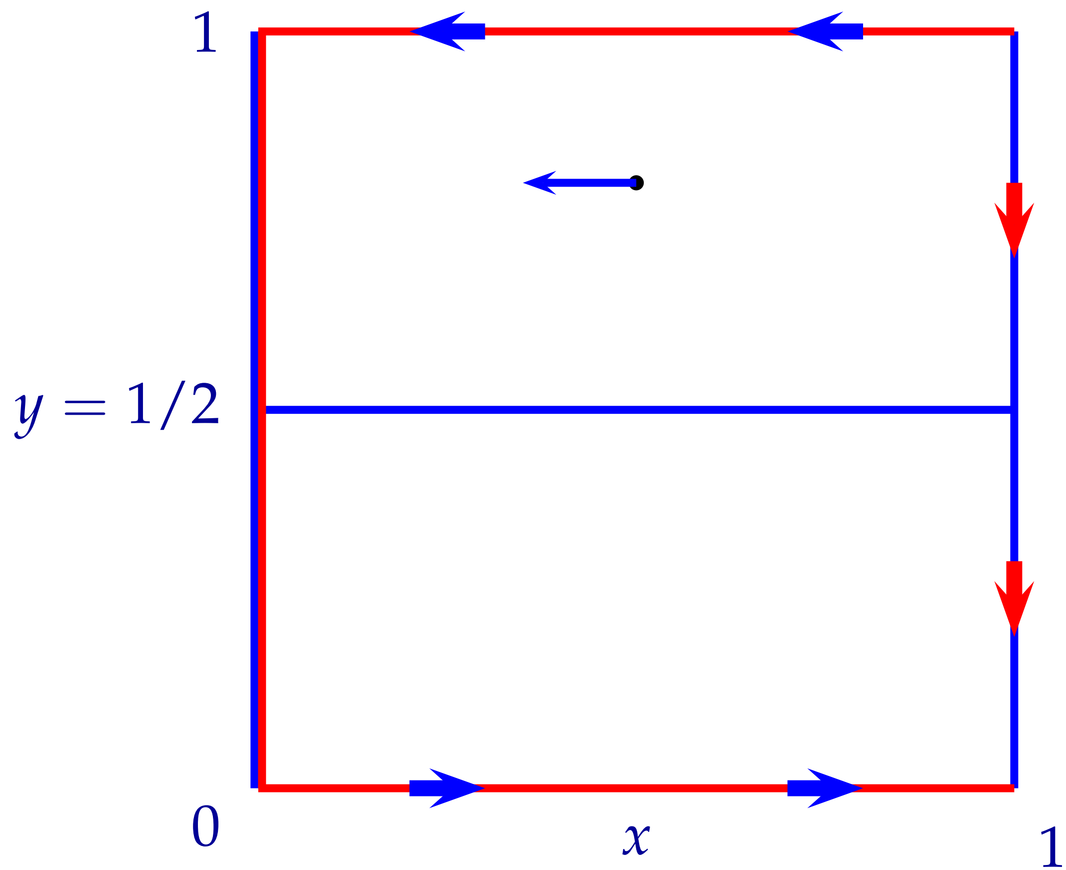

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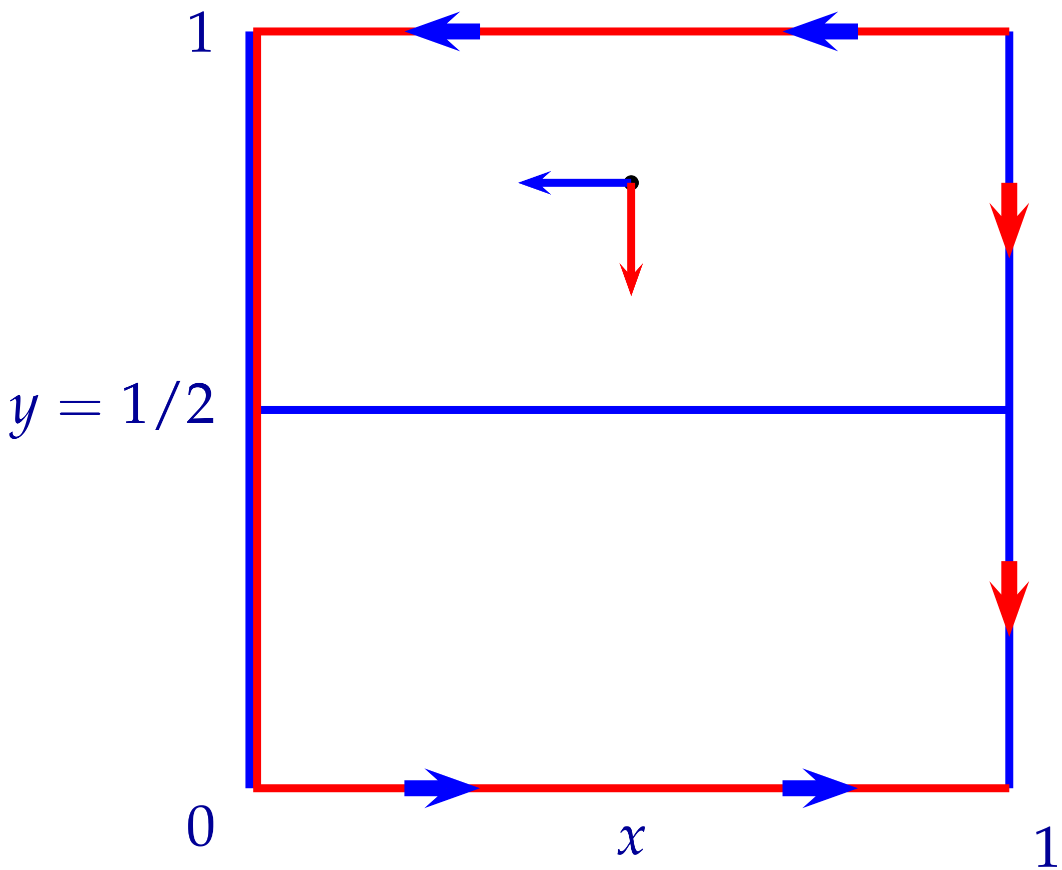

