Multi-agent learning

Bayesian play

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Wednesday 2nd June, 2021

Plan for today

Preparation

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- Bayes' rule.

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- Examples of Bayesian play.

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- Demo.

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Formalism

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- Domination of measures (a.k.a. absolute continuity).

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- Domination of measures (a.k.a. absolute continuity).
- Theorem of Blackwell and Dubins (1992).

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- Domination of measures (a.k.a. absolute continuity).
- Theorem of Blackwell and Dubins (1992).
- Notion of ϵ-closeness.
- Theorem of Kalai and Lehrer (1993): If a player gives all potential play paths a small positive probability ("grain of truth"), then, eventually, his/her subjective beliefs will be ϵ-close to the actual realisation of play.

Literature

Key publication

Kalai & Lehrer (1993). "Rational learning leads to Nash equilibrium". *Econometrica*, Vol. **61**, No. 5, pp 1019-1045.

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Kalai & Lehrer (1993). “Rational learning leads to Nash equilibrium”. *Econometrica*, Vol. **61**, No. 5, pp 1019-1045.

Scholarly resources

Young (2004): *Strategic Learning and it Limits*, Oxford UP. Ch. 7: “Bayesian Learning”.

Shoham *et al.* (2009): *Multi-agent Systems*. Ch. 7: “Learning and Teaching”. Sec. 7.3: “Rational Learning”.

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Shoham *et al.* (2009): *Multi-agent Systems*. Ch. 7: "Learning and Teaching". Sec. 7.3: "Rational Learning".

Practical computer science / AI application

Zeng & Sycara (1996): *Bayesian Learning in Negotiation* in: Working Notes of the AAAI Spring Symposium on Adaptation, Co-Evolution and Learning in Multiagent Systems, Stanford, CA.

Part I:
Elementary probability and Bayes' theorem

Conditional probability

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Typical exercise: "given $\Pr\{E\}$, $\Pr\{F\}$, and $\Pr\{F|E\}$, compute $\Pr\{E|F\}$". (E.g., E = "influenza", F = "fever".)

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$$\begin{aligned}\Pr\{F\} &= \Pr\{F|E\}\Pr\{E\} \\ &\quad + \Pr\{F|E^c\}\Pr\{E^c\},\end{aligned}$$

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so that

$$\Pr\{E|F\} = \frac{\Pr\{F|E\}\Pr\{E\}}{\Pr\{F|E\}\Pr\{E\} + \Pr\{F|E^c\}\Pr\{E^c\}}.$$

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- $f_Y(y|X=x)$ is the posterior density of Y given $X=x$.
- $f_X(x)$ and $f_Y(y)$ are marginal densities of X and Y.

Part Ia: Bayes' rule applied to guess opponent's reply rules

Use Bayes' rule to compute $\Pr\{g = g_i \mid h\}$ for every i

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- Bayes' rule for $g = g_i$'s posterior probability:

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- Because of identical priors and normalization, effectively $\Pr\{g = g_i \mid h\} \propto \Pr\{h \mid g = g_i\}$.

Part II: Demo and examples

Demo and examples

Demo and examples

1. **Demo**. Learn reactive reply rules, such as

   - All-C: always cooperate.
   - Unforgiving ("unforgiving"): cooperate until opponent defects, then defect forever.
   - C-90%: cooperate 90% of the time (randomly).
   - Tit-for-tat: mimic opponent's moves.
   - Josh 10%: play tit-for-tat 90% of the time, defect 10% of the time.
   - Majority: respond with the action most played by the opponent.
   - Eatherly: mirror the (projected) mixed strategy of the opponent.
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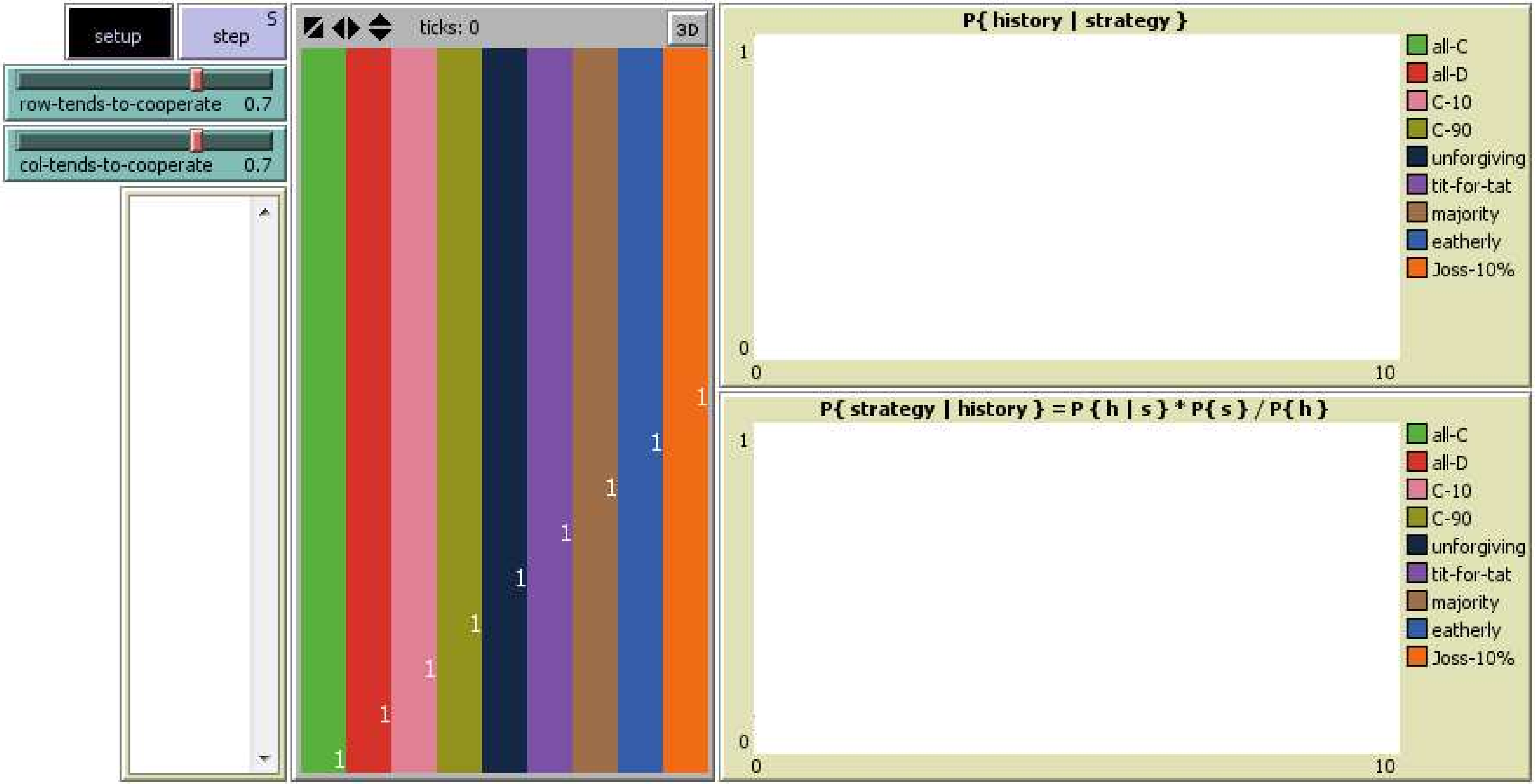
2. **Examples**. Learn reply rules in the repeated prisoners' dilemma; learn reply rules in the coordination game.

