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  a software / hardware entity that displays a certain degree of autonomy / initiative, and is proactive/goal-directed.
- Academic research studies strategic interaction among agents from an abstract point of view.

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- Game theory is about strategically interacting individuals.
- Therefore, game theory is an important prerequisite of multi-agent learning.

Normal-form games vs. extensive-form games

The two most important game types are: extensive-form games and normal-form games.

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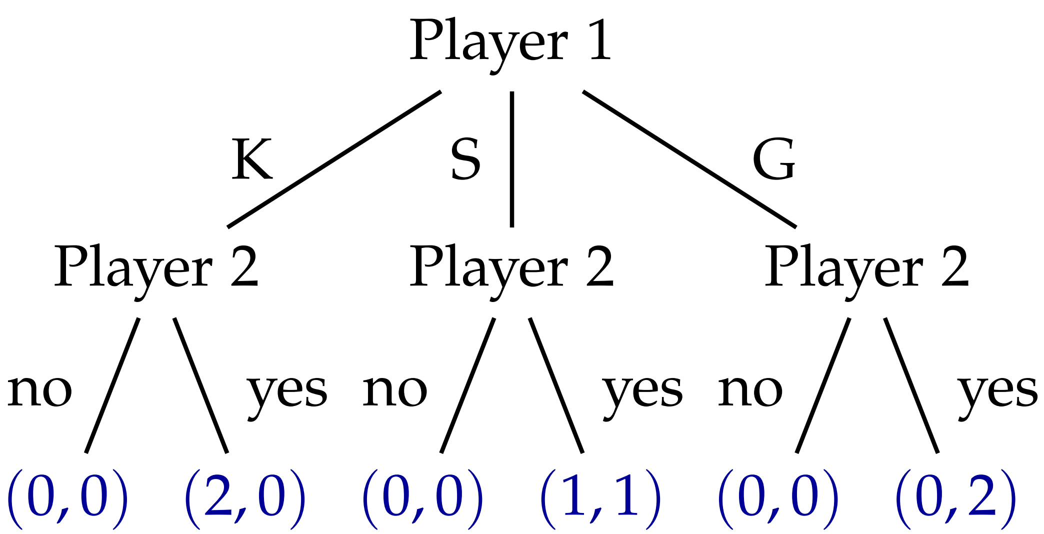

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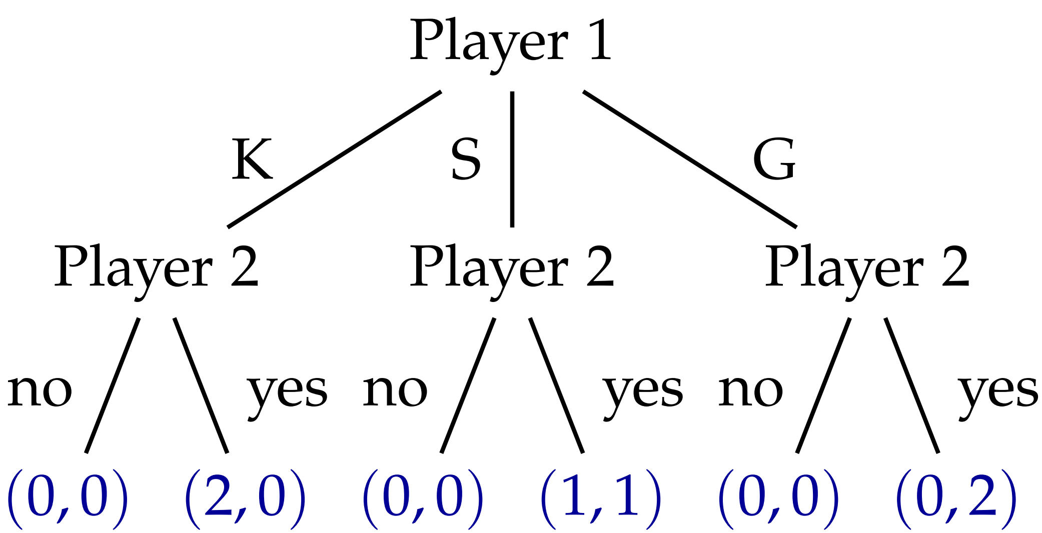

In a normal-form game a.k.a. matrix game, actions are taken **simultaneously**:

Player 1 \ Player 2	no	yes
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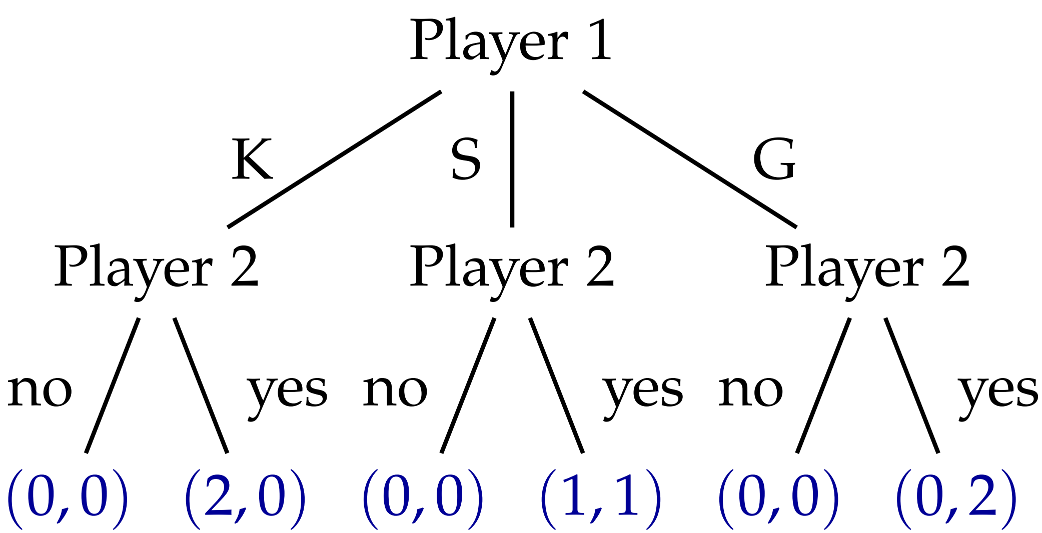

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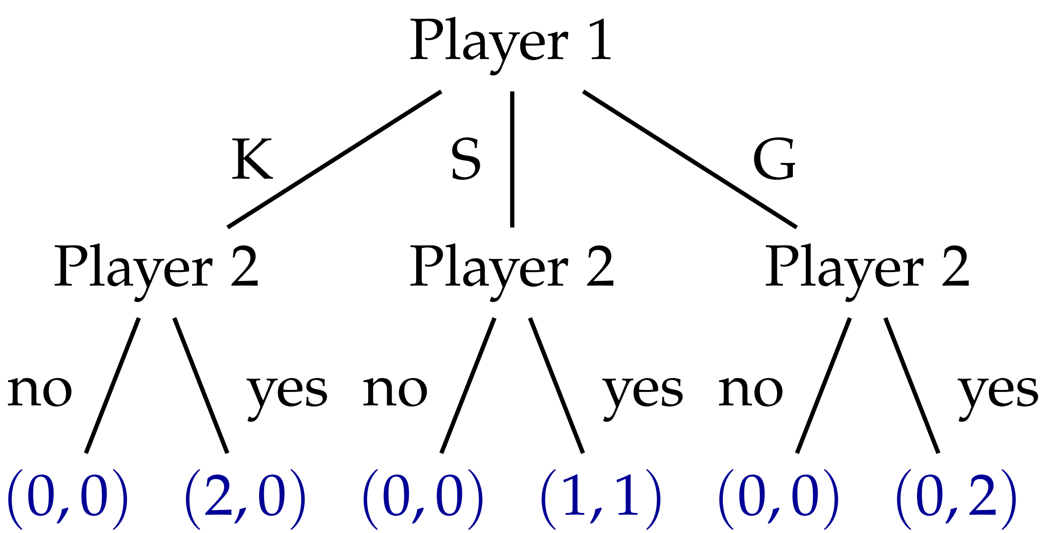

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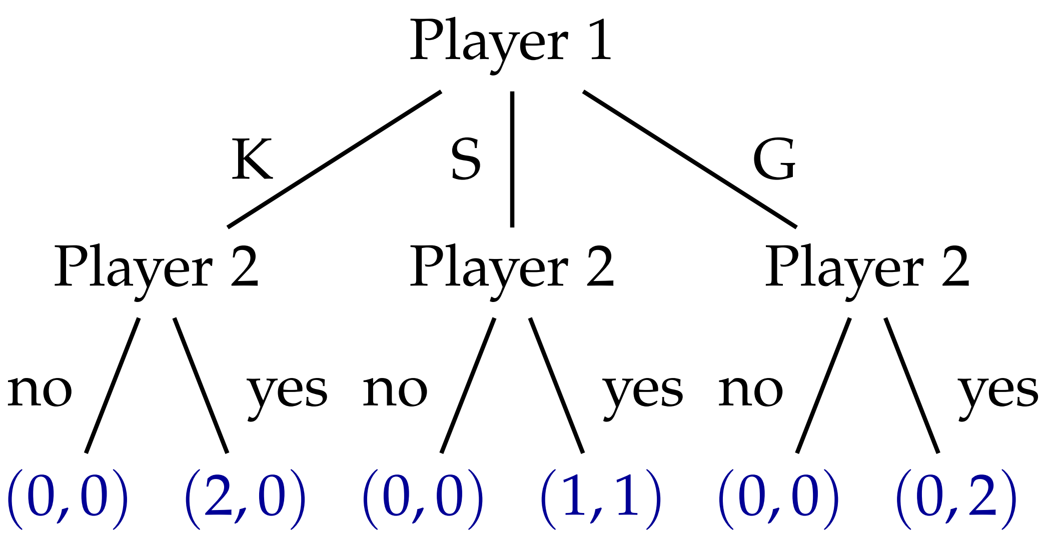

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- Strategies that are not dominated by other strategies.
- Rationalisable strategies.

Games in normal form

The Prisoner's dilemma

. . . oh no, not again

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C	3,3	0,5
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Other two-person / two-strategy normal form games

Chicken:

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- The remaining 132 are a-symmetric and players can be interchanged.
- We end up with $12 + 132/2 = 78$ essentially different games.

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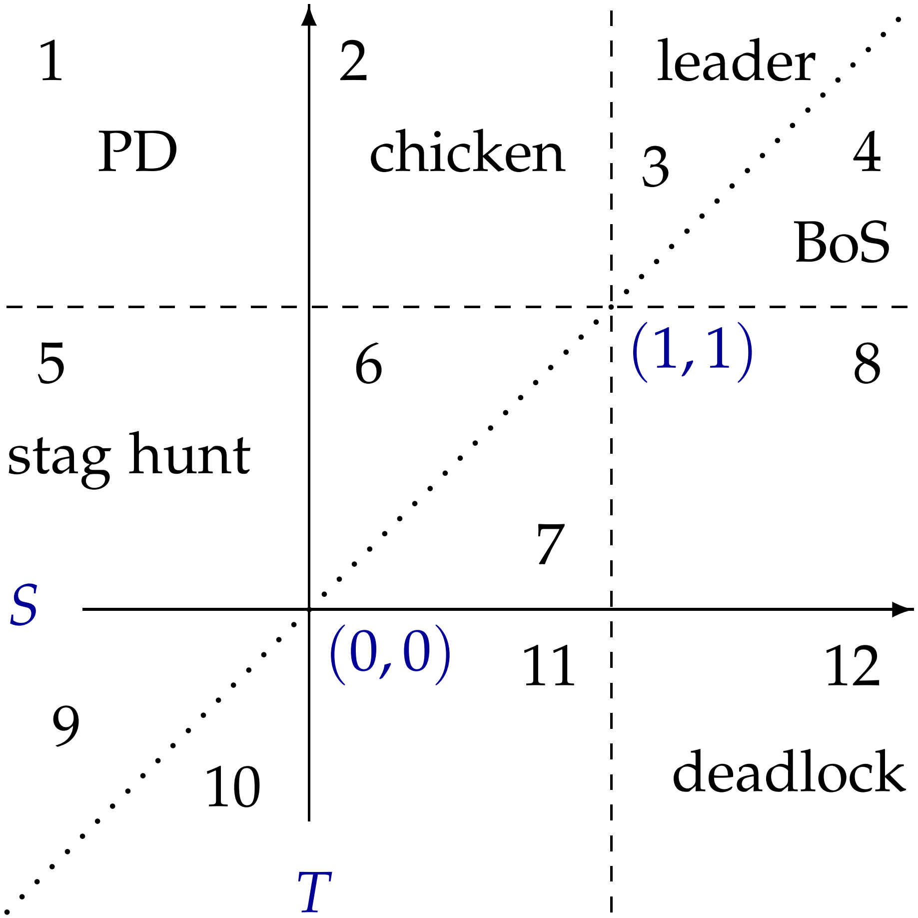
- For simplicity, we take $\epsilon = 1$ so that (S, T) is in $[-1, 2] \times [-1, 2]$.

The (S,T) plane



Partition of the (S, T) plane which displays various symmetric 2×2 games.

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Question: Determine the Pareto front of the following game.

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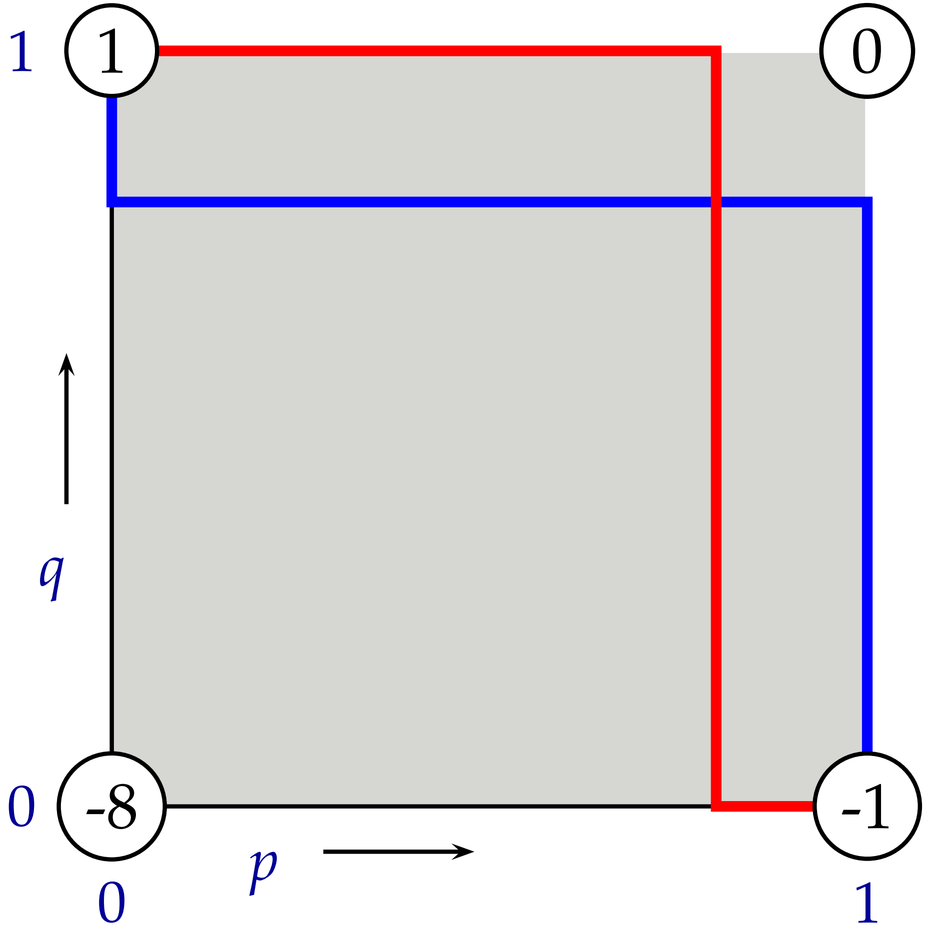
Mixed strategies

Chicken: mixed strategies



	S	D
S	$0, 0$	$-1, 1$
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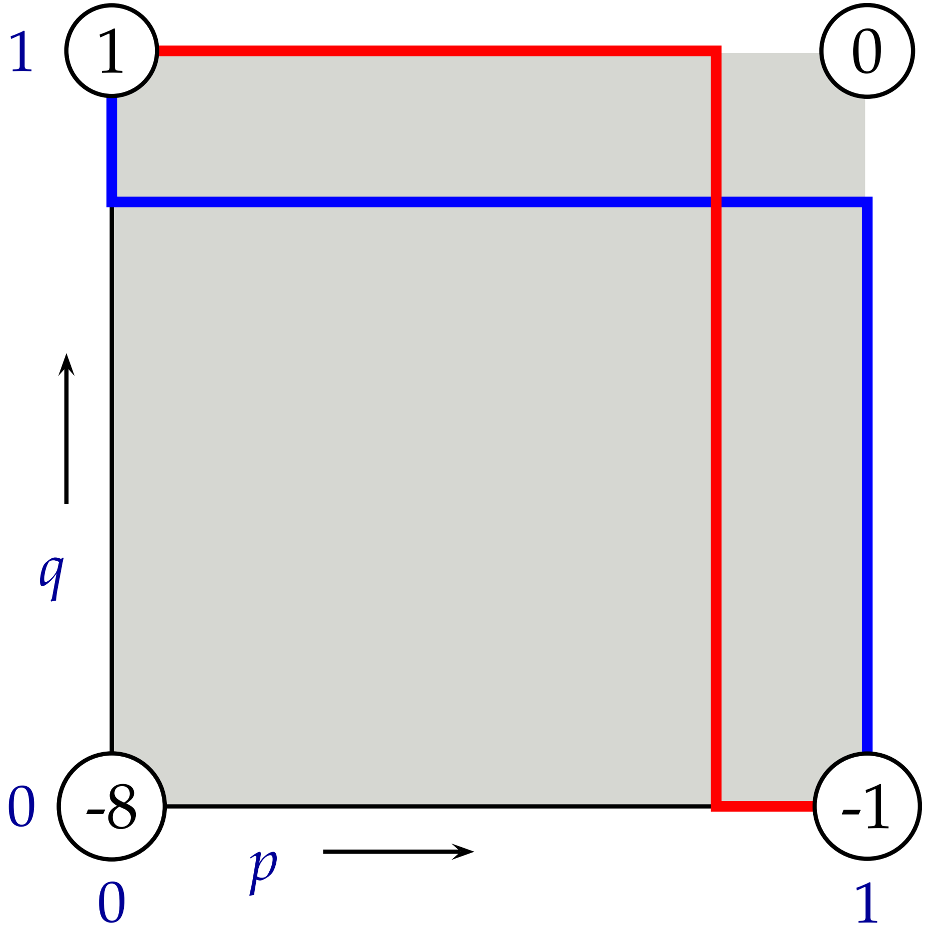

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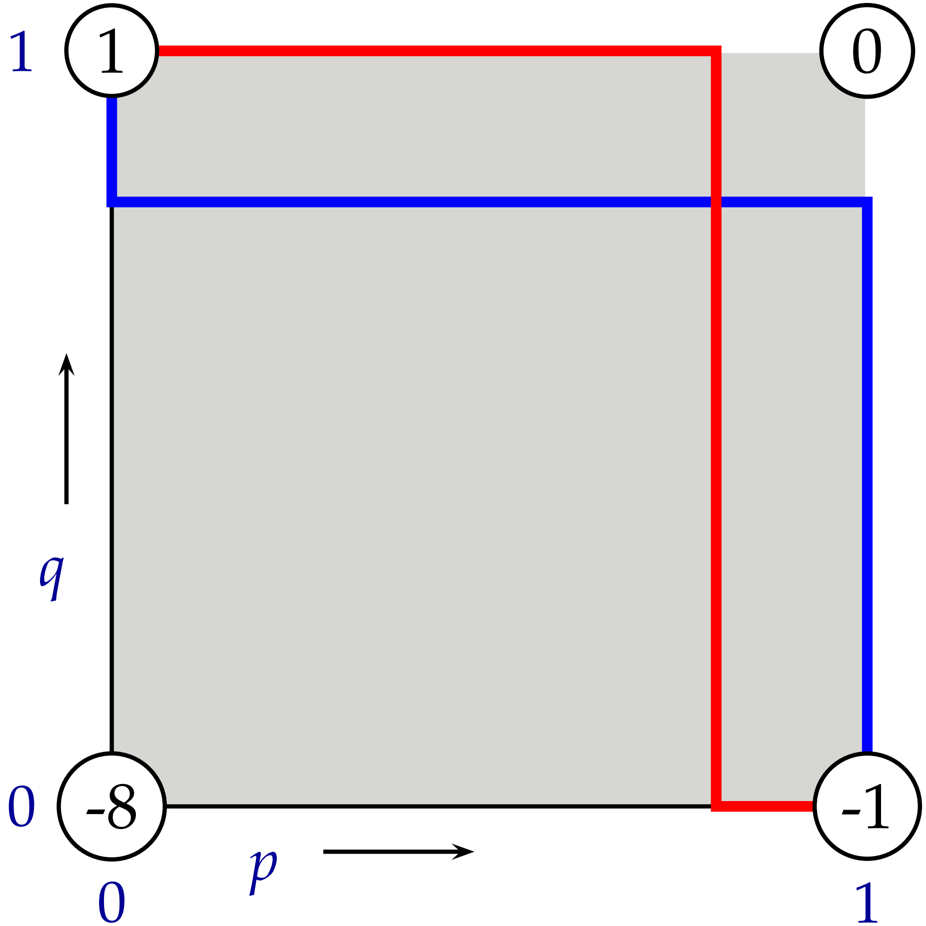

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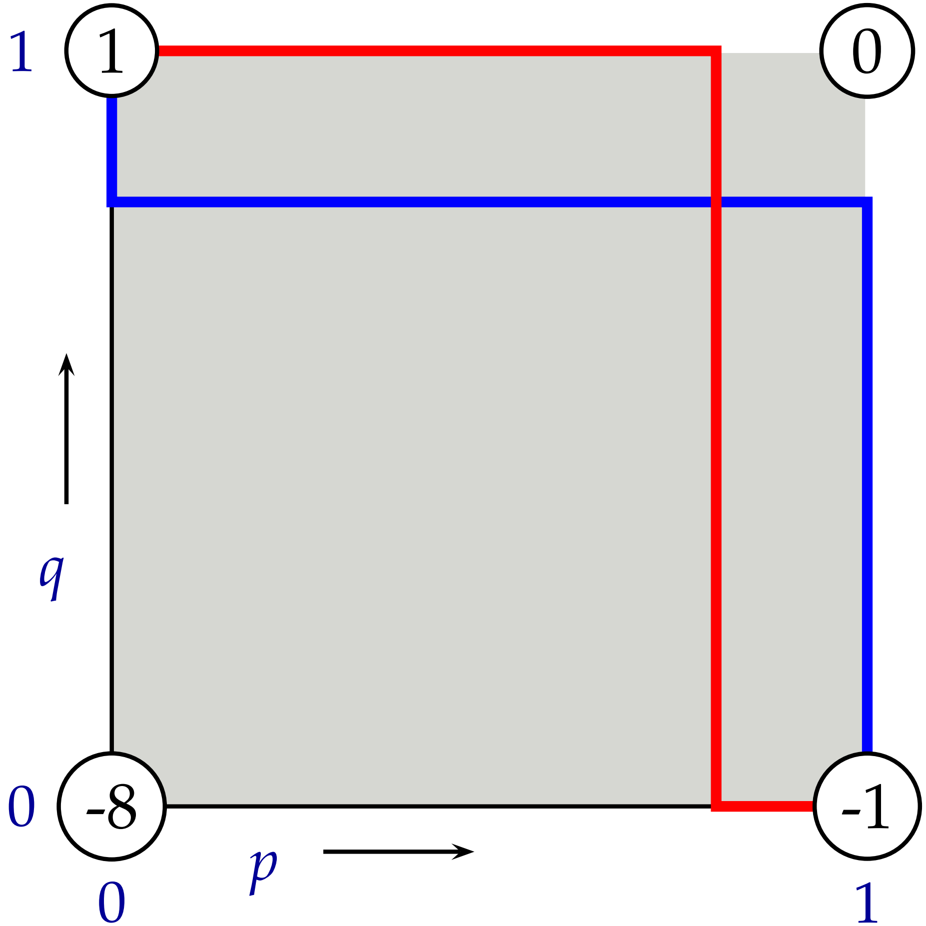

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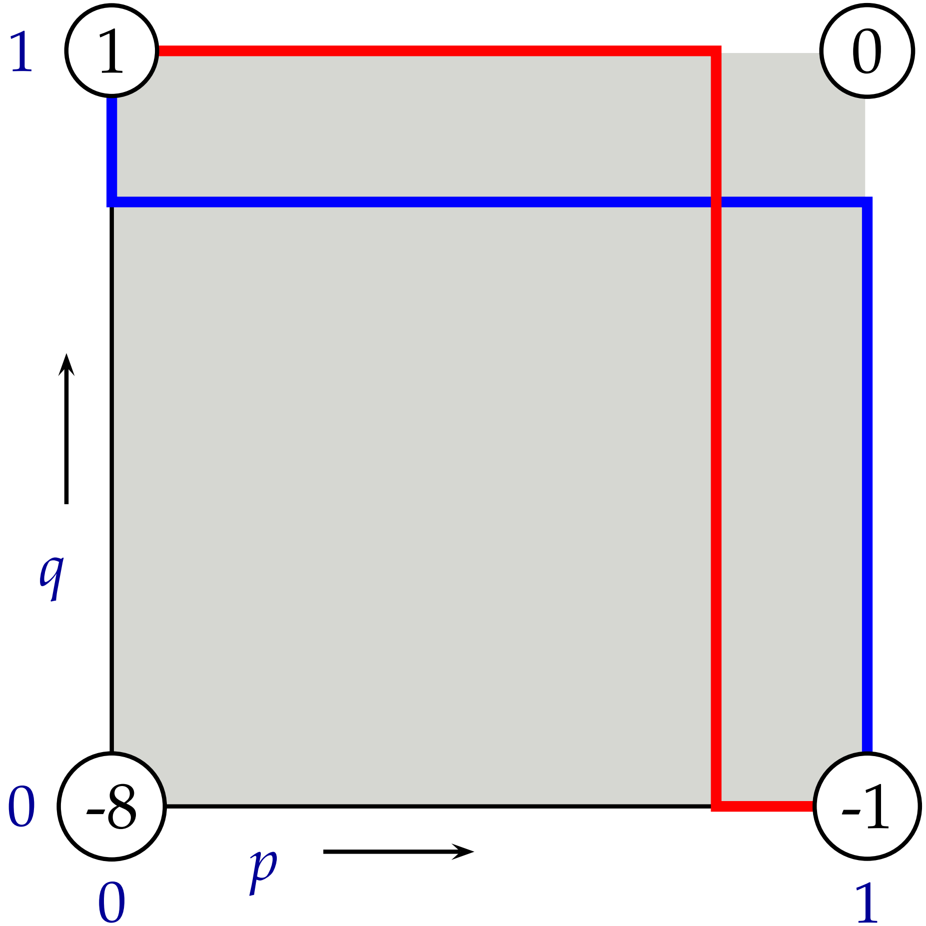

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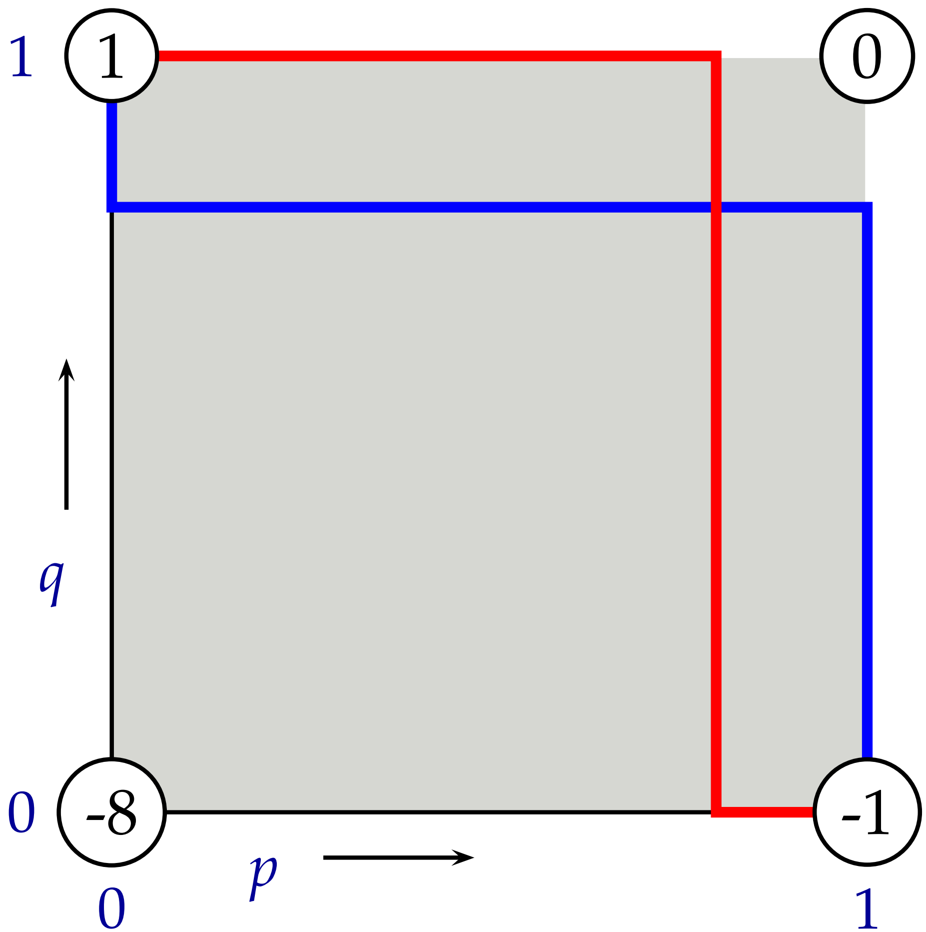

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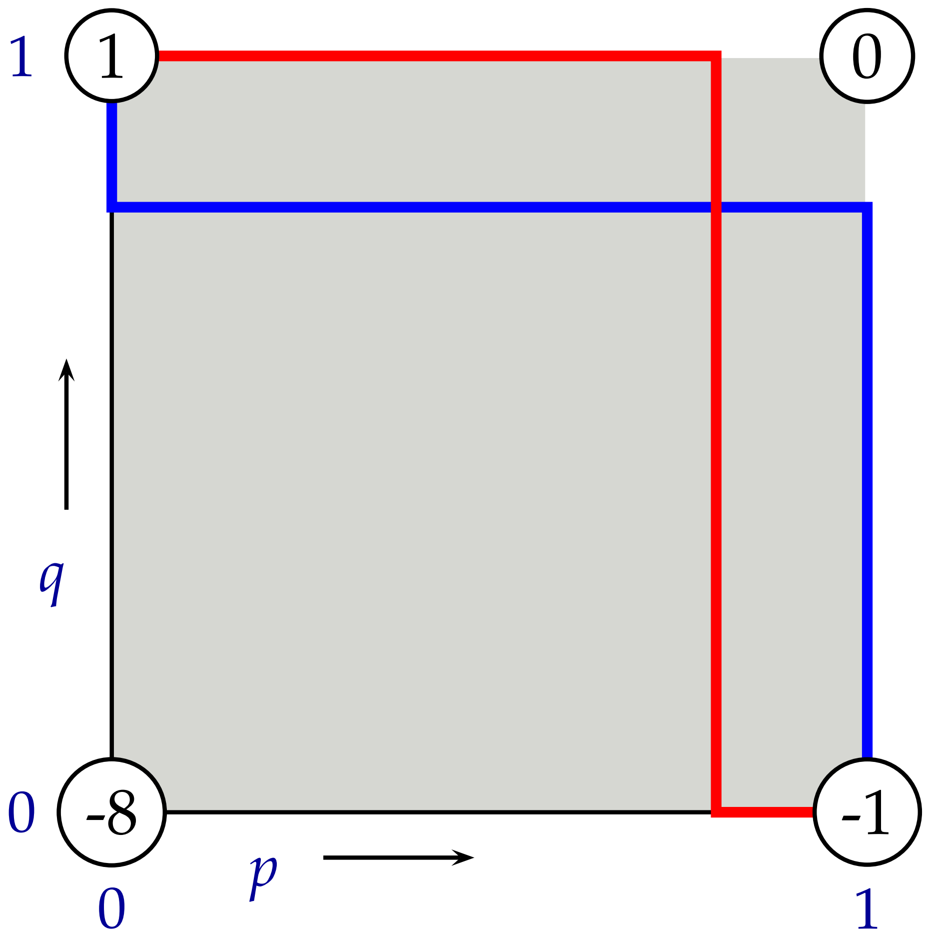

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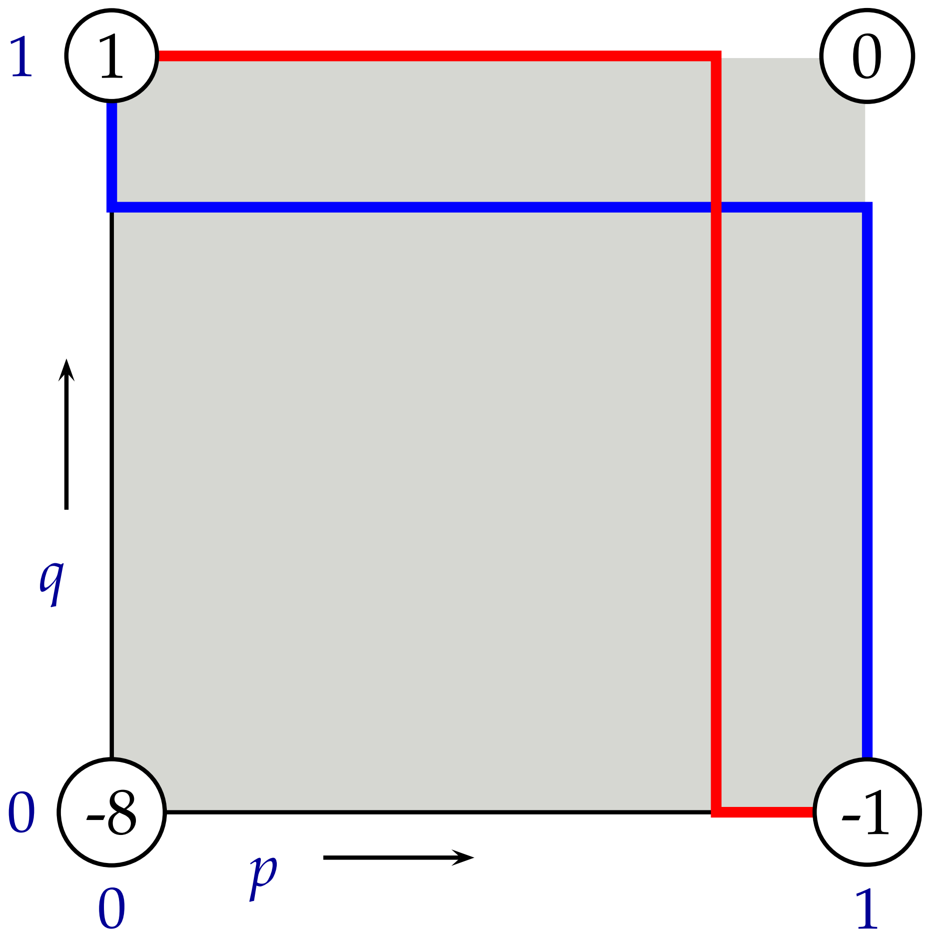

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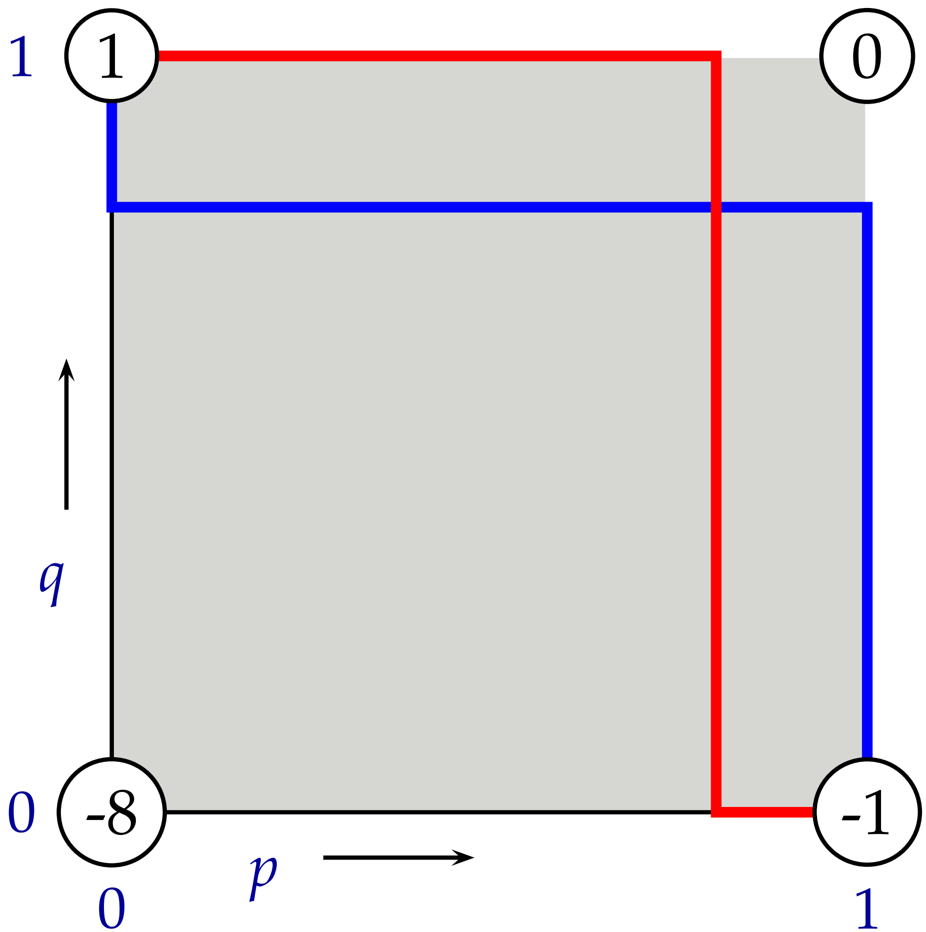

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- Analogous considerations for the column player.

Payoffs for mixed strategies in the chicken game

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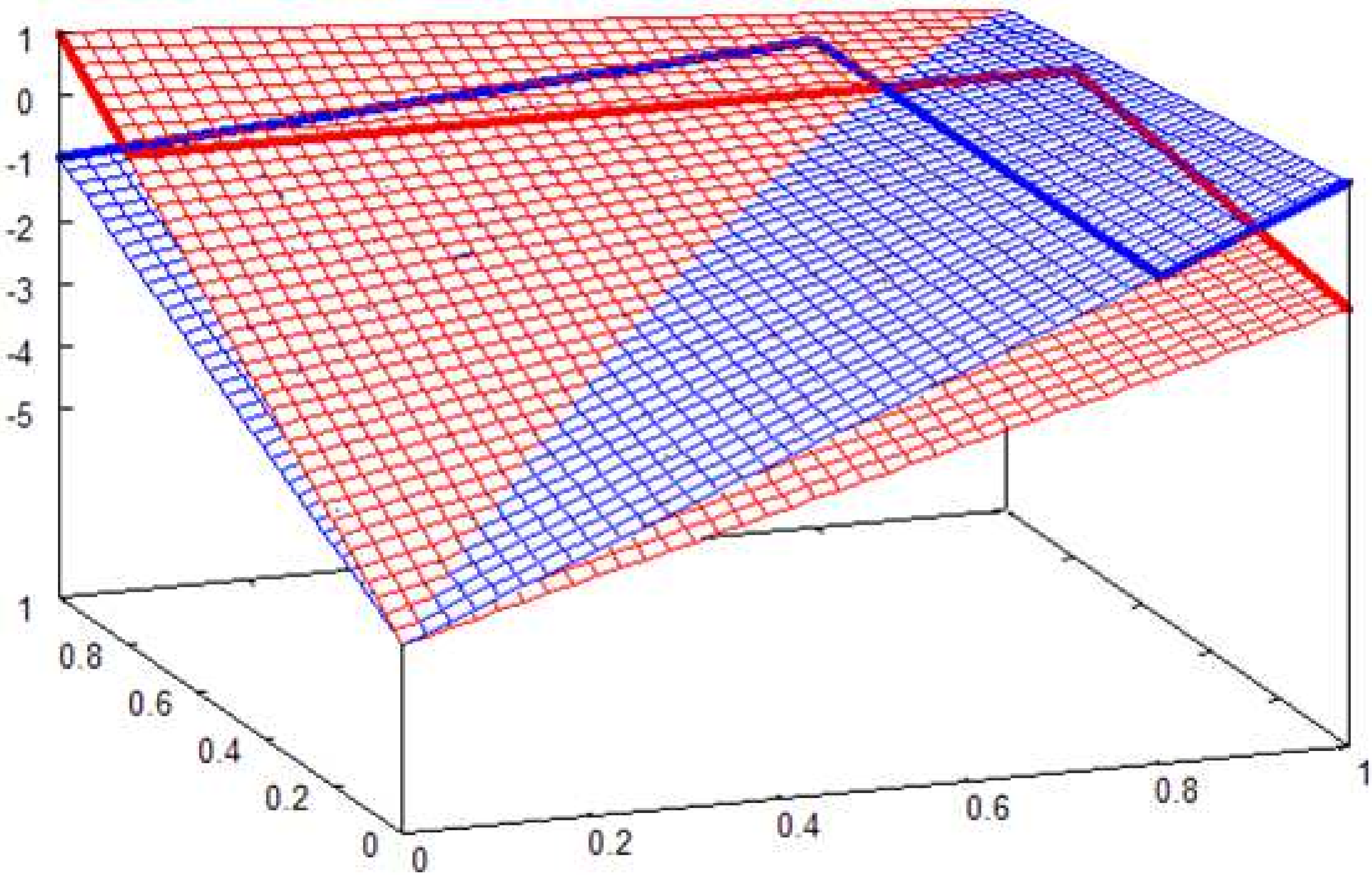

Player 1 may only move "back – front"; Player 2 may only move "left – right".

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- An action profile, a, is a tuple $(a_1, \dots, a_n)$ of actions in G.
- Let $s_j(a)$ be the probability that Player j plays his part of a. Then, for each action profile a,

$$\prod_{j=1}^{n} s_j(a)$$

is the probability that a is played.

- Let $u_i(a)$ be the payoff for Player i when a is played. Then

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j=1}^{n} s_j(a),$$

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  where A is the set of all action profiles of G.

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It follows that a best response is obtained as long as indices are chosen from s_i's support. □

Best responses

Example. Matching pennies:

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H	$1,-1$	$-1,1$
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  Intuitively, this is because the opponent is completely unpredictable. There is nothing to coordinate, it does not matter what you do.
- If your opponent is predictable, i.e, if

$$q \neq \frac{1}{2},$$

  there is only one best response.

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Spot a Nash equilibrium

Question:

Spot a Nash equilibrium in the following game. Players may use mixed strategies.

	A	B	C	D
A	1,8	4,7	3,0	1,3
B	1,8	3,0	4,7	4,6
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- Great!
- How do you know it is a Nash equilibrium?!

  I.e., how do you know it is a Nash equilibrium if players may used mixed strategies as well?

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Answer: (D,R) is Nash and ϵ-Nash; (U,L) is ϵ-Nash.

Security level and punishment strategies

Maxmin strategies

Maxmin strategies

- The maxmin or security level strategy for player i is a strategy for which the minimum payoff against all counter strategies is maximal:

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- Exercise: give **pure** maxmin strategies for the row player

	A	B	C	D
A	$-1, 2$	$2, 0$	$1, 0$	$-2, 0$
B	$3, 0$	$-1, 3$	$0, -1$	$1, 3$
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For the column player.

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  For the column player.

- Mixed maxmin strategies may have higher payoffs than pure maxmin strategies.

Minmax strategies

The minmax or punishment counter strategy profile against player i is a counter strategy profile for which the maximum payoff against all counter strategies is minimal:

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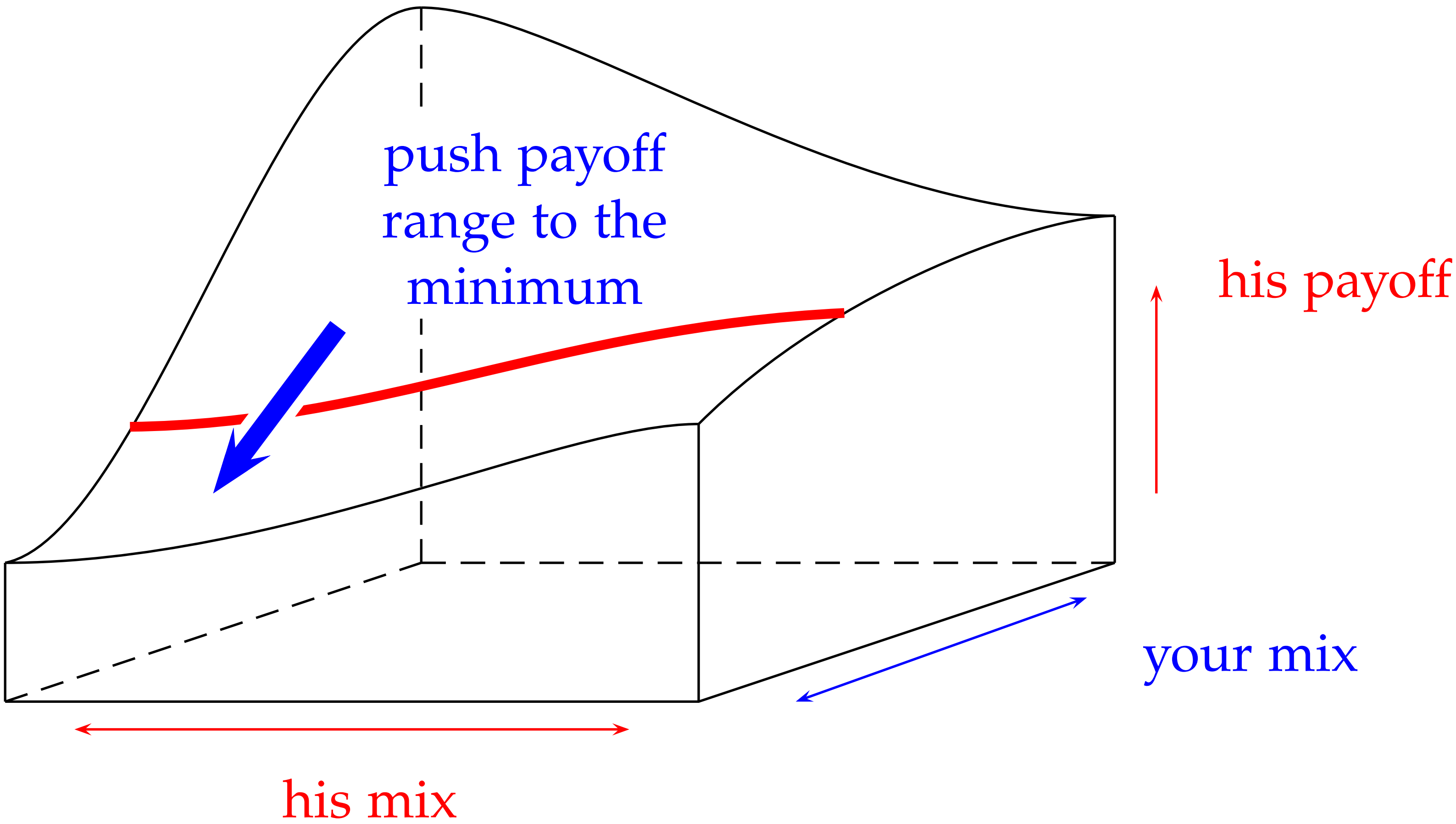
Minmax theorem (von Neumann, 1928). In any finite two-player zero-sum game, in any Nash equilibrium, each player receives a payoff that is equal to both his maxmin value and his minmax value.

Minmax: payoff surface of the opponent



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- Strategy s_i very weakly dominates s'_i if for all $s_{-i} \in S_{-i}$ we have $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.
- If one strategy dominates all others, we say that it is (strongly, weakly or very weakly) dominant.

Domination

- Strategy s_i stricly dominates s'_i if for all $s_{-i} \in S_{-i}$ we have $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.
- Strategy s_i weakly dominates s'_i if for all $s_{-i} \in S_{-i}$ we have $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$, and strictly for at least one s_{-i}.
- Strategy s_i very weakly dominates s'_i if for all $s_{-i} \in S_{-i}$ we have $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.
- If one strategy dominates all others, we say that it is (strongly, weakly or very weakly) dominant.

Exercise: eliminate dominated actions.

	L	C	R
U	3,1	0,1	0,0
M	1,1	1,1	5,0
D	0,1	4,1	0,0

Removal of dominated actions

	L	C	R
U	3, 1	0, 1	0, 0
M	1, 1	1, 1	5, 0
D	0, 1	4, 1	0, 0

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Answer: action R is strictly dominated by, for instance, action C:

	L	C
U	3, 1	0, 1
M	1, 1	1, 1
D	0, 1	4, 1

Removal of dominated actions

	L	C	R
U	3, 1	0, 1	0, 0
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Answer: action R is strictly dominated by, for instance, action C:

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Action M is now strictly dominated by a mix of U and D.

	L	C
U	3, 1	0, 1
D	0, 1	4, 1

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Exercise: remove strictly dominated strategies.

	L	C	R
U	0, 2	3, 1	2, 3
M	1, 4	2, 1	4, 1
D	2, 1	4, 4	3, 2

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	L	C
M	1, 4	2, 1
D	2, 1	4, 4

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D	2,1	4,4	3,2

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M	1,4	2,1
D	2,1	4,4

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D	2, 1	4, 4	3, 2

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D	2, 1	4, 4

	L	C
D	2, 1	4, 4

	L
D	4, 4

Rationalisability

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- Strategies that are strictly dominated are certainly not rational.
- Strategies from a NE are rational, and are correct in a NE.

Rationalisability (example)

Matching pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

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Rationalisability (example)

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	H	T
H	$1, -1$	$-1, 1$
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- ...

Games in extensive form

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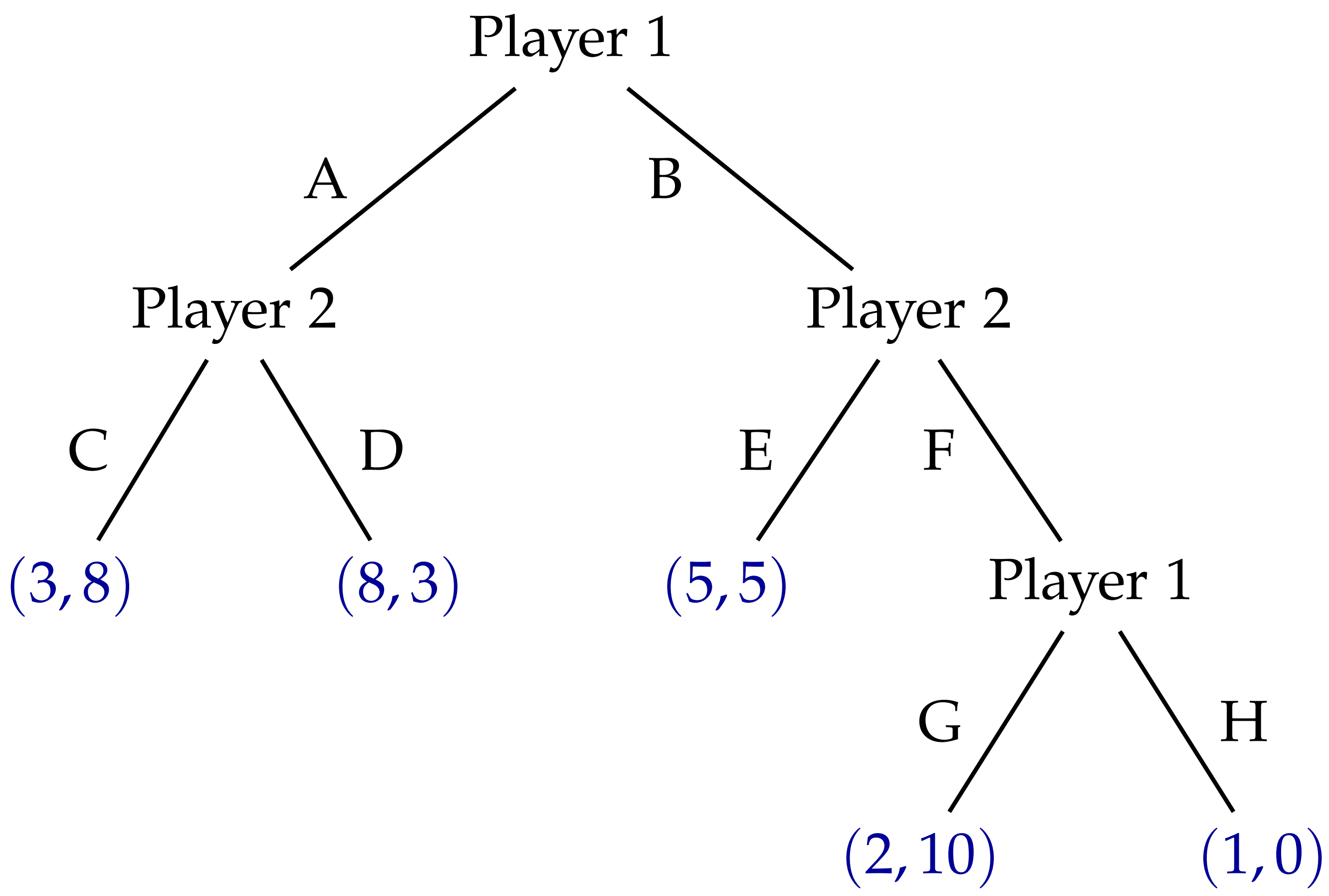

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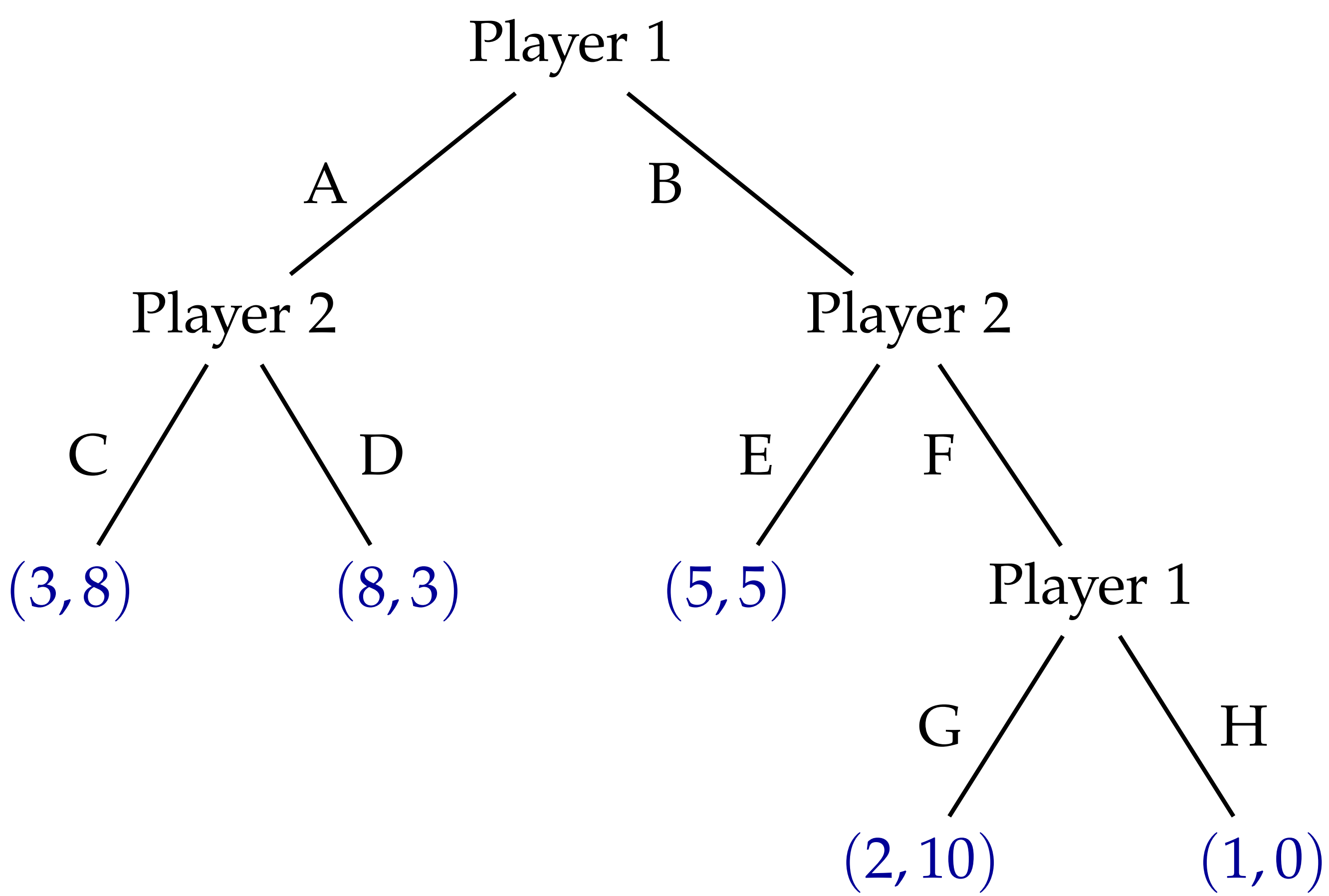

- A pure strategy for Player 1 could be: B, H.

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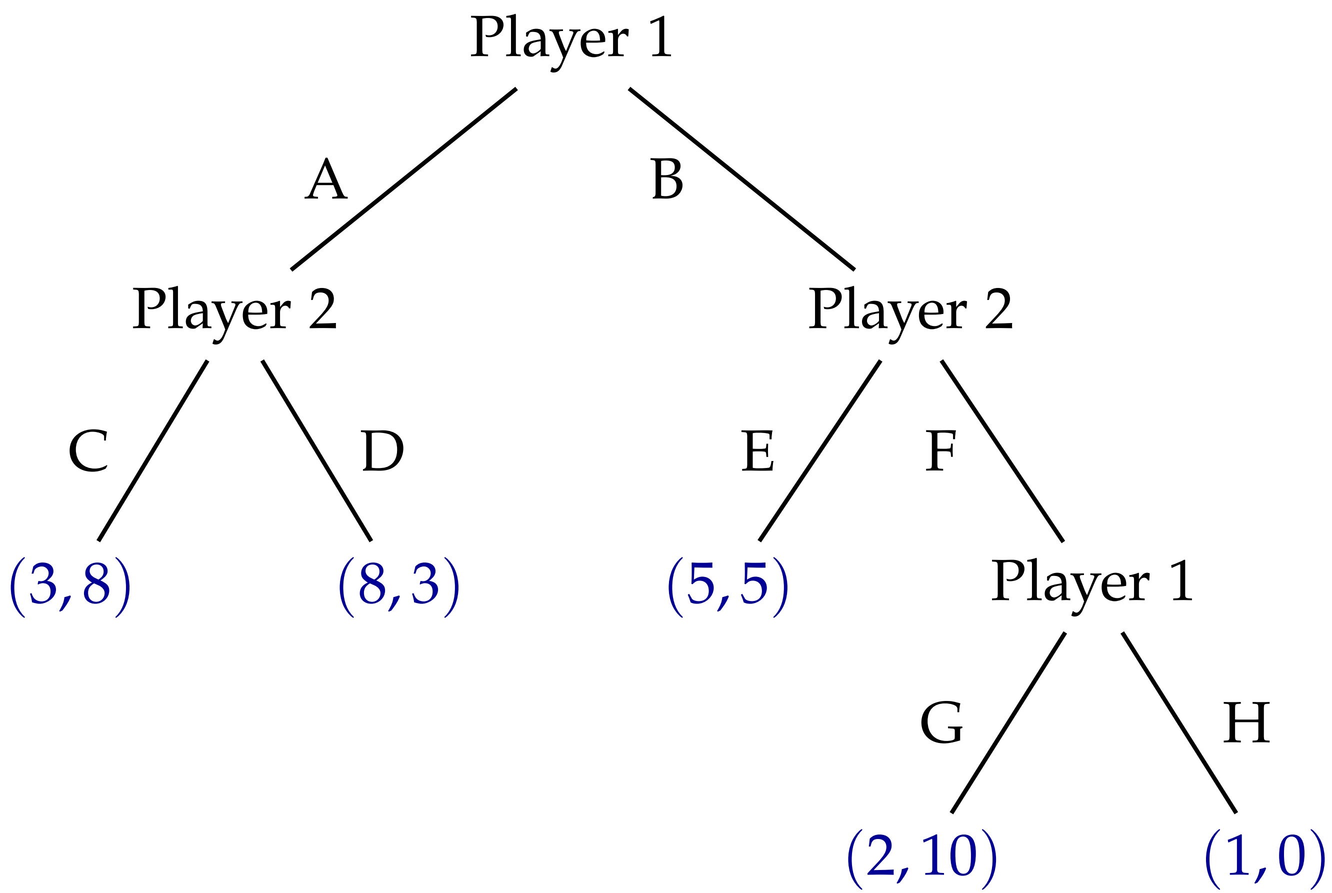

- A pure strategy for Player 1 could be: B, H.
- A pure strategy for Player 2 could be: D, E.

Backward induction

Theorem (Kuhn, 1952). Every finite game in extensive form has a pure strategy Nash equilibrium.

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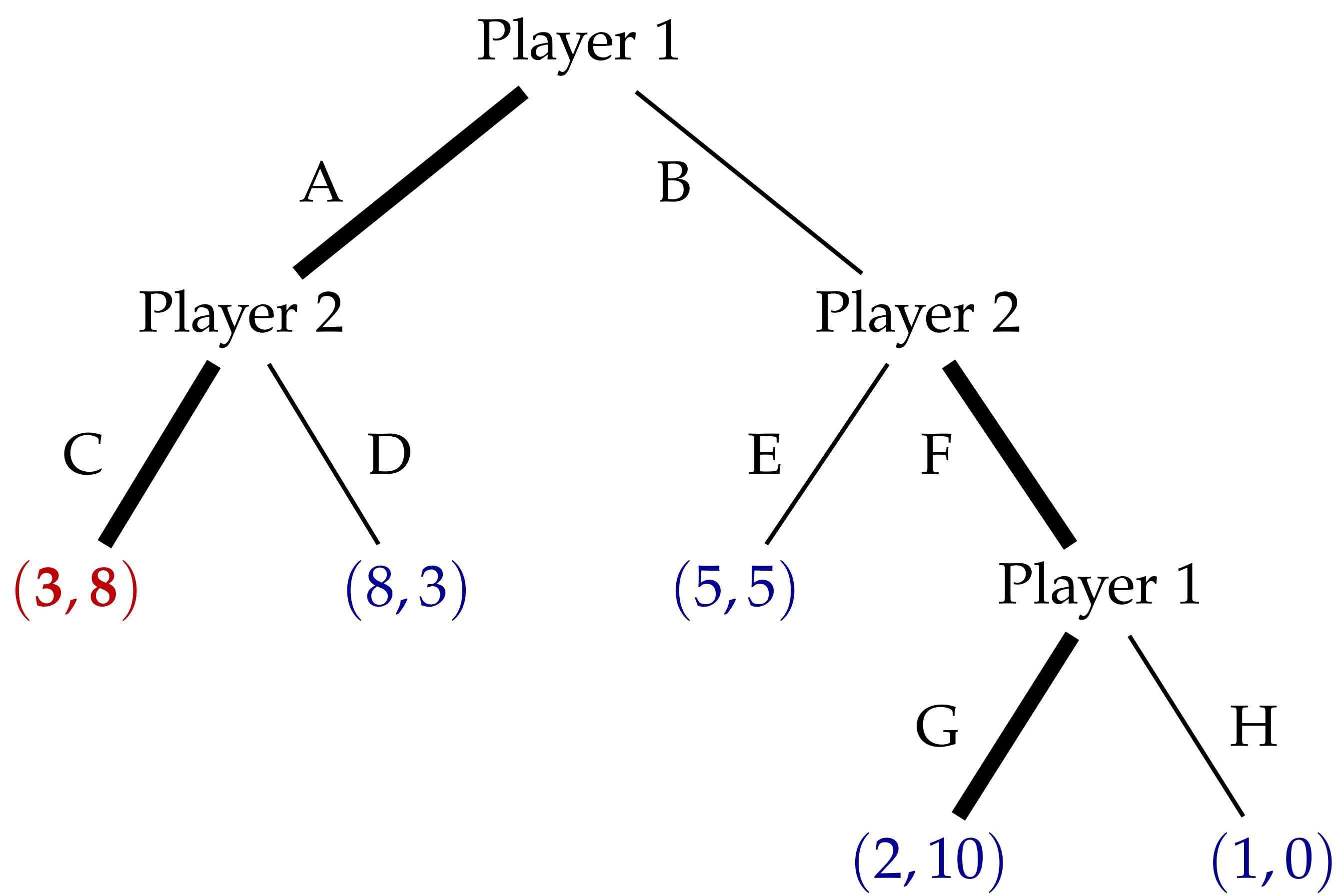
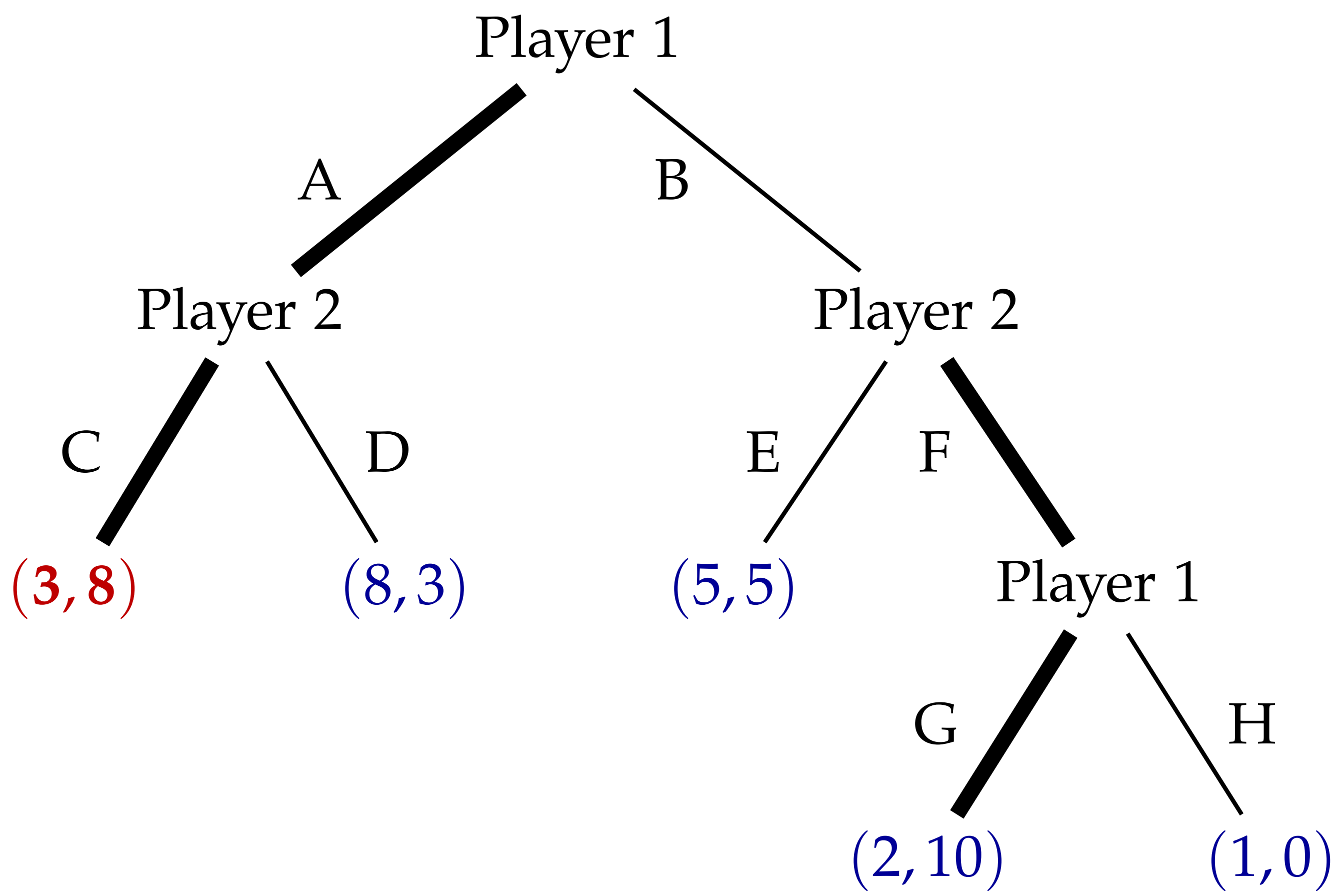
Proof: by means of so-called backward induction.



Subgames, and subgame-perfect equilibrium

Consequence of backward induction: subgame-perfectness: the main NE restricted to subgames are NE for those subgames as well.



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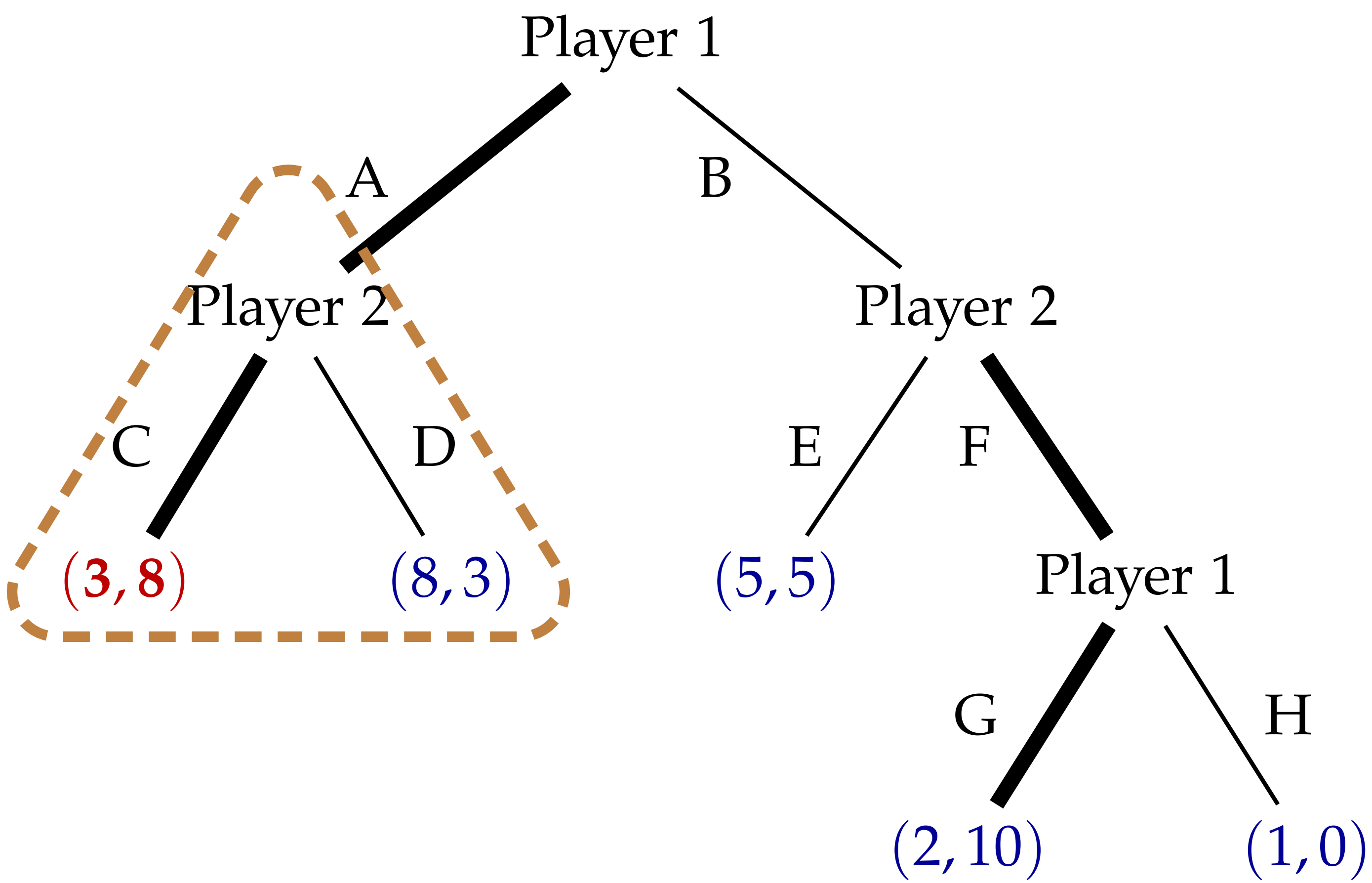

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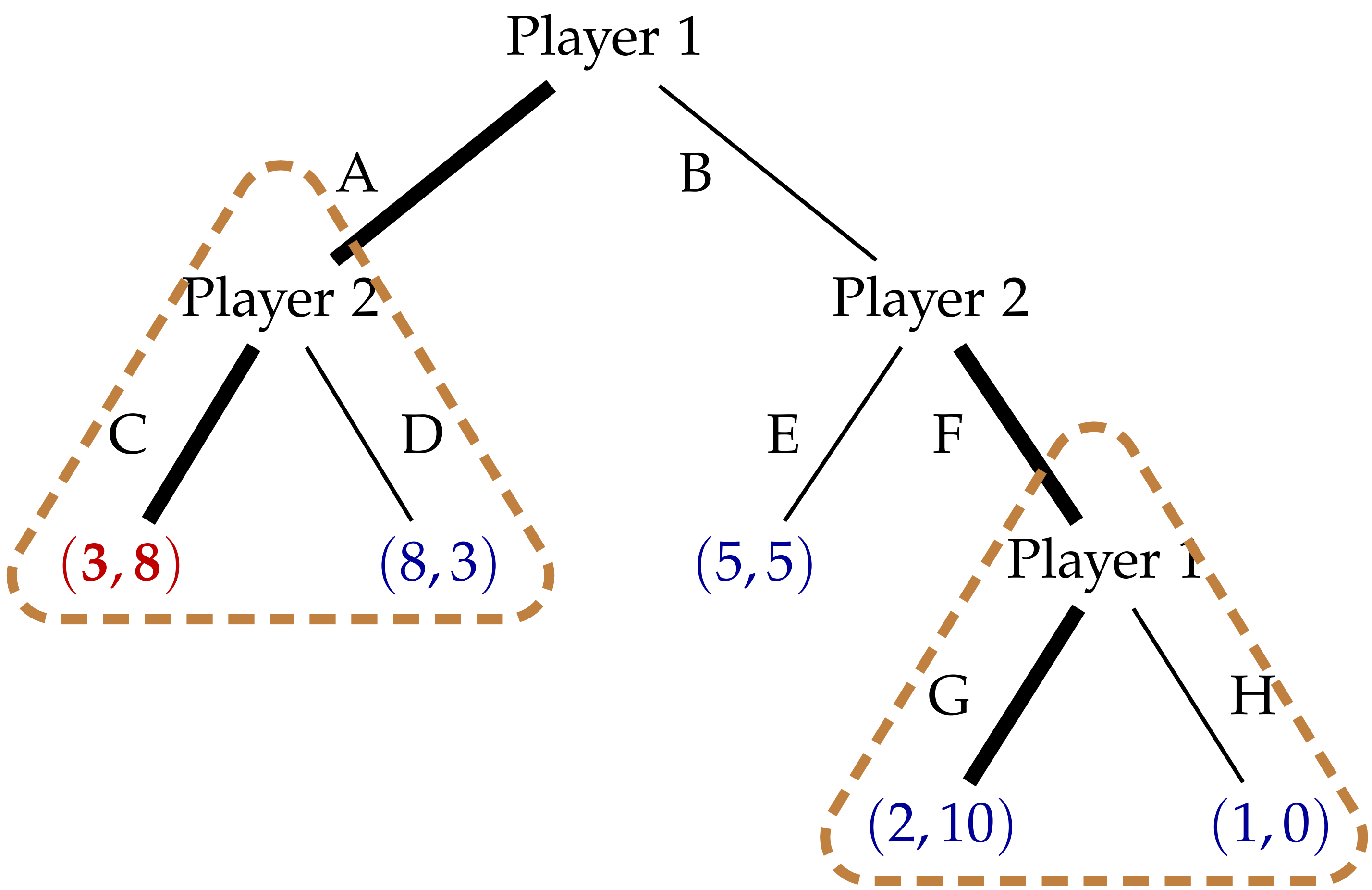

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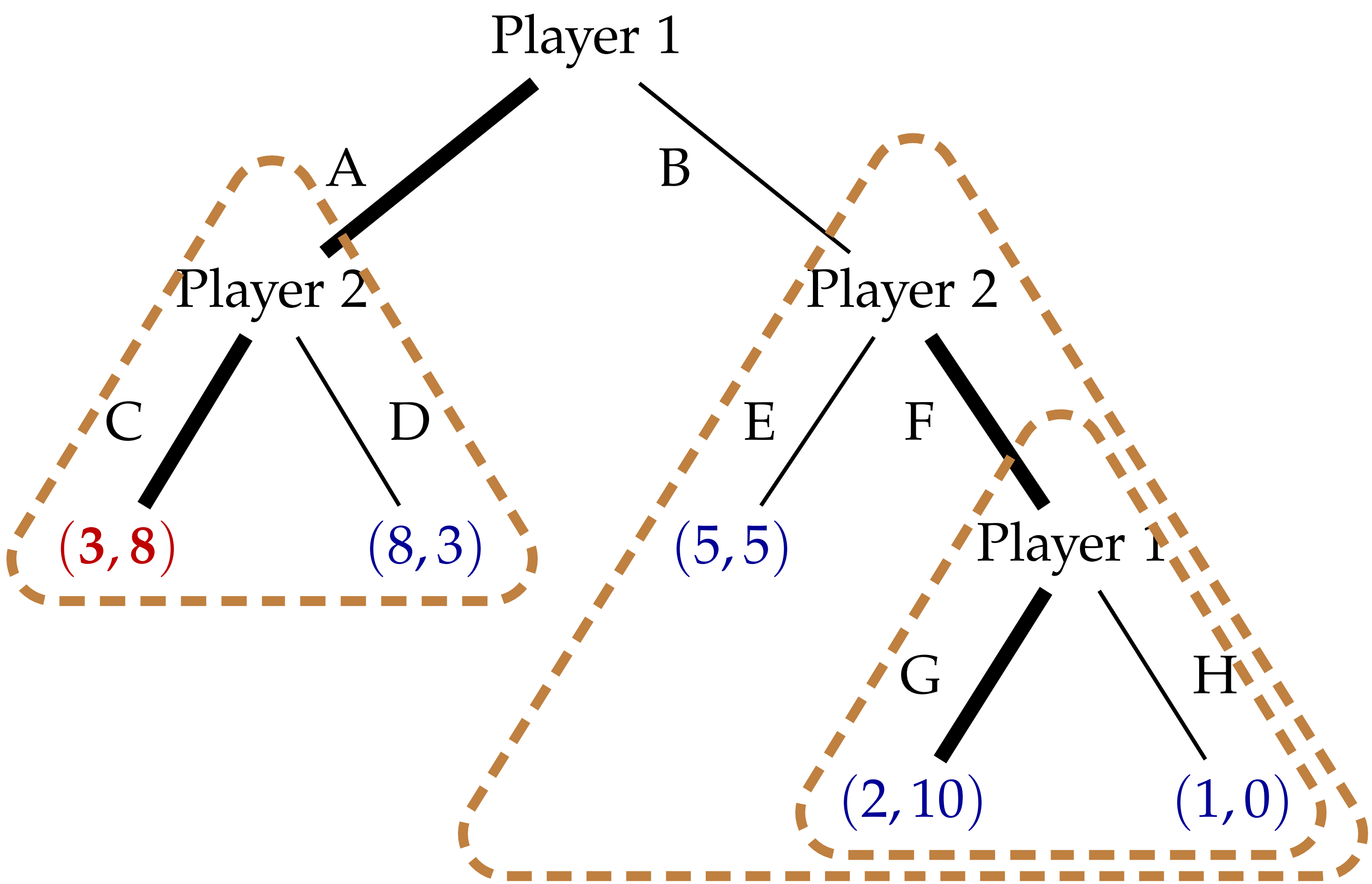

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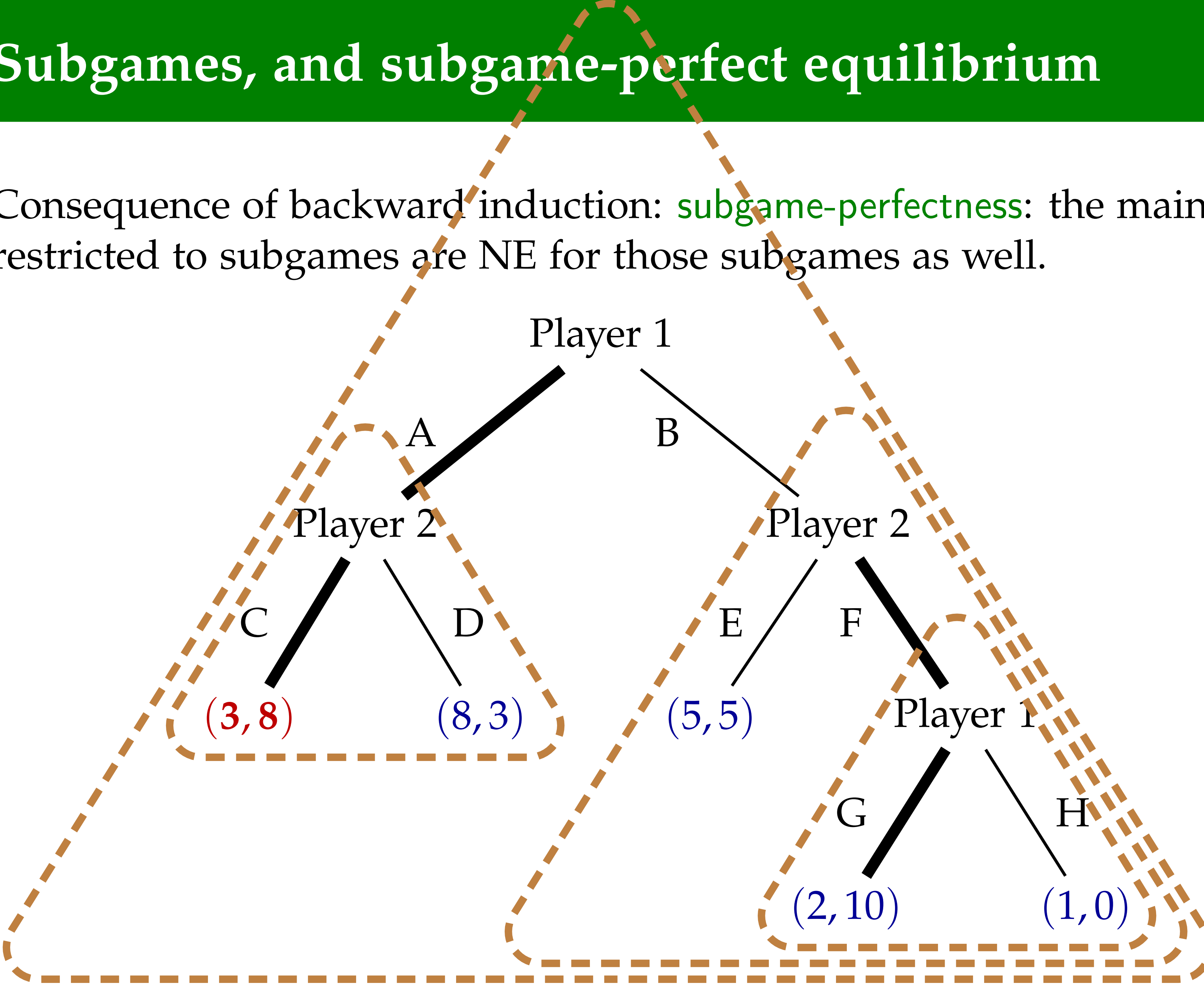
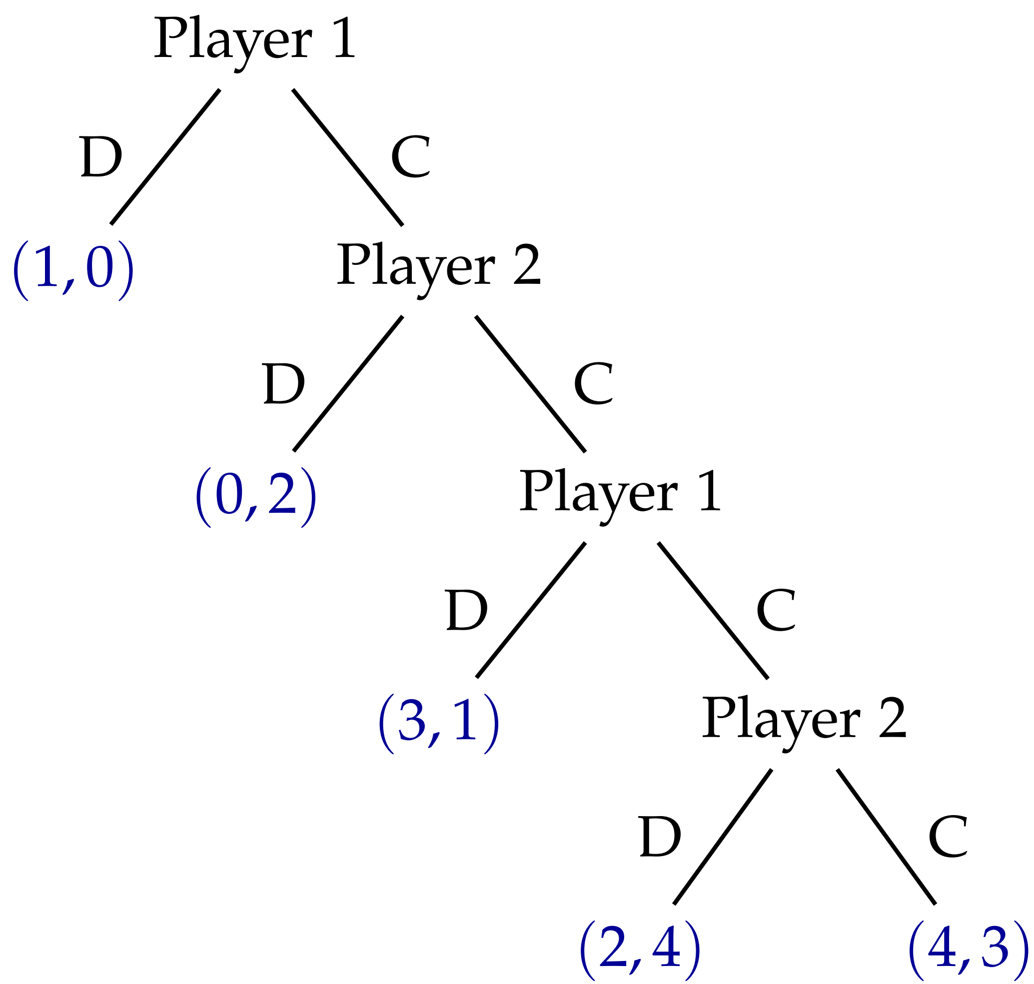
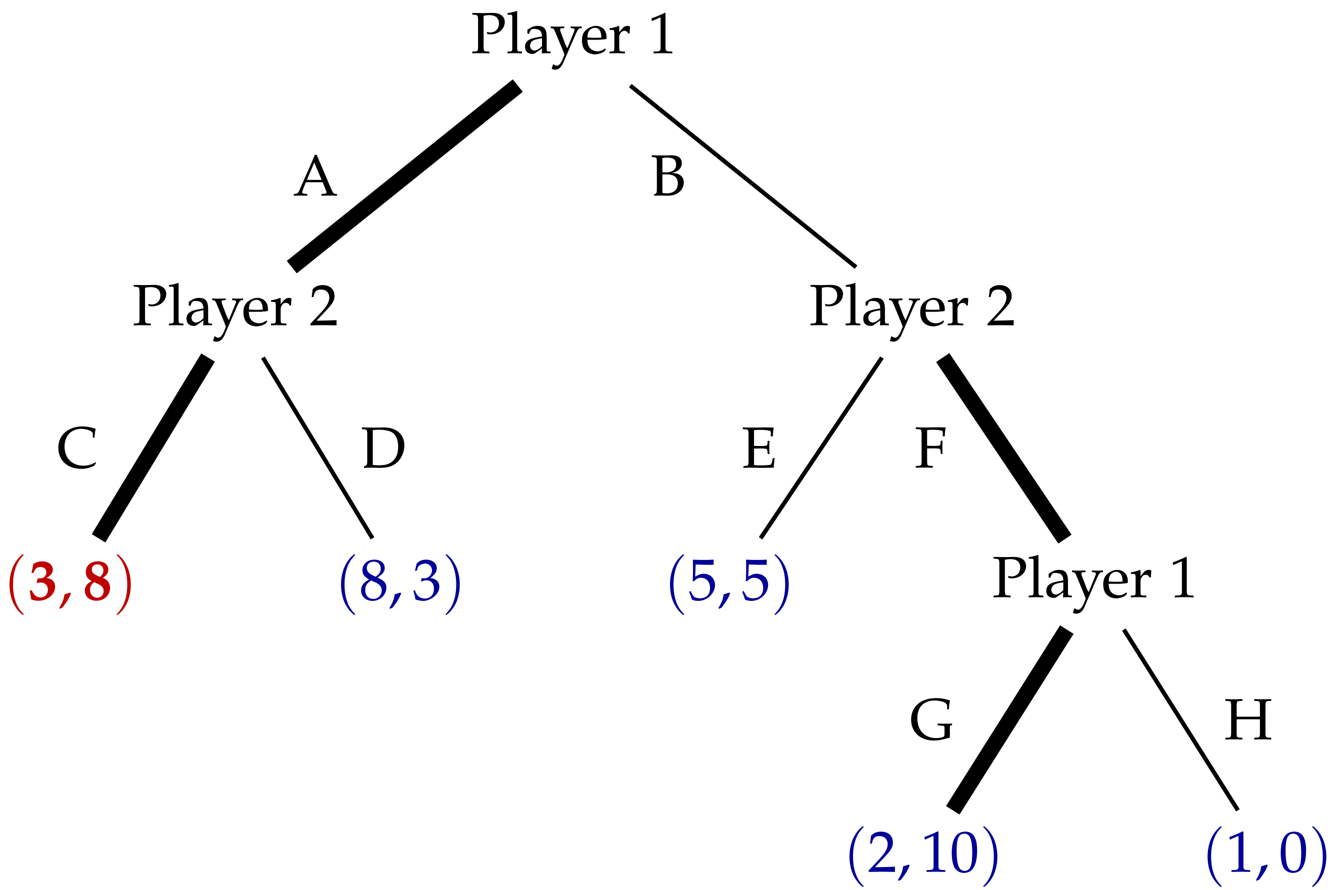


Backward induction is not always intuitive

The centipede game:



Nash equilibrium through backward induction



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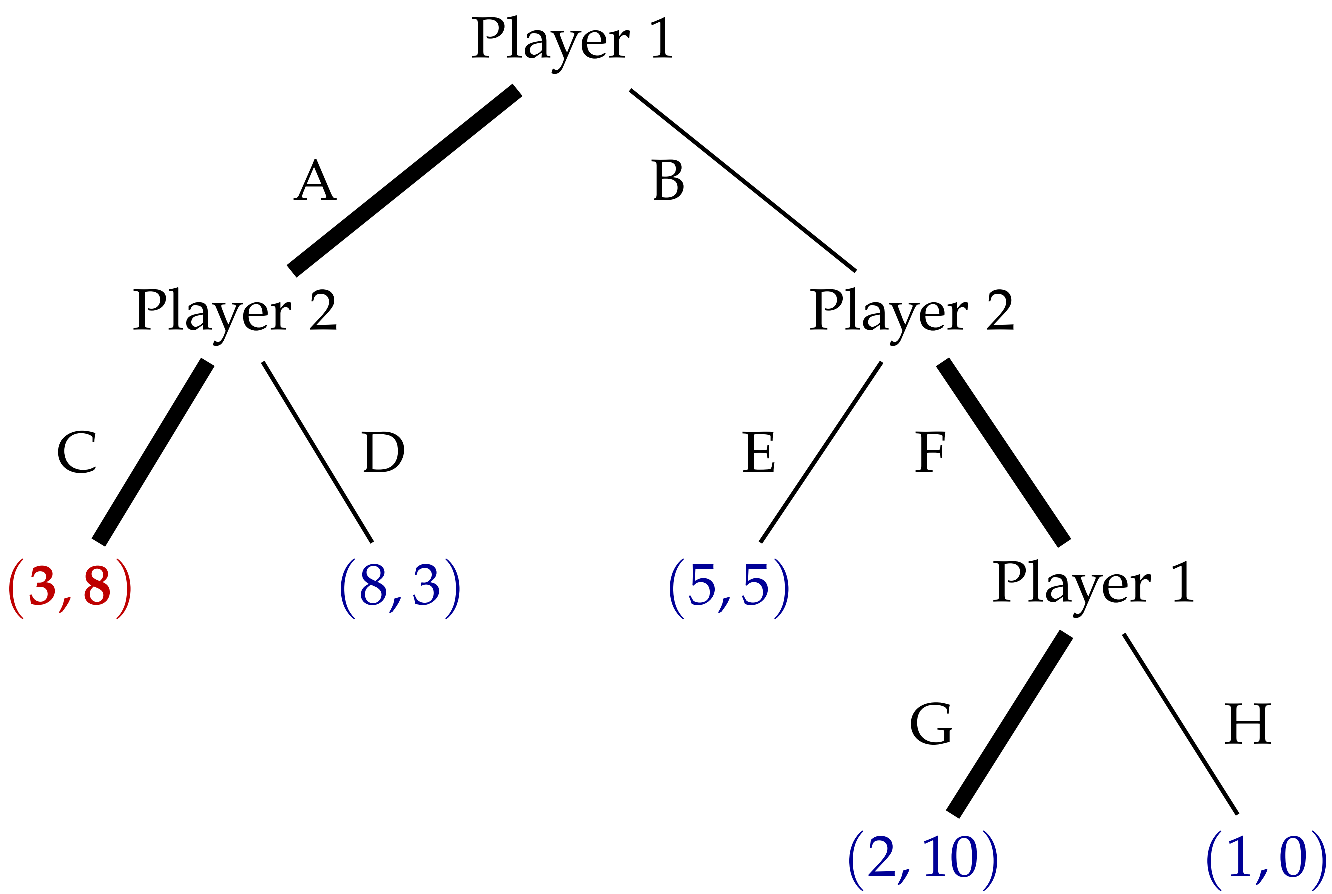
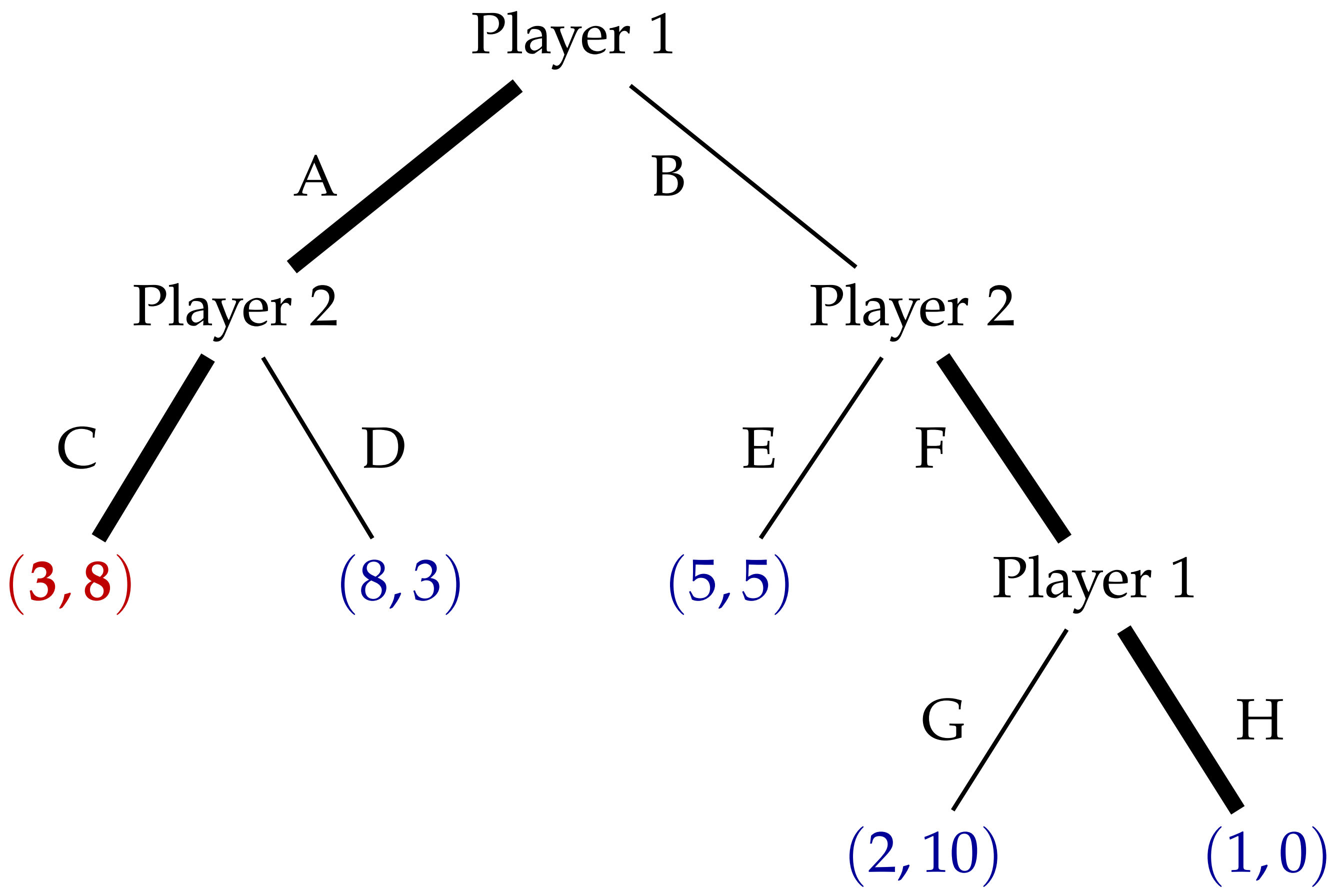


The action profile {(A,H), (C,F)} is a Nash equilibrium.

Nash equilibrium in an extensive game



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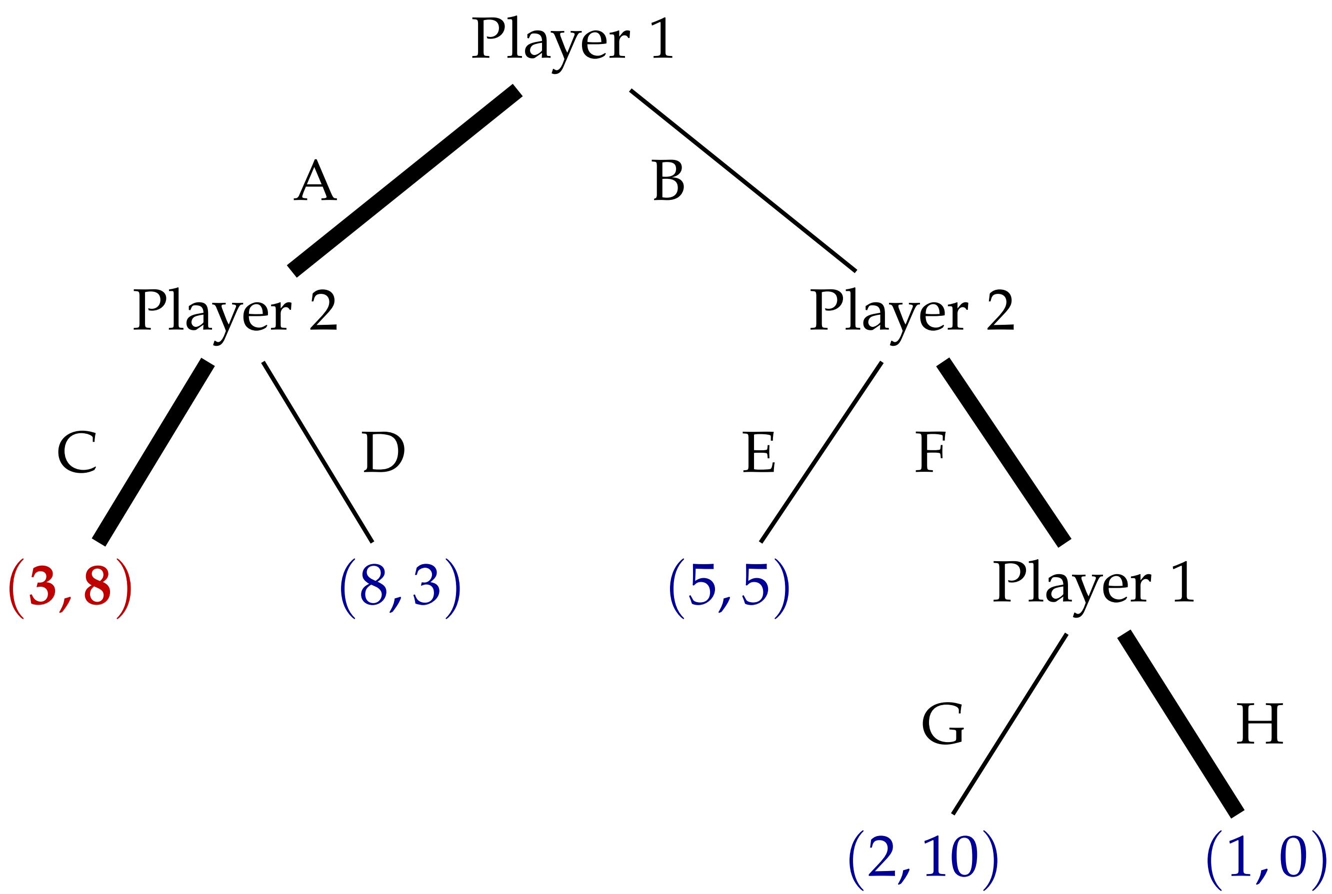

The action profile {(A,H), (C,F)} is also a valid Nash equilibrium.

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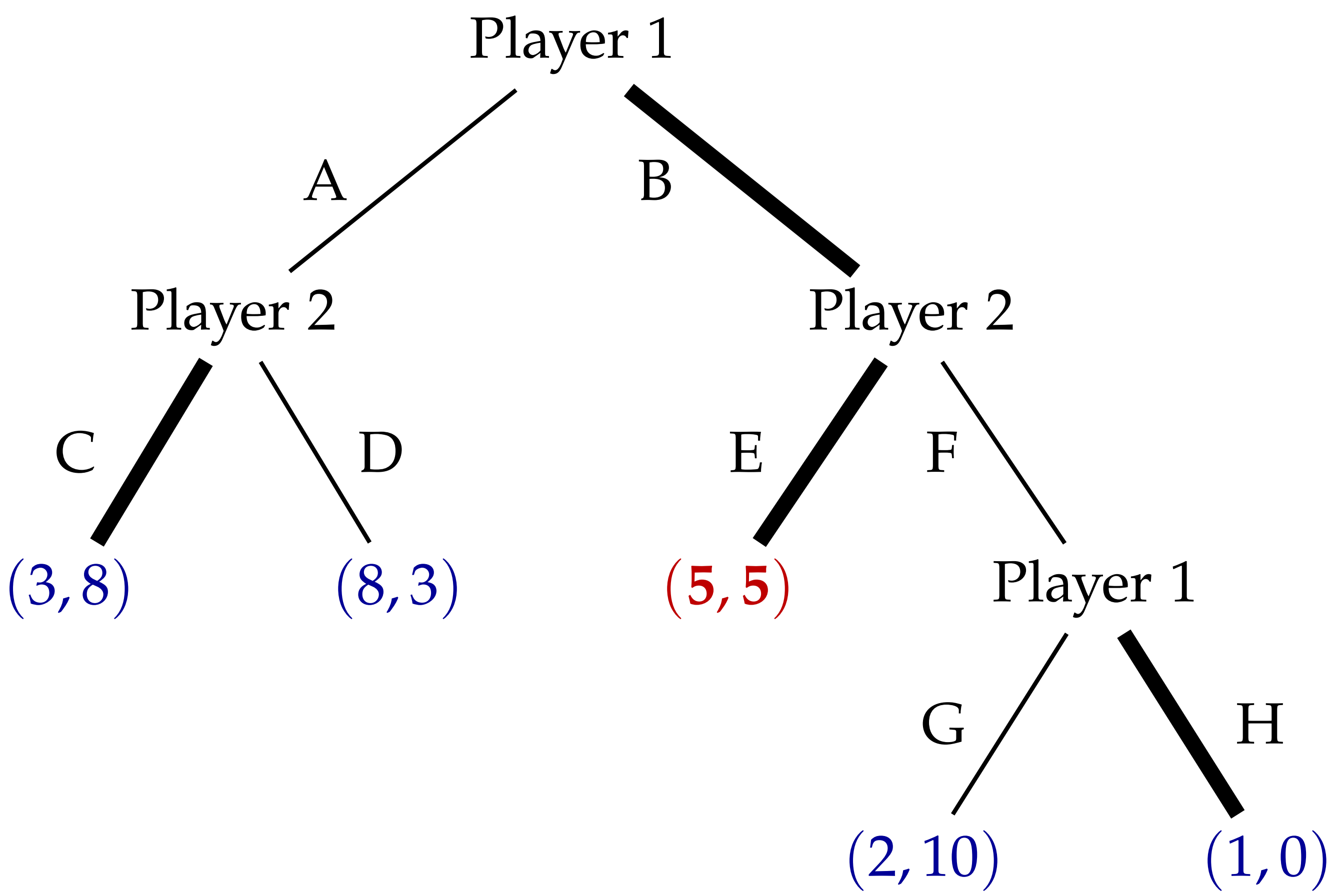

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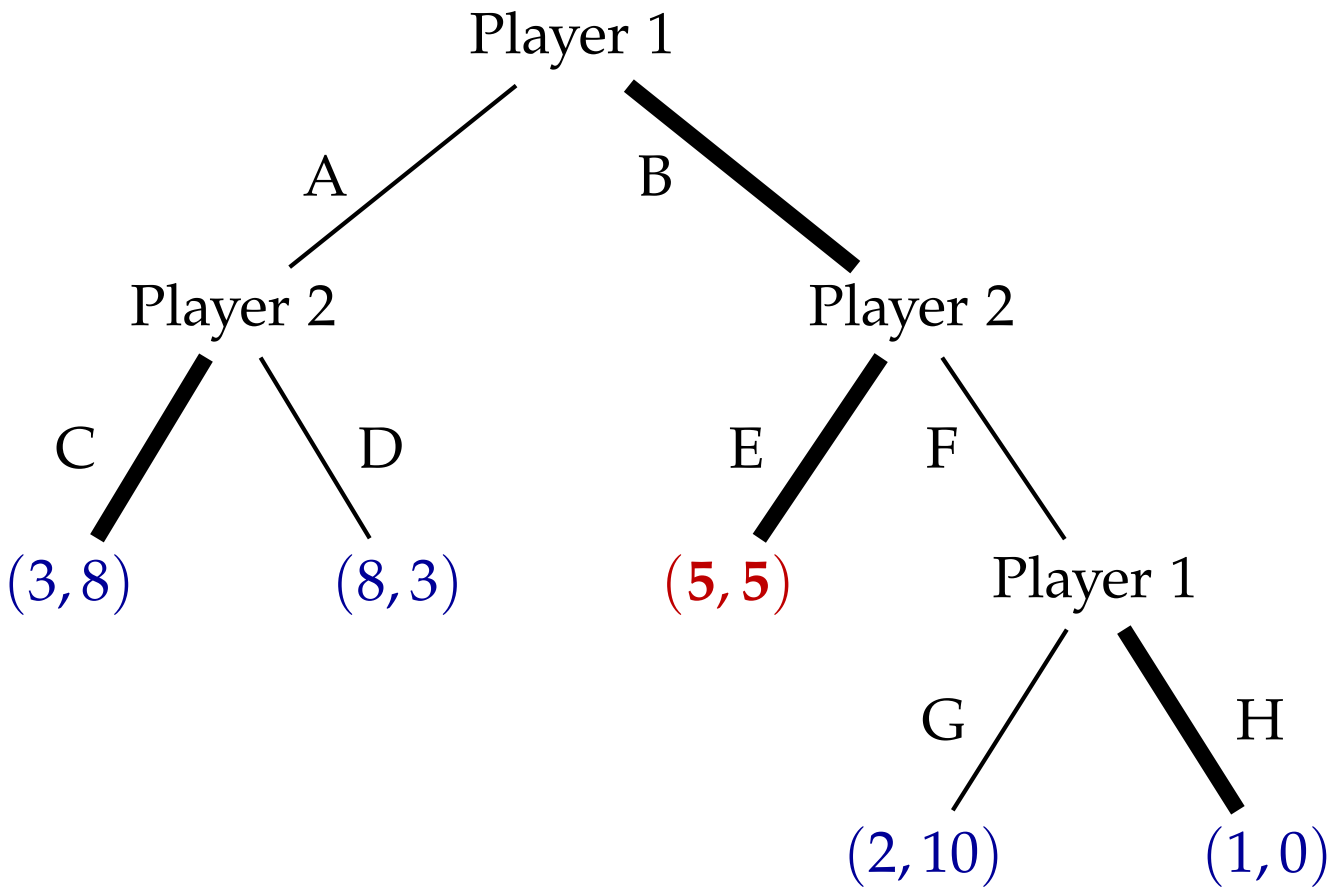
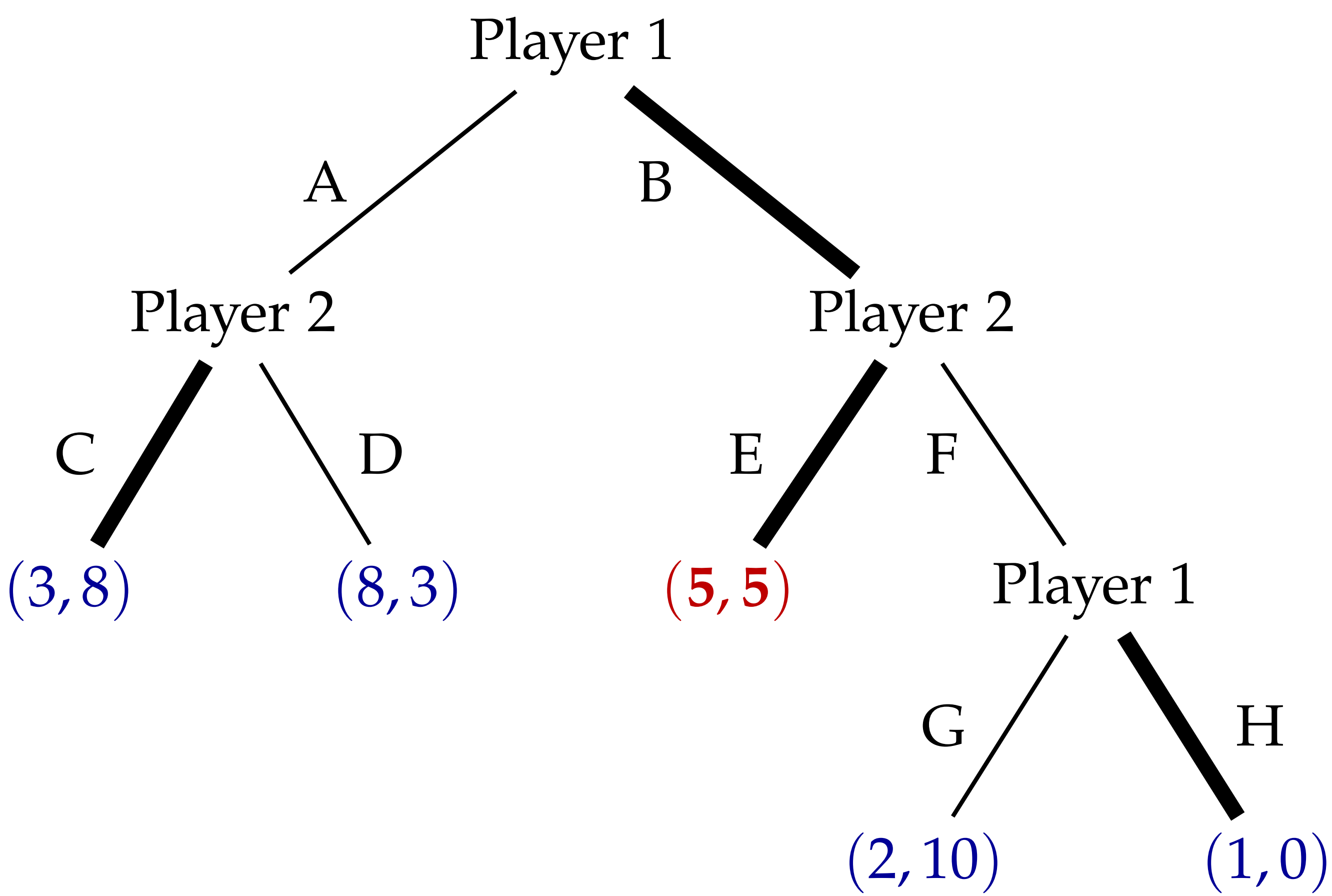


The action profile {(B,H), (C,E)} is also a Nash equilibrium.

Subgame-imperfect equilibrium



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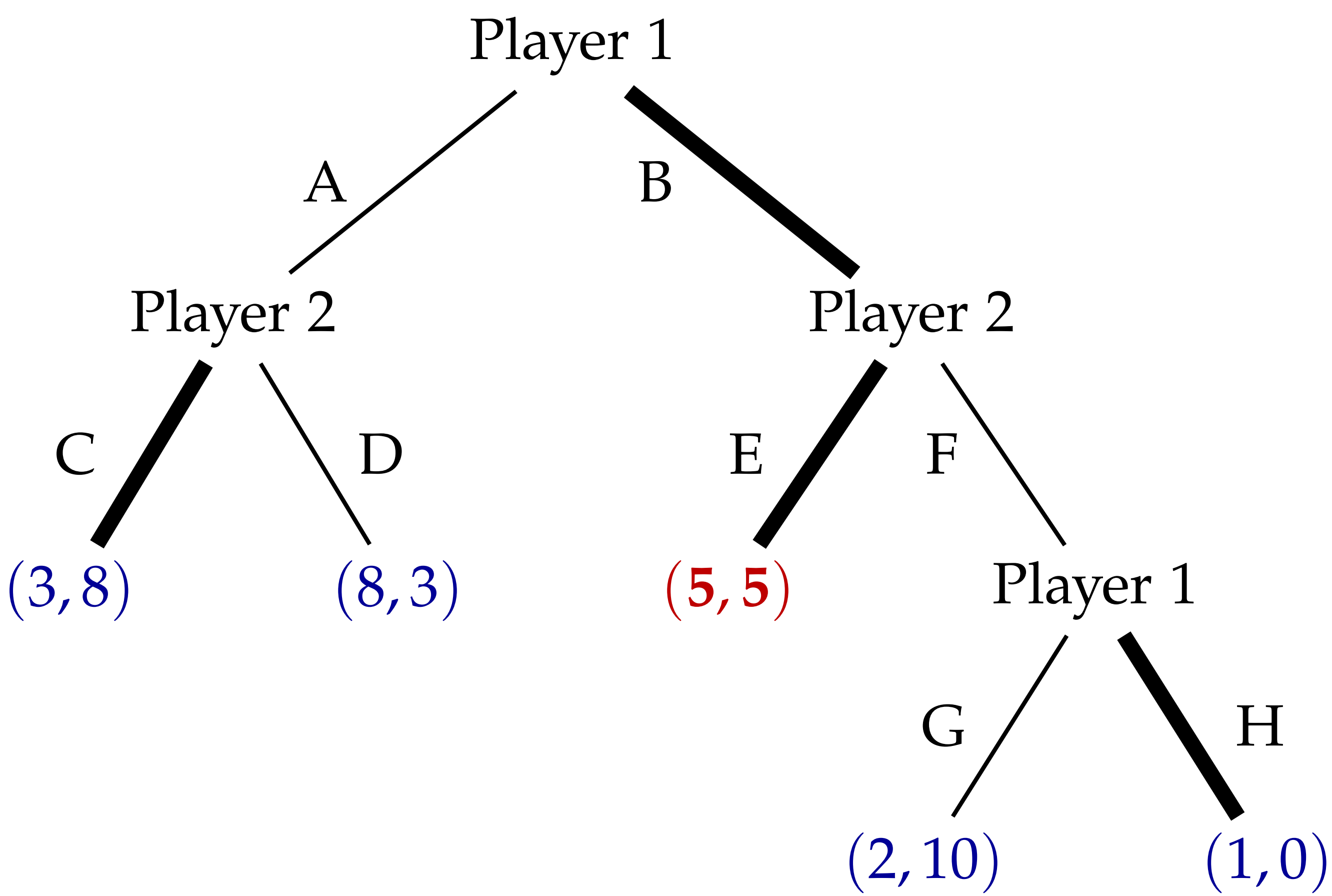

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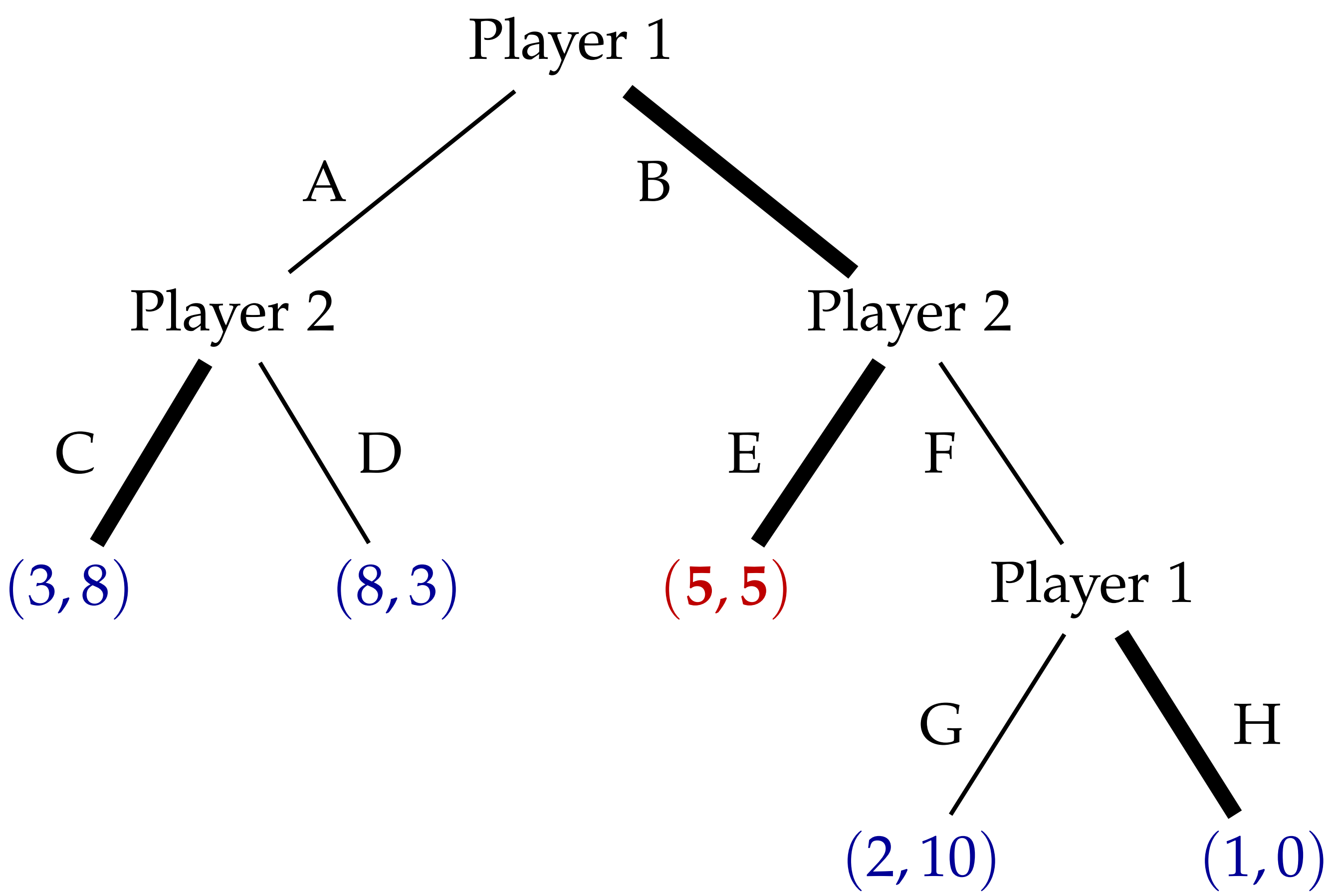

The action profile {(B,H), (C,E)} is a Nash equilibrium, but it does not induce a NE on all subgames.

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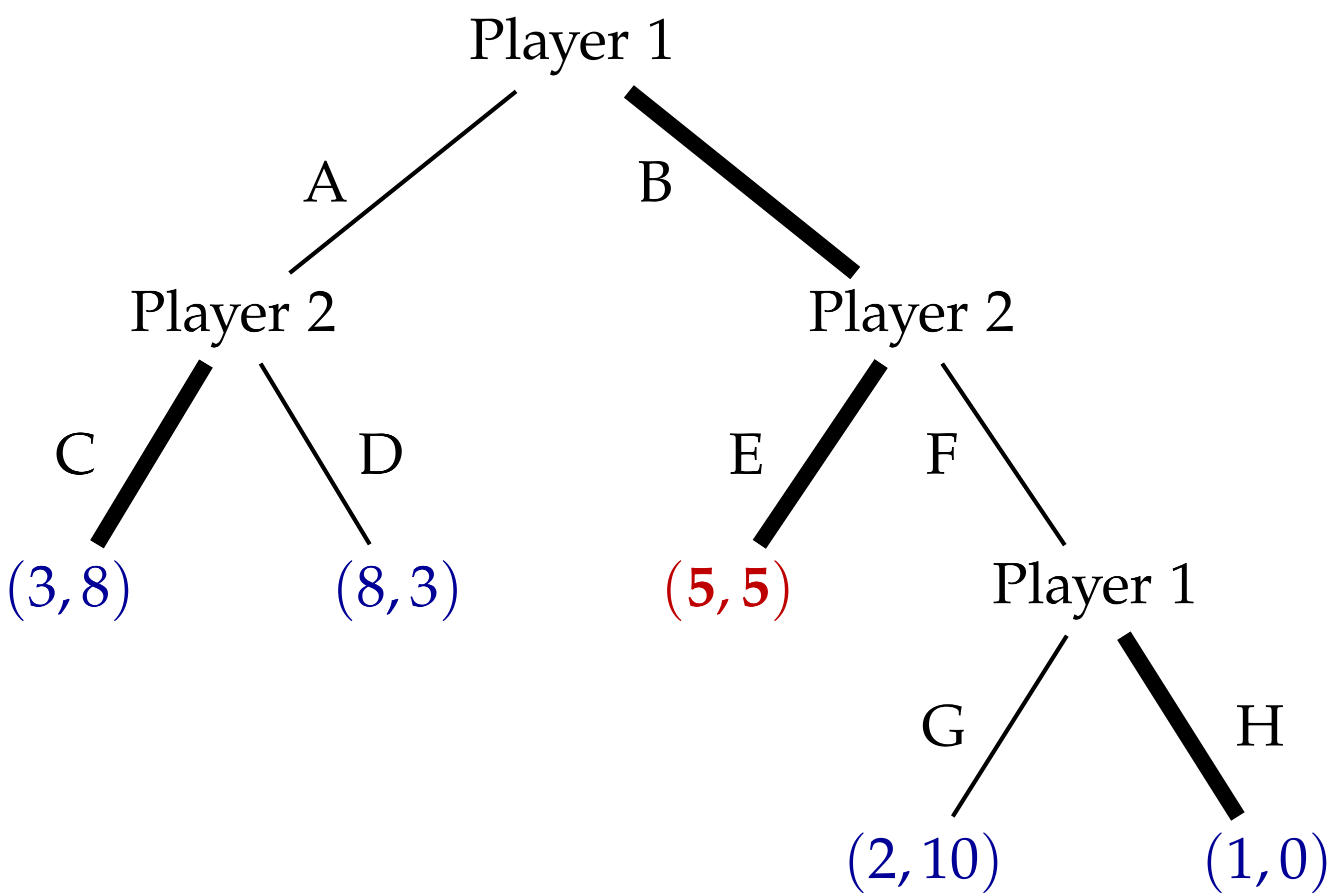

The action profile {(B,H), (C,E)} is a Nash equilibrium, but it does not induce a NE on all subgames. H is a non-credible threat.

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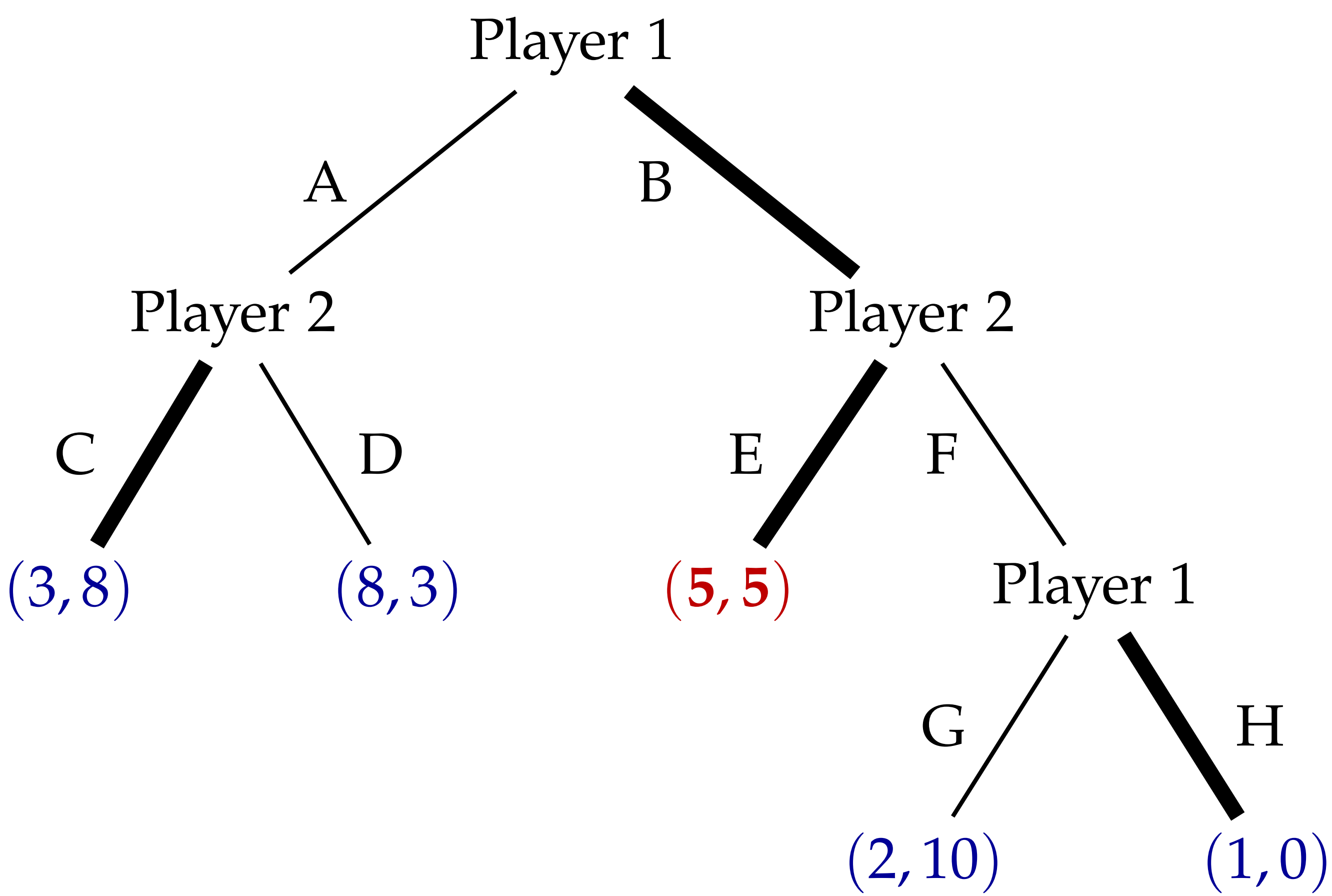

The action profile {(B,H), (C,E)} is a Nash equilibrium, but it does not induce a NE on all subgames. H is a non-credible threat. We have a subgame-imperfect Nash equilibrium.

From extensive form to normal form

Handy fact. Every extensive-form game can be put into normal-form with (evidently) identical pure and mixed Nash equilibria.

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	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	$3, 8$	$3, 8$	$8, 3$	$8, 3$
(A,H)	$3, 8$	$3, 8$	$8, 3$	$8, 3$
(B,G)	$5, 5$	$2, 10$	$5, 5$	$2, 10$
(B,H)	$5, 5$	$1, 0$	$5, 5$	$1, 0$

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	(C,E)	(C,F)	(D,E)	(D,F)
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(A,H)	3, 8	3, 8	8, 3	8, 3
(B,G)	5, 5	2, 10	5, 5	2, 10
(B,H)	5, 5	1, 0	5, 5	1, 0

- There are exactly three pure Nash equilibria.

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- There are exactly three pure Nash equilibria. All weak

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(B,H)	5, 5	1, 0	5, 5	1, 0

- There are exactly three pure Nash equilibria. All weak (= non-strict).

From extensive form to normal form

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	(C,E)	(C,F)	(D,E)	(D,F)
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(A,H)	3, 8	3, 8	8, 3	8, 3
(B,G)	5, 5	2, 10	5, 5	2, 10
(B,H)	5, 5	1, 0	5, 5	1, 0

- There are exactly three pure Nash equilibria. All weak (= non-strict).
- Exactly one pure NE is obtained by backward induction.

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	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3,8	3,8	8,3	8,3
(A,H)	3,8	3,8	8,3	8,3
(B,G)	5,5	2,10	5,5	2,10
(B,H)	5,5	1,0	5,5	1,0

- There are exactly three pure Nash equilibria. All weak (= non-strict).
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(B,G)	5, 5	2, 10	5, 5	2, 10
(B,H)	5, 5	1, 0	5, 5	1, 0

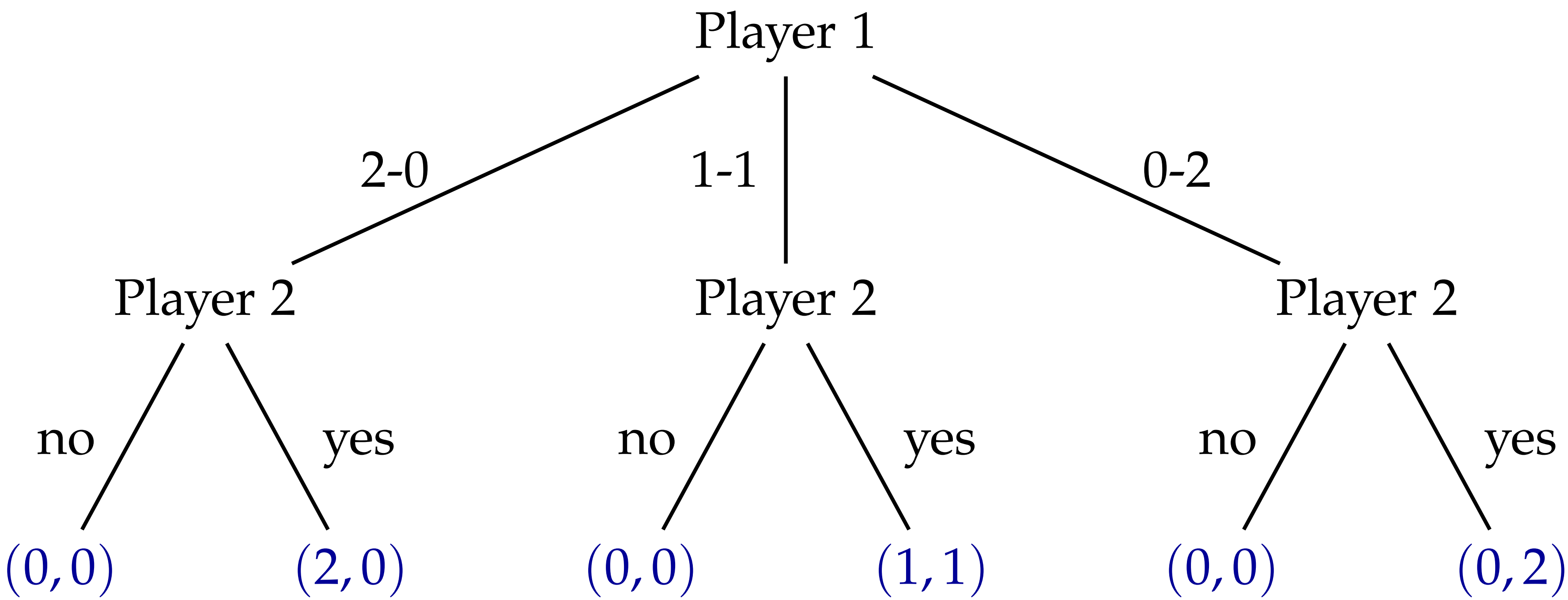
- There are exactly three pure Nash equilibria. All weak (= non-strict).
- Exactly one pure NE is obtained by backward induction.
- Exactly two pure NE are subgame-perfect.
- There are five more NE! (Found with `http://banach.lse.ac.uk/`.)

Games in extensive form

Example: the sharing game.

Games in extensive form

Example: the sharing game. Player 1 distributes two entities, then Player 2 accepts or not.



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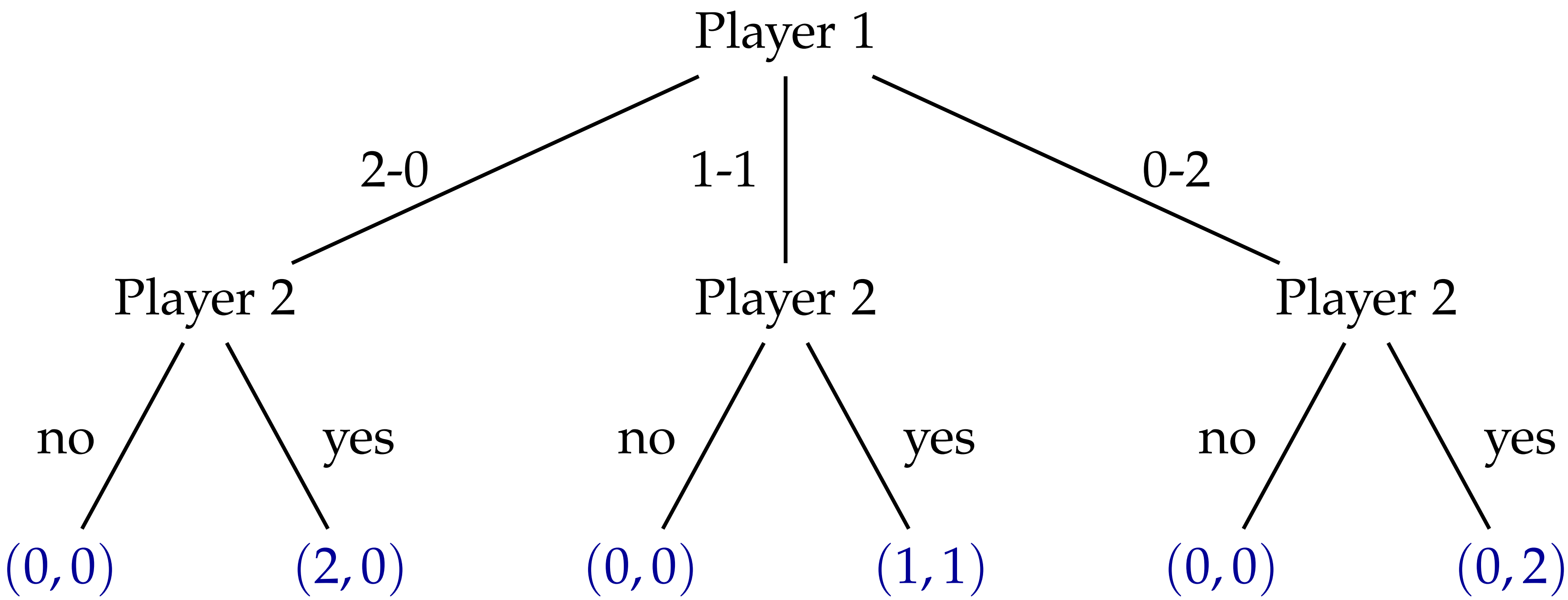
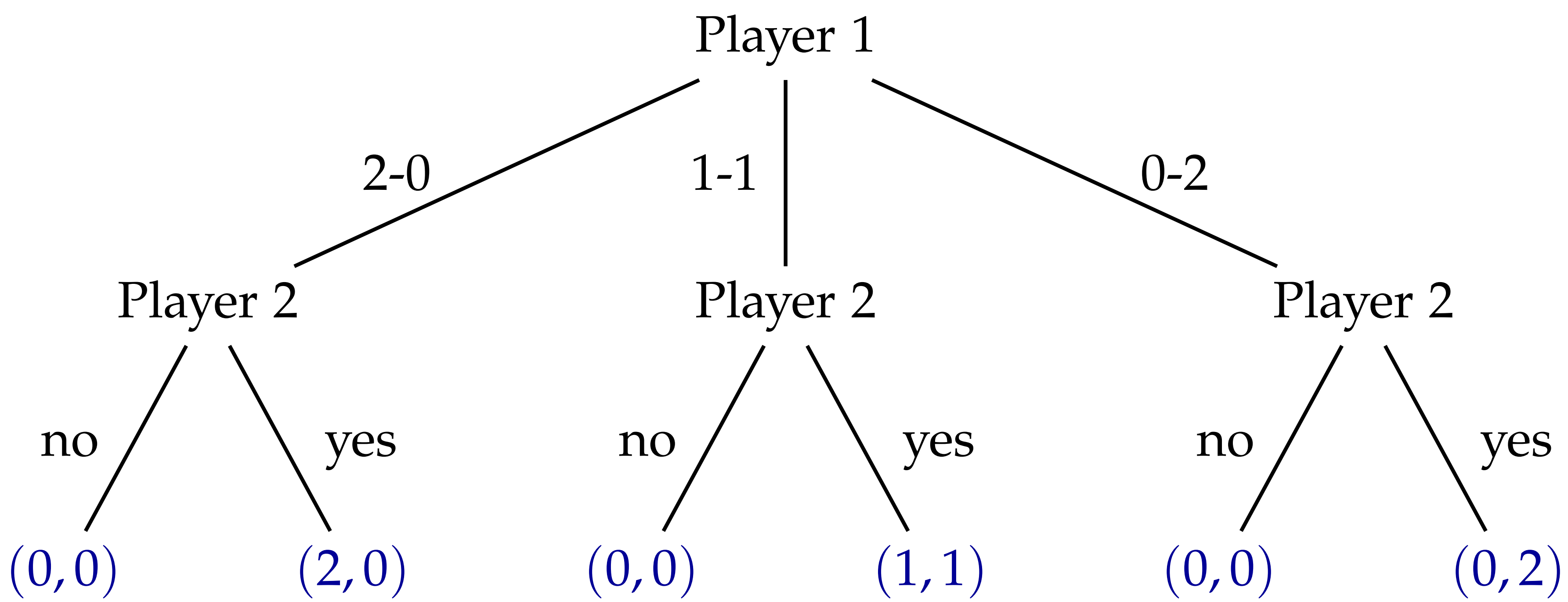


- A pure strategy for Player 1 could be: 1-1.

Games in extensive form

Example: the sharing game. Player 1 distributes two entities, then Player 2 accepts or not.



- A pure strategy for Player 1 could be: 1-1.
- A pure strategy for Player 2 could be: no, yes, yes.

From extensive form to normal form

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0, 0	0, 0	0, 0	0, 0	2, 0	2, 0	2, 0	2, 0
1-1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1
0-2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2

Pure Nash equilibria:

From extensive form to normal form

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0, 0	0, 0	0, 0	0, 0	2, 0	2, 0	2, 0	2, 0
1-1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1
0-2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2

Pure Nash equilibria:

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0, 0	0, 0	0, 0	0, 0	2, 0	2, 0	2, 0	2, 0
1-1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1
0-2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2

From extensive form to normal form

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0,0	0,0	0,0	0,0	2,0	2,0	2,0	2,0
1-1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1
0-2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2

Pure Nash equilibria:

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0,0	0,0	0,0	0,0	2,0	2,0	2,0	2,0
1-1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1
0-2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2

- Then there there are twelve (partially / fully) mixed equilibria.

From extensive form to normal form

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0,0	0,0	0,0	0,0	2,0	2,0	2,0	2,0
1-1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1
0-2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2

Pure Nash equilibria:

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	**0,0**	**0,0**	0,0	0,0	**2,0**	**2,0**	**2,0**	**2,0**
1-1	0,0	0,0	**1,1**	**1,1**	0,0	0,0	1,1	1,1
0-2	0,0	**0,2**	0,0	0,2	0,0	0,2	0,0	0,2

- Then there there are twelve (partially / fully) mixed equilibria.
- Some (not all) are obtained by backward induction.

From extensive form to normal form

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0,0	0,0	0,0	0,0	2,0	2,0	2,0	2,0
1-1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1
0-2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2

Pure Nash equilibria:

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0,0	0,0	0,0	0,0	2,0	2,0	2,0	2,0
1-1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1
0-2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2

- Then there there are twelve (partially / fully) mixed equilibria.
- Some (not all) are obtained by backward induction.
- Some (not all) are subgame-perfect.

From extensive form to normal form

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0, 0	0, 0	0, 0	0, 0	2, 0	2, 0	2, 0	2, 0
1-1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1
0-2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2

Pure Nash equilibria:

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0, 0	0, 0	0, 0	0, 0	2, 0	2, 0	2, 0	2, 0
1-1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1
0-2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2

- Then there there are twelve (partially / fully) mixed equilibria.
- Some (not all) are obtained by backward induction.
- Some (not all) are subgame-perfect.

Conclusion: extensive games allow for an embarrassing richness of NE.

Behavioural strategies

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0, 0	0, 0	0, 0	0, 0	2, 0	2, 0	2, 0	2, 0
1-1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1
0-2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2

Behavioural strategies

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0, 0	0, 0	0, 0	0, 0	2, 0	2, 0	2, 0	2, 0
1-1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1
0-2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2

- It is possible to search for all NE in the corresponding normal-form representation

Behavioural strategies

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0, 0	0, 0	0, 0	0, 0	2, 0	2, 0	2, 0	2, 0
1-1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1
0-2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2

- It is possible to search for all NE in the corresponding normal-form representation, but the actions in those strategies are correlated among nodes

Behavioural strategies

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0, 0	0, 0	0, 0	0, 0	2, 0	2, 0	2, 0	2, 0
1-1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1
0-2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2

- It is possible to search for all NE in the corresponding normal-form representation, but the actions in those strategies are correlated among nodes, which is a somewhat unnatural assumption.

Behavioural strategies

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0, 0	0, 0	0, 0	0, 0	2, 0	2, 0	2, 0	2, 0
1-1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1
0-2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2

- It is possible to search for all NE in the corresponding normal-form representation, but the actions in those strategies are correlated among nodes, which is a somewhat unnatural assumption.
- An alternative is work with so-called *behavioural strategies*.

Behavioural strategies

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0,0	0,0	0,0	0,0	2,0	2,0	2,0	2,0
1-1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1
0-2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2

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Behavioural strategy. A behavioural strategy for Player i puts a probability distribution on actions on all the nodes that i owns.

Behavioural strategies

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0, 0	0, 0	0, 0	0, 0	2, 0	2, 0	2, 0	2, 0
1-1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1
0-2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2

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Behavioural strategy. A behavioural strategy for Player i puts a probability distribution on actions on all the nodes that i owns.

- Pure strategy profiles and pure behavioural strategies coincide.

Behavioural strategies

	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
2-0	0,0	0,0	0,0	0,0	2,0	2,0	2,0	2,0
1-1	0,0	0,0	1,1	1,1	0,0	0,0	1,1	1,1
0-2	0,0	0,2	0,0	0,2	0,0	0,2	0,0	0,2

- It is possible to search for all NE in the corresponding normal-form representation, but the actions in those strategies are correlated among nodes, which is a somewhat unnatural assumption.
- An alternative is work with so-called *behavioural strategies*.

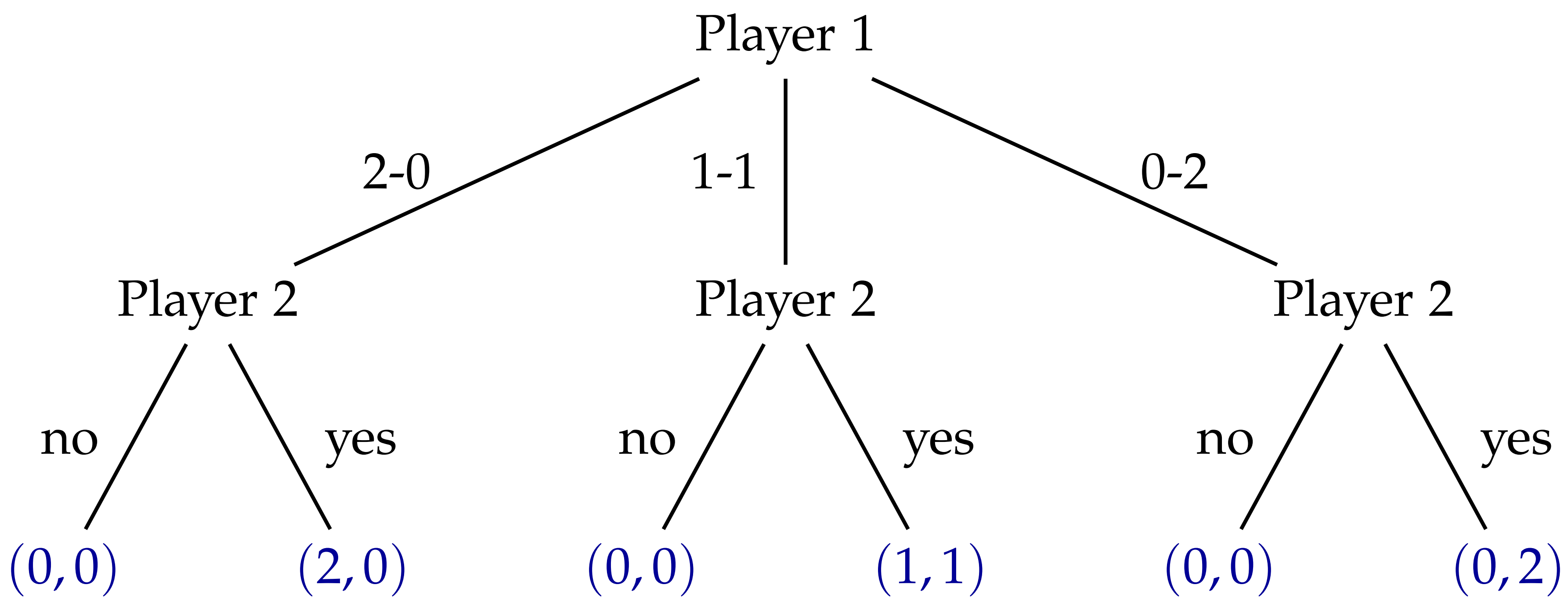
Behavioural strategy. A behavioural strategy for Player i puts a probability distribution on actions on all the nodes that i owns.

- Pure strategy profiles and pure behavioural strategies coincide.

  However, mixed strategy profiles $\neq$ mixed behavioural strategies.

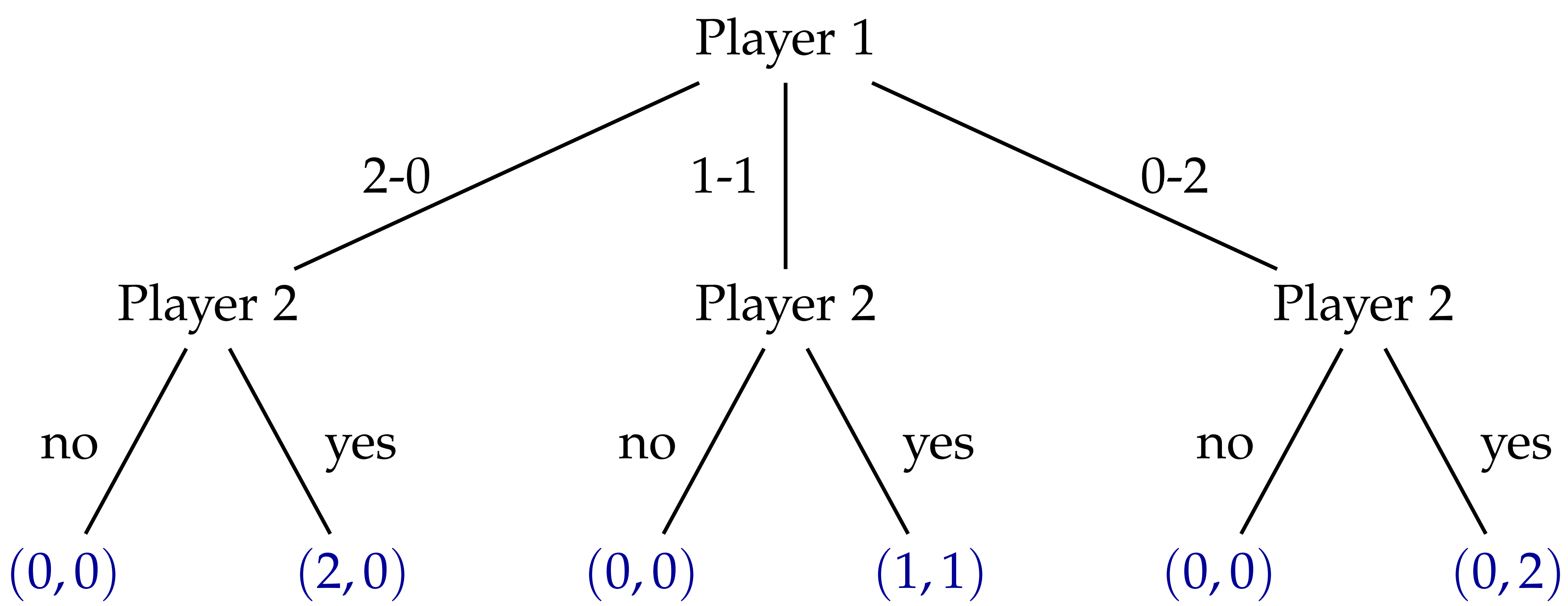
Behavioural strategies

Example: the sharing game.



Behavioural strategies

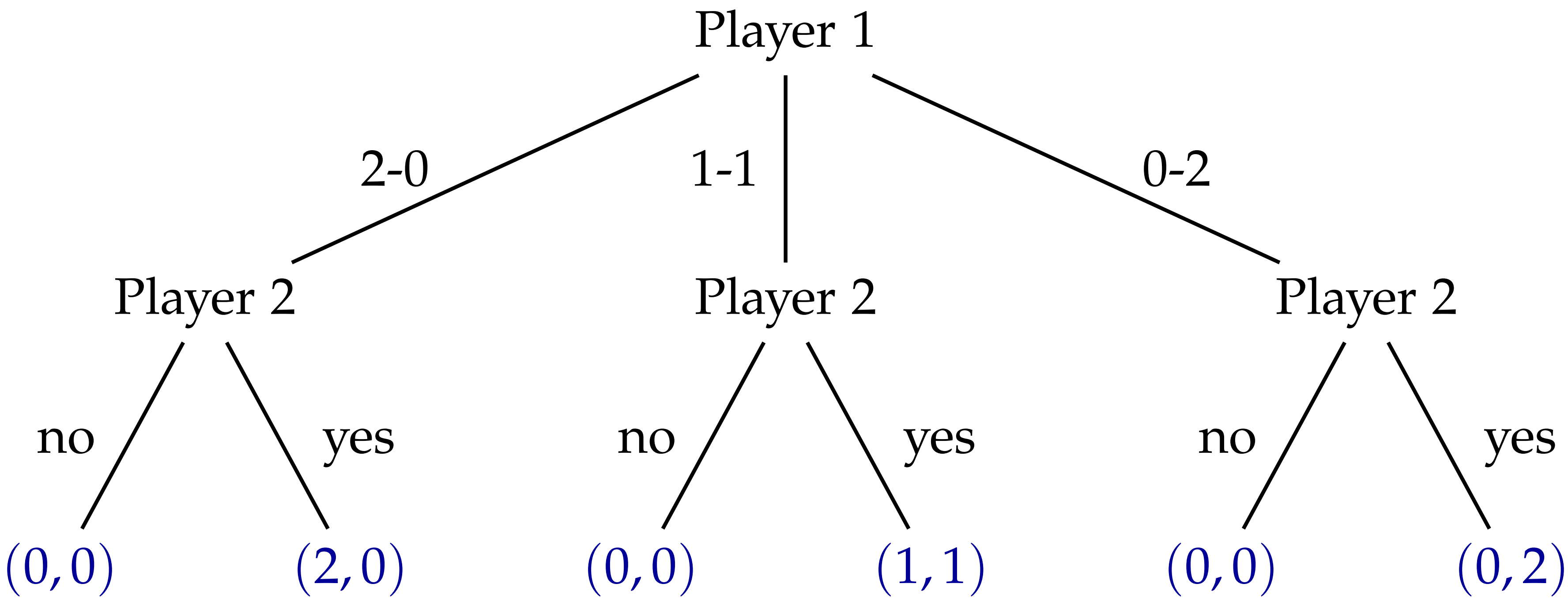
Example: the sharing game.



- A behavioural strategy for Player 1 could be: $(0.2, 0.3, 0.5)$.

Behavioural strategies

Example: the sharing game.



- A behavioural strategy for Player 1 could be: $(0.2, 0.3, 0.5)$.
- A behavioural strategy for Player 2 could be: $(0.4, 0.6)$, $(0.7, 0.3)$, $(0.1, 0.9)$.

Kuhn's theorem

Theorem (Kuhn, 1953). In an extensive-form game with perfect information,

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Remarks:

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Remarks:

1. Two strategies are considered equivalent if they induce the same probabilities on outcomes, for every fixed counter strategy profile.

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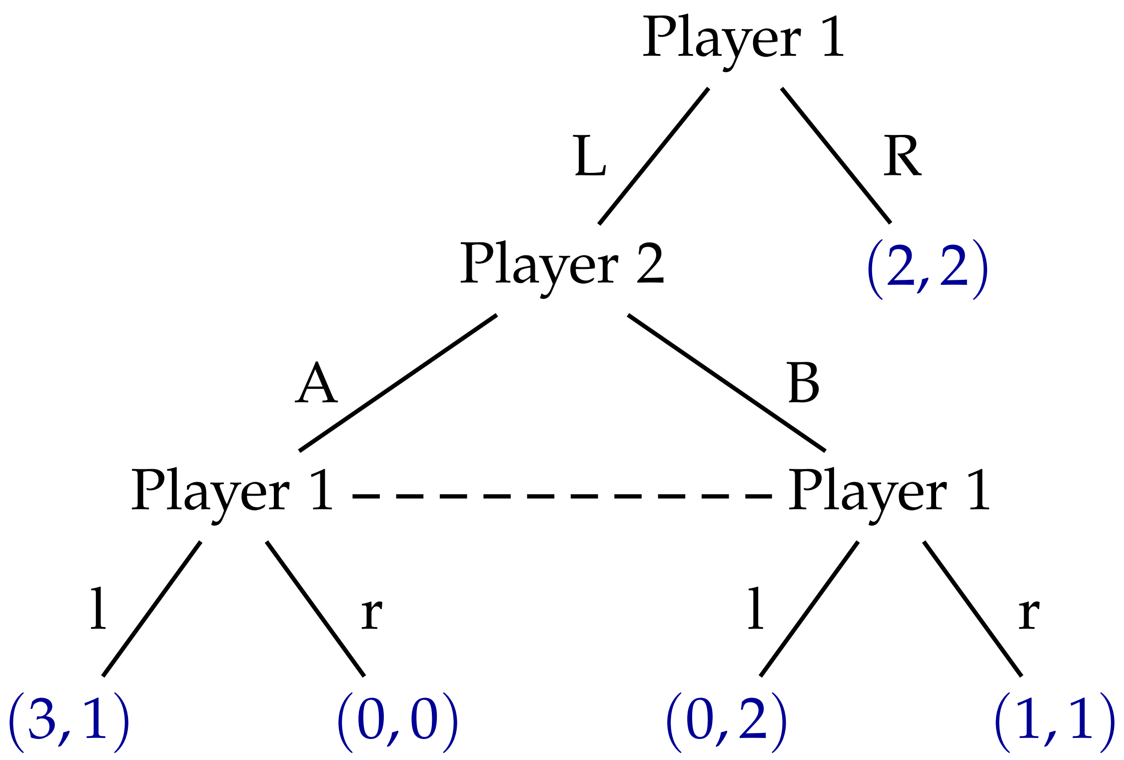
Remarks:

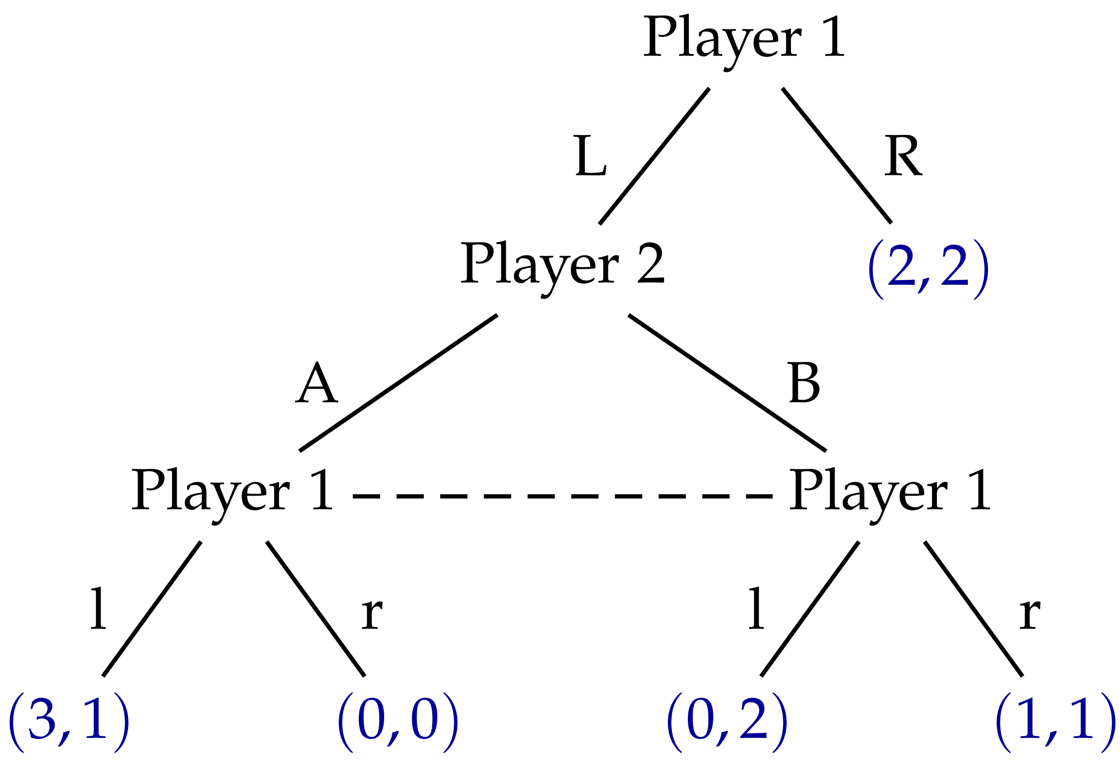
1. Two strategies are considered equivalent if they induce the same probabilities on outcomes, for every fixed counter strategy profile.
2. Induces the same equilibria.

Imperfect information games

An imperfect information game

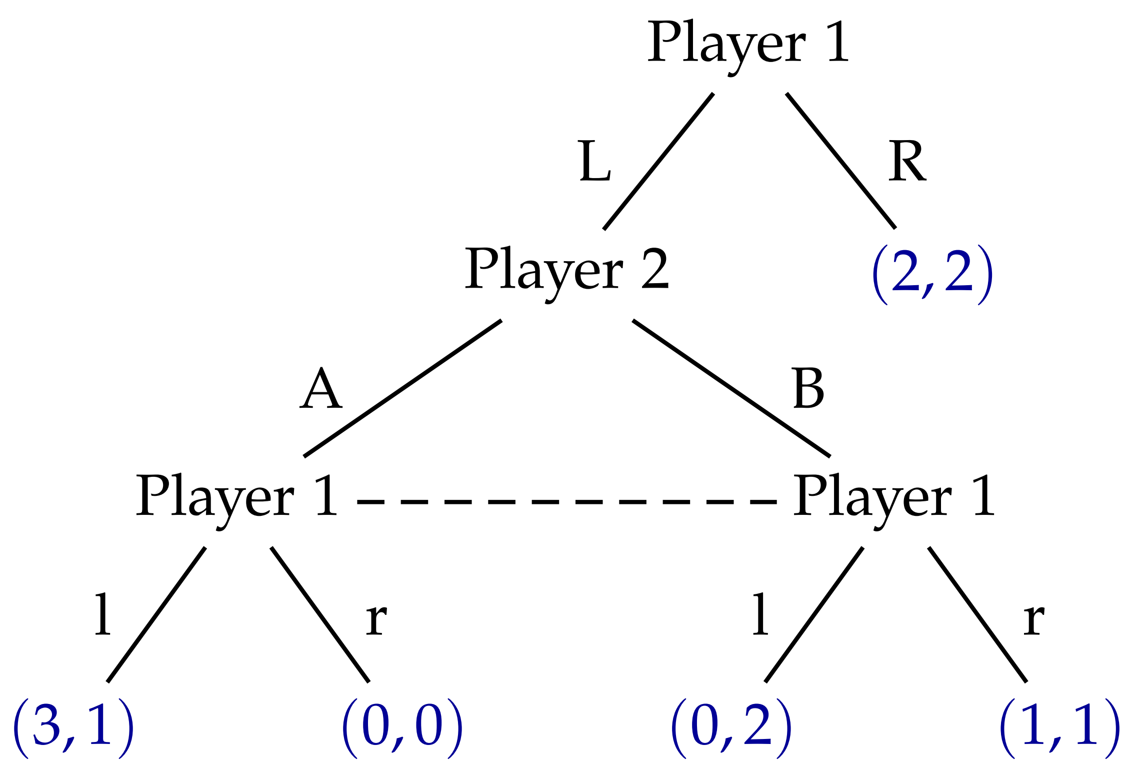


An imperfect information game



	A	B
Ll	3, 1	0, 2
Lr	0, 0	1, 1
Rl	2, 2	**2, 2**
Rr	2, 2	**2, 2**

An imperfect information game



	A	B
Ll	3, 1	0, 2
Lr	0, 0	1, 1
Rl	2, 2	**2, 2**
Rr	2, 2	**2, 2**

The Nash equilibrium concept (both pure and mixed) remains the same for imperfect-information extensive-form games.

The prisoner's dilemma

Exercise: represent the prisoner's dilemma as an imperfect information game.

The prisoner's dilemma

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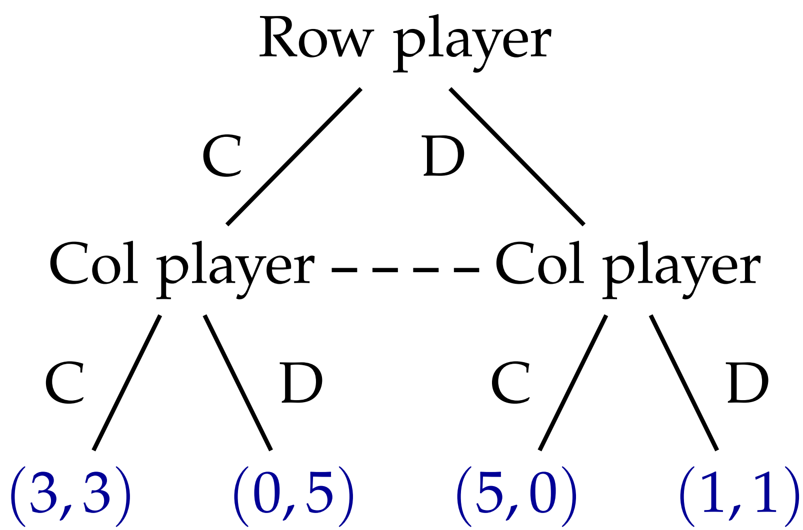
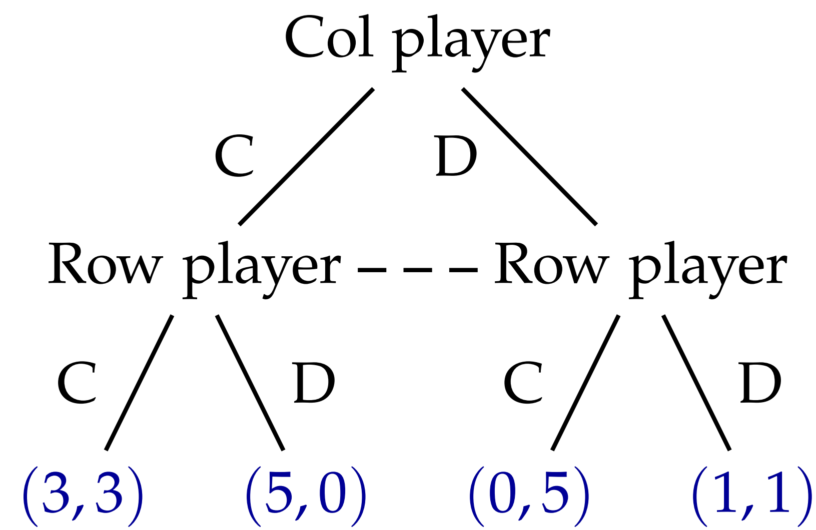
Solution:



or:



Mixed strategies differ from behavioural strategies

Player 1 has bad memory. He must makes two decisions: whether to **C**heck his front door after leaving the house, whether to **R**e-check his front door.

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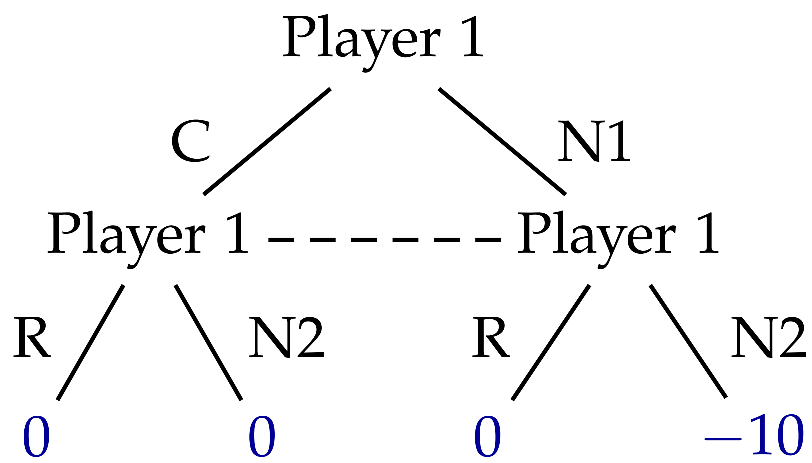

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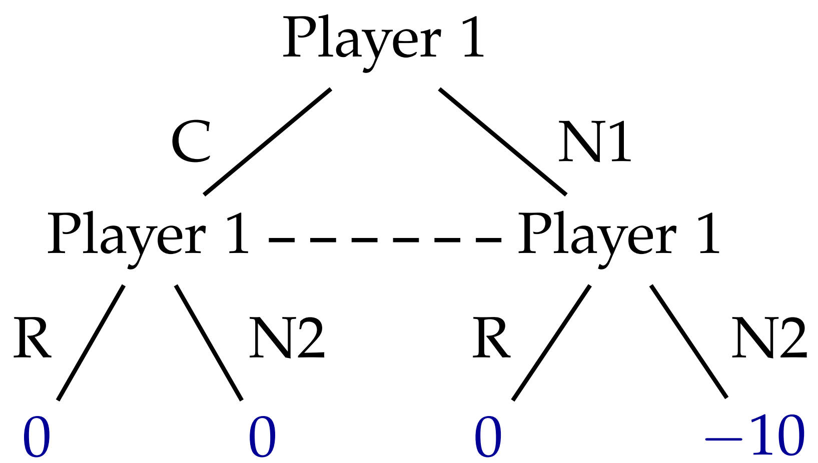

Pure (mixed and behavioural) strategies: CR, CN2, N1R, N1N2.

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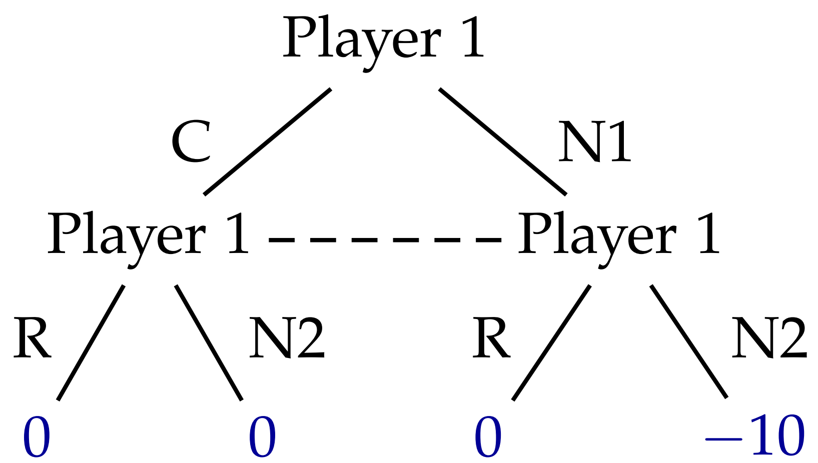

Pure (mixed and behavioural) strategies: CR, CN2, N1R, N1N2. Suppose we want a truly mixed strategy that maximises expected utility:

Mixed strategies differ from behavioural strategies

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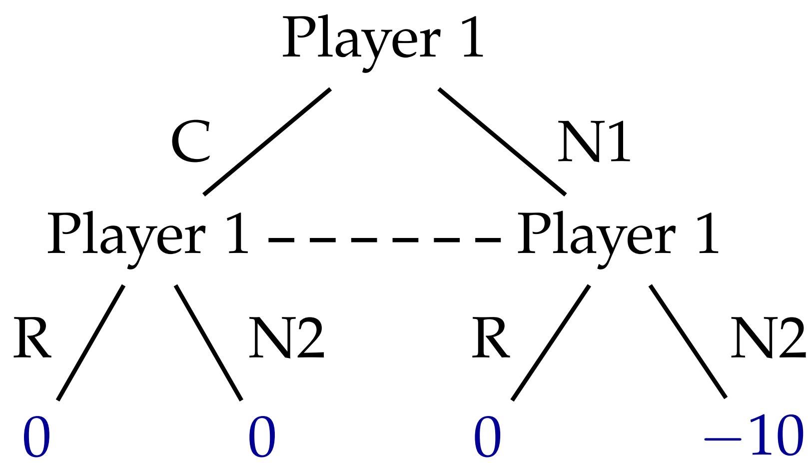

Pure (mixed and behavioural) strategies: CR, CN2, N1R, N1N2. Suppose we want a truly mixed strategy that maximises expected utility:

- **Mixed strategy**. For any $0 < p < 1$ the mix $(p, 1-p, 0, 0)$ does the job: $\text{EU} = p \cdot 0 + (1-p) \cdot 0 = 0$.

Mixed strategies differ from behavioural strategies

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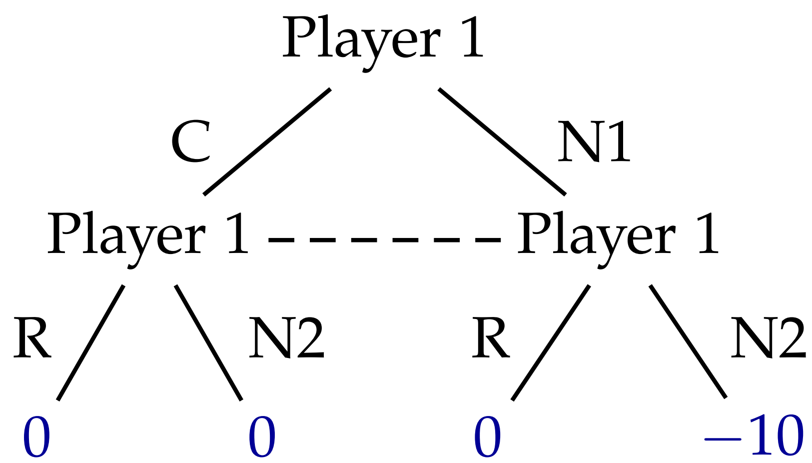

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- **Behavioural strategy**. Check with prob p and re-check second time with prob q.

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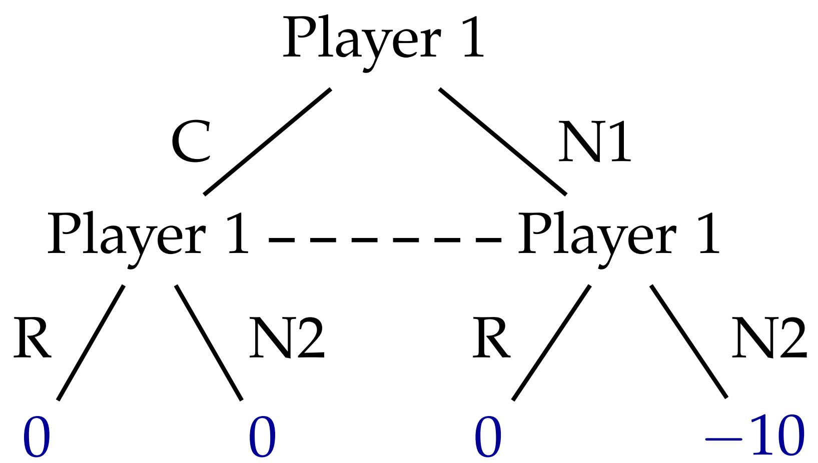

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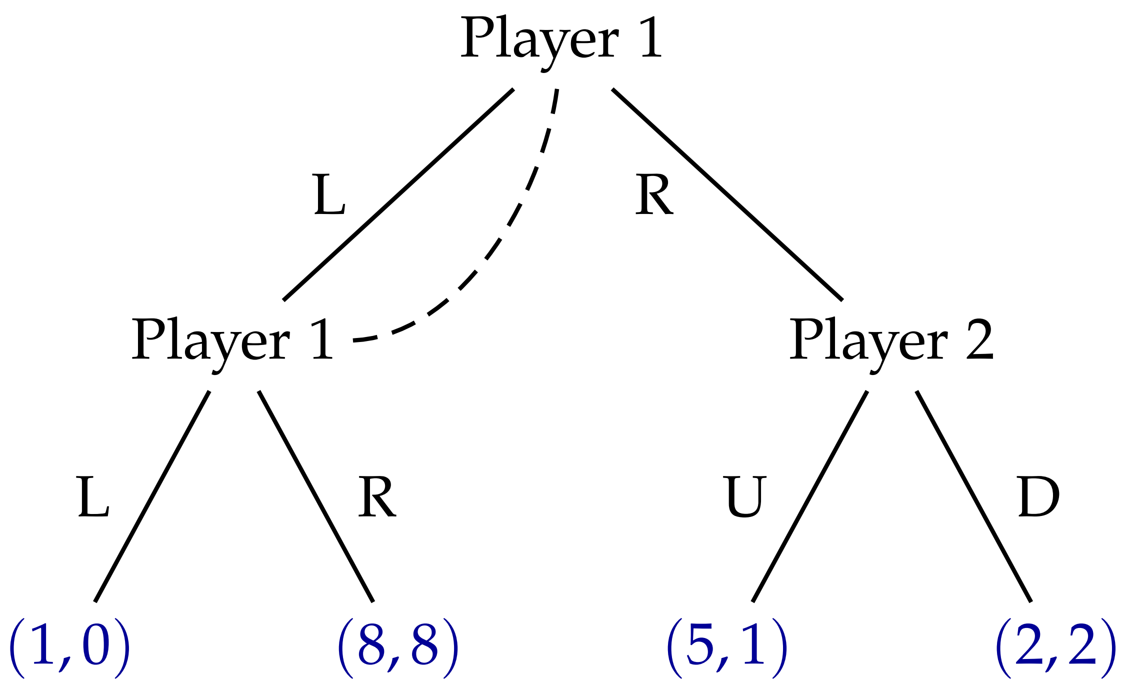
- **Mixed strategy**. For any $0 < p < 1$ the mix $(p, 1 - p, 0, 0)$ does the job: $\mathrm{EU} = p \cdot 0 + (1 - p) \cdot 0 = 0$.
- **Behavioural strategy**. Check with prob p and re-check second time with prob q. For $0 < p, q < 1$ we have $\mathrm{EU} = (1 - p)(1 - q) \cdot -10 < 0$.

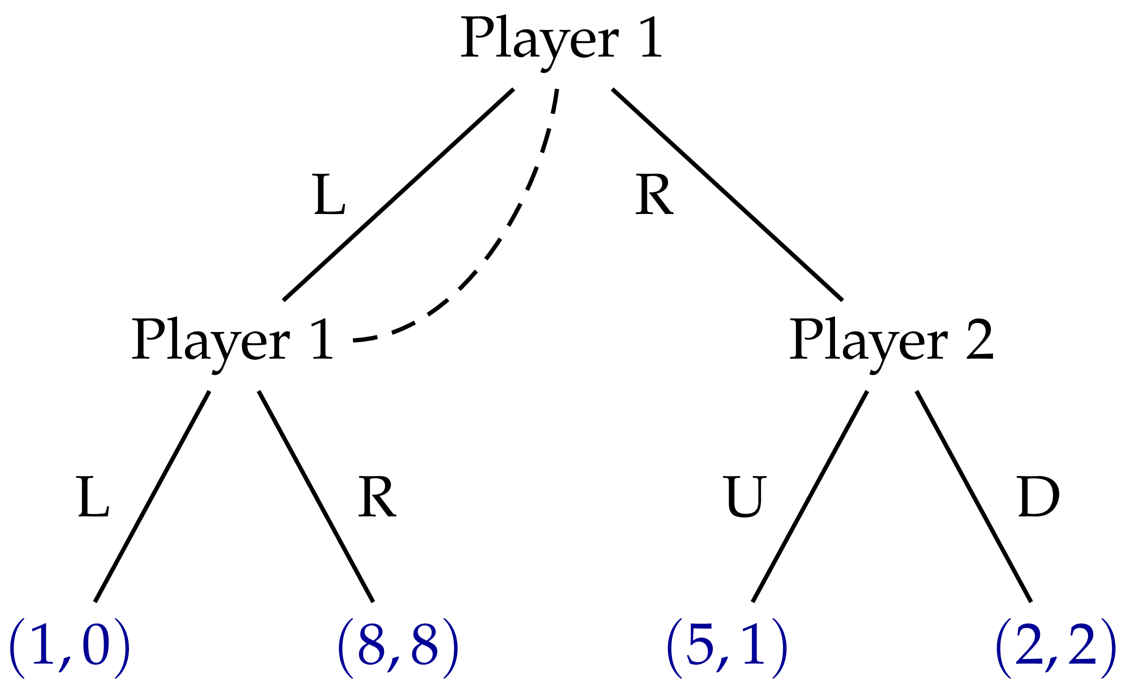
A game with imperfect recall



	L	R
U	1, 0	5, 1
D	1, 0	2, 2

Considering mixed strategies:

A game with imperfect recall



	L	R
U	1,0	5,1
D	1,0	2,2

Considering mixed strategies:

- R is the dominant strategy for Player 1.

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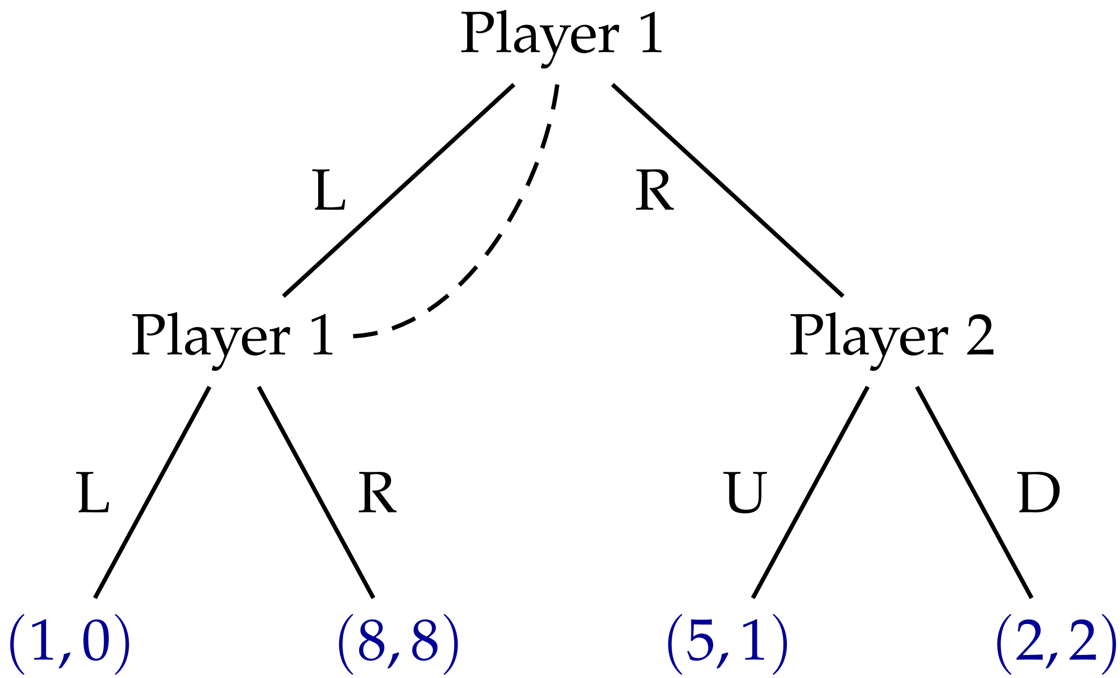
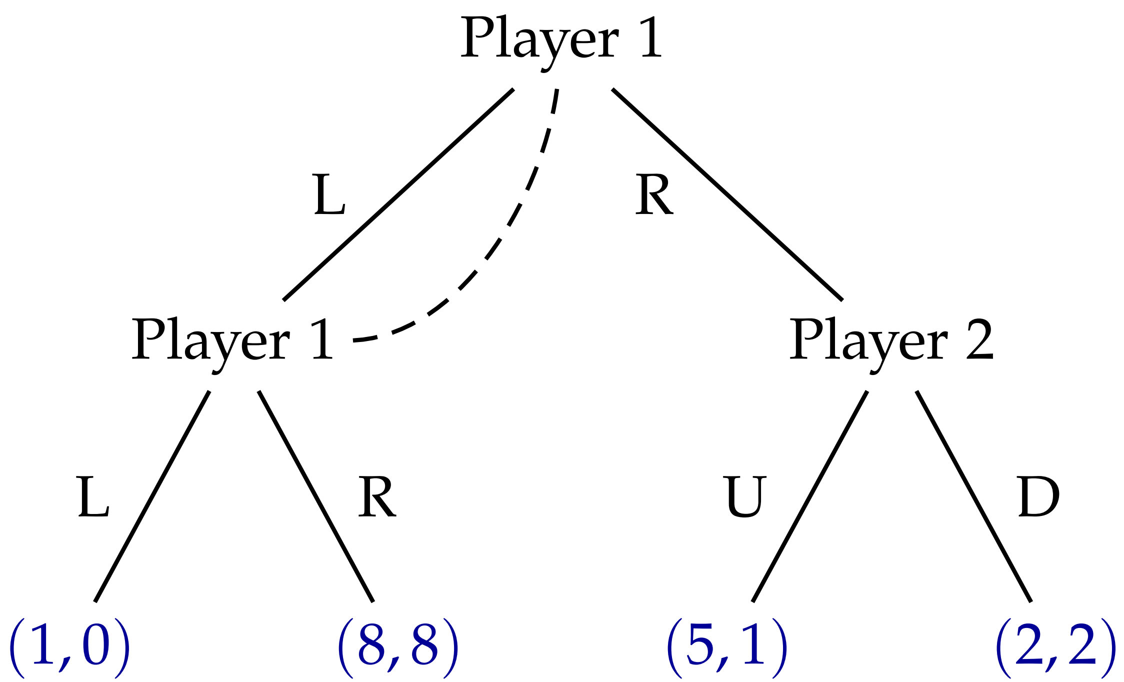


	L	R
U	1,0	5,1
D	1,0	2,2

Considering mixed strategies:

- R is the dominant strategy for Player 1.
- D is Player 2's best response.

A game with imperfect recall

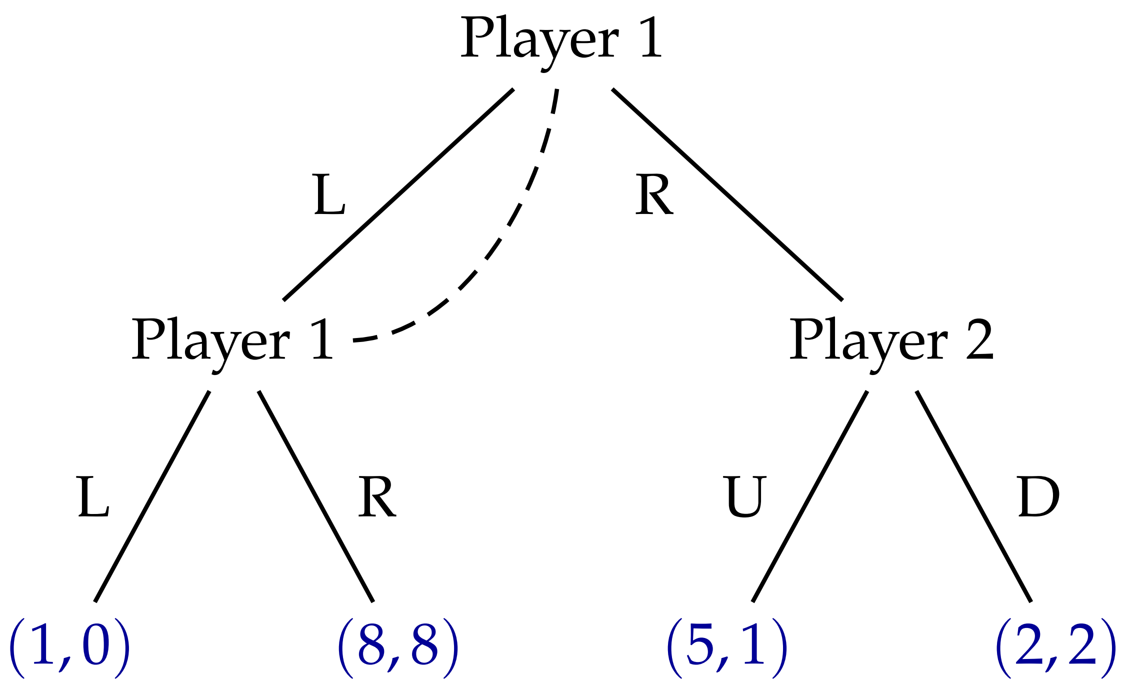


	L	R
U	1, 0	5, 1
D	1, 0	2, 2

Considering mixed strategies:

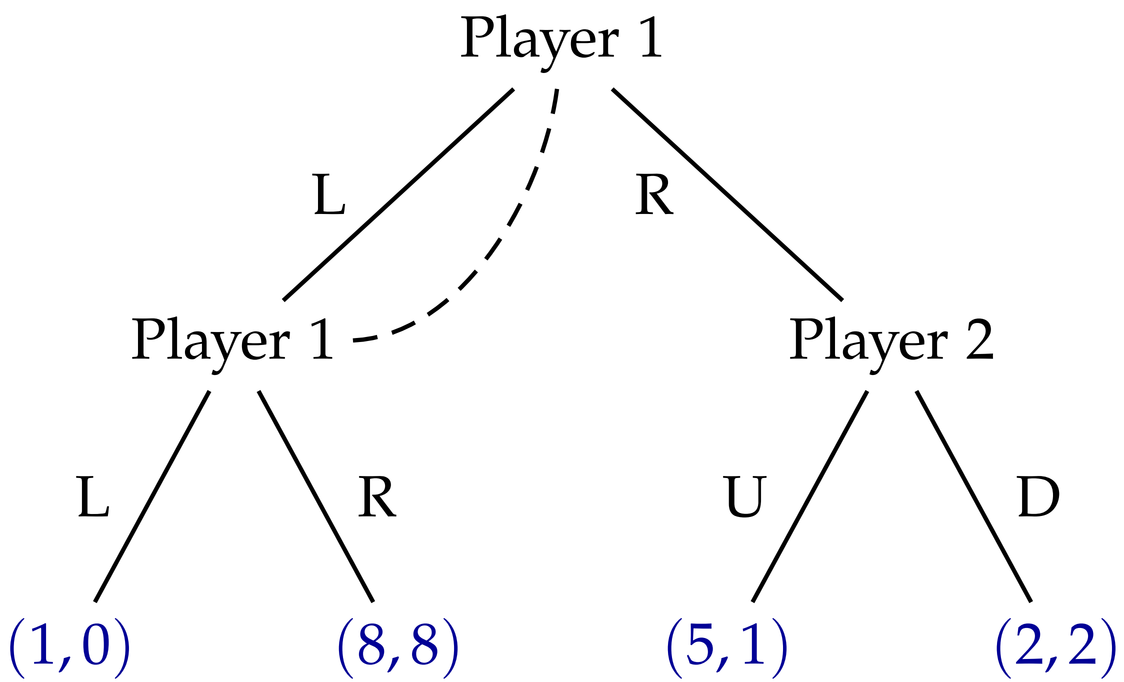
- R is the dominant strategy for Player 1.
- D is Player 2's best response.
- So (R, D) is the unique Nash equilibrium.

A game with imperfect recall



Considering behavioural strategies, suppose Player 1 chooses L with probability p.

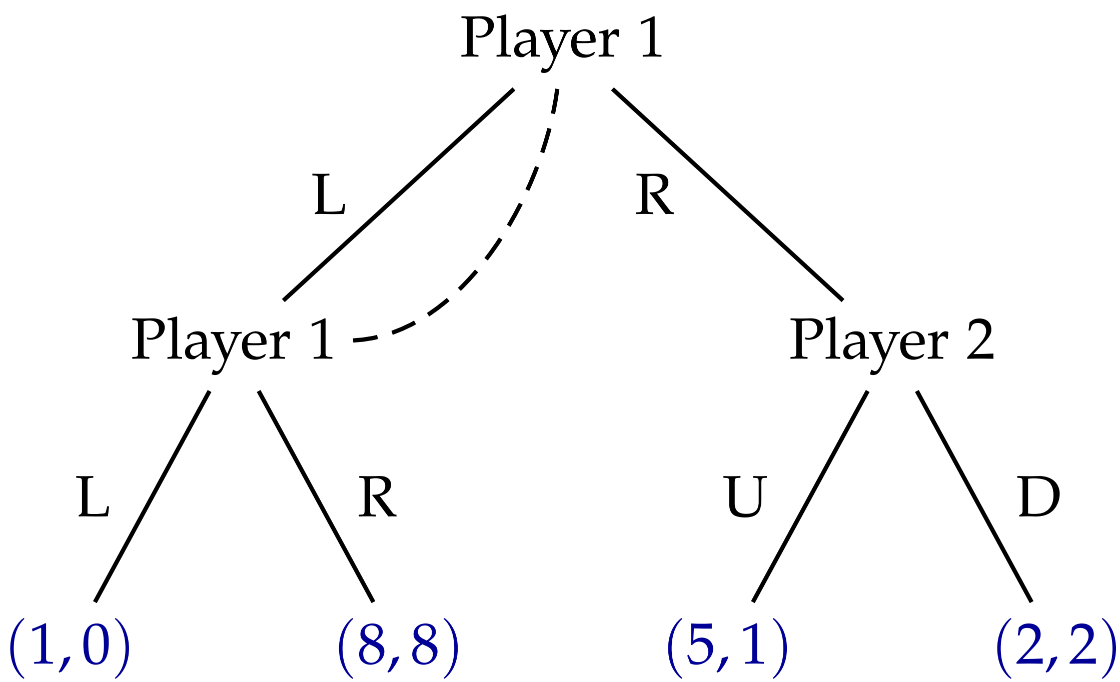
A game with imperfect recall



Considering behavioural strategies, suppose Player 1 chooses L with probability p.

- D is weakly dominant for Player 2 (c.f. $p = 1$).

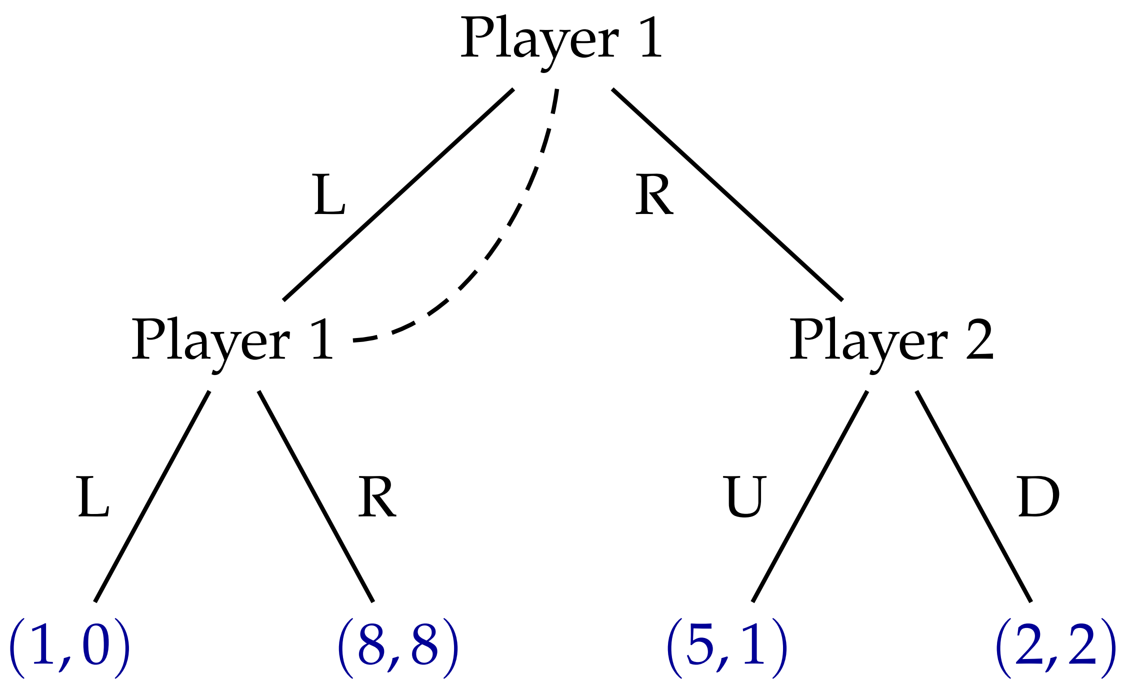
A game with imperfect recall



Considering behavioural strategies, suppose Player 1 chooses L with probability p.

- D is weakly dominant for Player 2 (c.f. $p = 1$).
- Player 1's expected payoff: $1p^2 + 8p(1-p) + 2(1-p)$.

A game with imperfect recall



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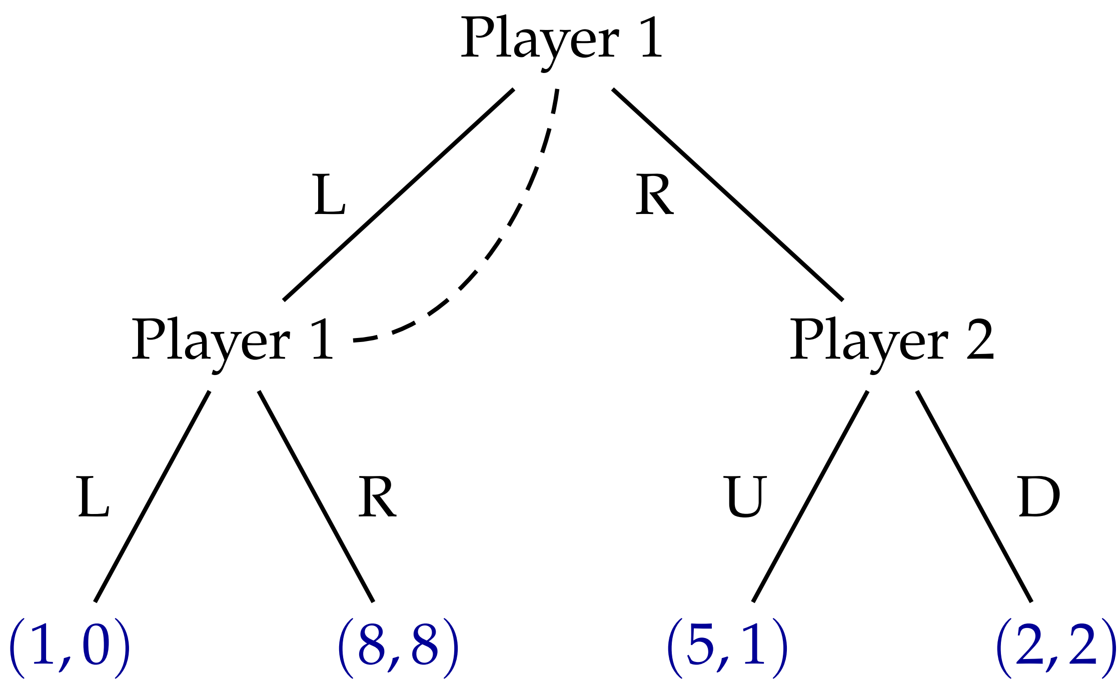
A game with imperfect recall



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- So $((3/7, 4/7), (0, 1))$ is the unique behavioural equilibrium.

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2. The translation works because conditional probabilities depend on information sets and not on the particular moves therein.

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- Learning in games $\sim$ to adapt strategies in time $\sim$ multi-agent learning.