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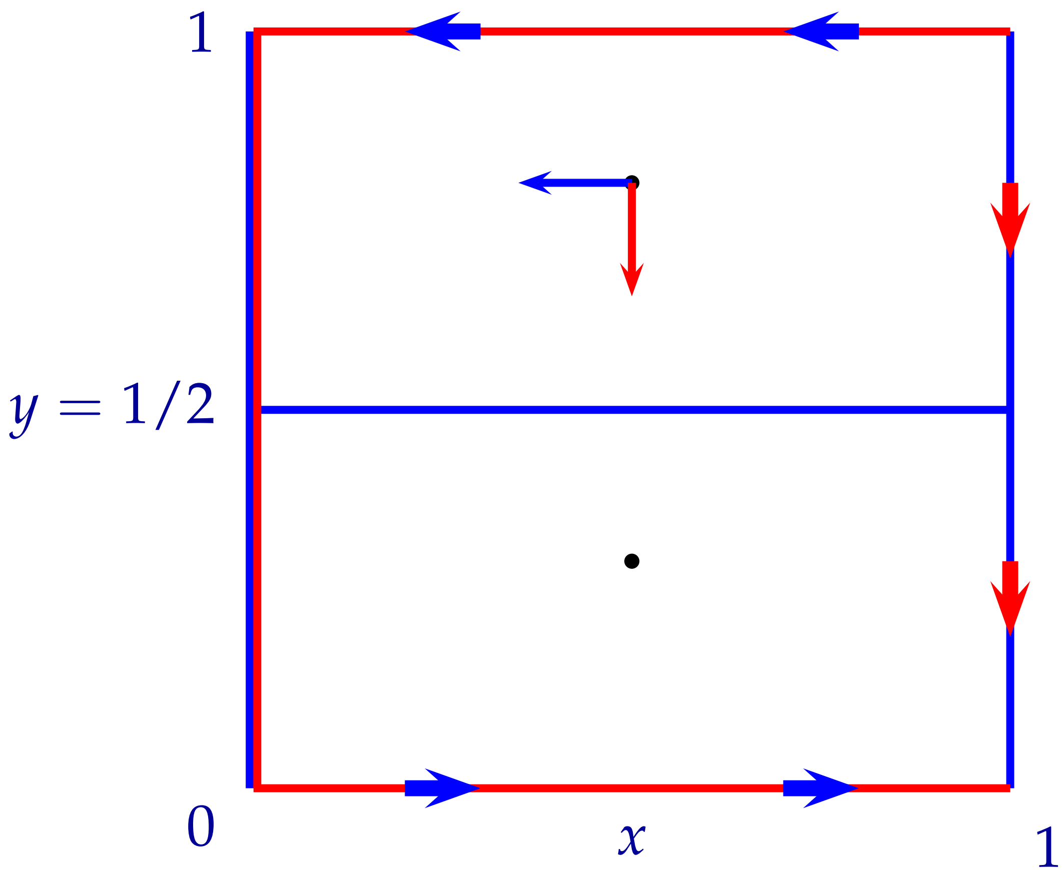