Example: learning on reactive strategies



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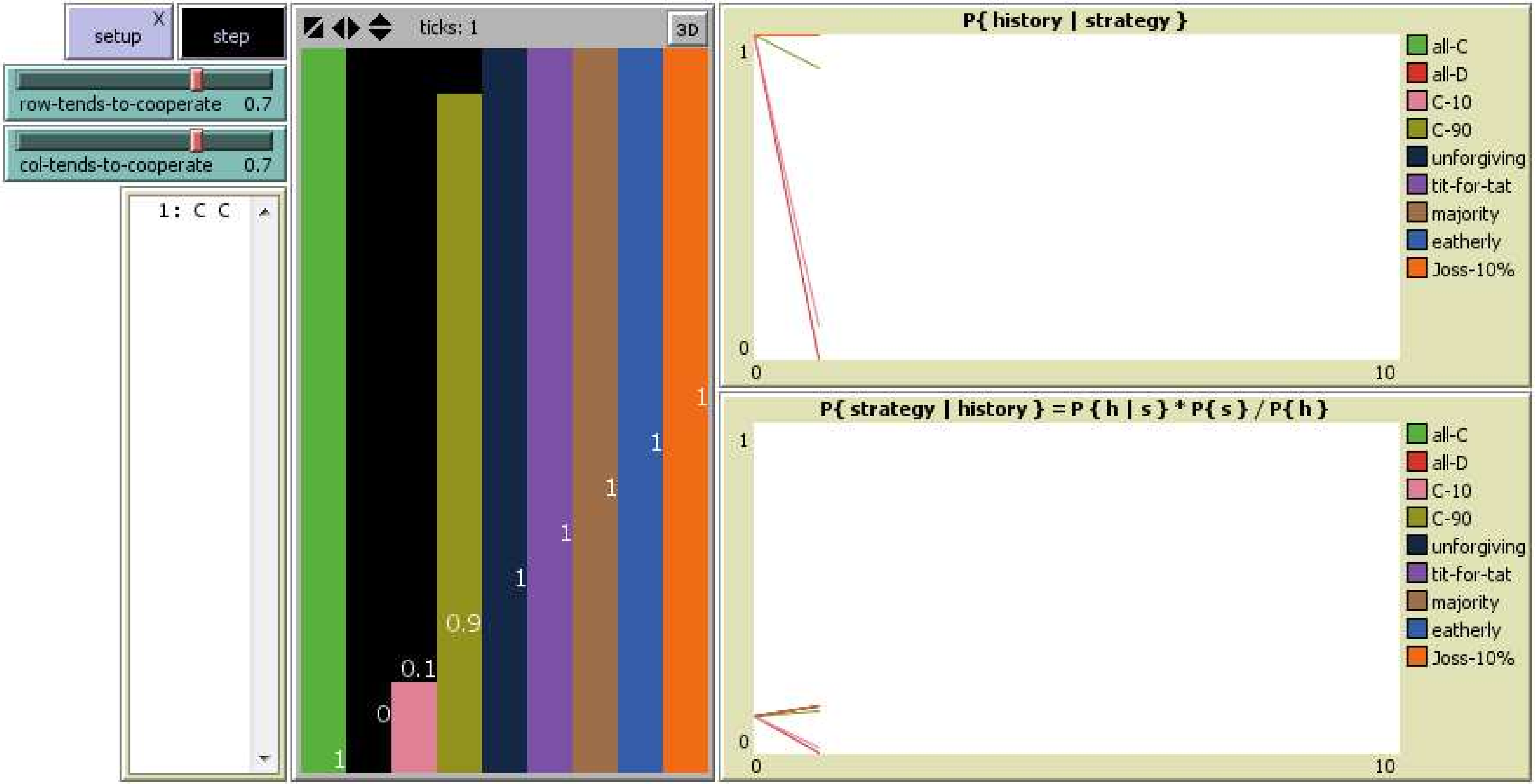

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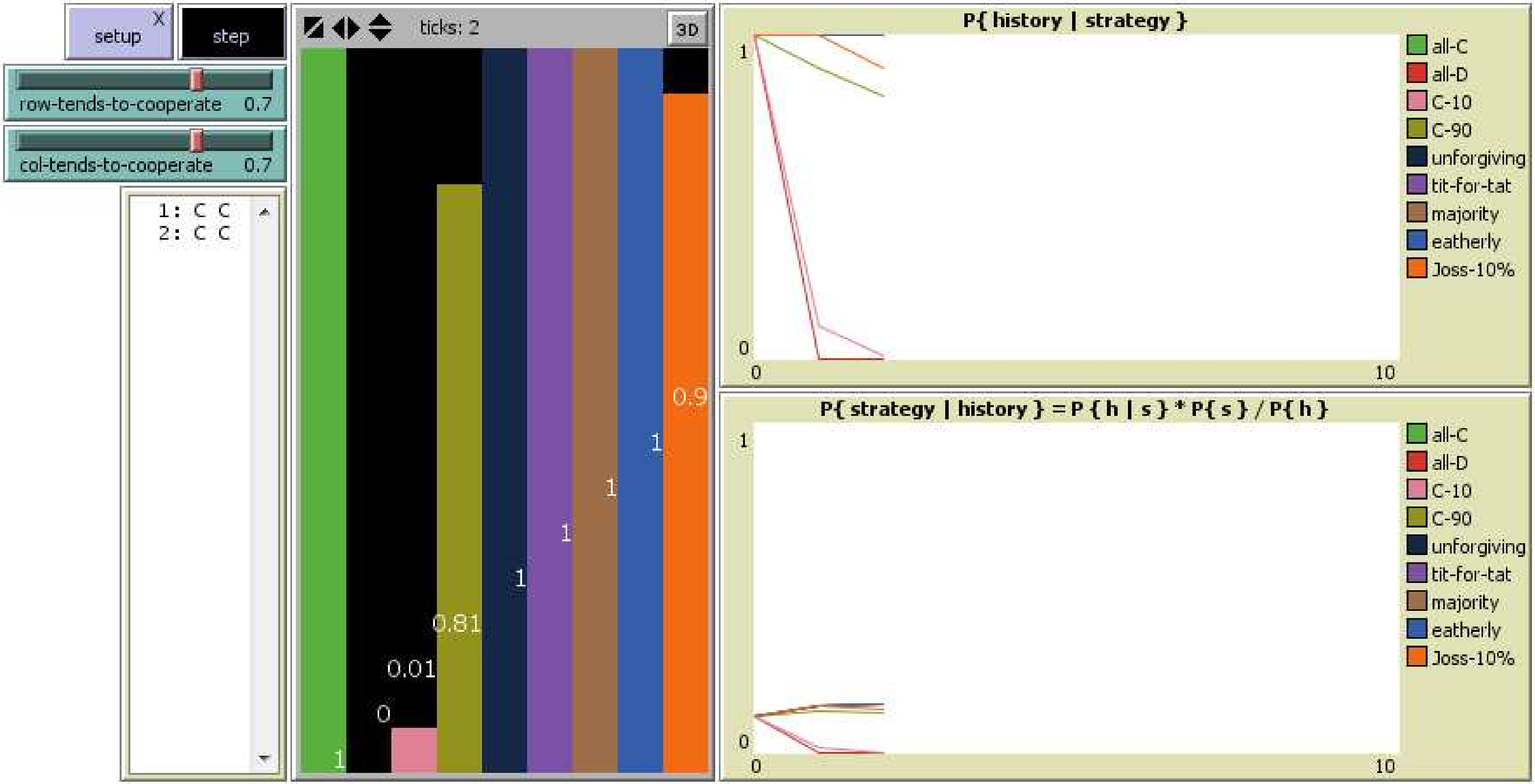

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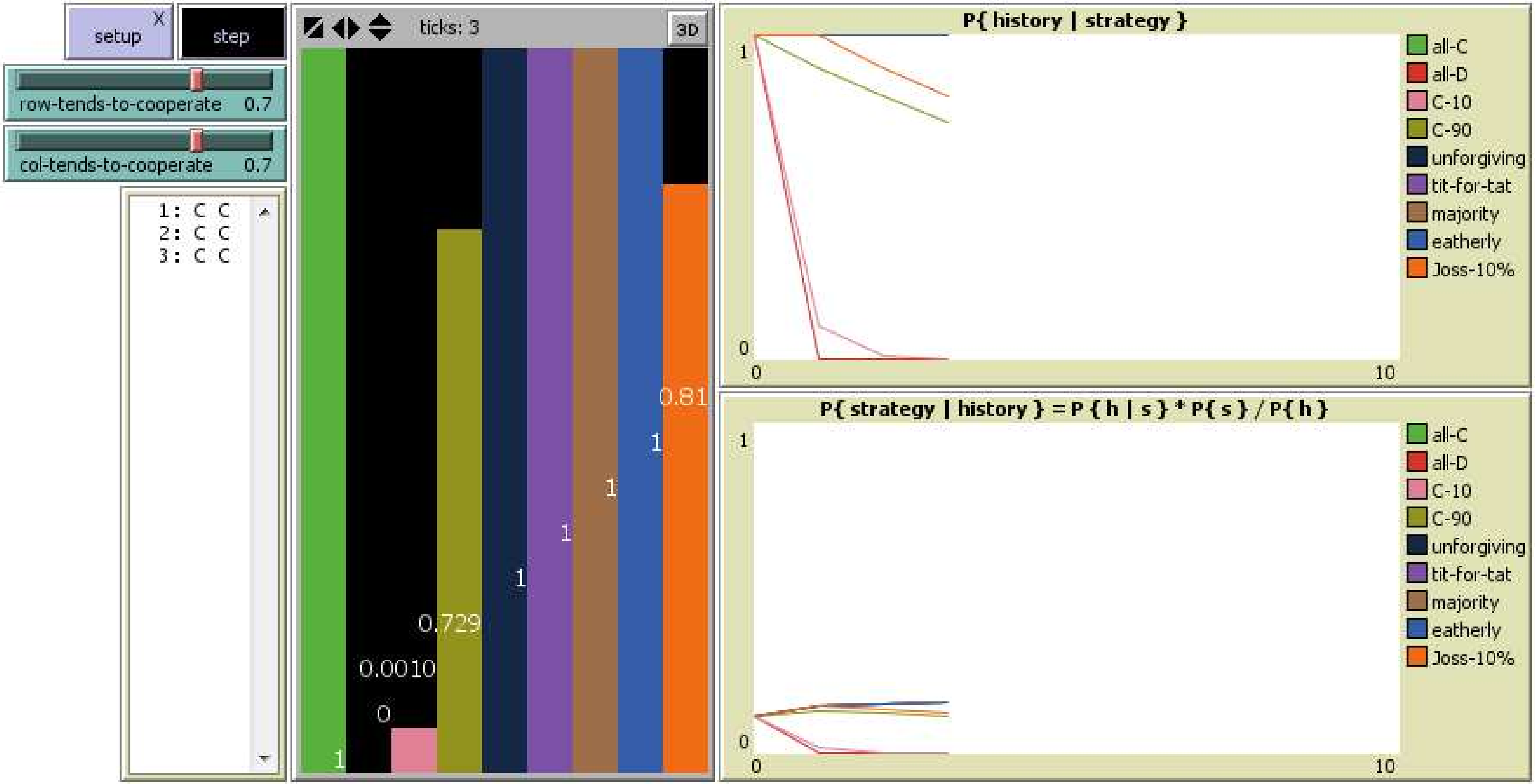

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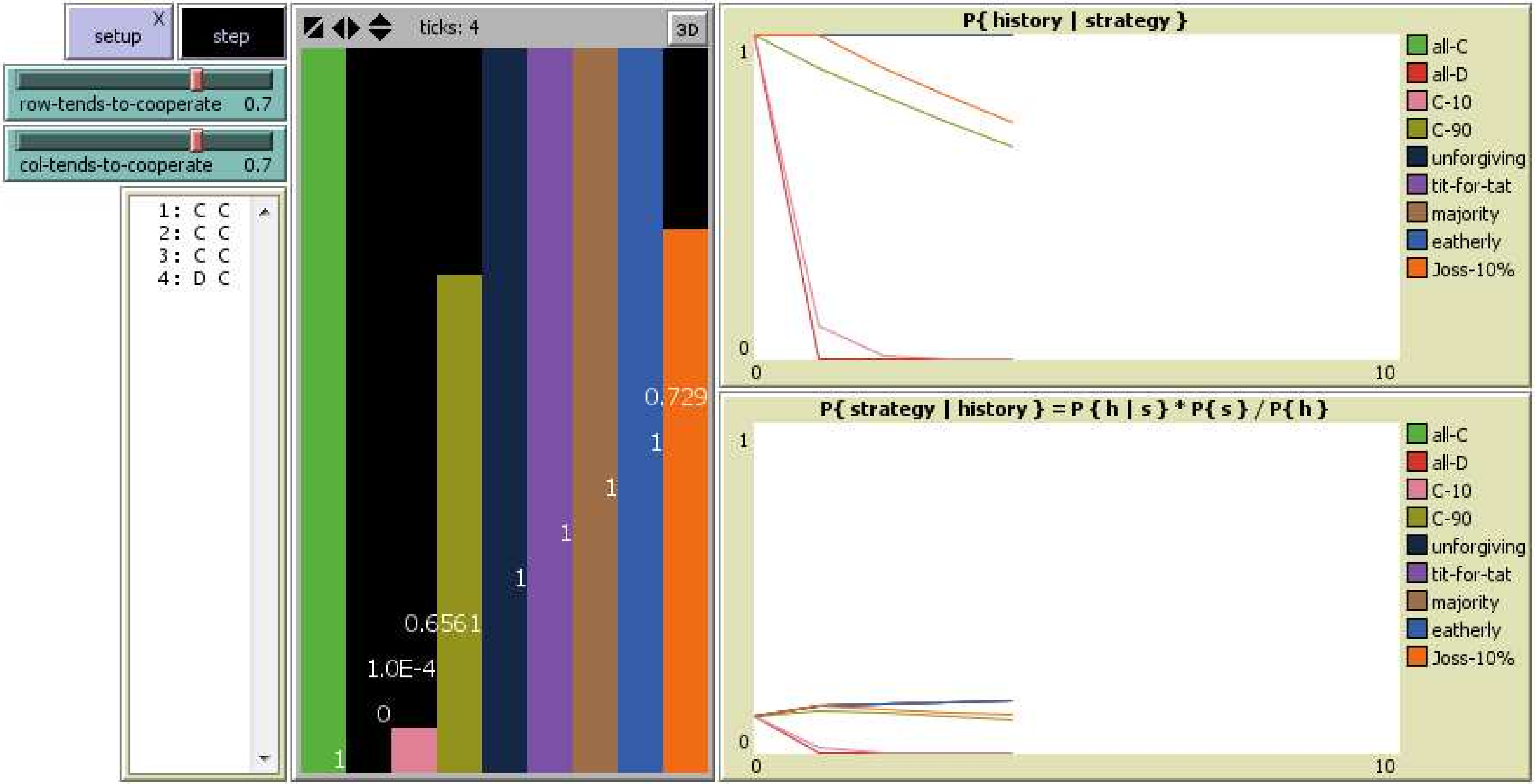

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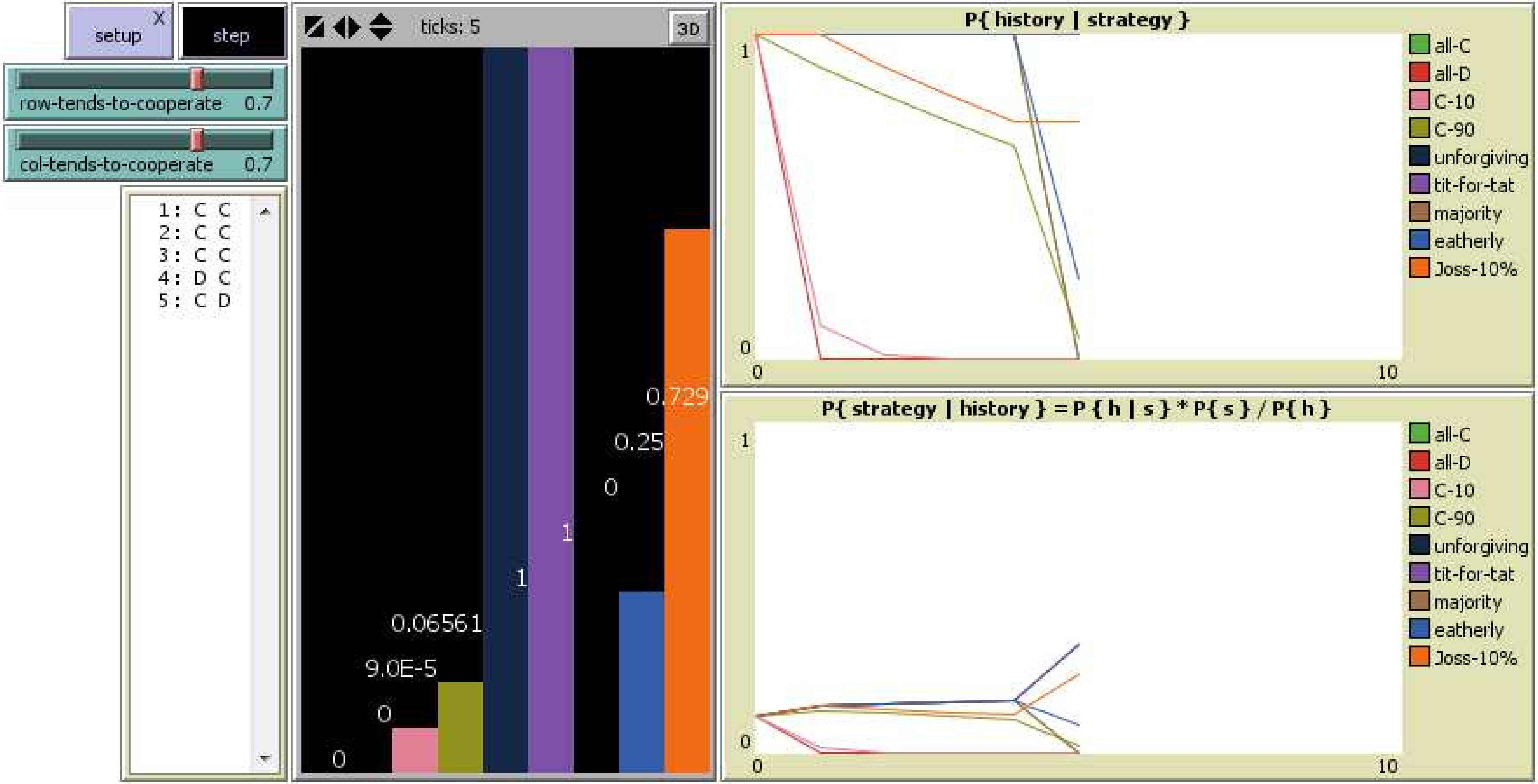

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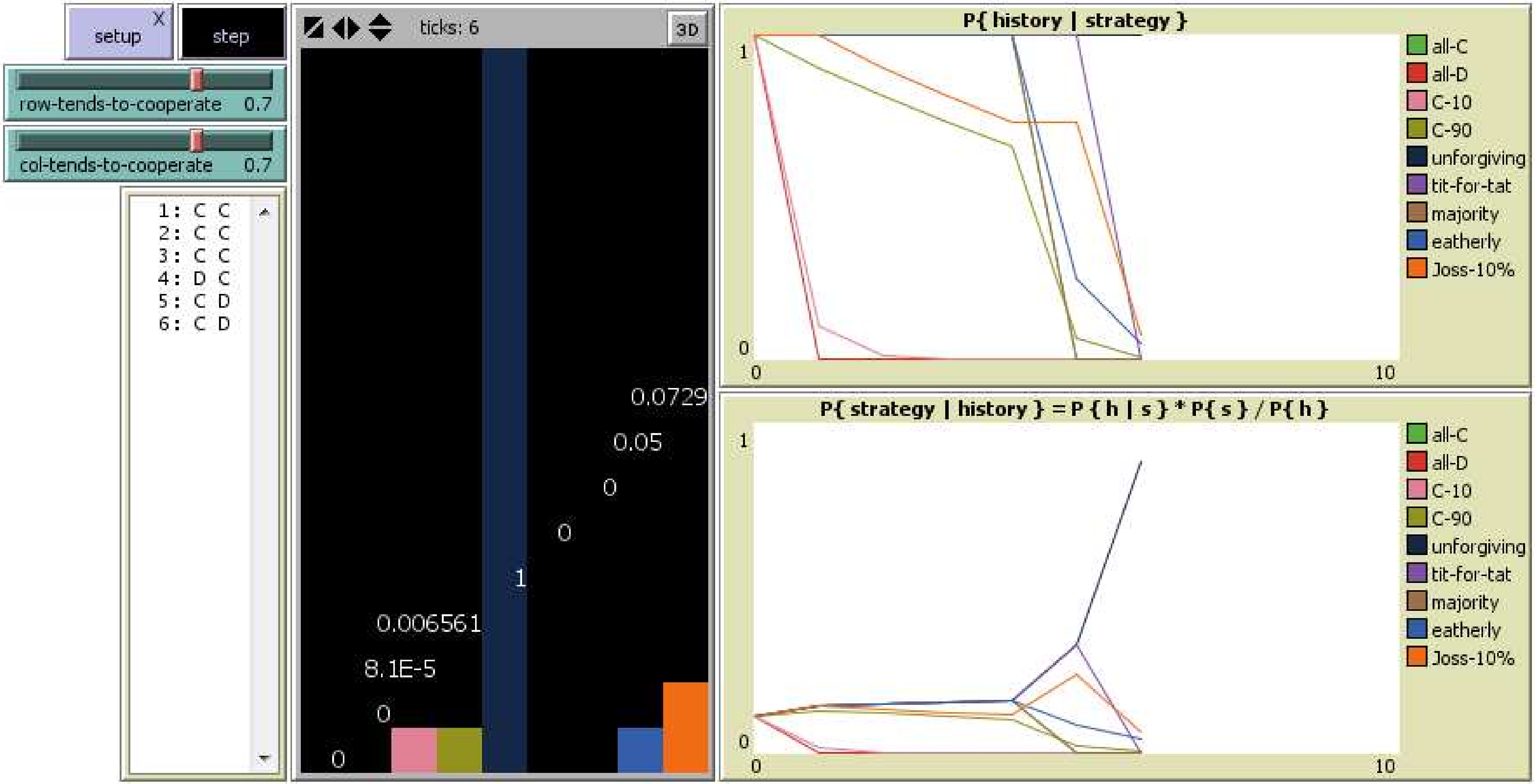

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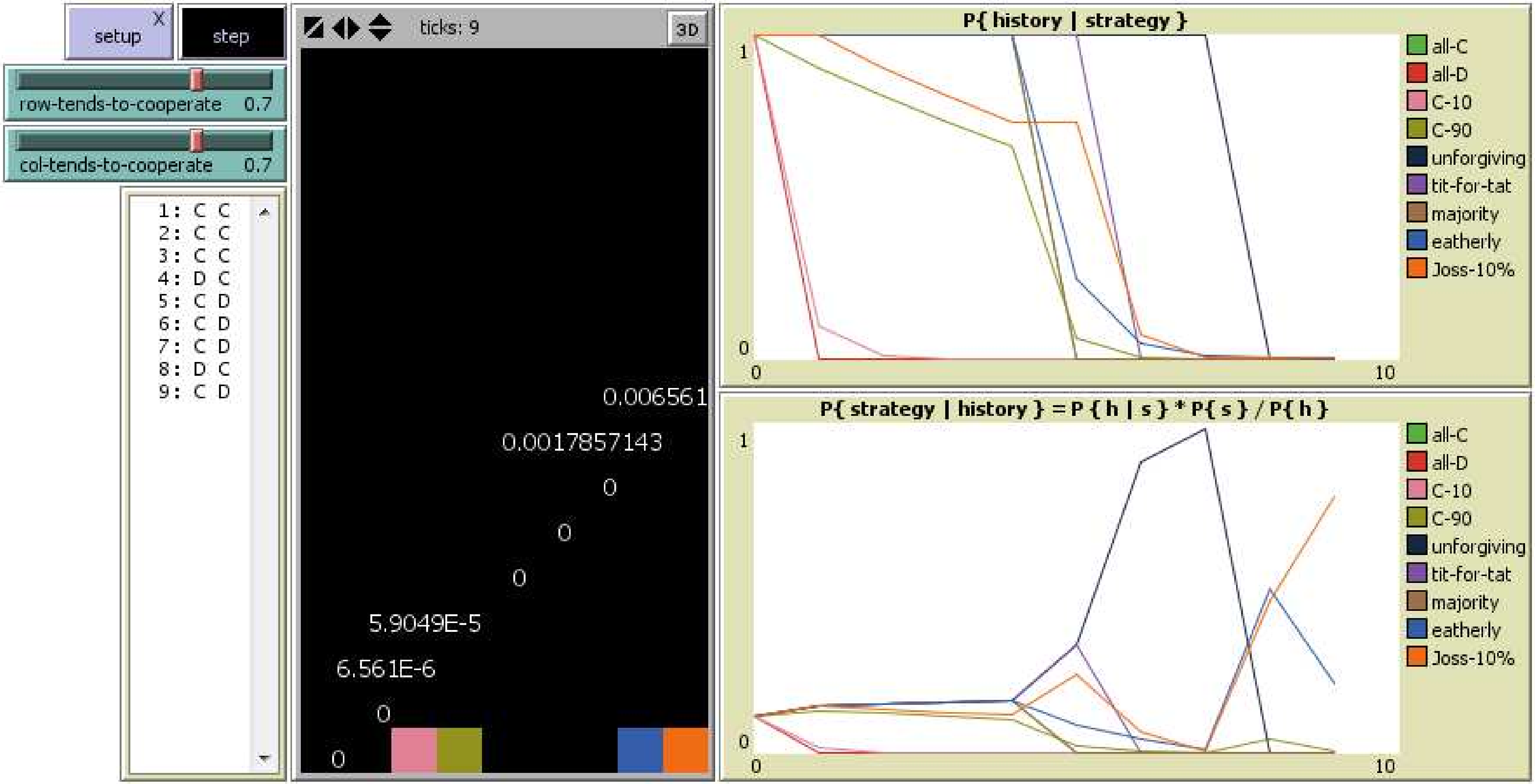

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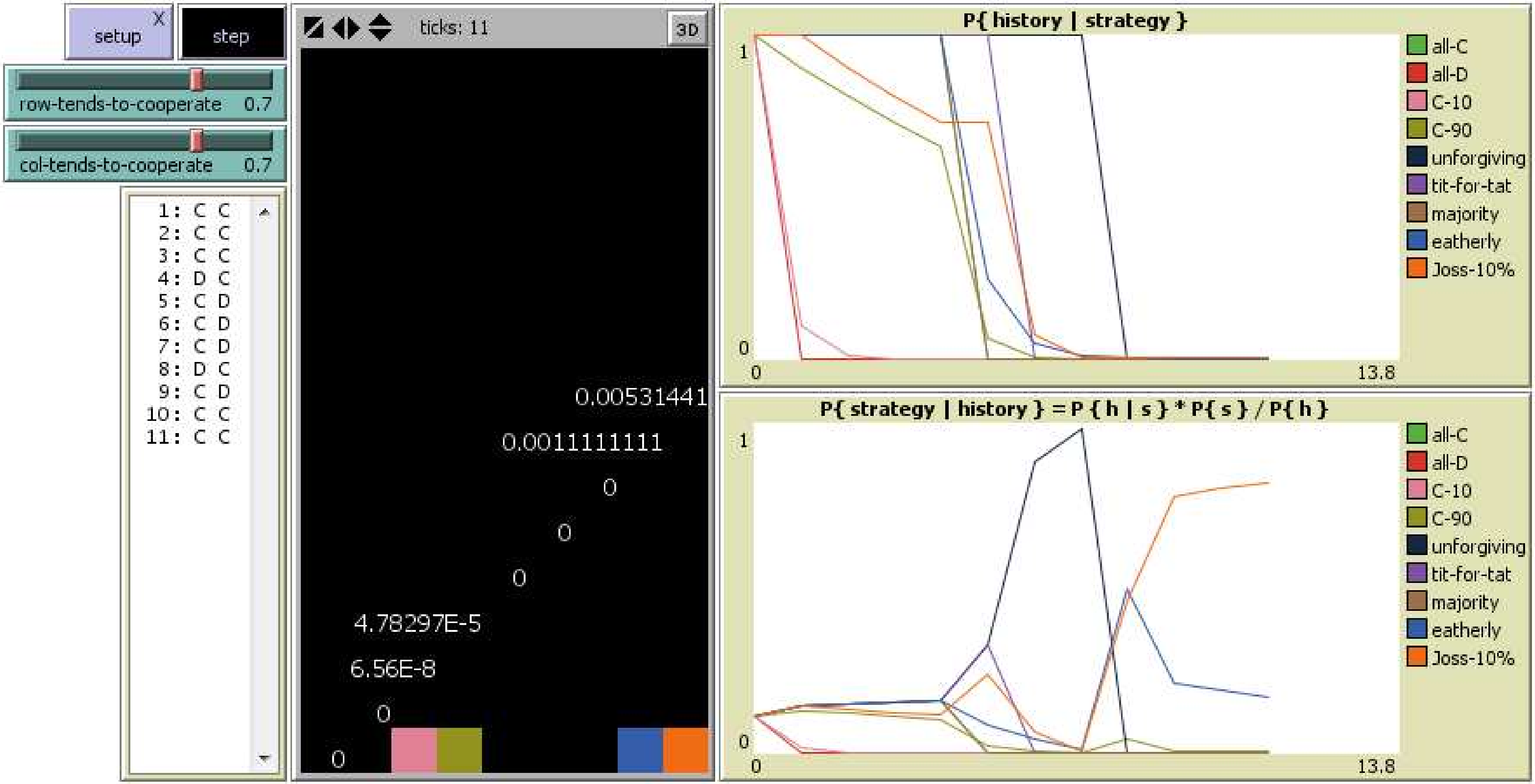

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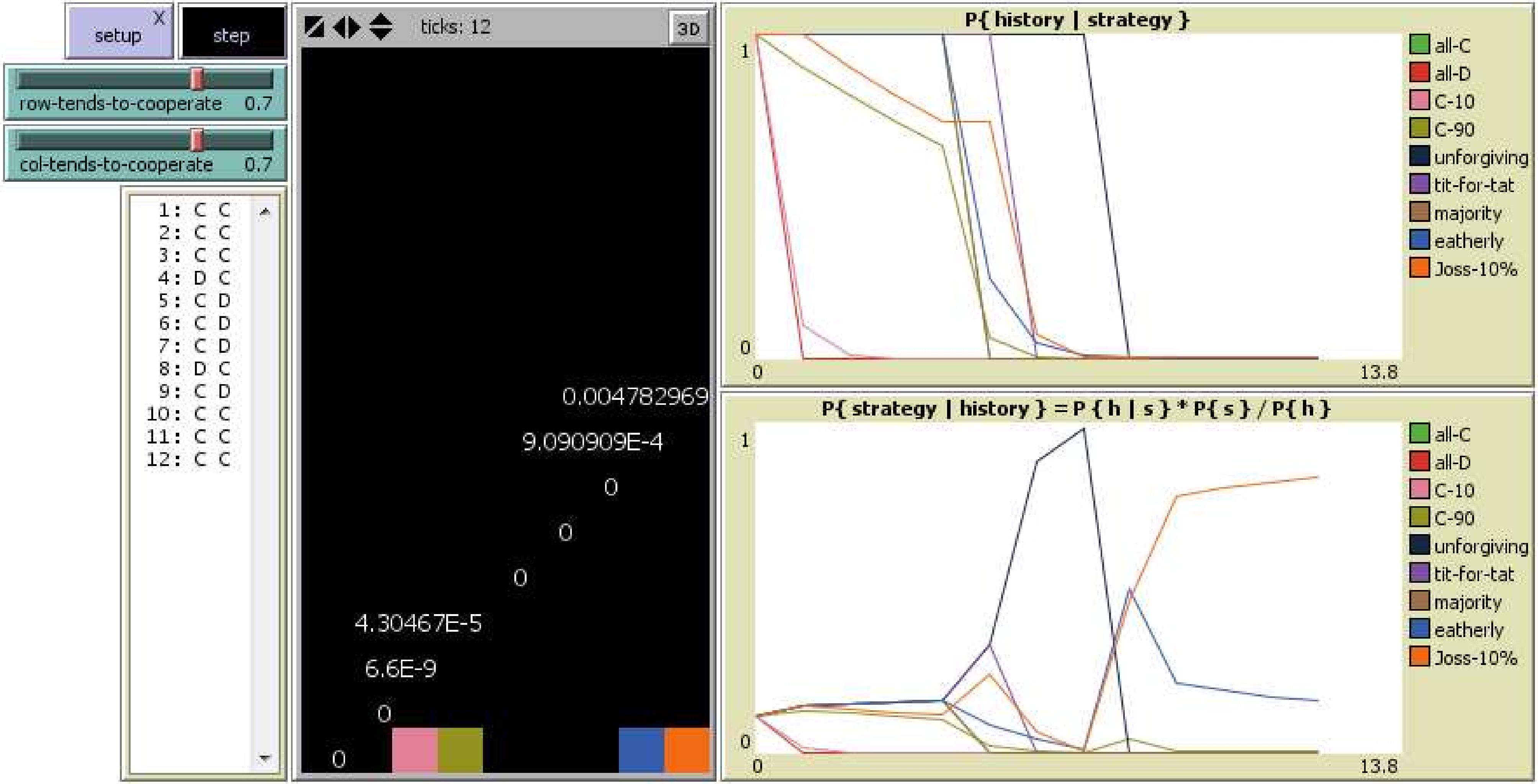

Example: learning on reactive strategies

Explanation of $\Pr\{h \mid s_2 = \text{Joss-10\%}\}$

Event	Description
$s_2 = \text{Joss-10\%}$	Player 2's strategy is Joss-10%
$X_1^{t-1} = C$	Player 1 cooperated in the previous round
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$\Pr\{X_2^t = x \mid s_2 = \text{Joss-10\%}\}$:	0.9	0.1	0.9	0.9	1	0	0
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\Pr\{h_2^t = C, h \mid X_1^{t-1} = C\} &= (\Pr\{X_2^t = C \mid X_1^{t-1} = C, \xi^t\}\Pr\{\xi^t\} + \\
&\qquad \Pr\{X_2^t = C \mid X_1^{t-1} = C, \bar{\xi}^t\}\Pr\{\bar{\xi}^t\})\Pr\{h\} \\
&= (0 \cdot 0.1 + 1 \cdot 0.9)\Pr\{h\} \\
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Suppose Player 1 and 2 play the coordination game, and deem the following strategies possible:

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$$h = [(L, R), (R, L)]$$

Player 1 reasons:

$$\Pr\{R^* | h\} = \frac{\Pr\{h | R^*\}\Pr\{R^*\}}{\sum_{j=1}^{3} \Pr\{h | s_j\}\Pr\{s_j\}} = \frac{0 \cdot 0.3}{0 \cdot 0.6 + 0 \cdot 0.3 + (\frac{1}{2})^2 \cdot 0.1} = 0.$$

$[\Pr\{(L, \cdot\}, (R, \cdot) | F^*\} = \left(\frac{1}{2}\right)^2]$ and

$$\Pr\{F^* | h\} = \frac{\Pr\{h | F^*\}\Pr\{F^*\}}{\sum_{j=1}^{3} \Pr\{h | s_j\}\Pr\{s_j\}} = \frac{(\frac{1}{2})^2 \cdot 0.1}{0 \cdot 0.6 + 0 \cdot 0.3 + (\frac{1}{2})^2 \cdot 0.1} = 1.$$

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Now, $\mathrm{BR}(s_p|h) \cap S \neq \emptyset$, so that from round to round play vs. prediction is a closed system.

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- **Solution:** extend μ of H to μ on a σ-algebra of Ω.

Part III: Formalism

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  Often, (2) is meant.

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Often, this is abbreviated to Δ_{-i}^H.

Idea of Kuhn's proof

$$
\begin{array}{rr|cccccc|c}
\text{distribution} & & \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_r & \dots & \\
\textbf{strategies} & \mathbf{G} & \mathbf{g_1^j} & \mathbf{g_2^j} & \mathbf{g_3^j} & \dots & \mathbf{g_r^j} & \dots & \mathbf{g^j} \\
\hline
\text{histories} & h_1 & q_1^1 & q_2^1 & q_3^1 & \dots & q_r^1 & \dots & \lambda_1 q_1^1 + \dots + \lambda_r q_r^1 + \dots \\
 & h_2 & q_1^2 & q_2^2 & \dots & \dots & q_r^2 & \dots & \lambda_1 q_1^2 + \dots + \lambda_r q_r^2 + \dots \\
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histories	h_1	q^1_1	q^1_2	q^1_3	$\dots$	q^1_r	$\dots$	$\lambda_1 q^1_1 + \dots + \lambda_r q^1_r + \dots$
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Remarks:

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Behavioural strategy and conditional forecast

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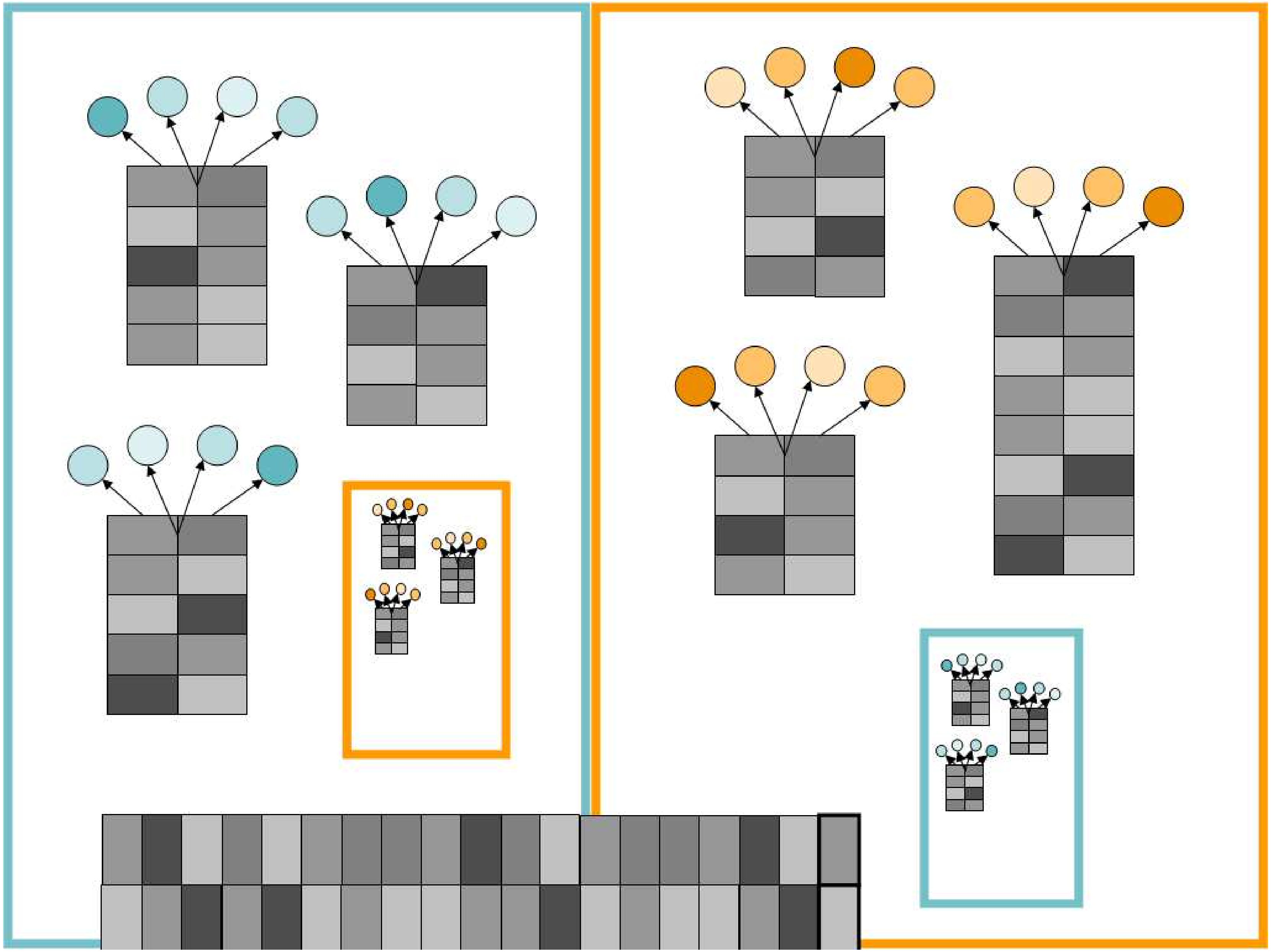
Summary:

$$\begin{array}{ccc} H \times \Delta_{-i} \xrightarrow{\tau_i} & \Delta_{-i} & \\ & \vdots & \\ & H \times \Delta_{-i} & \xrightarrow{\sigma_i} \Delta_i \end{array}$$

Behav. strategy and belief of opponent’s behav. strategy



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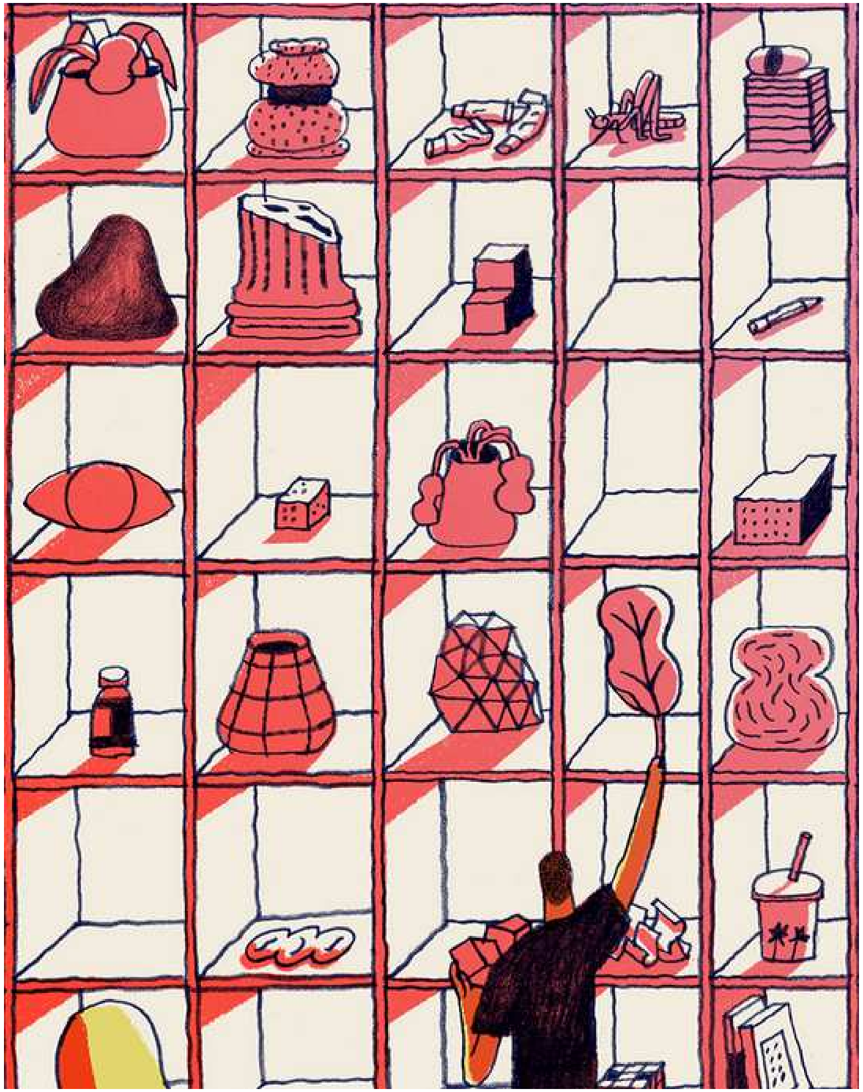

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- Beliefs ν_i are maintained through Bayesian updating.

Part IV:
True distribution of play vs. subjective distribution of play

The true distribution of play μ on H

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- The reply profile $g \in \Delta^H$ induces a probability distribution μ on H, inductively:

$$\mu\{hx\} = \begin{cases} 1 \cdot g(x) & \text{if } h = \emptyset, \\ \mu\{h\} \cdot g(x|h) & \text{otherwise.} \end{cases}$$

  The probability measure μ on H, on its turn, induces a probability measure on (a σ-algebra of) Ω.

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Standard results in probability theory[1]now imply that μ' can be extended to a proper probability measure on (a σ-algebra of) Ω.

[1]Carathéodory's extension theorem.

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- Similarly, players i's conditionalised predictive learning rule, (ν_i, g_i), generates a subjective distribution of play, μ_i, on Ω:

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_n \end{pmatrix} \text{ is generated by } (\nu, g) = \begin{pmatrix} g_1 & \nu_1^2 & \nu_1^3 & \dots & \nu_1^n \\ \nu_2^1 & g_2 & \nu_2^3 & \dots & \nu_2^n \\ \nu_3^1 & \nu_3^2 & g_3 & \dots & \nu_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu_n^1 & \nu_n^2 & \nu_n^3 & \dots & g_n \end{pmatrix}.$$

Part V: Main results

The subjective view on distribution of play

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First stab at the formulation of a theorem:

If $t \to \infty$, then i's subjective view on play at t, namely,

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approximates the true distribution of play at t, namely,

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So, if i's beliefs contain a grain of truth on actual play, then μ_i is, roughly put, "as expressive as" μ.

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Theorem (Blackwell and Dubins, 1962). *Let μ represent the true distribution of play, and let μ_i represent i's view on the distribution of play. If $\mu \ll \mu_i$, then i's beliefs merge with the true distribution of play.*

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- Very good predictor $\Rightarrow$ good predictor.

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- Very good predictor $\Rightarrow$ good predictor.
- From (Blackwell and Dubins, 1962) the following can be proven.

Corollary. If $\mu \ll \mu_i$ then i is a very good predictor a.s.

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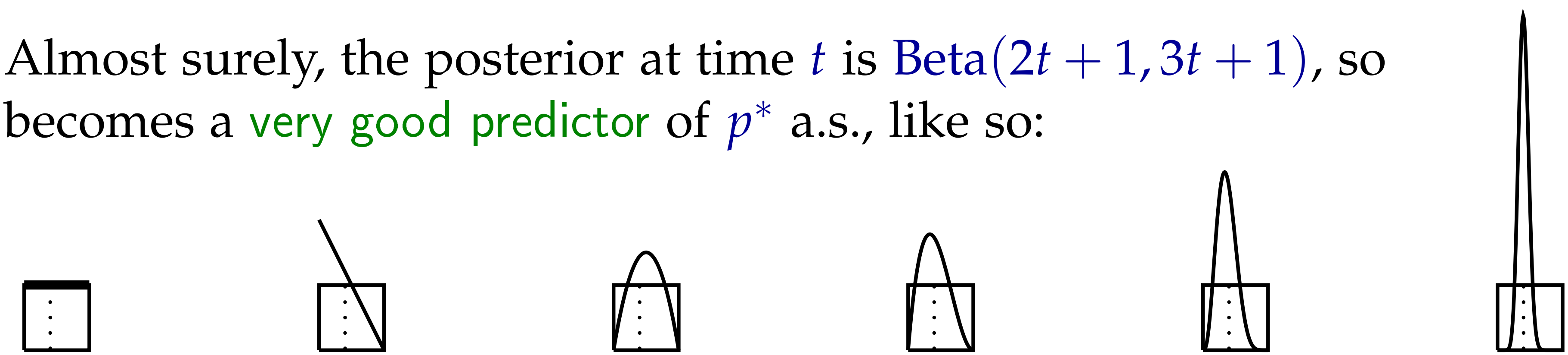

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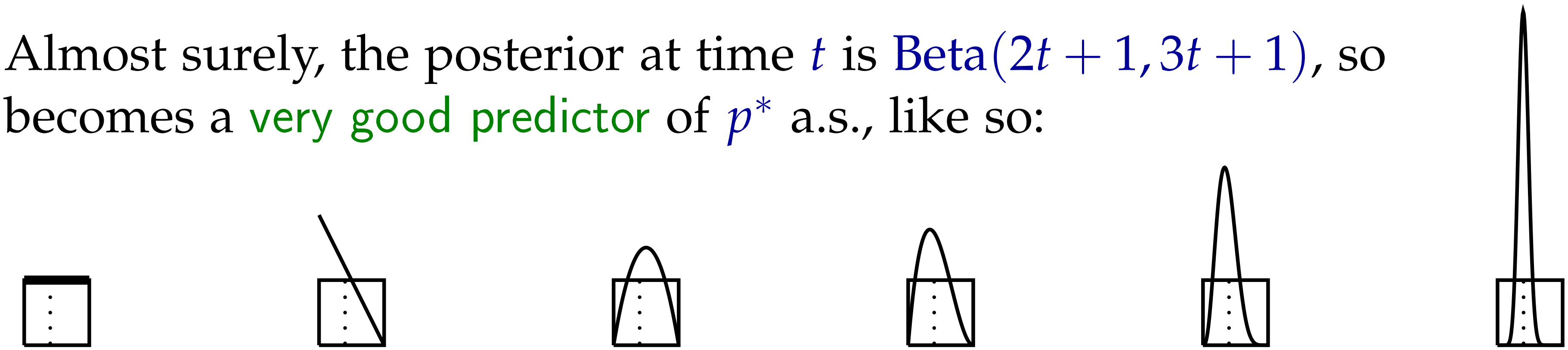

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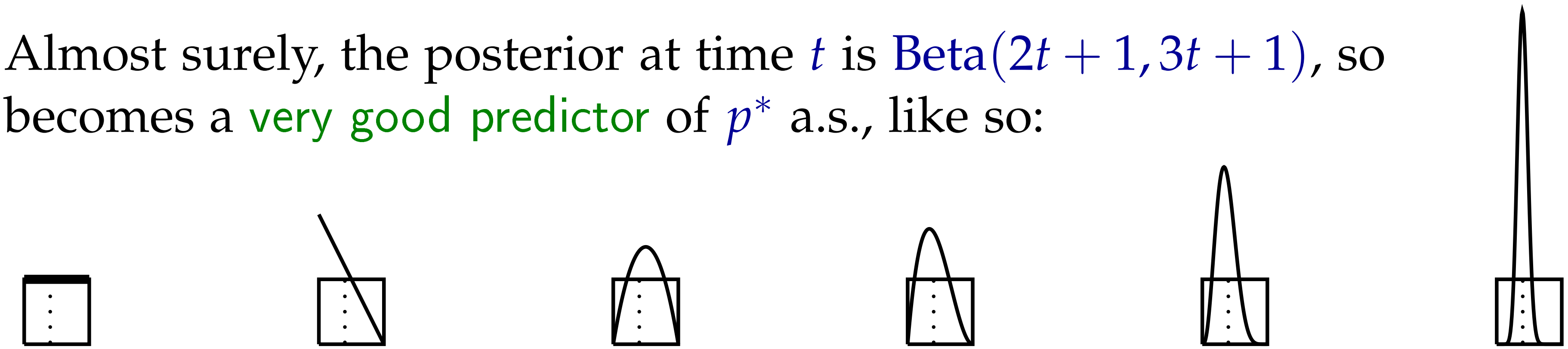

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Definition. Let μ and τ probability distributions. μ and τ are said to be ϵ-close, written $\mu \sim_\epsilon \tau$, if for some $Y \subseteq X$ we have $\mu\{Y\} > 1 - \epsilon$ and $\tau\{Y\} > 1 - \epsilon$ and for all measurable $E \subseteq Y$:

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- Thus, being ϵ-close means that not only does the player assess the future correctly, *it even assesses developments following unlikely histories correctly*.

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- This theorem implies closeness of probabilities also for courses of play that are extremely unlikely.
- The theorem does not state that a player learns to predict other players' off path strategies. (Recall Player 2's beliefs in unforgiving constellation.)

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  - It be analysed *quantitatively* (by indicating the rate of convergence). This is much harder.

In Chapter 22 of the *Handbook on Computational Economics* (2006), Young describes how social dynamics can drift to so-called stochastically stables states, provided individuals act rationally most of the time.

The following slides were not used.

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Clearly, The mappings are each other's inverse. Both mappings are surjective.
Further, fundamental math defines 0 as $\emptyset$, and $n+1$ as $n \cup \{n\}$:

$$\begin{aligned}
0 &= \emptyset \\
1 &= 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{\emptyset\} \\
2 &= 1 \cup \{1\} = \{\emptyset, \{\emptyset\}\} \\
3 &= 2 \cup \{2\} = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\} \\
&\vdots
\end{aligned}$$

So

$$\begin{aligned}
B^n &= B^{\{0,1,\dots,n-1\}} \\
&= \{f \mid f : \{0,1,\dots,n-1\} \to B\}.
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- $(Z^X)^Y \sim (Z^Y)^X$
- $X^Y \times X^Z \sim X^{(Y \cup Z)}$, provided Y and Z are disjoint

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The factor $\lambda_r|h$ represents the posterior probability of choosing g^j_r given h:

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Playing against the strategies g_r^j with the probabilities λ_r and playing against an equivalently constructed behaviour strategy g^j generate identical probability distributions on future play paths.

(Kuhn, 1953; Aumann, 1964.)

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1. The theorem om Radon-Nikodym (mentioned above), which (roughly!) says that if $\nu \ll \mu$, then ν can be expressed in terms of μ.

2. Lévy's theorem, which (very roughly!) is about continuity of expectation through a so-called *filter*.[2]

Theorem. (Lévy's Convergence Theorem). *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\{\mathcal{F}_n\}_n$ be a non-decreasing family of σ-algebras contained in $\mathcal{F}$. Let $\mathcal{F}_\infty$ be the smallest σ-algebra around $\cup\{\mathcal{F}_n\}_n$. Let X be a random variable with finite expectation. Then, both P-a.s.*[3] *and in the L_1-sense,*[4]

$$E[X|\mathcal{F}_n] \to E[X|\mathcal{F}_\infty] \text{ as } n \to \infty.$$

[2] Sequence of monotone non-decreasing σ-algebras.
[3] $P\{|\ldots - \ldots| > \epsilon\} \to 0$.
[4] $E|\ldots - \ldots| \to 0$.

The Radon-Nikodym theorem

Given X *and* σ*-finite probability measures* μ *and* ν *on* X *such that* $\nu \ll \mu$*, then there is a measurable function* $f : X \to [0, \infty)$*, such that for all* μ*-measurable sets* E*,*

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- The theorem tells if and how it is possible to change from one probability measure to another.
- Specifically, the probability density function of a random variable is the Radon-Nikodym derivative of the induced measure with respect to some base measure (usually the Lebesgue measure for continuous random variables).

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Essentially, it is a sufficient condition for the almost sure convergence to imply L1-convergence. The condition $|X_n| < Y$, $\mathrm{E}Y < \infty$ could be relaxed. Instead, the sequence $\{X_n\}_{n=1}^{\infty}$ should be uniformly integrable.

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The theorem is simply a special case of Lebesgue's dominated convergence theorem in measure theory.

Part VI: An Impossibility Result

Suppose players respond rationally

Definition. *Given a predictive learning rule* (f_i, g_i), *rule* g_i *is said to be* rational *given* f_i *if, for each* $h \in H$, *rule* g_i *maximises expected discounted payoffs over all continuations of* h.

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- Also, g_i cannot depend on u_{-i} because i's payoffs do not give information on the payoffs of opponents.

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Definition. A vector of n profiles

$$(v_i, g_i) = \begin{pmatrix} g_1 & v_1^2 & v_1^3 & \dots & v_1^n \\ v_2^1 & g_2 & v_2^3 & \dots & v_2^n \\ v_3^1 & v_3^2 & g_3 & \dots & v_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n^1 & v_n^2 & v_n^3 & \dots & g_n \end{pmatrix}$$

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There's no catch: recall that players converge even if they do not play best replys but play, e.g., maxmin.

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   From here, G''s matrix is fixed, and G' is ready to be played.

Example. "Uncertain matching pennies". Property: when λ is sufficiently small, G' still possesses a unique (mixed) Nash equilibirum.

$$M = \begin{array}{c c} & \begin{array}{cc} \mathrm{H} & \mathrm{T} \end{array} \\ \begin{array}{c} \mathrm{H} \\ \mathrm{T} \end{array} & \begin{pmatrix} 1+\epsilon_{11}, -1+\epsilon'_{11} & -1+\epsilon_{12}, 1+\epsilon'_{12} \\ -1+\epsilon_{21}, 1+\epsilon'_{21} & 1+\epsilon_{22}, -1+\epsilon'_{22} \end{pmatrix} \end{array}$$

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- This is far more stronger than the negation of asymptotic closeness, in the sense of Kalai and Lehrer.

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- Explanation:
  - Kalai *et al.*'s main theorem states that players will (almost) correctly predict the on-path portions of the other players' strategies.
  - Kalai *et al.*'s main theorem does not state that players will (almost) learn the true strategies of their opponents.