Multi-agent learning

No-regret learning

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Monday 11th May, 2020

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   - Smoothed fictitious play. Give a soft-max response to the (recent) empirical frequency of opponents' actions.
   - Hypothesis testing with smoothed best responses. Give a best response to maintained beliefs about *patterns of play*.

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Part I: Basic concepts

No-regret: example

Payoffs Player A	0	0	0	1	1	0	0	0	1	0	0	
Actions Player A	L	R	L	L	R	R	L	R	R	R	R	?
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  So no-regret does not take the interactive nature of play into account.

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$$\Leftrightarrow \quad \lim_{t\to\infty} [\, \bar{r}_x^t(\omega) \,]_+ = 0.$$

Part II: proportional regret matching

The strategy of proportional regret matching

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Regret matching differs from reinforcement learning

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	Payoff	*Average regret*	*Regret matching*
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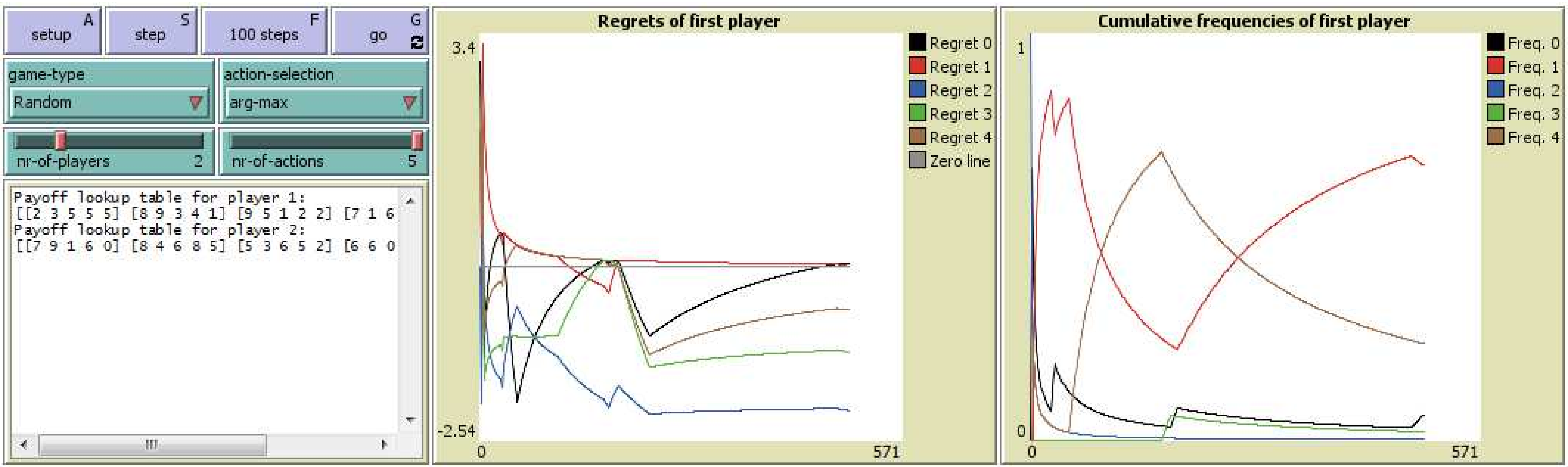
Cumulative payoff matching:

	Accumulated payoff	*Mixed strategy*
Action L:	1	$1/3$
Action R:	2	$2/3$

Regret matching in a 5-person 5-action game

Payoff matrix uninformative. Omitted . . .



Netlogo simulation of regret matching in a 5-person 5-action game.

Regret matching in Shapley's game

	R	Y	B
R	$(1,0)$	$(0,0)$	$(0,1)$
Y	$(0,1)$	$(1,0)$	$(0,0)$
B	$(0,0)$	$(0,1)$	$(1,0)$

Column is "fashion leader", row is "fashion follower". Column wants to wear a different color than row.

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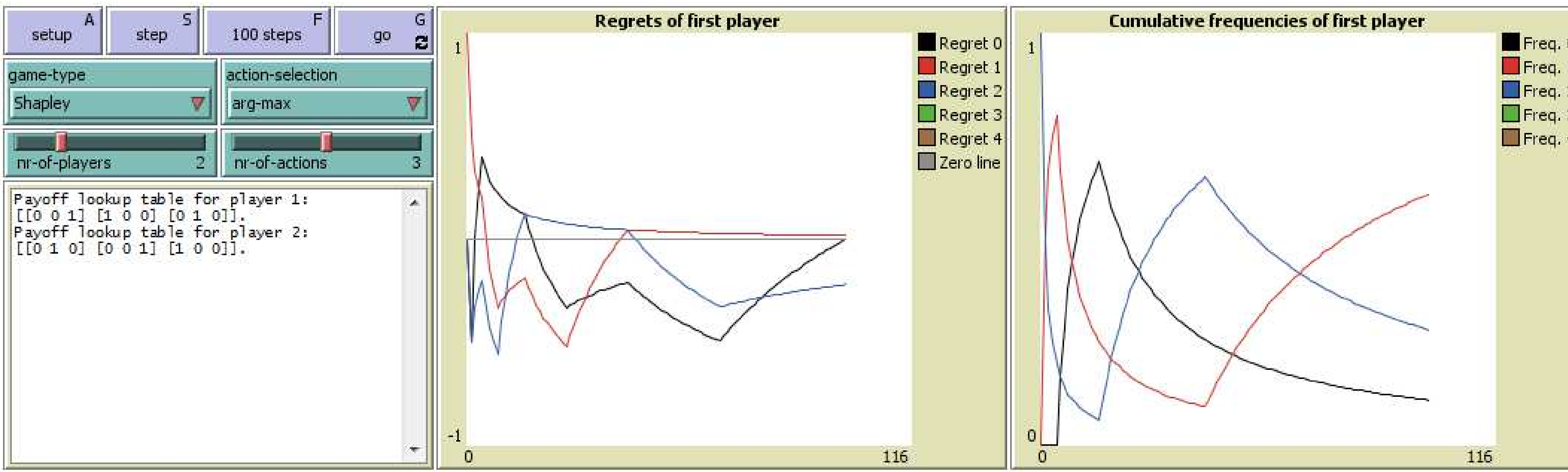

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\Delta r_x^t &=_{Def} \text{ incremental regret for not playing } x : r_x^t \; - \; r_x^{t-1}
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\Delta r_x^t &=_{Def} \text{ incremental regret for not playing } x : r_x^t \;-\; r_x^{t-1} \\
E[\Delta r_x^t] &= \text{ expected incremental regret for not playing } x
\end{aligned}
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  Vector versions: r^t, $\bar{r}^t$, $[\bar{r}^t]_+$, $\dots$, $E[r^t]$, $E[\Delta r^t]$.

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  i.e., the regret vector must approach the negative orthant with probability one.

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Why does regret matching work?

A strategy q such that $\lim_{t\to\infty}[\bar{r}^t]_+ = 0$: 1st attempt

Take $q_1^t = q_2^t = 1/2$ for all t.

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Inductively then

$$E[\bar{r}_1^t + \bar{r}_2^t] = 0,$$

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However, the two terms may neutralise each other.

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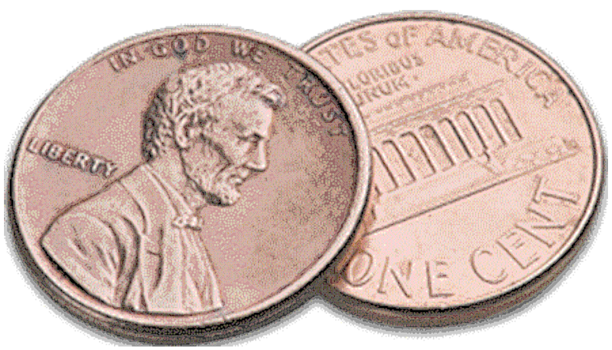

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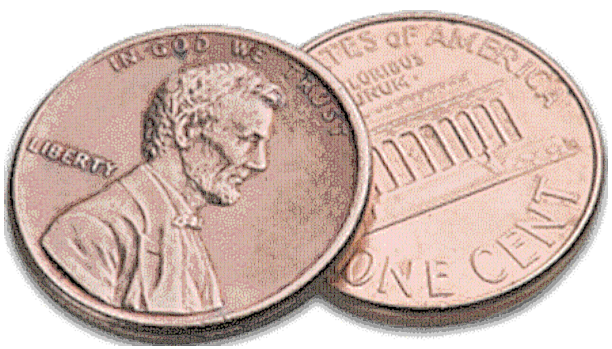

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  - Won't work: suppose you meet an opponent who happens to switch every round as well . . .
- Won't work in general: your corrections may by coincidence be out of phase with the path of play of your opponent. Peyton Young:

  > "Recall that no-regret must hold even when Nature is malevolent."
  >
  > (p. 26)

Decrease of expected regret

The objective is to find a (mixed) strategy $g : H \to \Delta(\{1,2\})$ such that

$$E[\,\bar{r}^{t+1} \mid r^t, \ldots, r^1] \, < \, \bar{r}^t \tag{1}$$

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\end{aligned}$$

Decrease of expected regret

The objective is to find a (mixed) strategy $g : H \to \Delta(\{1,2\})$ such that

$$E[\,\bar{r}^{t+1} \mid r^t, \ldots, r^1] \, < \, \bar{r}^t \tag{1}$$

because then Blackwell's approachability theorem can be applied to conclude $\lim_{t\to\infty}[\bar{r}^t]_+ = 0$. Since $\Delta E[r^{t+1}]$ is known, we have

$$\begin{aligned}
E[\,\bar{r}^{t+1} \mid r^t, \ldots, r^1] &= E[\,\frac{r^t + \Delta r^{t+1}}{t+1} \mid r^t, \ldots, r^1] \\
&= \frac{t}{t+1} E[\frac{r^t}{t} \mid r^t, \ldots, r^1] + \frac{1}{t+1} E[\Delta r^{t+1} \mid r^t, \ldots, r^1] \\
&= \frac{t}{t+1} \bar{r}^t + \frac{1}{t+1} E[\Delta r^{t+1} \mid r^t, \ldots, r^1] \\
&= \frac{t}{t+1} \bar{r}^t + \frac{1}{t+1} (-\alpha^{t+1} q_2^{t+1}, \, \alpha^{t+1} q_1^{t+1})
\end{aligned}$$

So, the objective is to find a strategy such that $\alpha^{t+1}(-q_2^{t+1}, \, q_1^{t+1}) < \bar{r}^t$.

A strategy q such that $\lim_{t\to\infty}[\bar{r}^t]_+ = 0$: 3rd attempt

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- Recall: our objective is $[\bar{r}^t]_+ \to 0$.
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$$E[\Delta r^{t+1}] \perp [\bar{r}^t]_+$$

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So:

$$\begin{aligned}
& E[\Delta r^{t+1}] \cdot [\bar{r}^t]_+ = 0 \\
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The last equation amounts to proportional regret matching.

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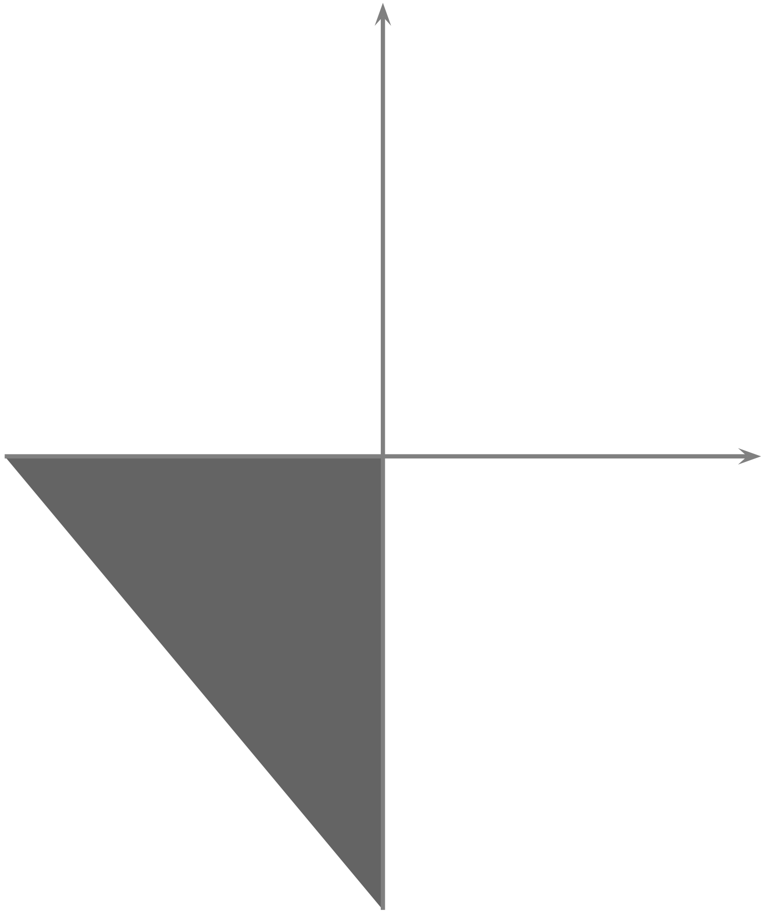
  - If all regret is non-positive, then play an action at random.

Stochastic dynamics of regret matching



- Expected incremental regret, $E[\Delta r^{t+1}]$ is made orthogonal to the current regret, independently of the unknown α^{t+1}.

  Because at A does not know what B will play next, this is crucial.

- $E[\bar{r}^{t+1}]$ is a convex combination of $\bar{r}^t_+$ and $E[\Delta r^{t+1}]$.

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- Ultimately, the result follows from Blackwell's approachability theorem (Strategic Learning and its Limits, 2004, Ch. 4).

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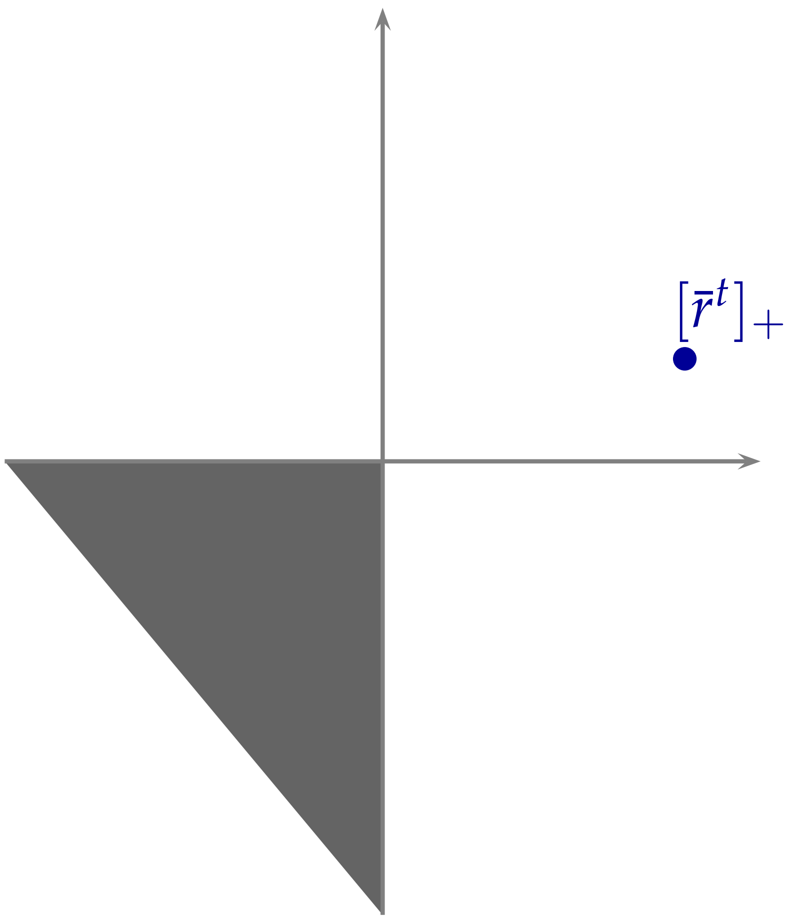

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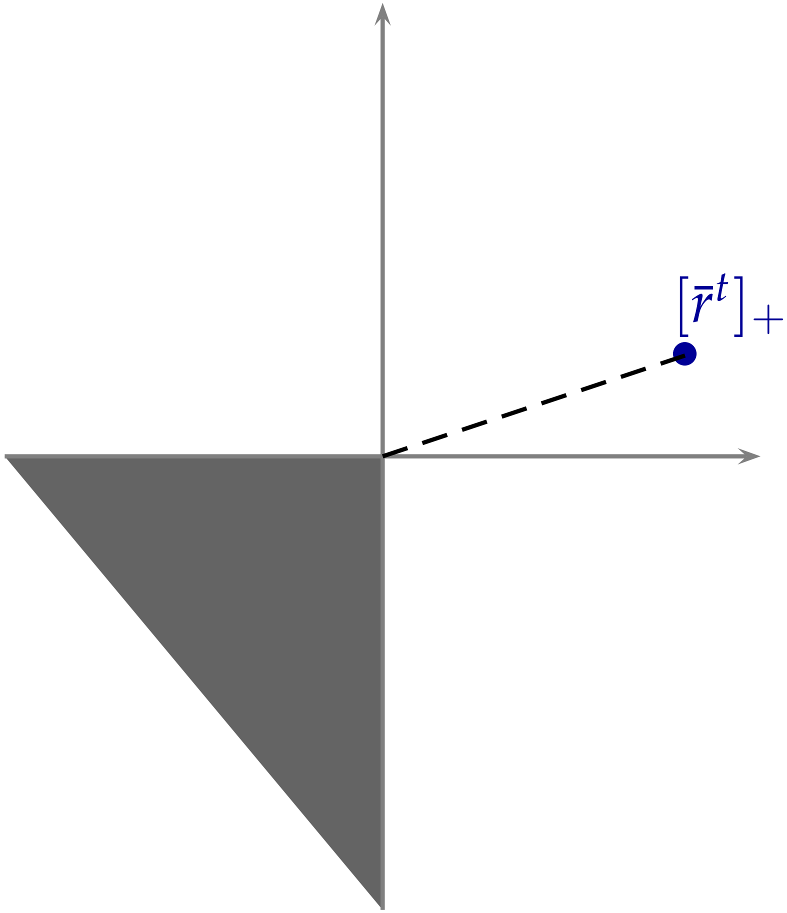

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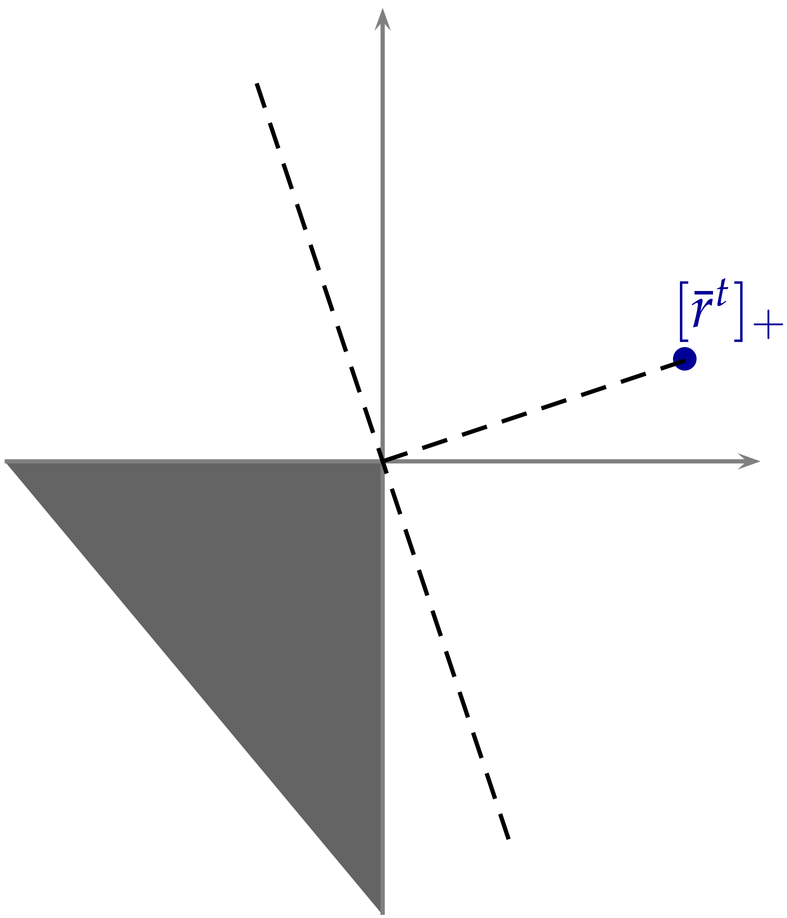

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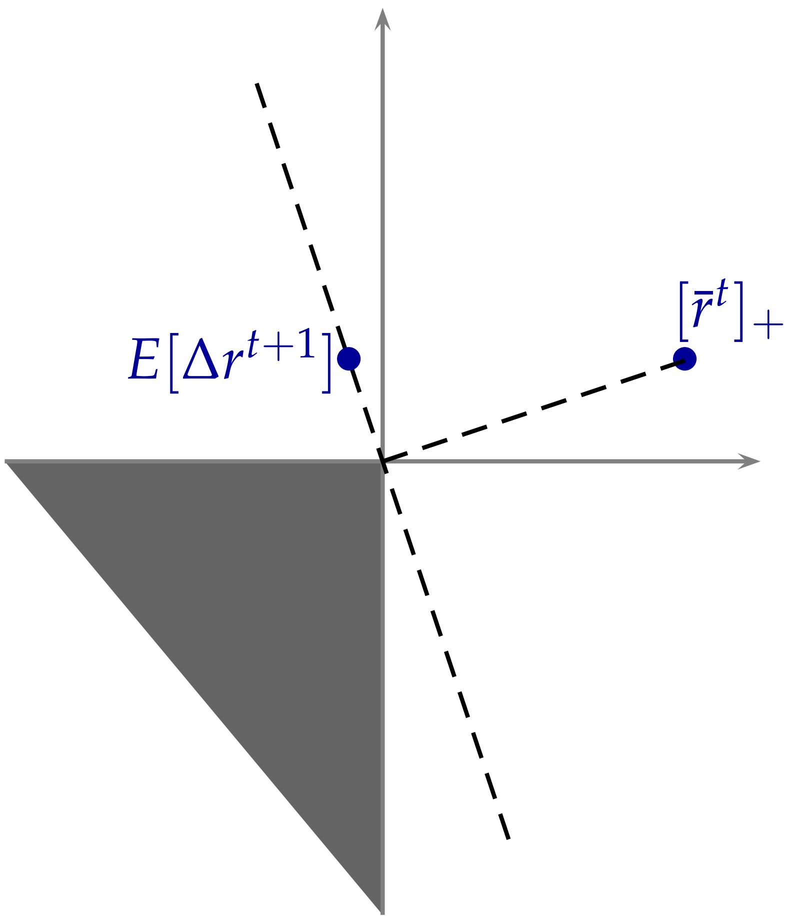

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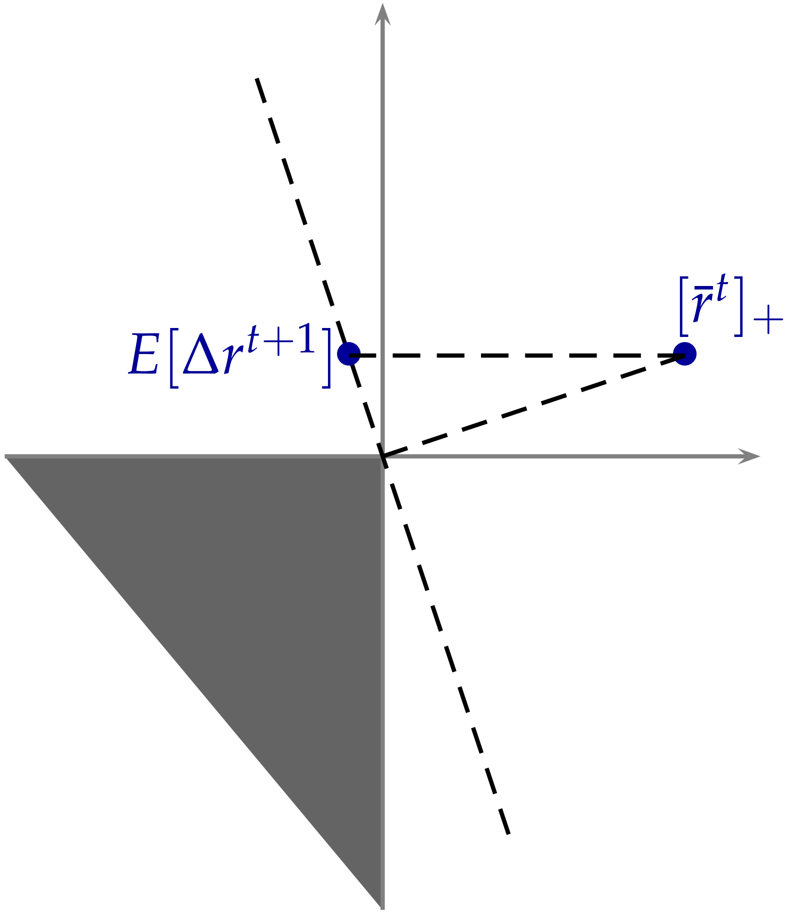

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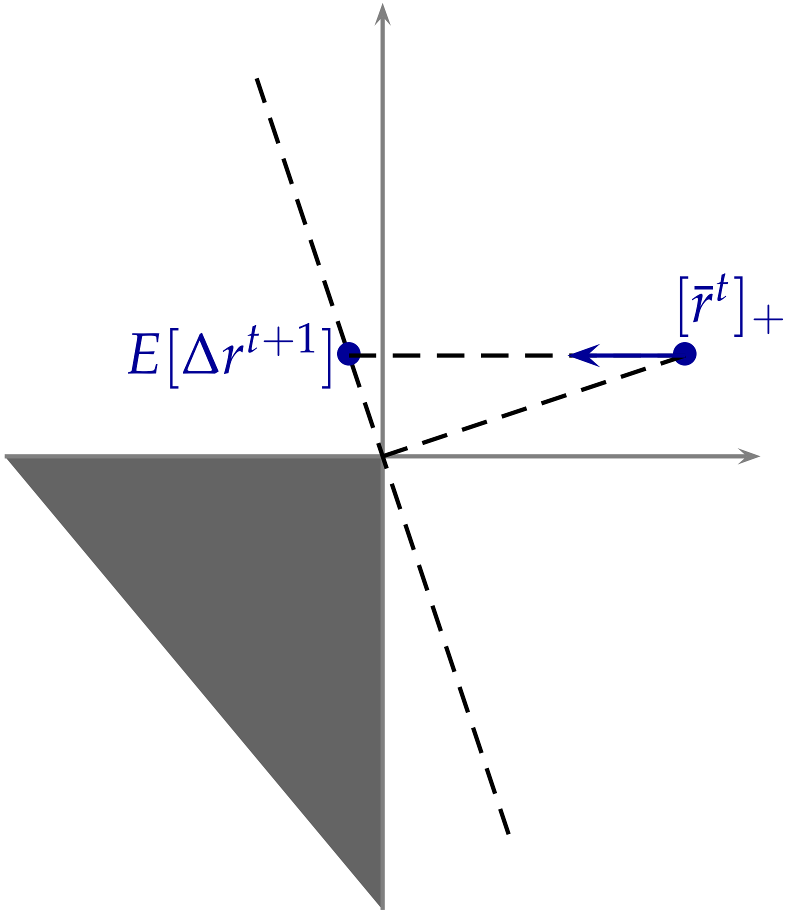

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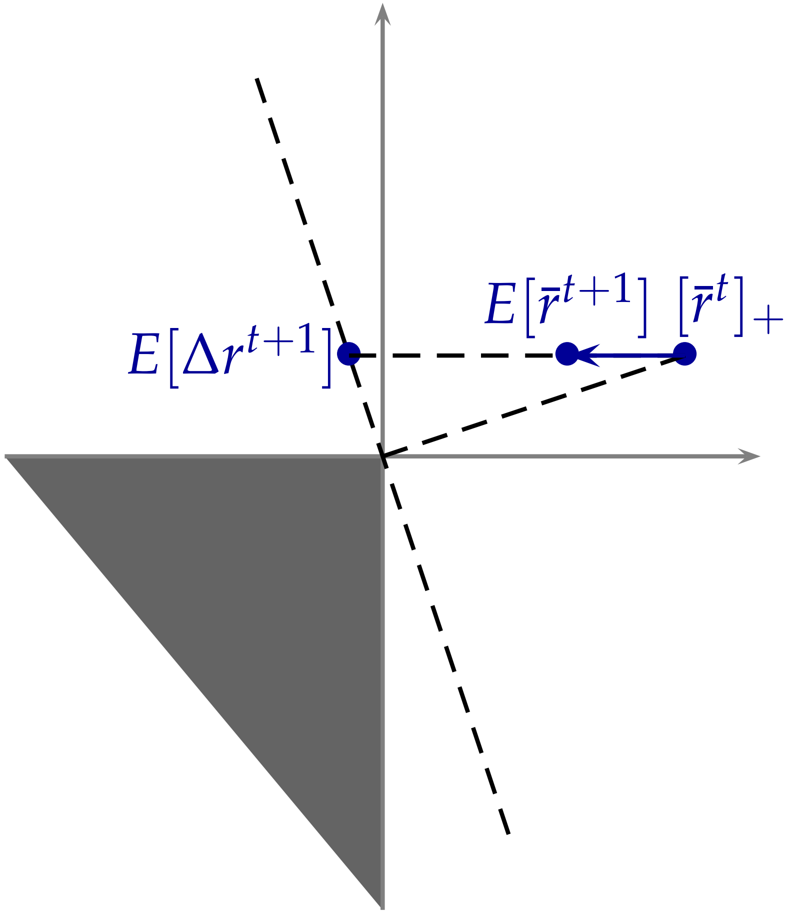

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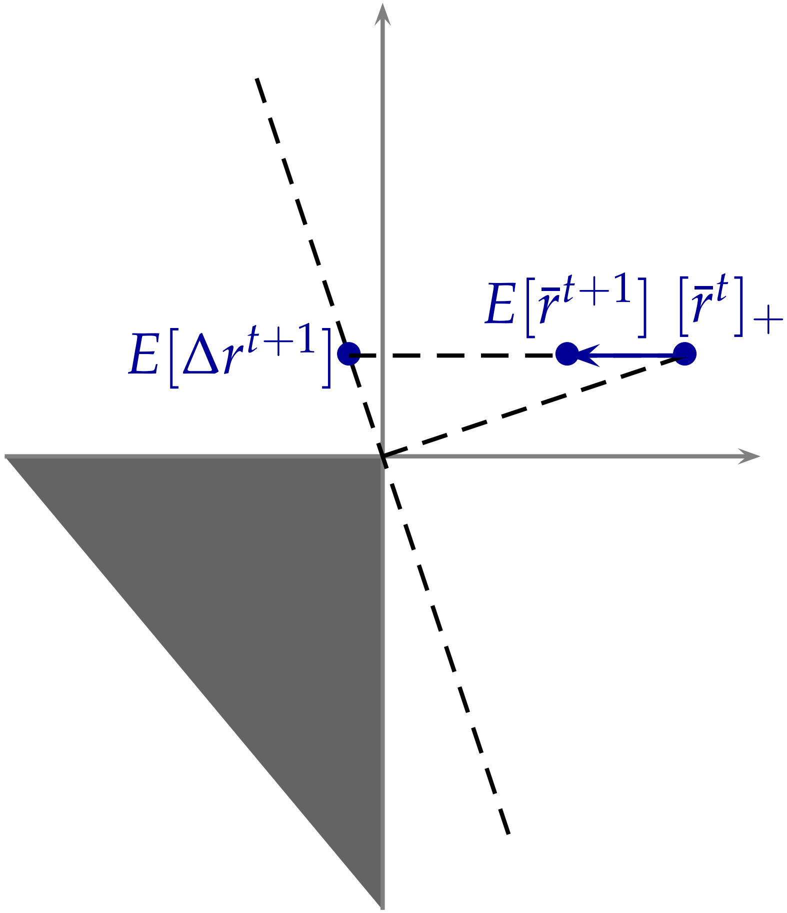

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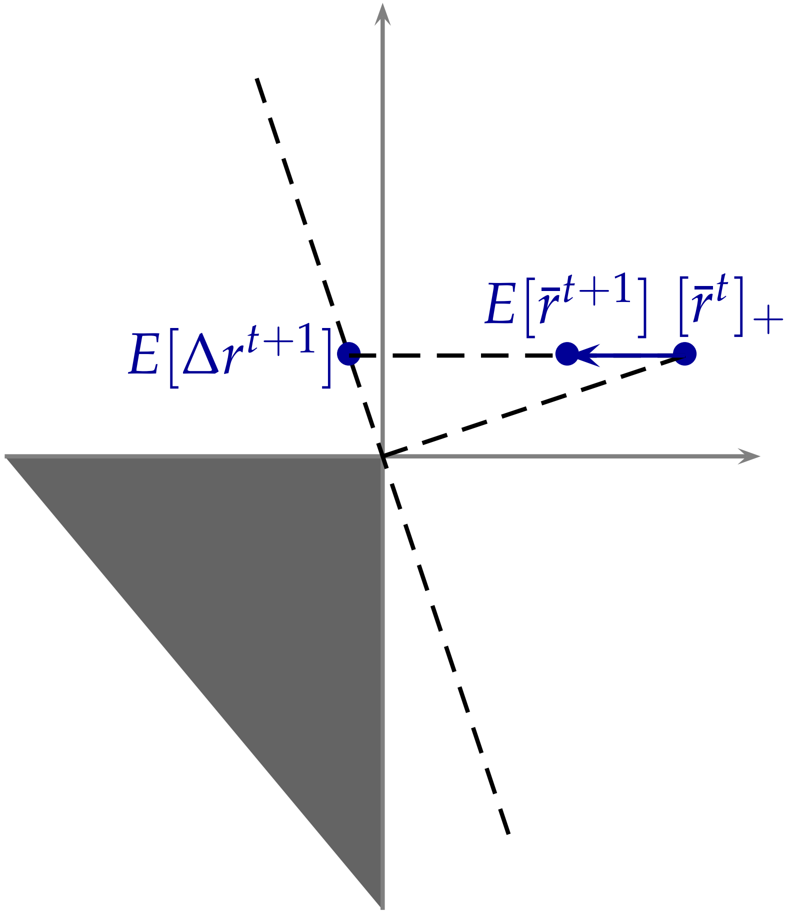

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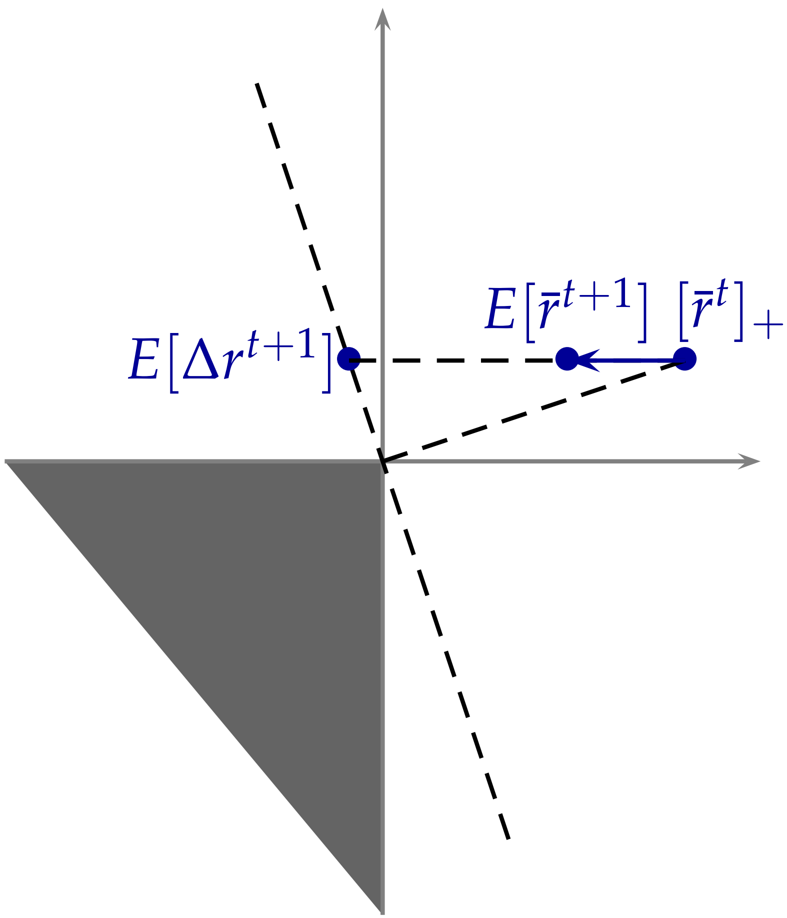

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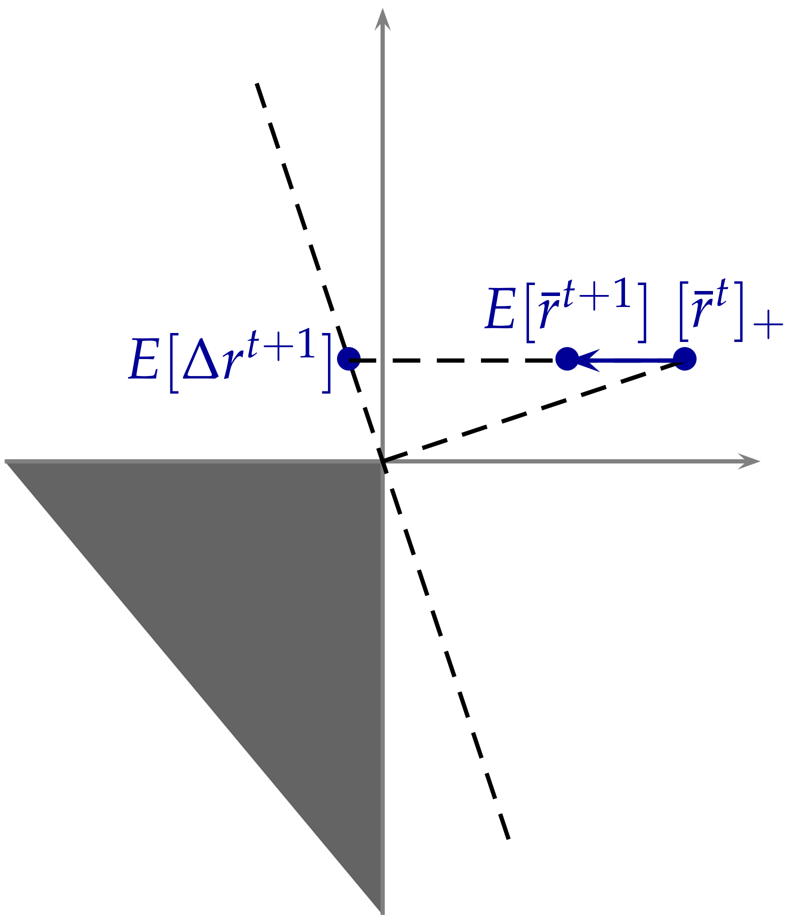

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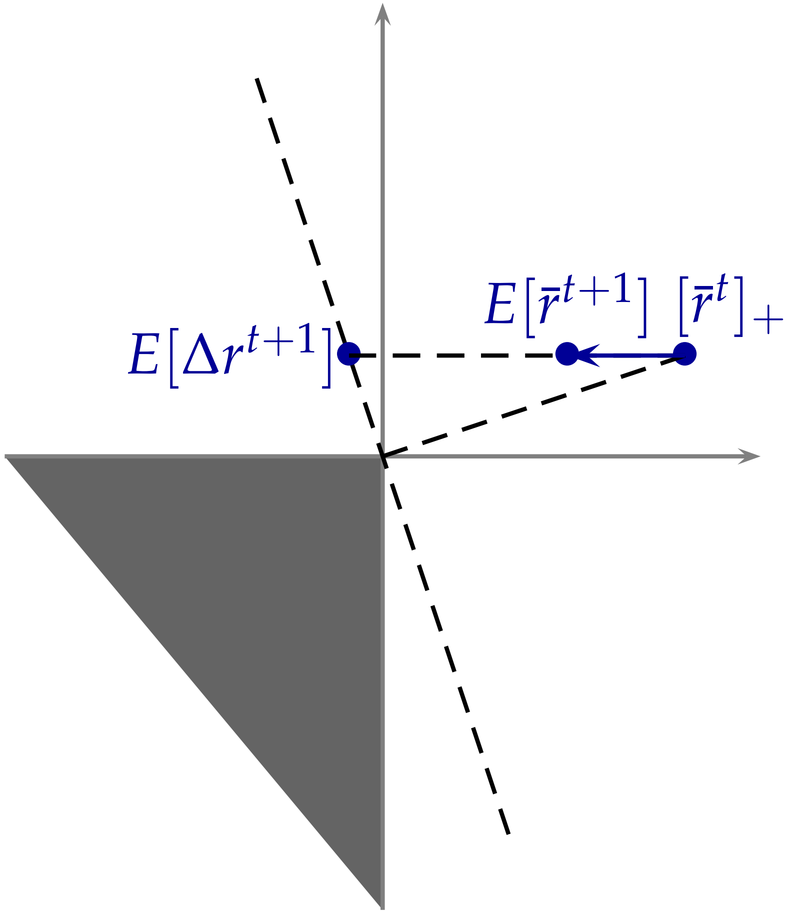

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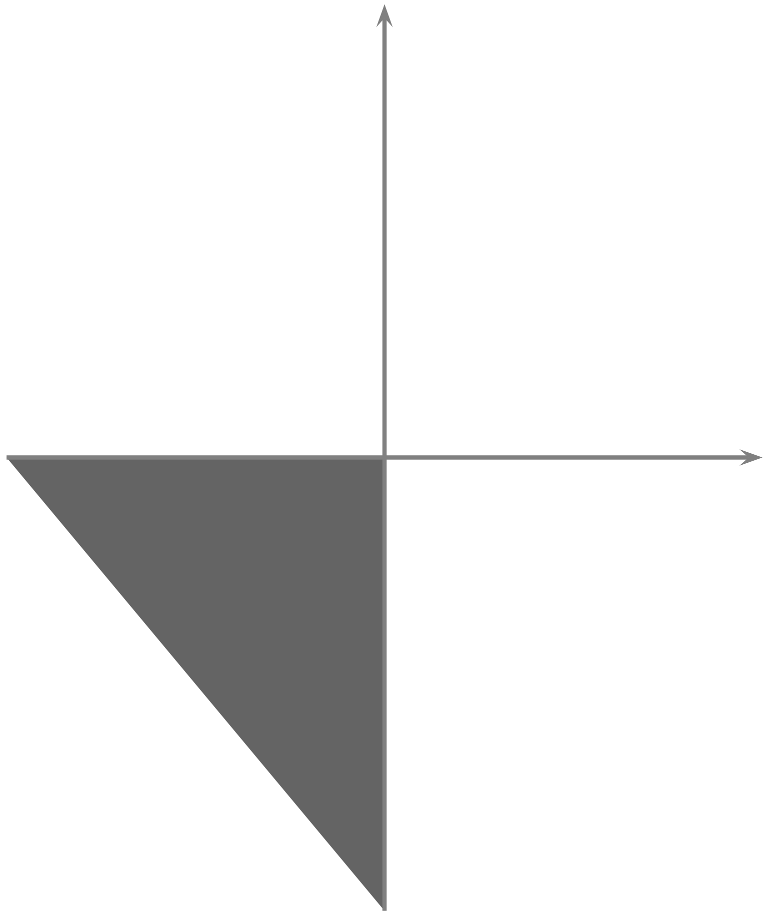

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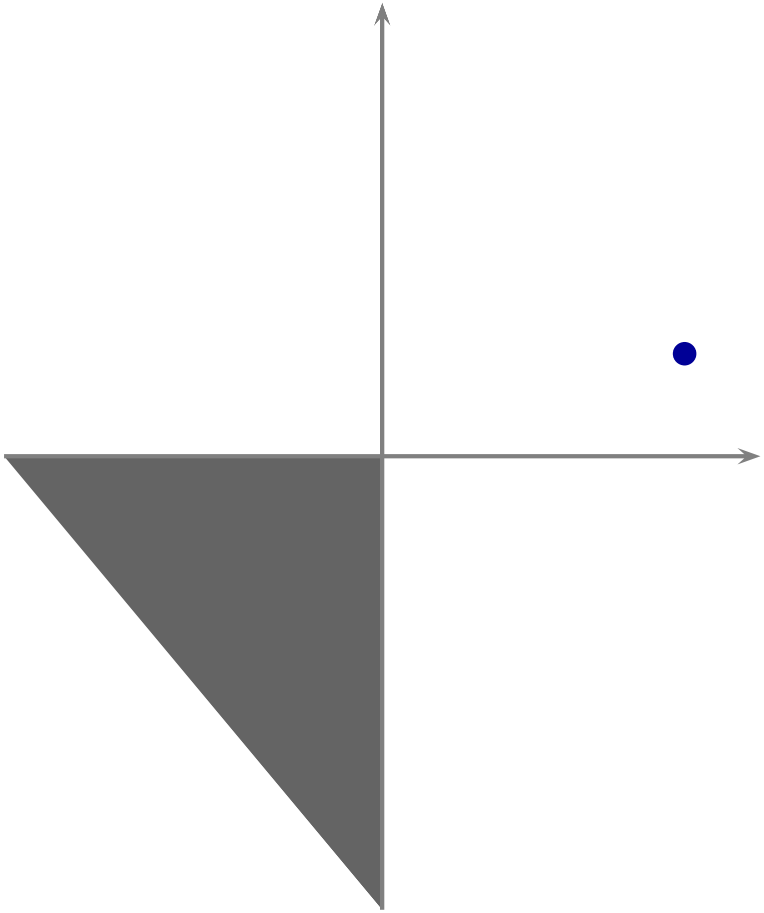

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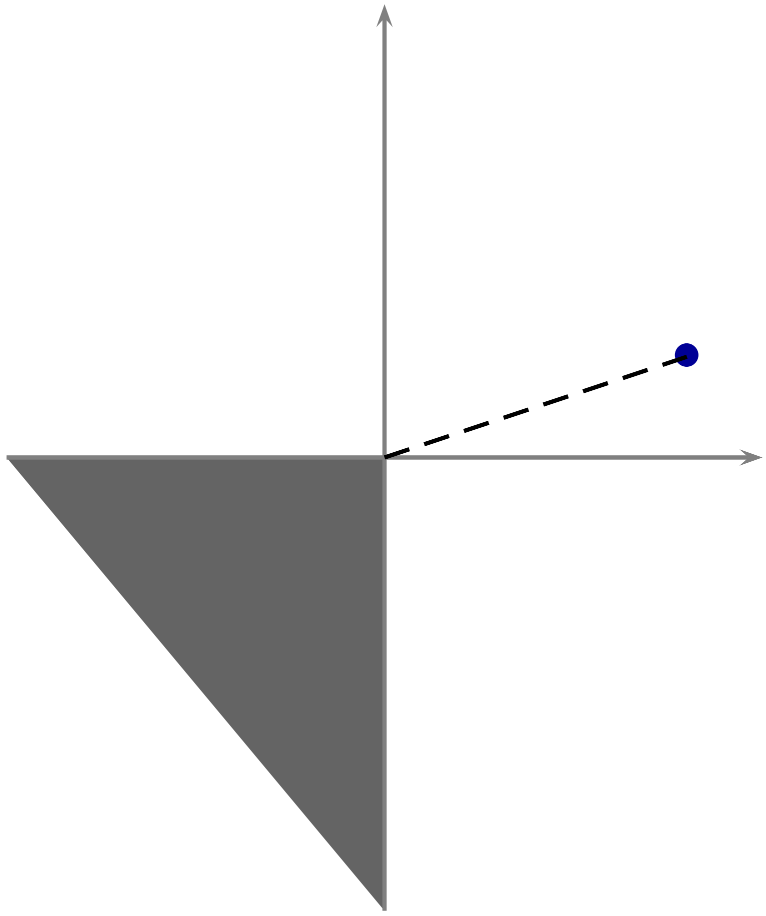

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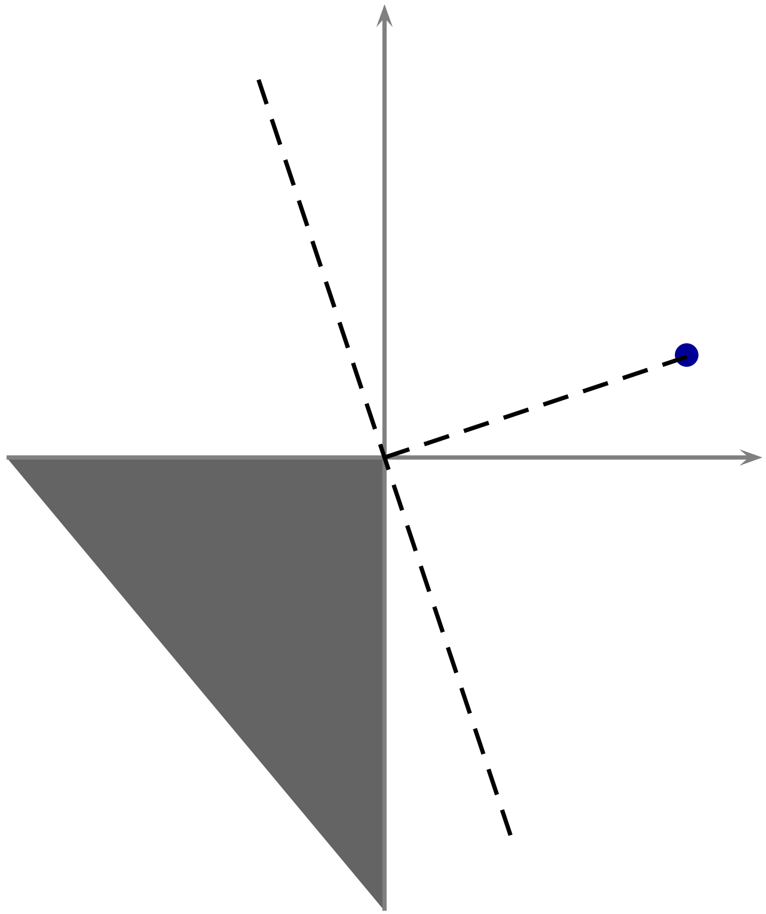

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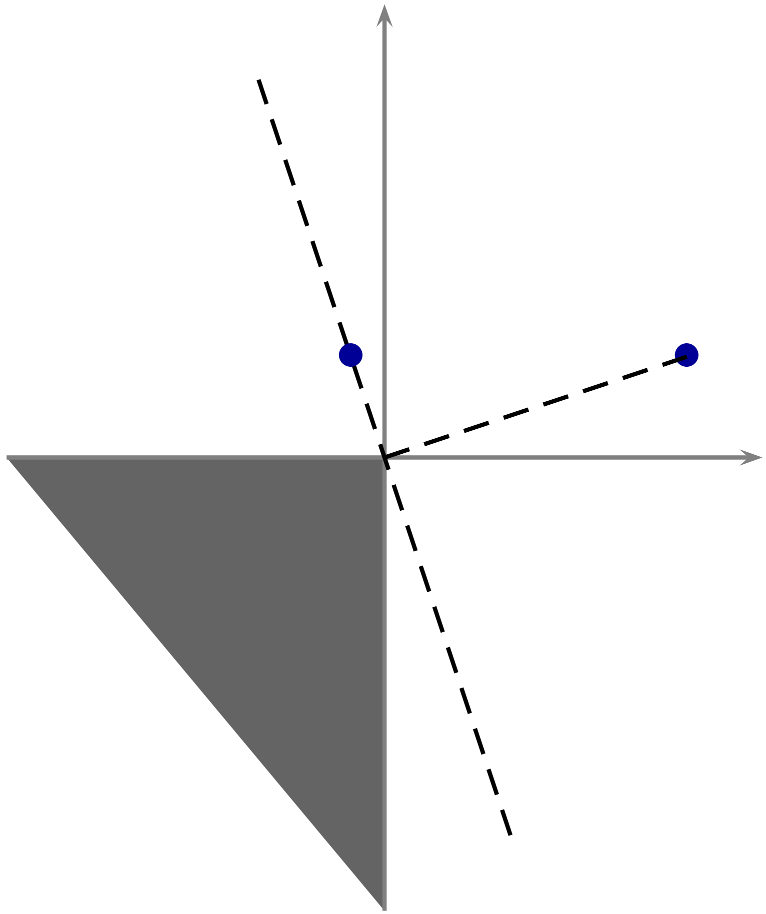

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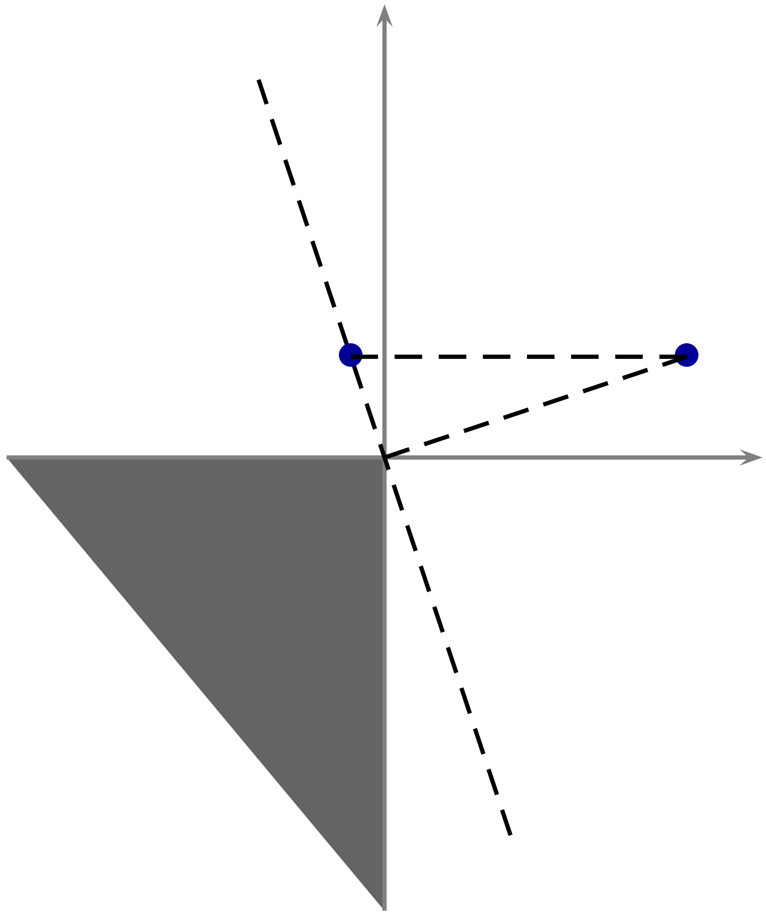

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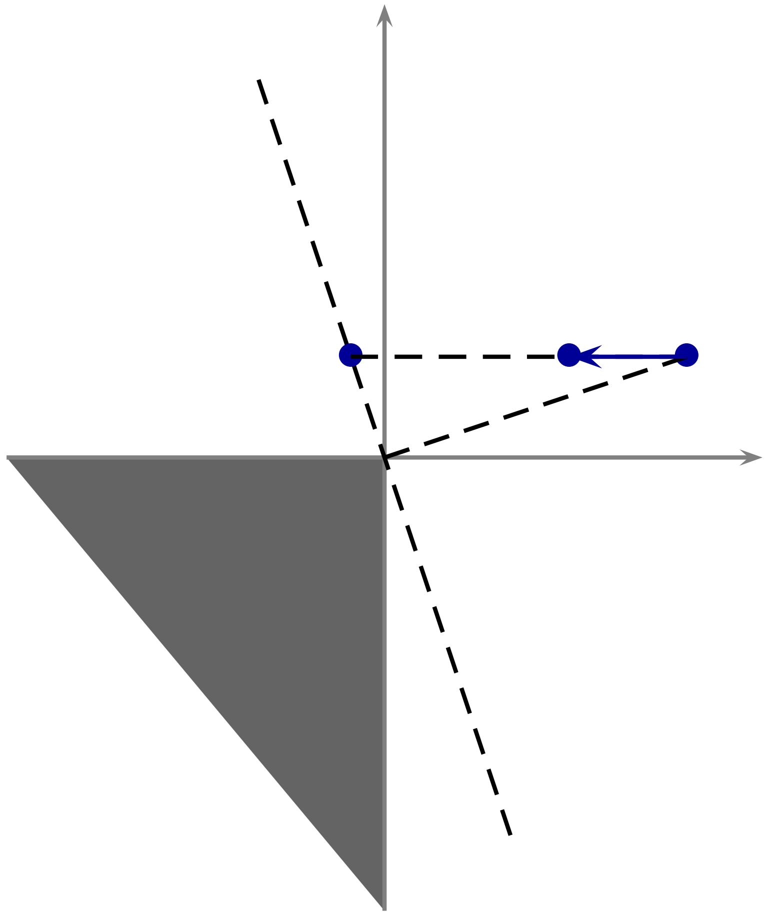

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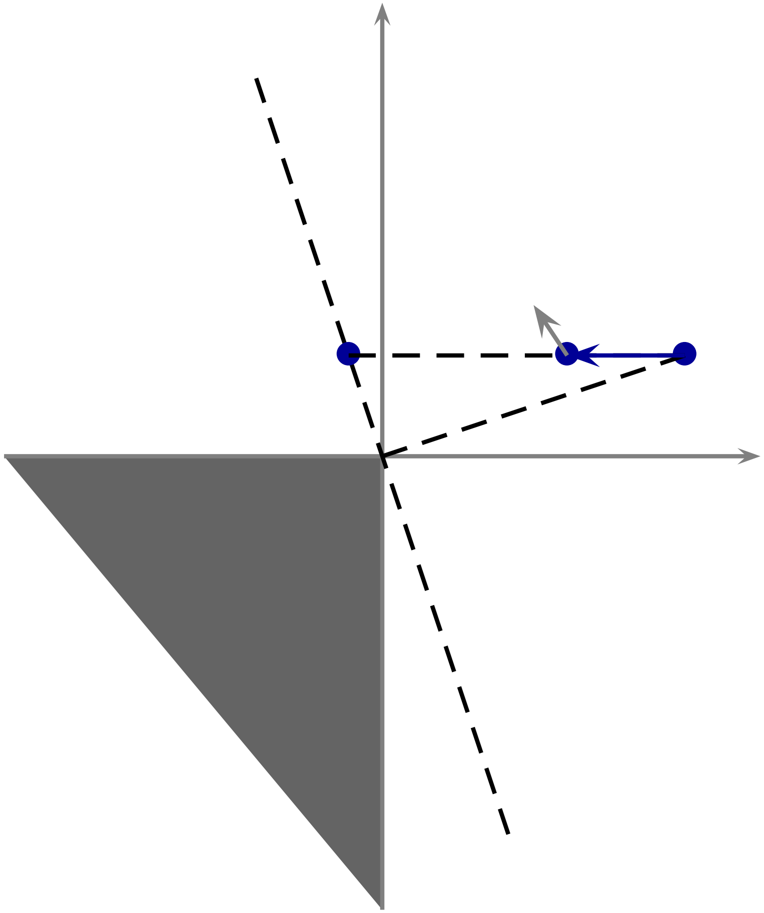

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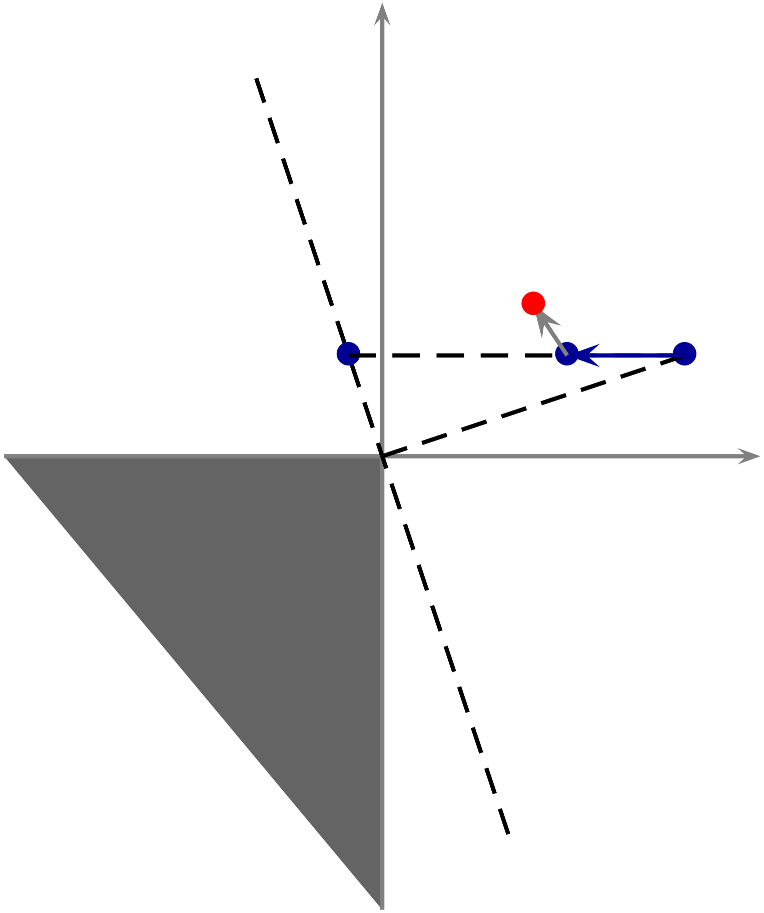

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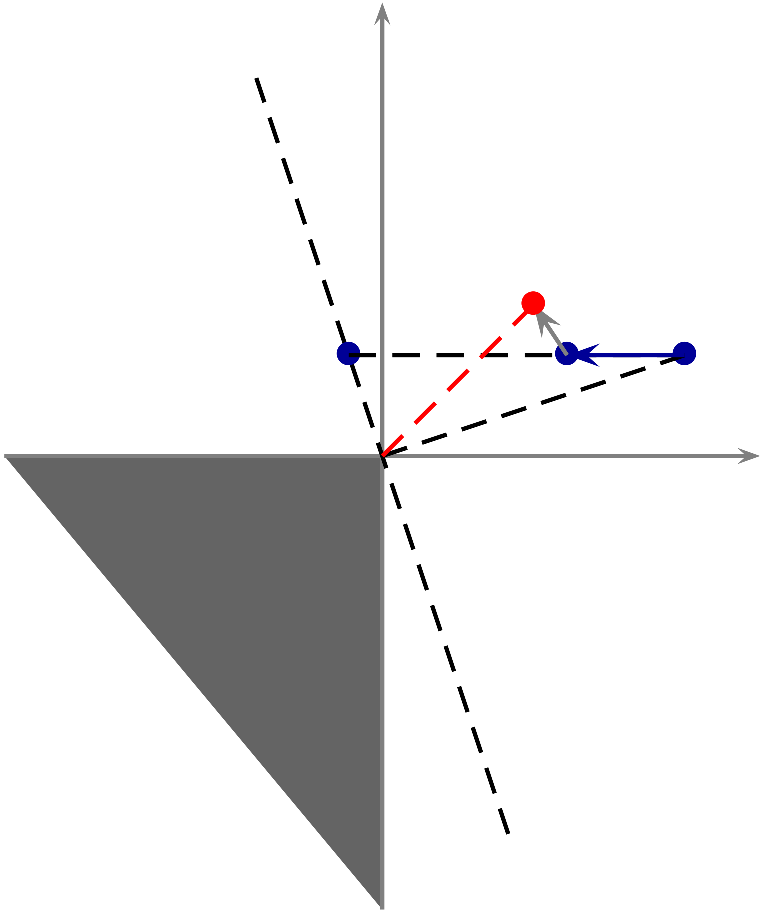

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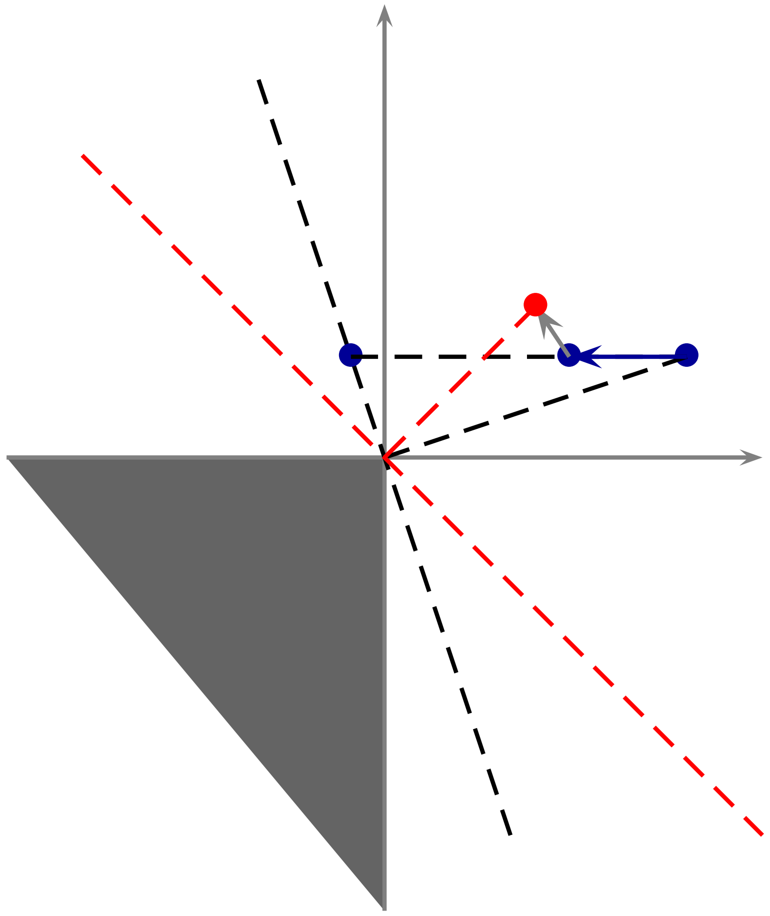

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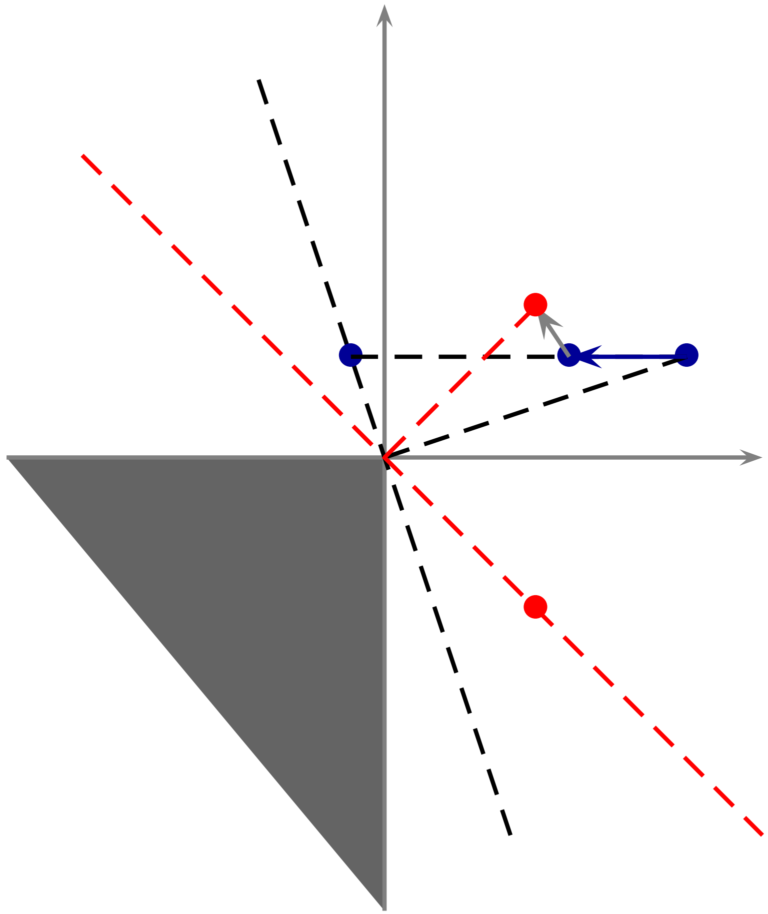

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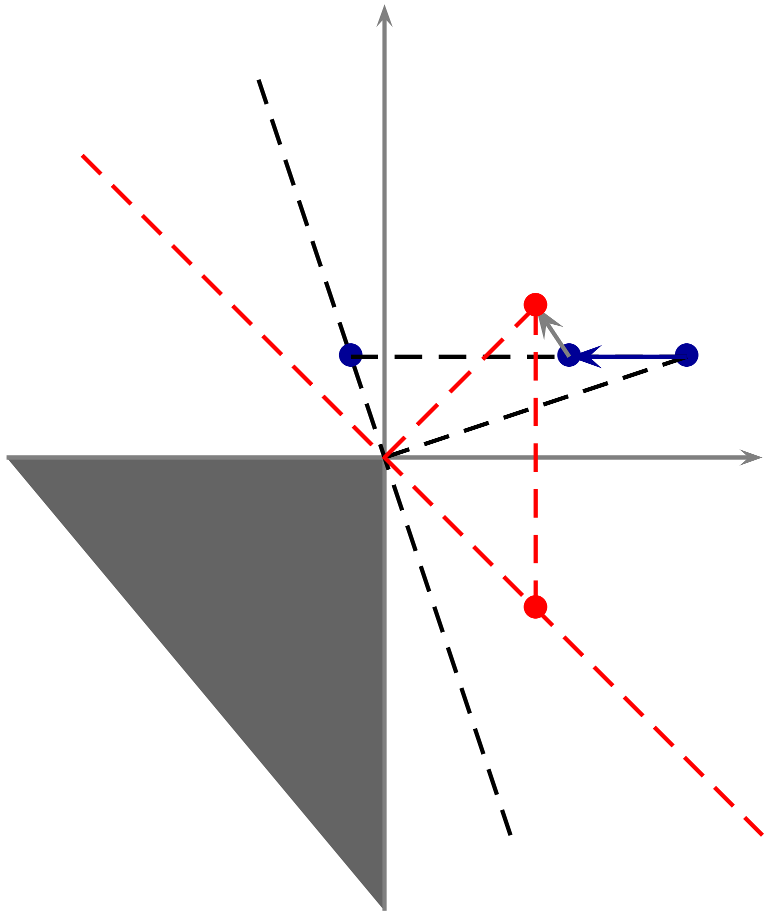

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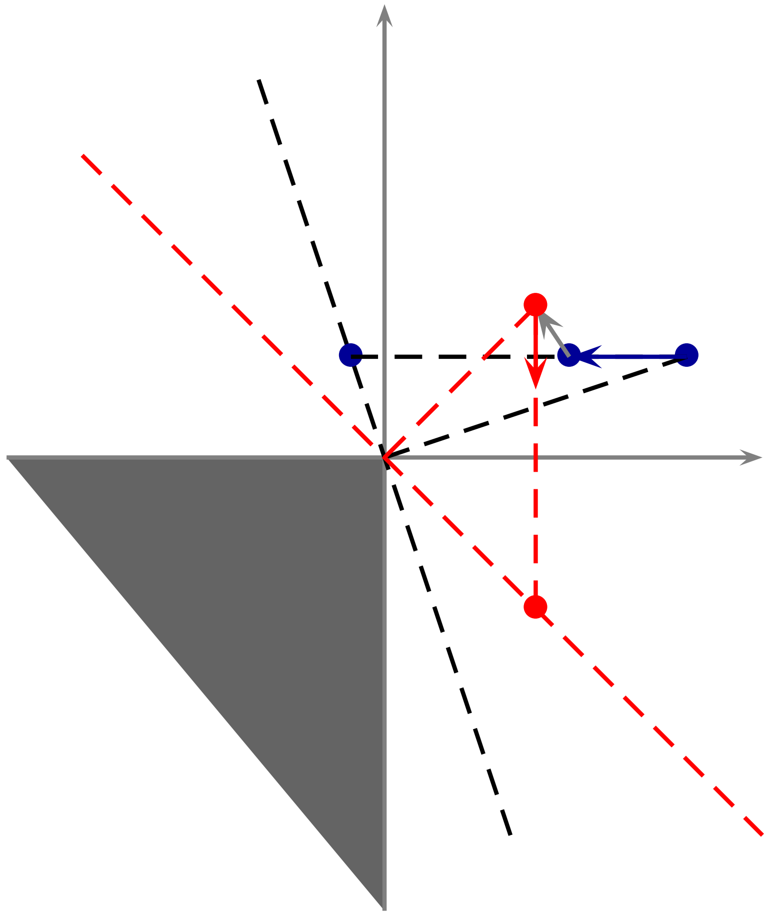

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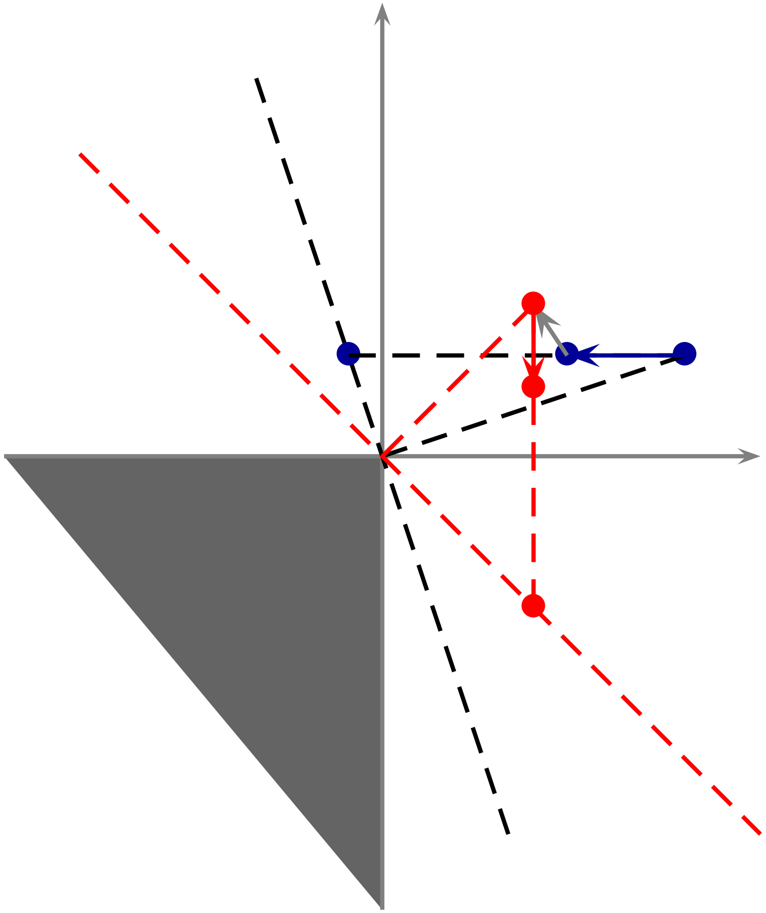

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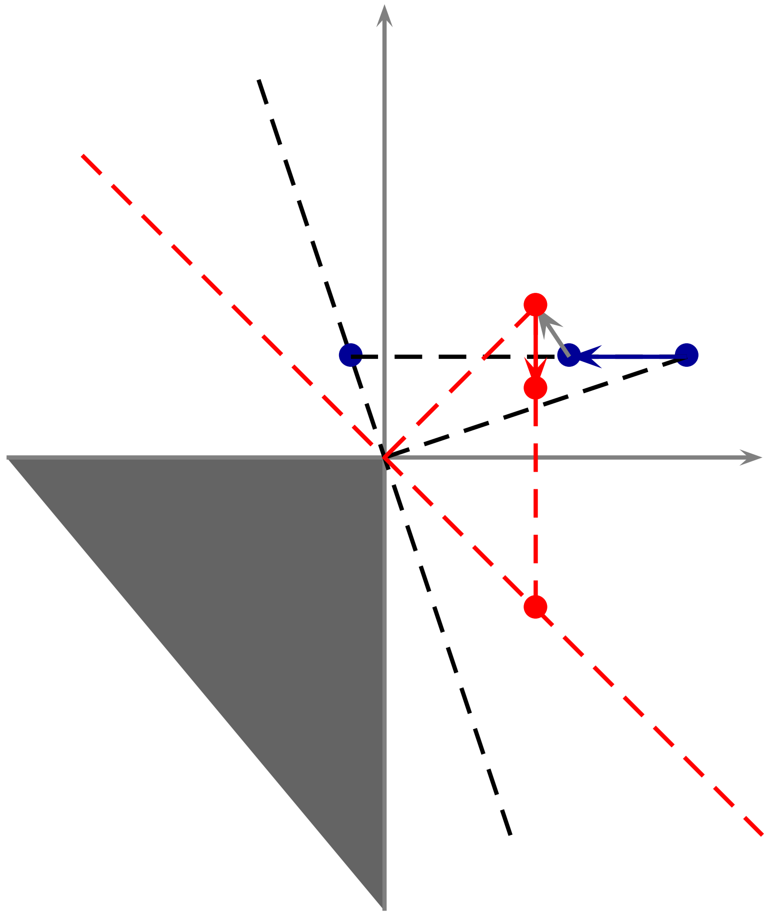

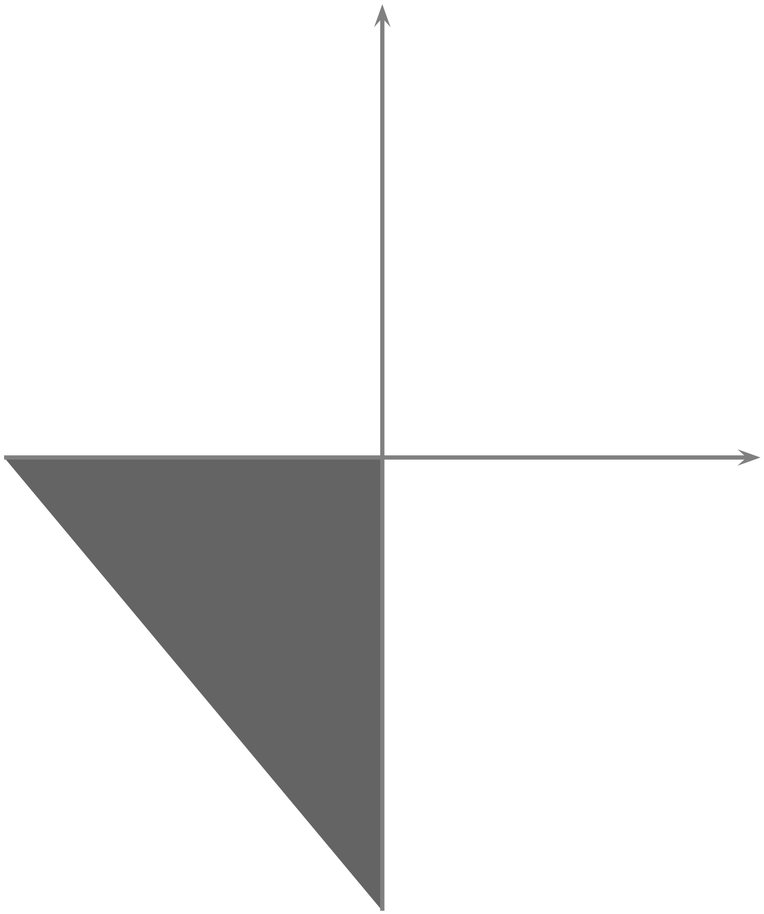

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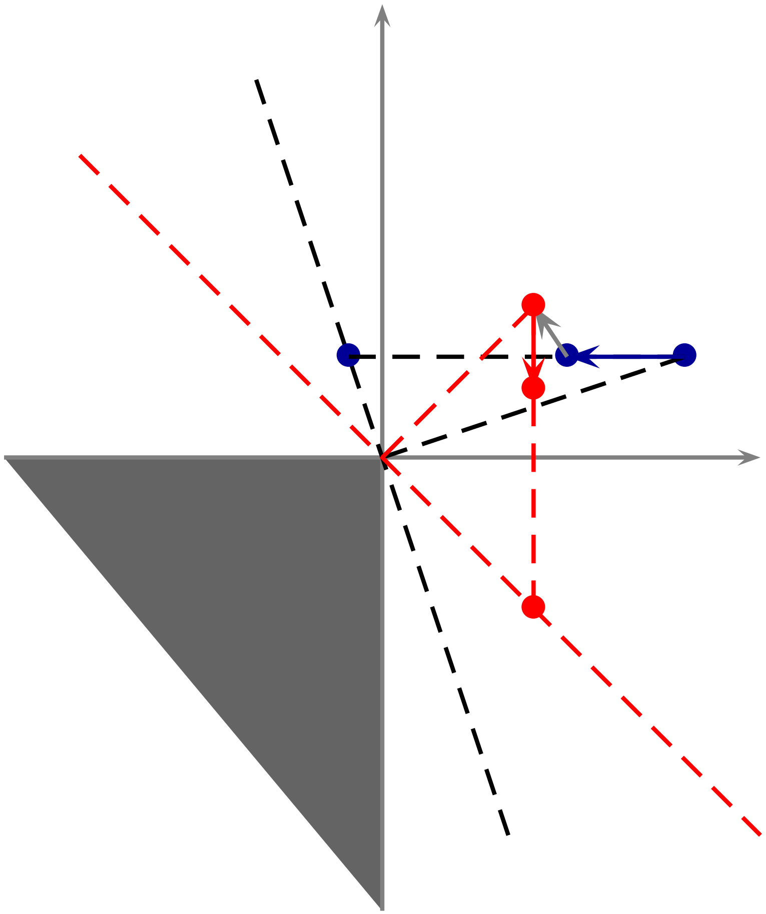

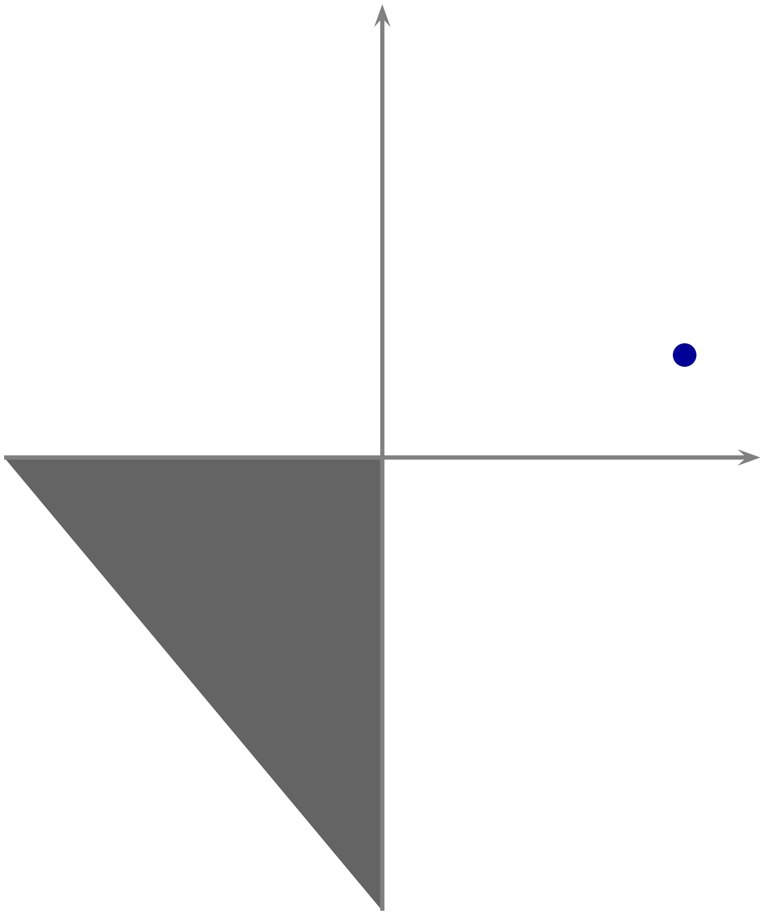

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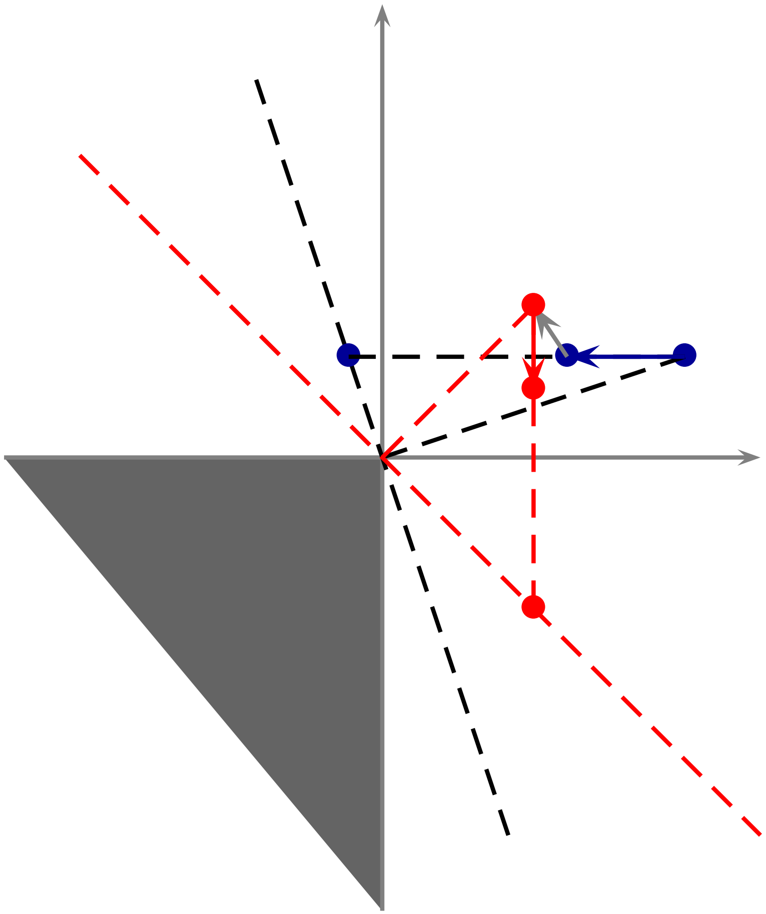

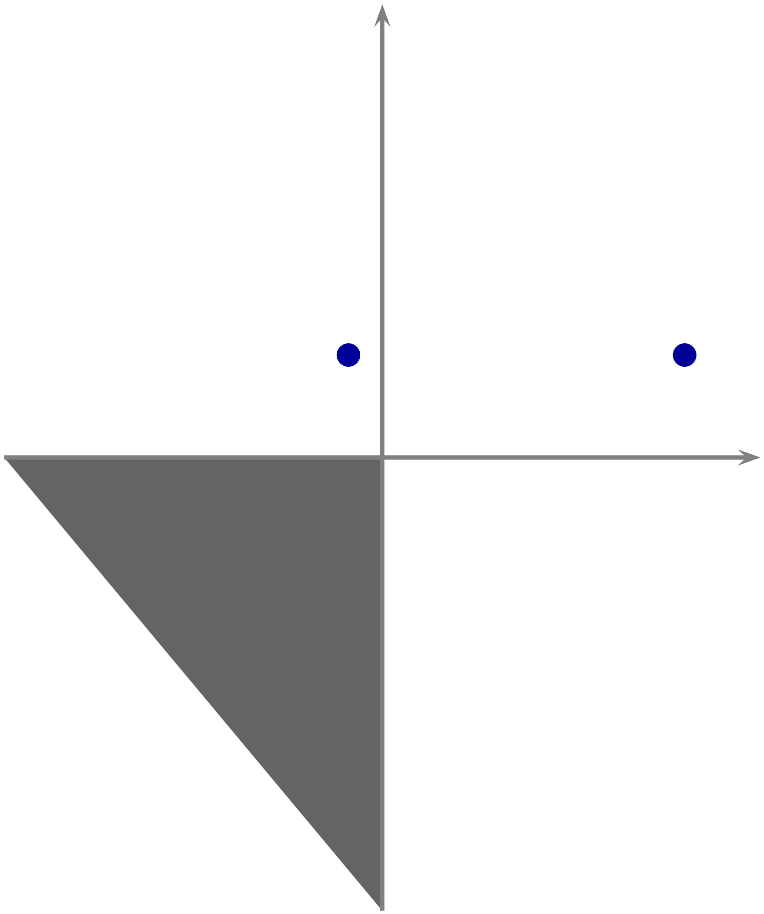

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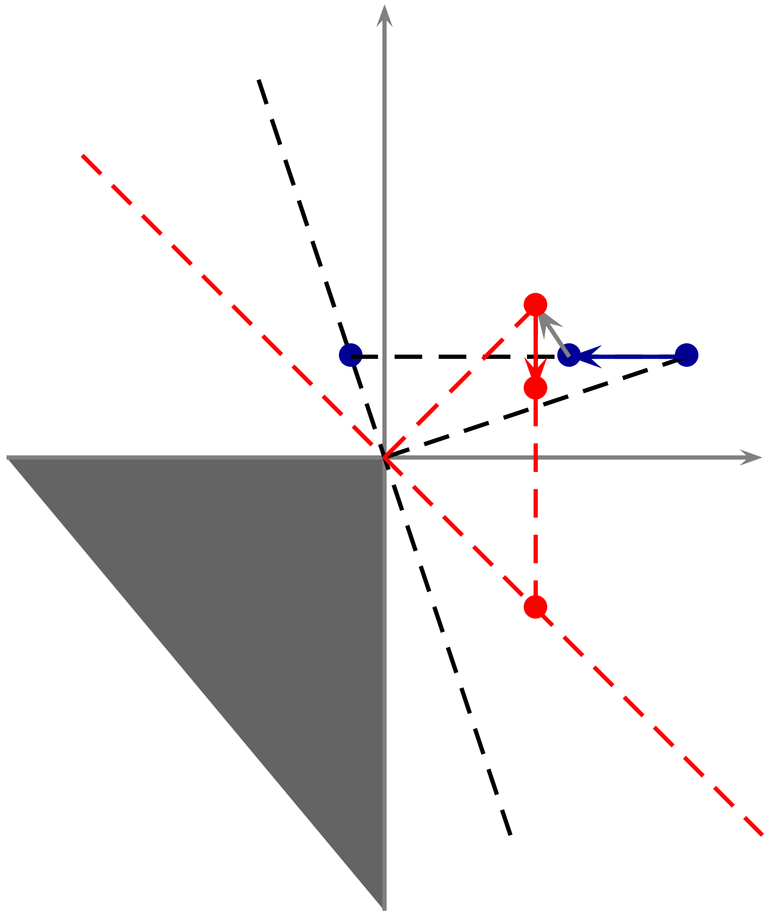

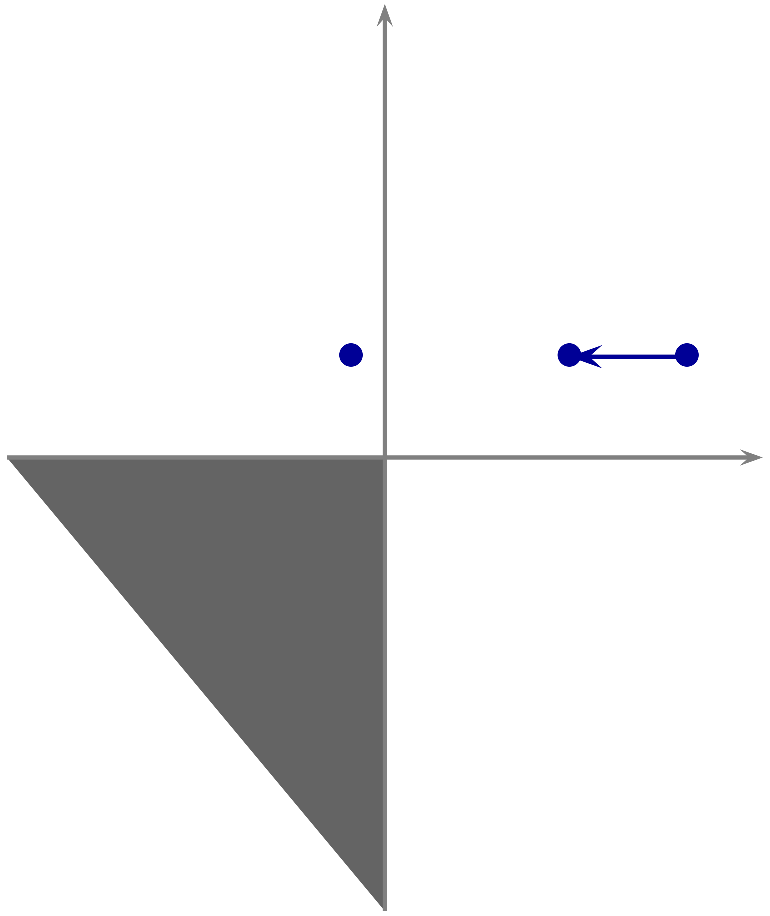

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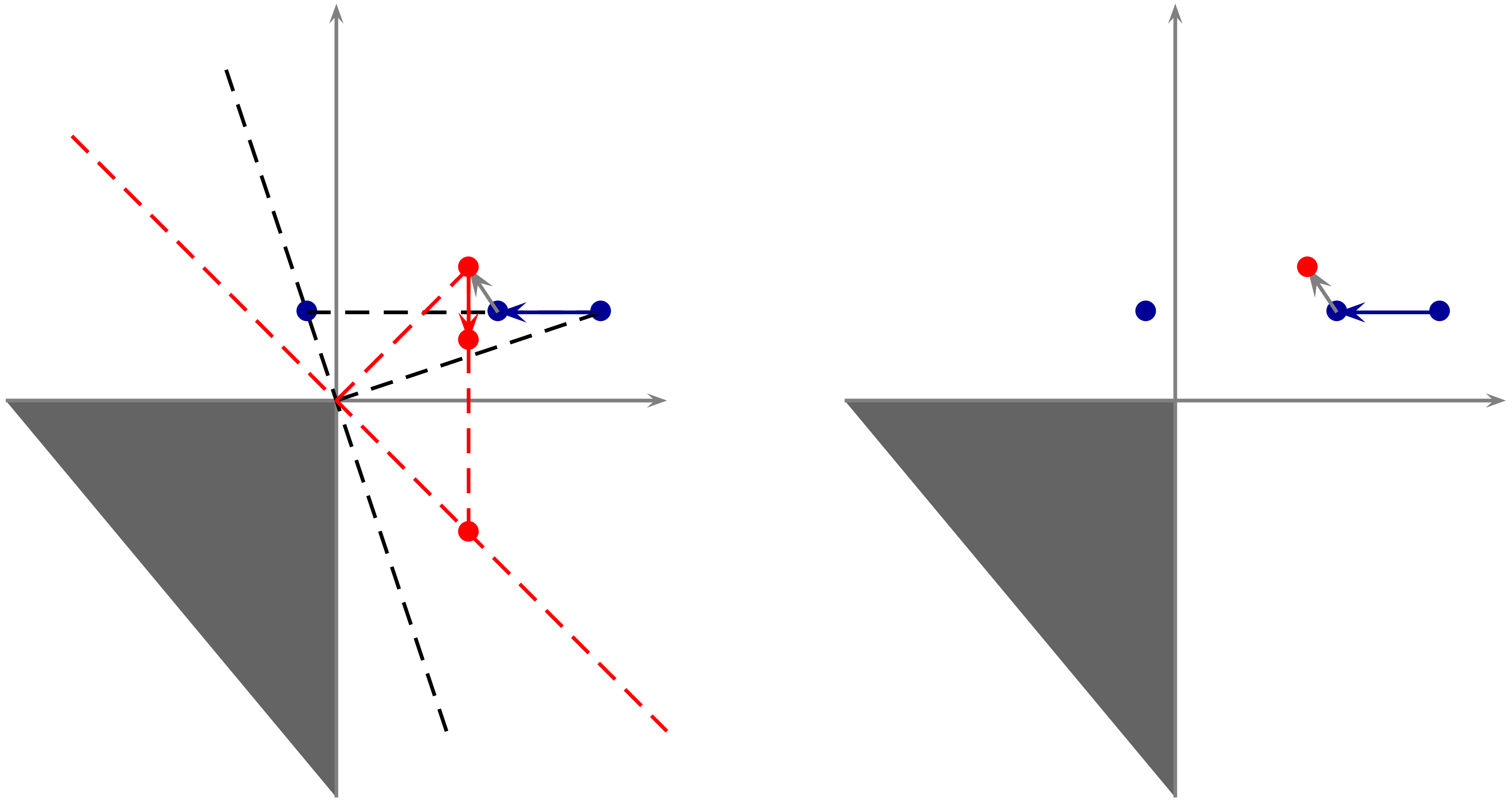

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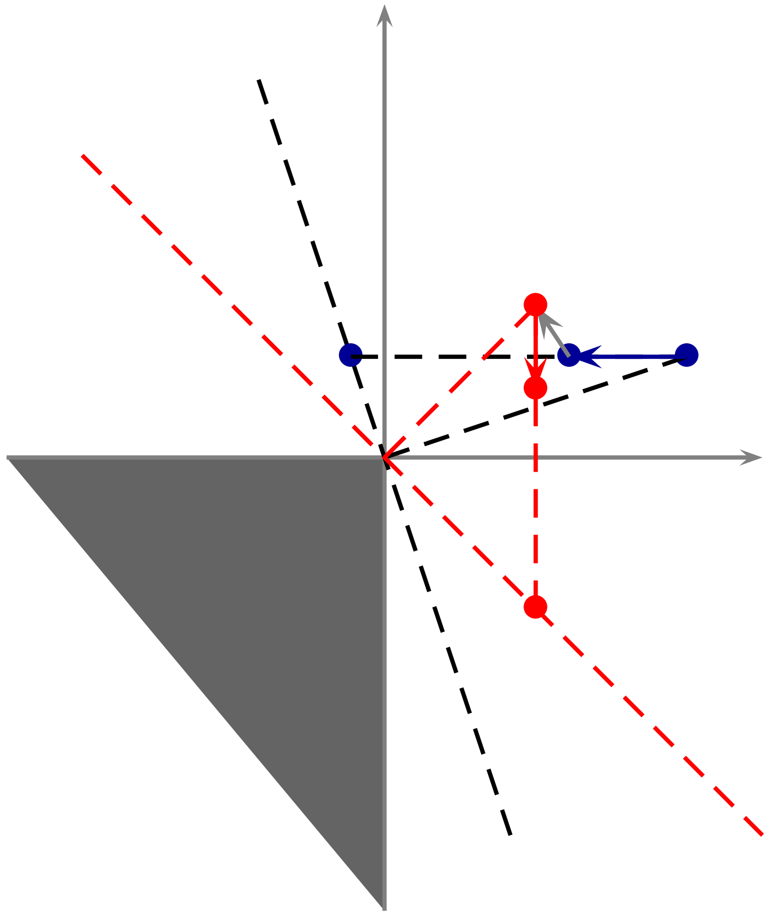

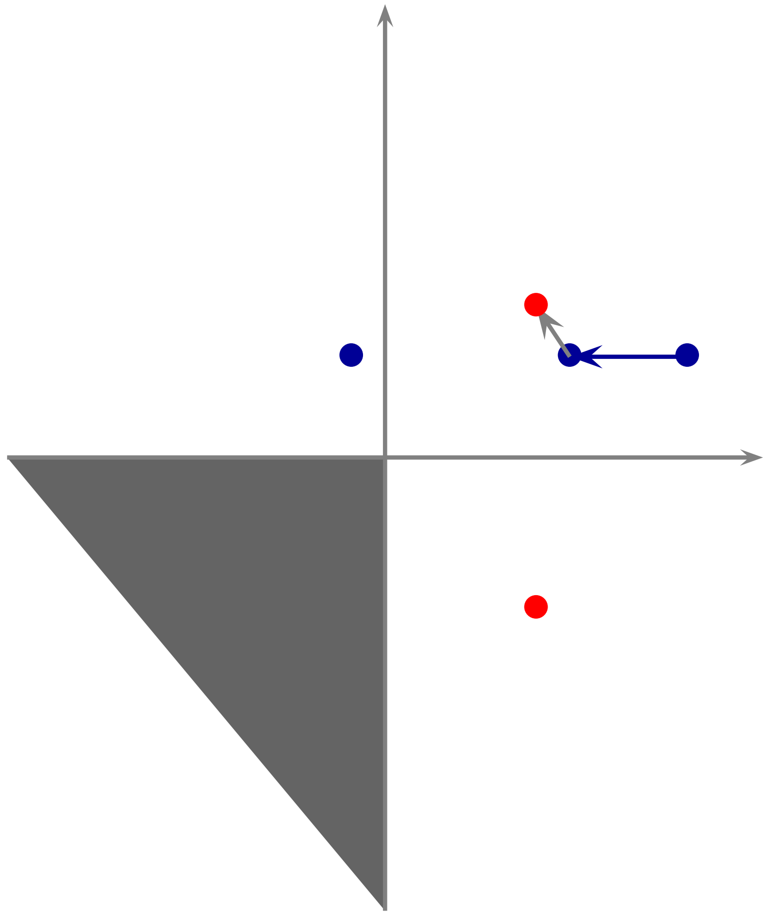

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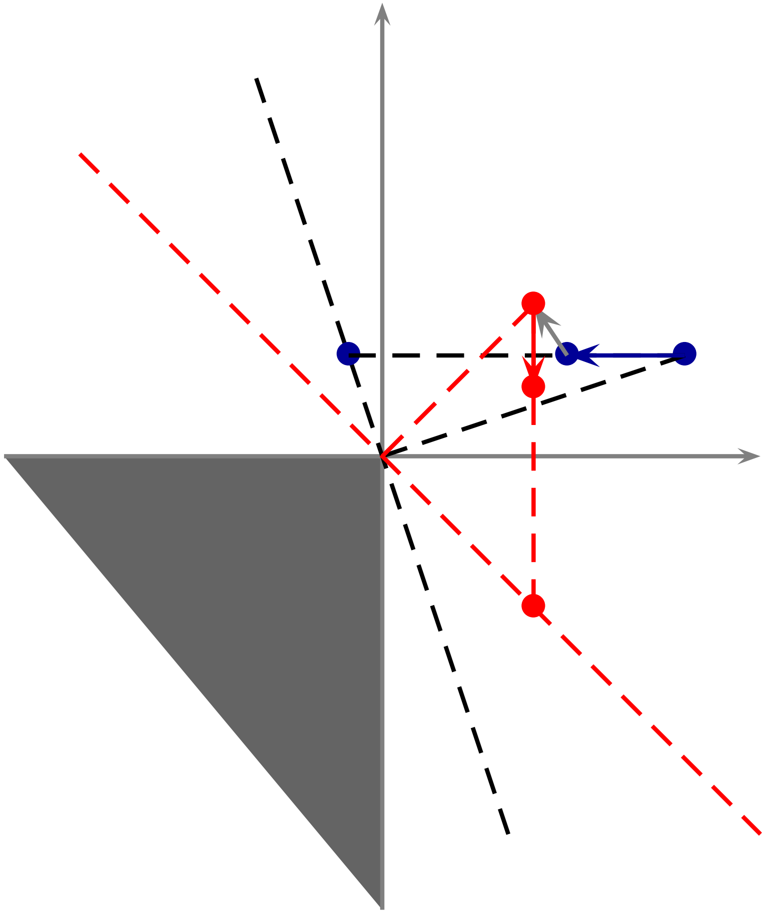

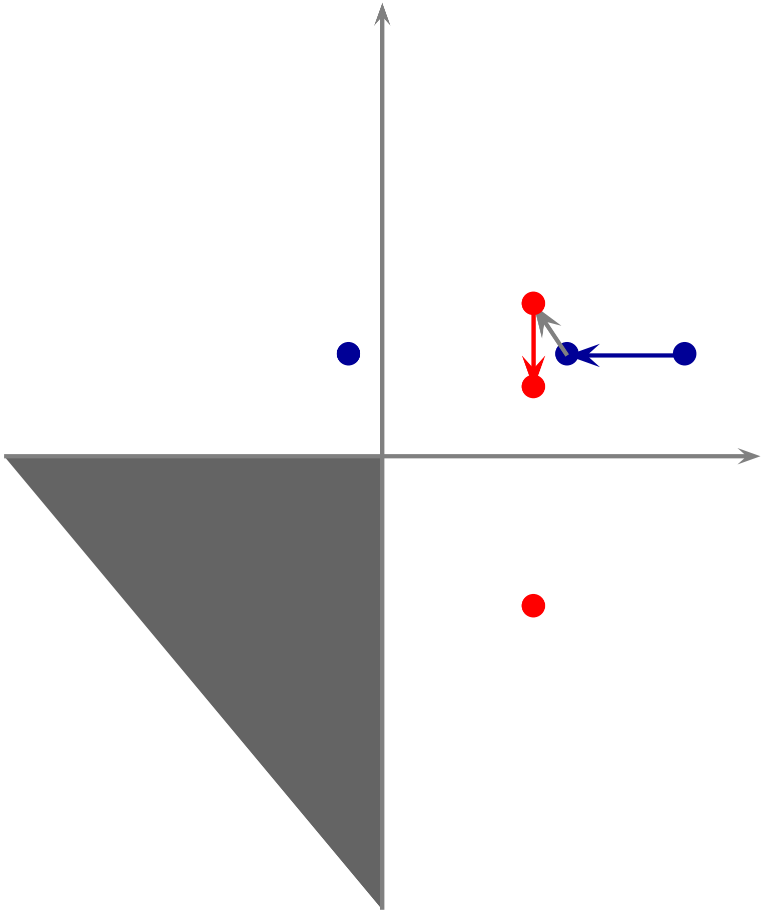

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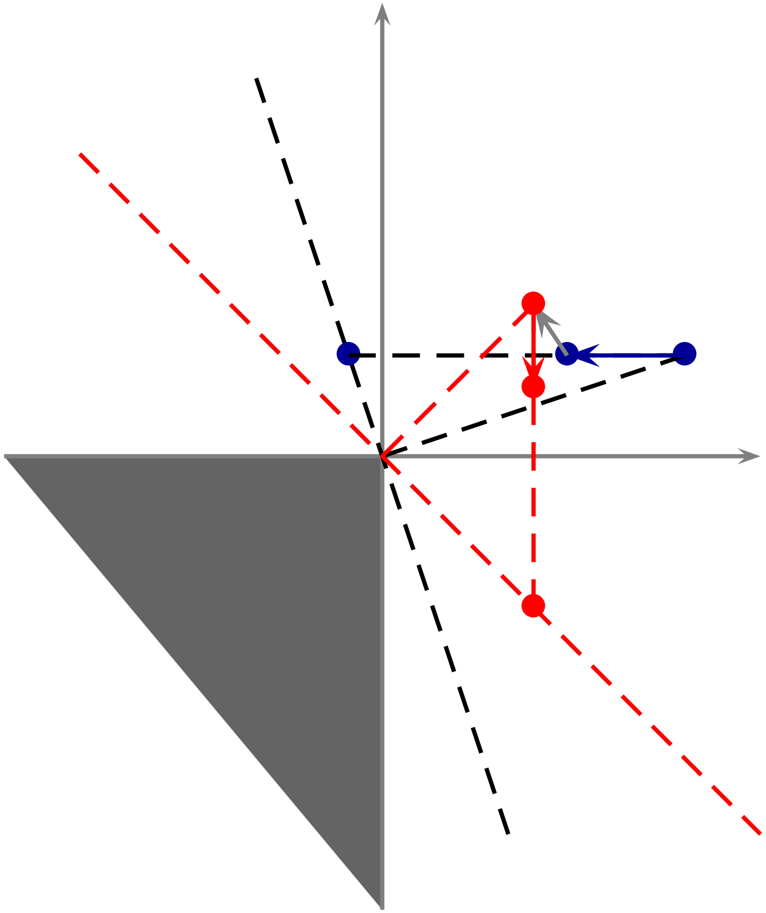

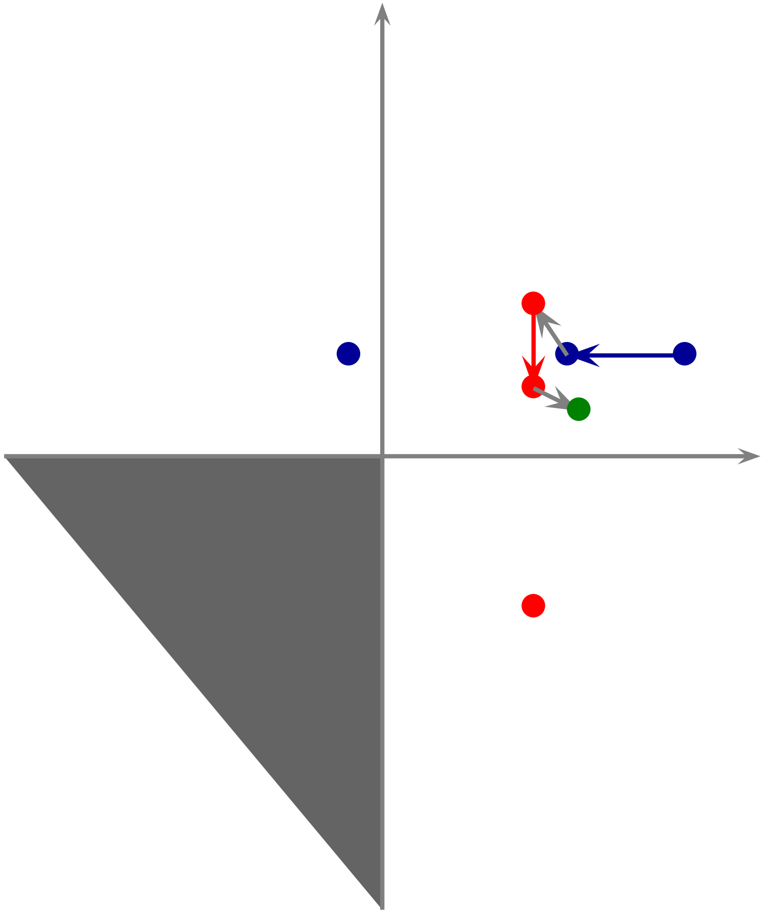

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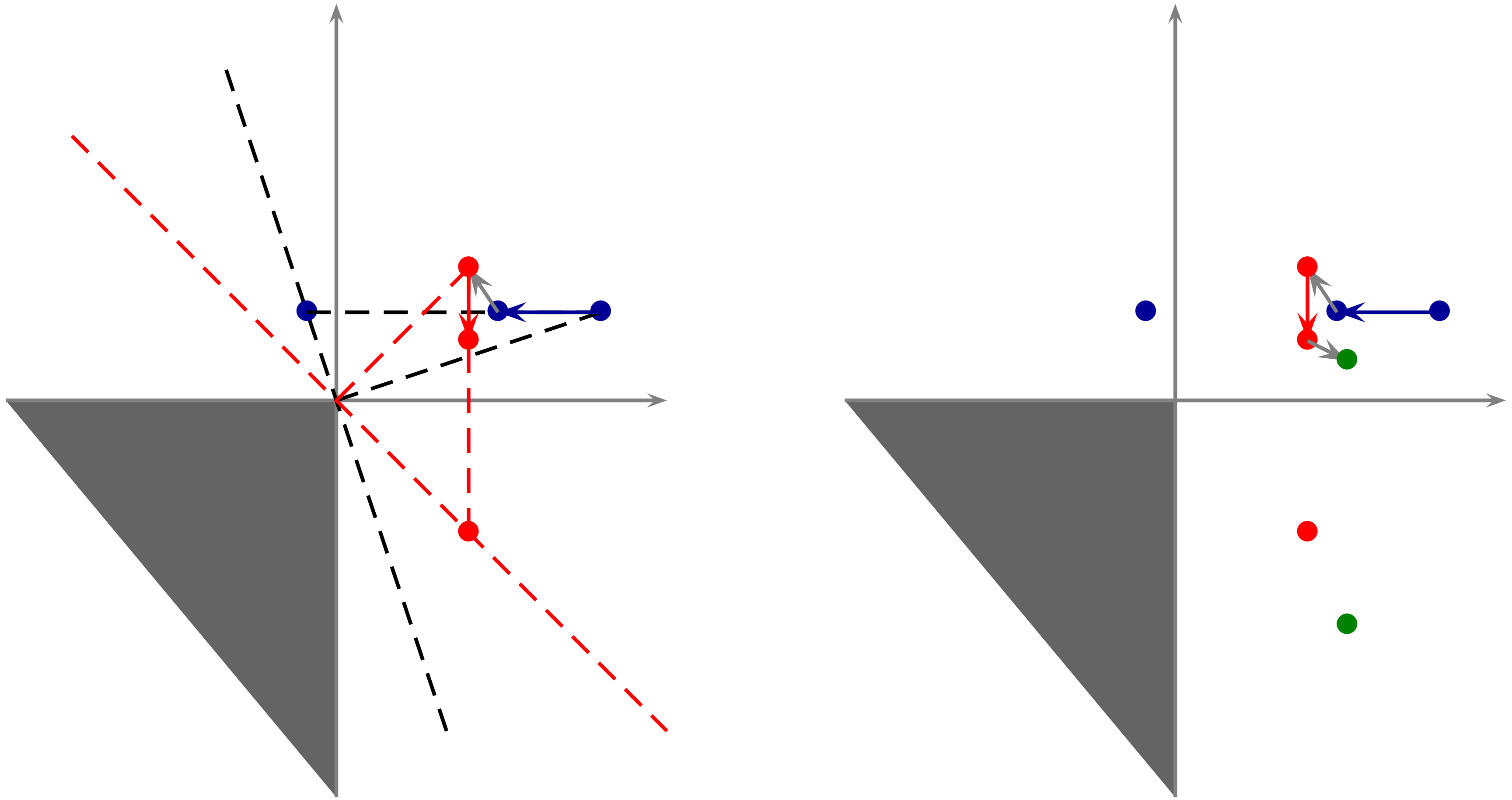

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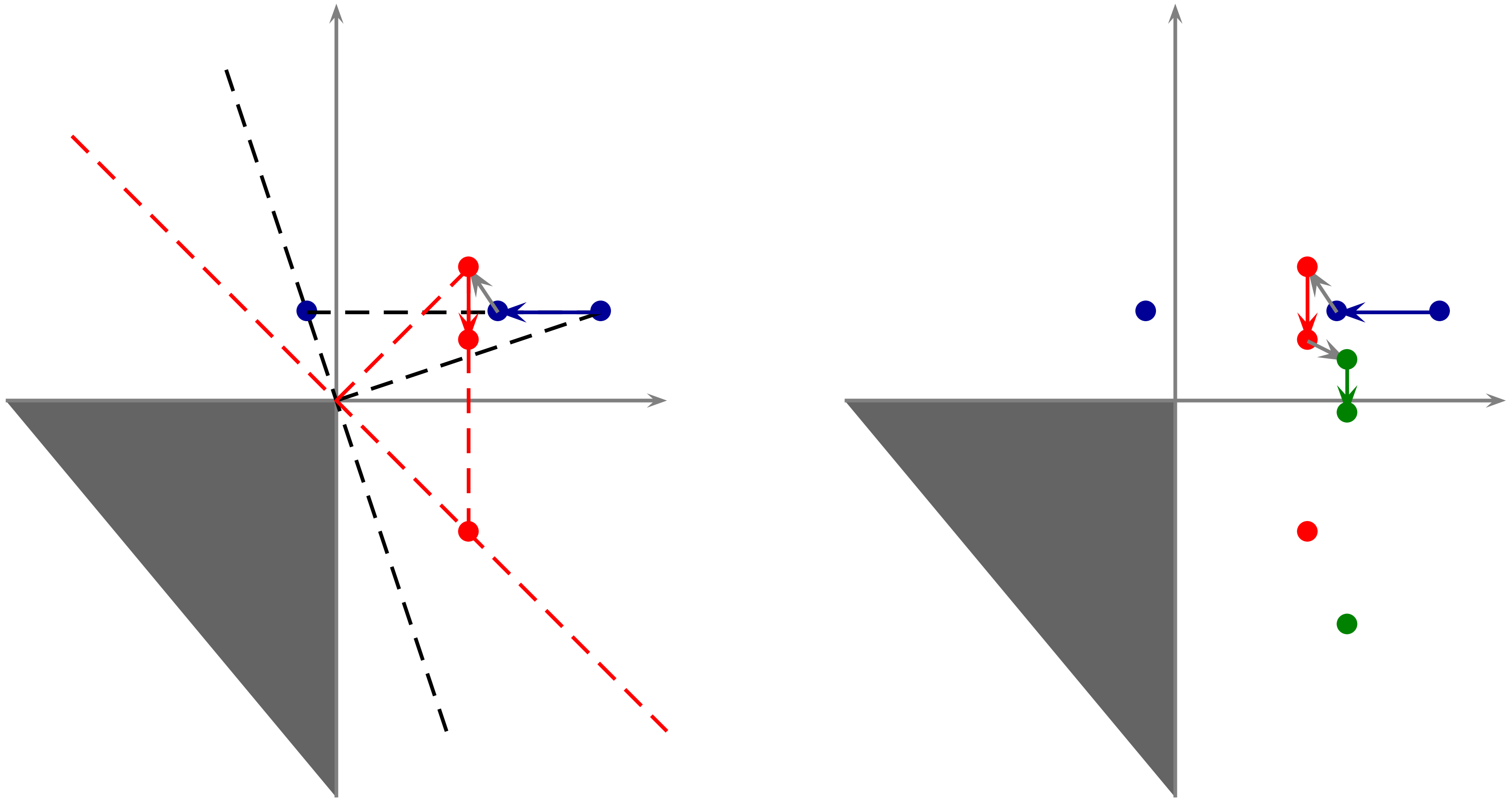

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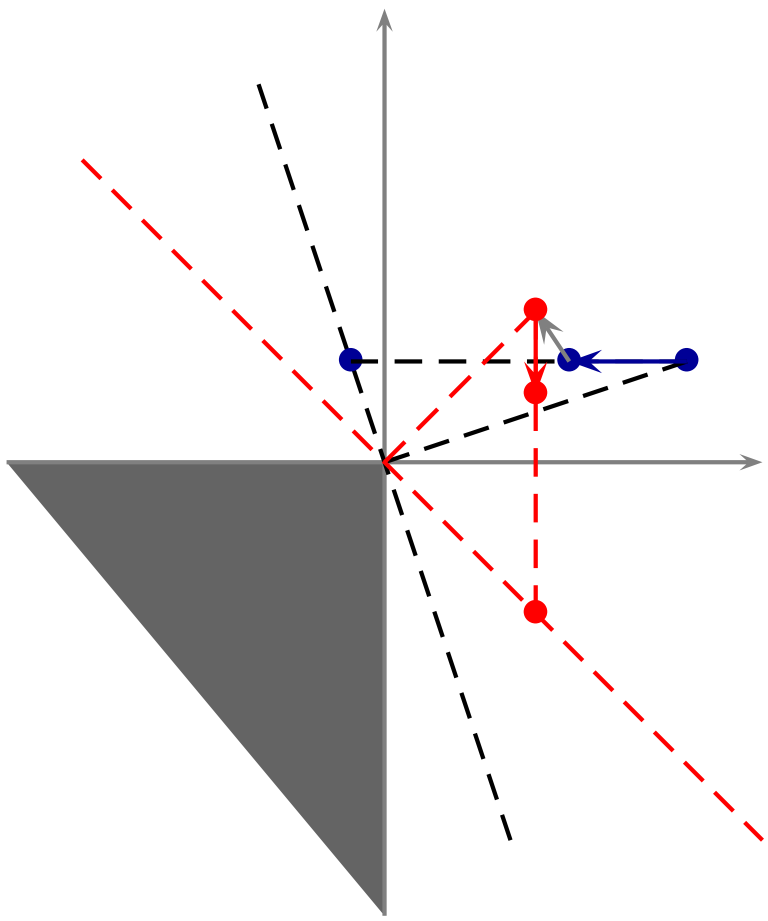

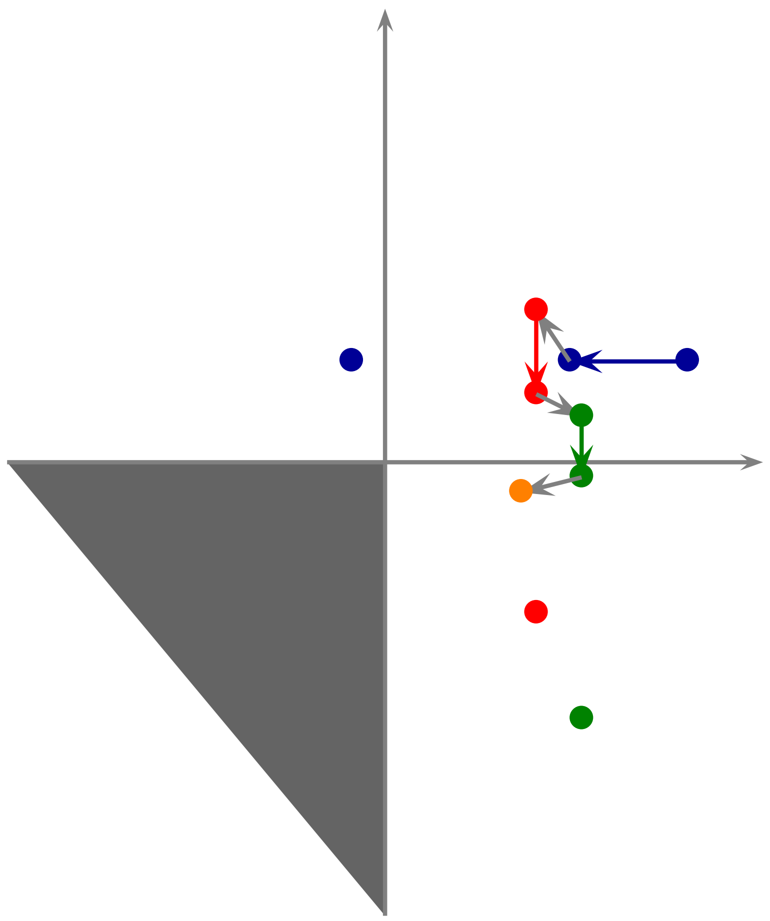

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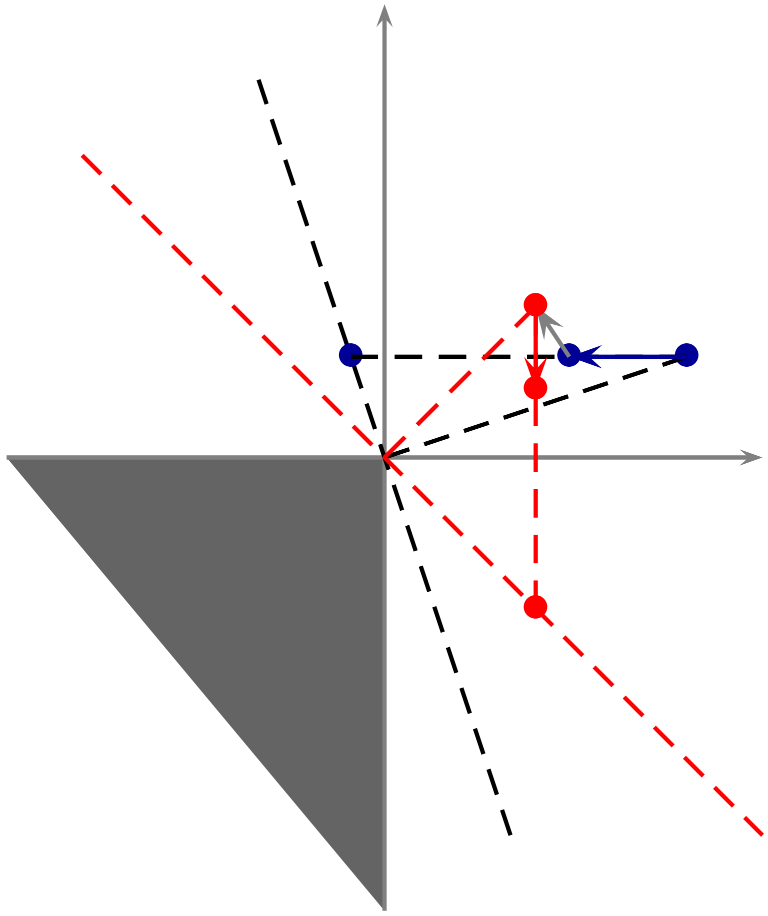

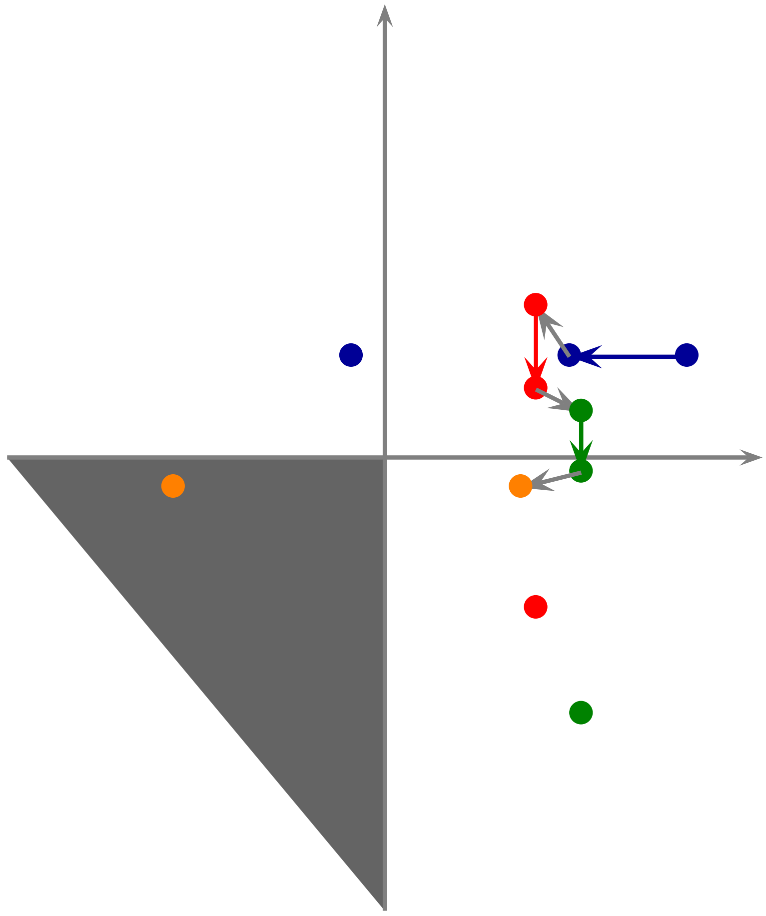

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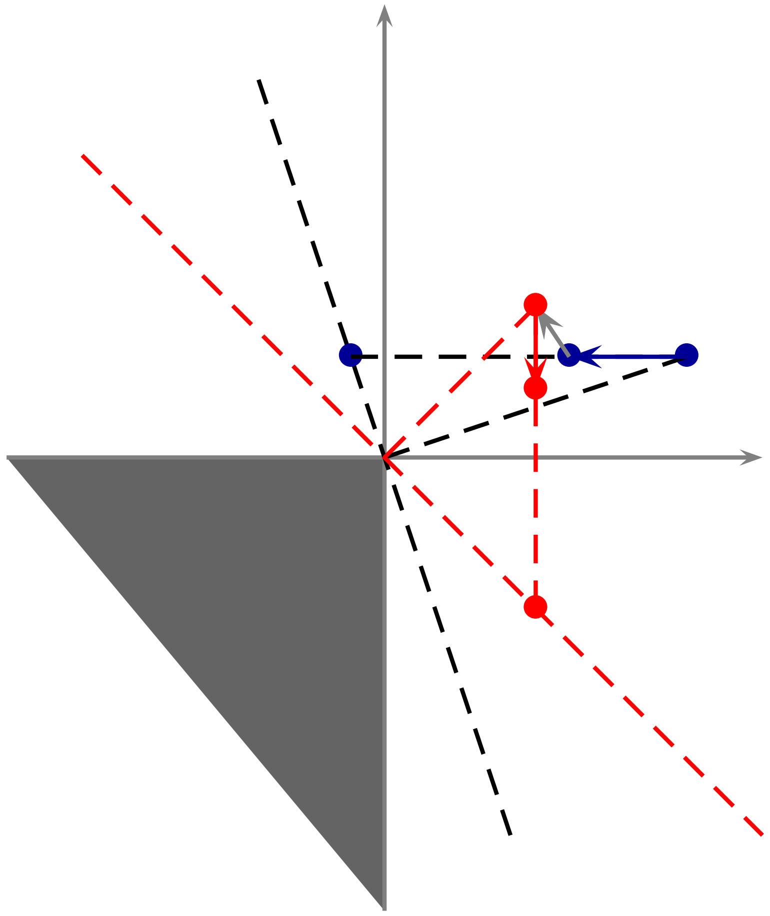

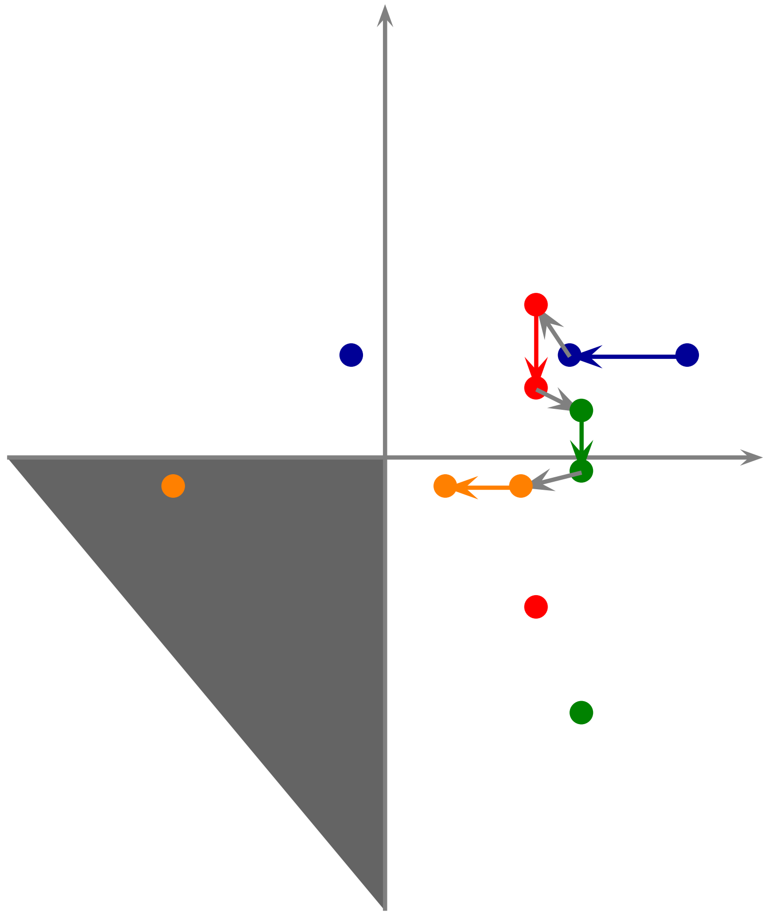

Part III:
ϵ-Greedy Off-policy Regret Matching

ϵ-Greedy regret matching (Foster & Vohra, 1998)

ϵ-greedy regret matching. Let $\epsilon > 0$ small.

1. **Explore**. Play randomly ϵ% of the time.
2. **Exploit**. Else, play off-policy regret matching.

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Define off-policy regret for x in round t as

$$\bar{r}_x^t =_{Def} \bar{u}_x^t(E) - \bar{u}^t, \qquad \text{where} \quad \bar{u}_x^t(E) = \left[\frac{1}{|E_x|} \sum_{t \in E_x} u(x^t, y^t) \right]$$

and $E_x = \{ \, t \mid \text{player } A \text{ experimented in round } t \text{ and played } x \, \}$.

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- Proposed as a forecasting heuristic by Foster and Vohra (1993).
- Does not need to know the actions of its opponents.
- Turns out to estimate regrets.

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Theorem (Foster *et al.*, 1998). *For all $\delta > 0$ there exists an $\epsilon > 0$ such that ϵ-greedy regret matching has at most δ regret.*

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It follows that

$$E[e^t] = \left(\frac{\epsilon}{k}, \ldots, \frac{\epsilon}{k}\right).$$

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&= \frac{k}{\epsilon} \cdot \frac{\epsilon}{k} \cdot E[u(x, y^t)] \; - \; E[u(x, y^t)]
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Strong law of large numbers for dependent random variables. Let $\{w^t\}^t$ be a bounded sequence of possibly dependent random variables in R^k. Let $z^t = E[\, w^t \mid w^{t-1}, w^{t-2}, \ldots, w^1 \,] - w^t$, and $\bar{z}^t$ the average of the z^t's. Then $\lim_{t\to\infty} \bar{z}^t = 0$ with probability one.[a]

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then from the strong law of large numbers for dependent random variables it follows that $\lim_{t\to\infty} \bar{z}^t = 0$ a.s.

Estimated vs. true regret

Now write $\bar{z}_x^t$ as follows (!):

$$\bar{z}_x^t = \underbrace{\frac{1}{t}\sum_{s=1}^{t} \frac{k}{\epsilon} \cdot e_x^s \cdot u(x, y^s) - \bar{u}^t}_{\text{scaled empirical regret}} - \underbrace{\frac{1}{t}\sum_{s=1}^{t} u(x, y^s) - \bar{u}^t}_{\text{true regret}}$$

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Now write $\bar{z}_x^t$ as follows (!):

$$\bar{z}_x^t = \underbrace{\frac{1}{t}\sum_{s=1}^{t}\frac{k}{\epsilon}\cdot e_x^s\cdot u(x,y^s) - \bar{u}^t}_{\text{scaled empirical regret}} - \underbrace{\frac{1}{t}\sum_{s=1}^{t}u(x,y^s) - \bar{u}^t}_{\text{true regret}}$$

1. Since $\lim_{t\to\infty}\bar{z}^t = 0$, scaled empirical regret converges to true regret a.s.
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2. ϵ% of the time A explores.
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4. In the long run, empirical regret is within 2ϵ from true regret.
5. If ϵ is set to $\delta/2$, then empirical regret remains within $2\cdot\delta/2$ from zero. $\square$

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- **Satisficing Play**. While payoffs equal or supersede the average of past payoffs, keep playing the same action.