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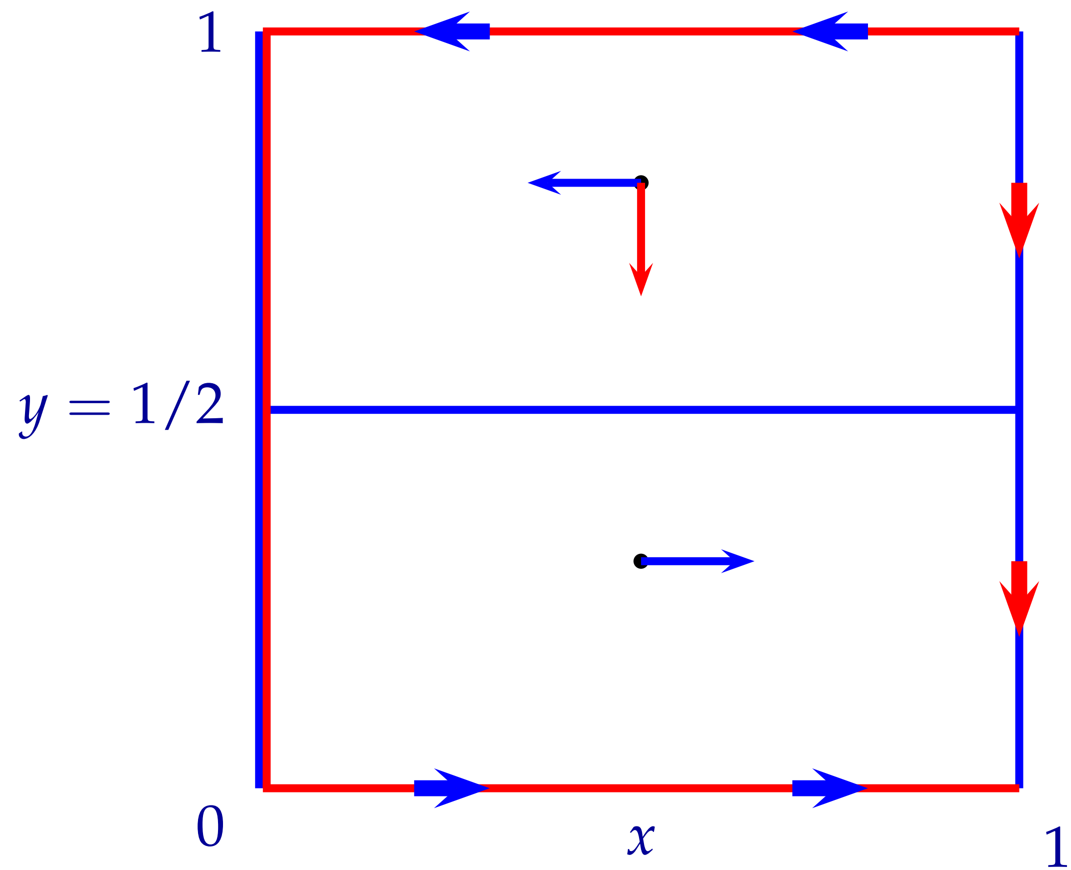

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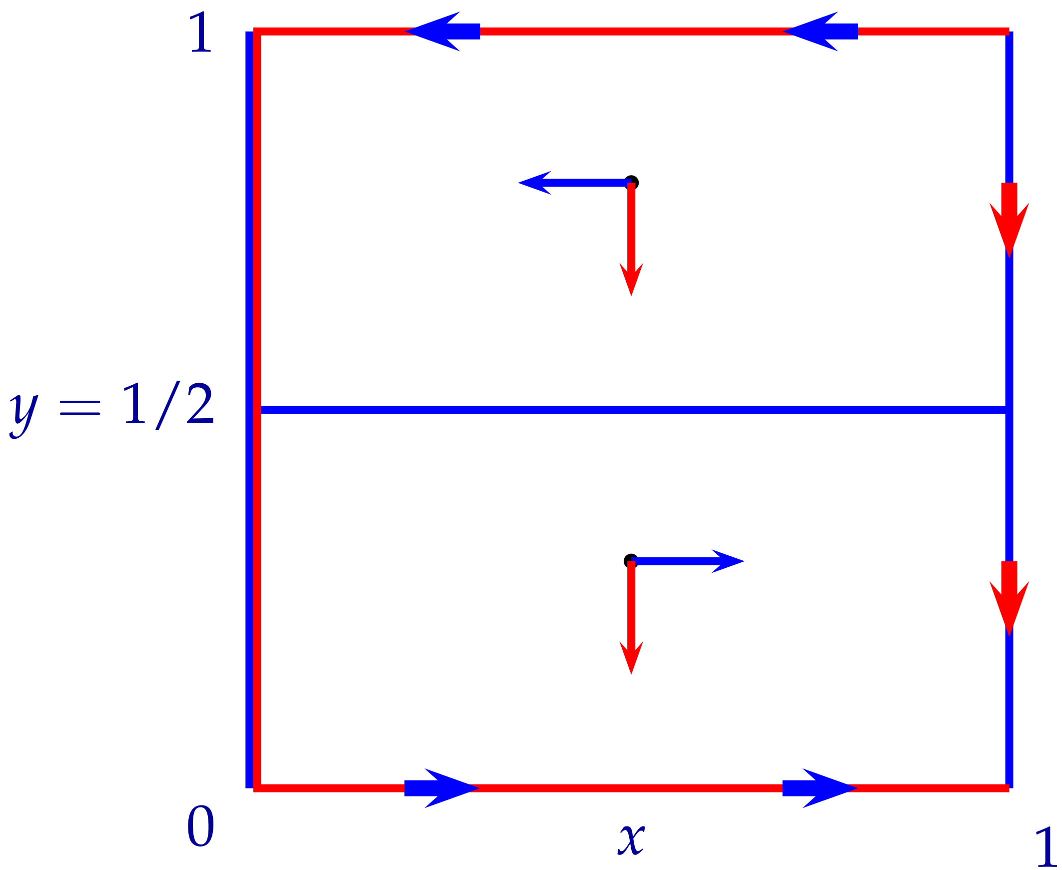

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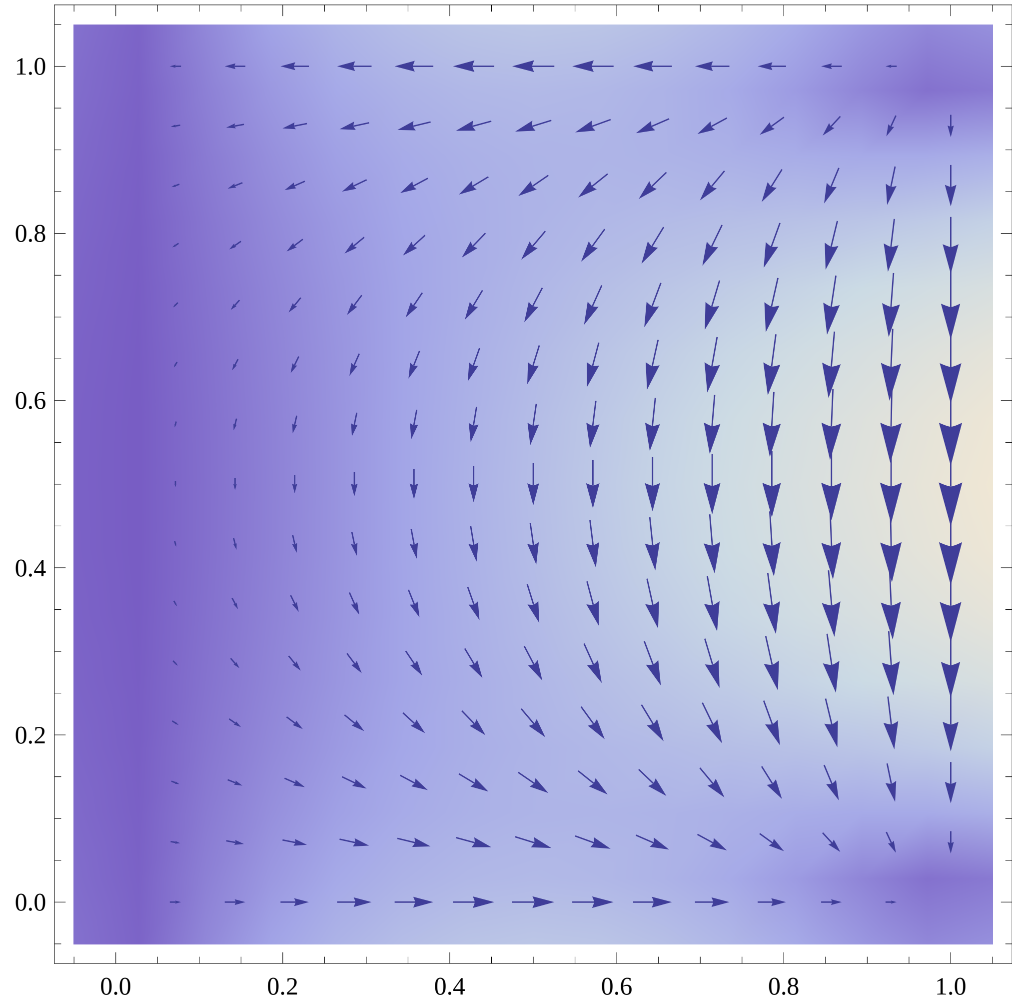
Rest points at $x = 0$, $(x, y) = (1, 0)$, $(x, y) = (1, 1)$.

Stable rest point at $(x, y) = (1, 0)$.

Corresponding vector density plot:



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Rest points at
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Stable rest point at
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Corresponding stream density plot: