Exercises Reasoning Part MAIR (extended, with solutions)

1. Consider the following statement.

   (a) For all sentences α and β: if $\models \alpha \vee \beta$, then $\models \alpha$ or $\models \beta$.

   (b) For all sentences α and β: if $\neg\alpha \models \beta$, then $\models \alpha \vee \beta$.

   Are these statements true? (Justify your answer.)

   Answer:

   (a) This statement is not true. Consider $\alpha = p$, $\beta = \neg p$. Then $\models \alpha \vee \beta$ (i.e., $\alpha \vee \beta$ is *valid*), but $\not\models \alpha$ and $\not\models \beta$

   (b) This statement is true. LetMod be the set of all the models of the logic, and let $M(F)$ denotes the set of models of F. Then $\neg\alpha \models \beta$ means $M(\neg\alpha) \subseteq M(\beta)$, so $Mod \setminus M(\alpha) \subseteq M(\beta)$ (because $M(\neg\alpha) = Mod \setminus M(\alpha)$). Adding the elements of $M(\alpha)$ to both sides of $\subseteq$, we obtain $Mod \subseteq M(\alpha) \cup M(\beta)$, i.e., $Mod \subseteq M(\alpha \vee \beta)$. Thus, every model is a model of $\alpha \vee \beta$, so $\models \alpha \vee \beta$. (This is not a candidate for an exam question)

2. Write a sentence of FOL whose models are:

   (a) all first-order models with *at most* two elements in the domain

   (b) all first-order models with *at least* two elements in the domain

   (c) all first-order models with *exactly* two elements in the domain.

   Answer:

   (a) $\forall x \forall y \forall z (x = y \vee x = z \vee y = z)$

   (b) $\exists x \exists y \neg(x = y)$

   (c) Conjunction of the formulas from (a) and (b)
   or
   $\exists x \exists y (\neg(x = y) \wedge \forall z (x = z \vee y = z))$

3. Using unary predicate symbols *Student*, *Green* and *Bicycle* ($Student(x)$ stands for "x is a student", $Green(x)$ for "x is green" and $Bicycle(x)$ stands for "x is a bicycle"), and a binary predicate symbol Has ($Has(x, y)$ stands for "x has y" - i.e., "x owns y"), translate the following sentences from English into first order logic.

   (a) Every bicycle is green.

   (b) Every student has a green bicycle.

   Answer:

   (a) $\forall x (Bicycle(x) \rightarrow Green(x)$

(b) $\forall x(Student(x) \to \exists y(Bicycle(y) \wedge Green(y) \wedge Has(x, y))$

4. Express the following sentences in first order logic using predicate symbols $Student$ (unary, $Student(a)$ means a is a student), $Tutor$ (binary, $Tutor(b, a)$ means b is a's tutor), $Lazy$ (unary), $Happy$ (unary):

   (a) Every student has a tutor.

   (b) There are no lazy students.

   (c) No student has two different tutors.

   (d) If a student is lazy, then the student's tutor is not happy.

   (e) There is a tutor all of whose tutees are lazy.

   Answer:

   (a) $\forall x(Student(x) \to \exists y Tutor(y, x))$
   or
   $\forall x \exists y(Student(x) \to Tutor(y, x))$

   (b) $\neg \exists x(Lazy(x) \wedge Student(x))$
   or
   $\forall x(Student(x) \to \neg Lazy(x))$

   (c) $\neg \exists x(Student(x) \wedge \exists y \exists z(\neg(y = z) \wedge Tutor(y, x) \wedge Tutor(z, x)))$
   or
   $\forall x \forall y \forall z(Student(x) \wedge Tutor(y, x) \wedge Tutor(z, x) \to (y = z))$

   (d) $\forall x \forall y(Student(x) \wedge Lazy(x) \wedge Tutor(y, x) \to \neg Happy(y))$

   (e) $\exists x \forall y(Tutor(x, y) \to Lazy(y))$

5. Consider an interpretation where the domain consists of 4 suitcases a, b, c, d where a and b are large and c and d are small. In other words, the predicate symbol $Large$ is interpreted as the set $\{a, b\}$ and $Small$ is interpreted as the set $\{c, d\}$. There is also a predicate symbol $FitsIn$ that is interpreted as the set of pairs $\{(c, a), (c, b), (d, a), (d, b)\}$ (small suitcases fit inside large ones). Are the following first order sentences true or false in this interpretation (and why):

   (a) $\forall x \forall y(Large(x) \wedge Small(y) \to FitsIn(x, y))$

   (b) $\forall x \forall y(Large(x) \wedge Small(y) \to FitsIn(y, x))$

   (c) $\exists x \forall y FitsIn(x, y)$

   (d) $\forall x \exists y \neg FitsIn(x, y)$

   (e) $\forall x \forall y(\neg FitsIn(x, y) \vee \neg FitsIn(y, x))$

   Answer:

   (a) False, because if x is large, y is small, then $FitsIn(x, y)$ requires that x fits into y, and this is false for all such pairs of values for x, y.

(b) True, because for any pair of values for x and y, if x is large and y is small, then y fits in x.

(c) False: there is no value for x such as $FitsIn(x, y)$ is true for all y. For example, no suitcase fits into itself, so the same value for x as for y constitutes a counterexample.

(d) True for the same reason as above is false.

(e) True: if x is a large suitcase, the $\neg FitsIn(x, y)$ is true for all y; if x is a small suitcase, $\neg FitsIn(y, x)$ is true for all y.

6. Which class of human beings is described by the following concept?

$$\neg\forall child.Male \sqcap \neg\forall child.Female$$

Answer: The human beings that have at least one daughter and at least one son.

7. Translate the concept in the previous exercise in FOL.

Answer:

$$(\neg\forall y(child(x, y) \rightarrow Male(y)) \wedge (\neg\forall y(child(x, y) \rightarrow Female(y)).$$

8. Translate the following FOL sentence in DL and describe the concept in natural language.

$$\exists y(neighbour(x, y) \wedge Old(y)) \wedge \forall y(neighbour(x, y) \rightarrow Friendly(y)).$$

Answer:

$$\exists \mathsf{neighbour.Old} \sqcap \forall \mathsf{neighbour.Friendly}$$

The concept of "those who have at least one old neighbour and only have friendly neighbours".

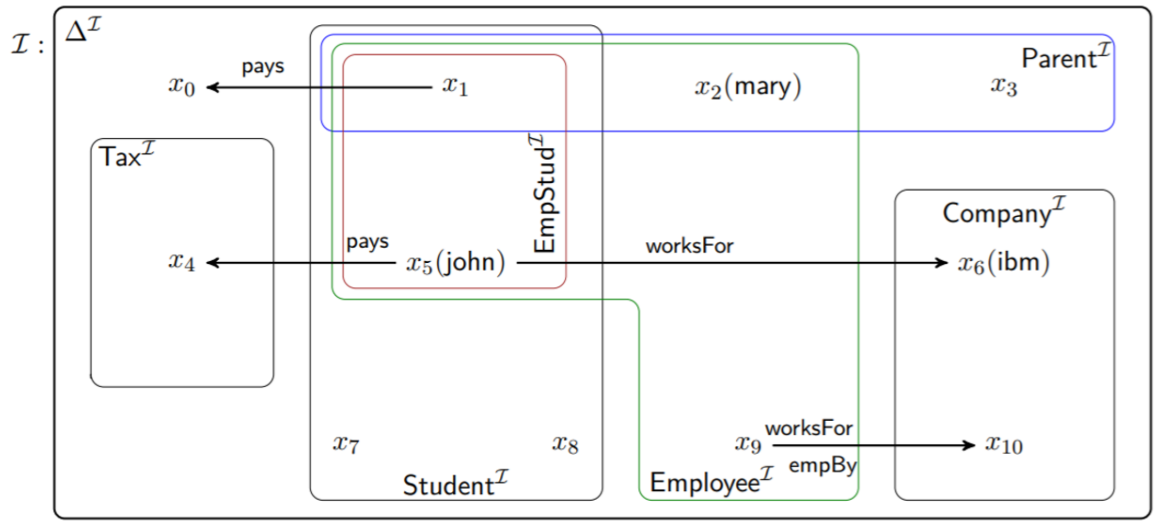
9. Translate the following FOL sentence in DL and describe the concept in natural language.

$$\exists y\big(neighbour(x, y) \wedge \forall z(neighbour(y, z) \rightarrow \neg Friendly(z))\big).$$

Answer:

$$\exists neighbour.\forall neighbour.\neg Friendly$$

The concept of "those who have a neighbour that only has unfriendly neighbours".



10. For the interpretation $\mathcal{I}$ from the picture, determine:

(a) $(\neg\mathsf{Employee})^{\mathcal{I}}$

(b) $(\neg\mathsf{EmpStud} \sqcap \forall\mathsf{empBy}.\mathsf{Company})^{\mathcal{I}}$

(c) $(\mathsf{Student} \sqcap \forall\mathsf{pays}.\bot)^{\mathcal{I}}$

Answer:

(a) $(\neg\mathsf{Employee})^{\mathcal{I}} = \{x_0, x_3, x_4, x_6, x_7, x_8, x_{10}\}$

(b) $(\neg\mathsf{EmpStud} \sqcap \forall\mathsf{empBy}.\mathsf{Company})^{\mathcal{I}} = \{x_0, x_2, x_3, x_4, x_6, x_7, x_8, x_9, x_{10}\}$ This might look unintuitive! Please note that $(\forall\mathsf{empBy}.\mathsf{Company})^{\mathcal{I}}$ is the set of all eleven elements of the domain (see the text after the concept (16) on page 6 of *A Description Logic Primer*)

(c) $(\mathsf{Student} \sqcap \forall\mathsf{pays}.\bot)^{\mathcal{I}} = \{x_7, x_8\}$

11. For the interpretation $\mathcal{I}$ from the picture, determine whether:

(a) $\mathcal{I} \Vdash \exists\mathsf{worksFor}.\top \sqsubseteq \mathsf{Employee}$

(b) $\mathcal{I} \Vdash \mathsf{Employee} \sqsubseteq \exists\mathsf{worksFor}.\top$

(c) $\mathcal{I} \Vdash \exists\mathsf{empBy}.\top \sqsubseteq \exists\mathsf{worksFor}.\mathsf{Company}$

Answer:

(a) $\mathcal{I} \Vdash \exists\mathsf{worksFor}.\top \sqsubseteq \mathsf{Employee}$ Yes! (since $\{x_5, x_9\} \subseteq \{x_1, x_2, x_5, x_9\}$)

(b) $\mathcal{I} \Vdash \mathsf{Employee} \sqsubseteq \exists\mathsf{worksFor}.\top$ No! For example, *mary* is an employee and she does not work for anyone.

(c) $\mathcal{I} \Vdash \exists \mathsf{empBy}.\top \sqsubseteq \exists \mathsf{worksFor}.\mathsf{Company}$ Yes! (since $\{x_9\} \subseteq \{x_5, x_9\}$)

12. For the interpretation $\mathcal{I}$ from the picture, determine whether:

    (a) $\mathcal{I} \Vdash (\mathsf{mary}, \mathsf{ibm}) : \mathsf{empBy}$

    (b) $\mathcal{I} \Vdash (\mathsf{ibm}, \mathsf{john}) : \mathsf{worksFor}$

    (c) $\mathcal{I} \Vdash \mathsf{john} : \forall \mathsf{empBy}.\mathsf{Company}$

    Answer:

    (a) $\mathcal{I} \Vdash (\mathsf{mary}, \mathsf{ibm}) : \mathsf{empBy}$ No!

    (b) $\mathcal{I} \Vdash (\mathsf{ibm}, \mathsf{john}) : \mathsf{worksFor}$ No! IBM does not work for John.

    (c) $\mathcal{I} \Vdash \mathsf{john} : \forall \mathsf{empBy}.\mathsf{Company}$ Yes!

13. Explain for each of the following FOL sentences why they are not expressible in DL (at least not in the standard way).

    (a) $\exists xyz P(x, y, z)$, where P is a 3-ary predicate.

    (b) $\forall x \forall y (P(x, y) \to P(y, x))$.

    (c) $\forall x P(x, x)$.

    Answer: (a) There are no 3-ary predicates in DL. (b) and (c) Statements in DL are about concepts, that is, unary predicates.

14. What does it mean for a logic to be decidable?

    Answer: See slides of Lecture 12.

15. Is propositional logic decidable? And FOL?

    Answer: Propositional logic is decidable — for any formula we can enumerate all valuations restricted to the propositional letters that appear in the formula, and for each valuation we can check if the formula holds. FOL is not decidable – see slides of Lecture 12.

16. Is any extension of a decidable logic decidable?

    Answer: No. FOL is an extension of propositional logic (see the previous question).

    New exercises

17. Translate the following statements from DL to English. The statements use atomic concepts $Vegetarian$, $Person$, $Meat$, $Fish$, $Animal$, $Vegan$, and roles $Eats$ and $ProductOf$.[1]

    (a) $Vegetarian \equiv Person \sqcap \neg \exists Eats.Meat \sqcap \neg \exists Eats.Fish$

[1] Please note that the concept language is different than the language of a similar sentence from the quiz about logics.

(b) $Vegan \equiv Person \sqcap \neg \exists Eats.\exists ProductOf.Animal$

(c) $\neg Vegetarian \sqcap Vegan \sqsubseteq \bot$

(d) $\exists Eats.Meat \sqsubseteq \exists Eats.\exists ProductOf.Animal$

Answer:

(a) A vegetarian is a person who does not eat meat and does not eat fish

(b) A vegan is a person who does not eat animal products

(c) The concepts of non-vegetarian and vegan are disjoint

(d) Everyone who eats meat also eats animal products

18. Translate the following statements from FOL to English.

(a) $\forall x \exists y Loves(x, y)$

(b) $\exists x \forall y Loves(x, y)$

(c) $\forall y \exists x Loves(x, y)$

(d) $\forall x \forall y(Brother(x, y) \rightarrow Sibling(x, y))$

(e) $\forall x \forall y(Mother(x, y) \leftrightarrow (Female(x) \wedge Parent(x, y)))$

(f) $\forall x \forall y(FirstCousin(x, y) \leftrightarrow \exists y \exists z Parent(y, x) \wedge Sibling(z, y) \wedge Parent(z, y))$

Answer:

(a) Everybody loves somebody

(b) Somebody loves everyone

(c) Everyone is loved by someone

(d) Brothers are siblings

(e) One's mother is one's female parent

(f) A first cousin is a child of a parent's sibling

Answers to Exercises in Book

These exercises are not essential, they are added for those students that asked for more material on FOL.

2.7.1 The properties expressed in (a) and (b) have names: (a) transitivity, (b) being antisymmetric.

In all three cases, let the domain be $D = \{r, s, t\}$.

(a) false, (b) and (c) true: $I(a) = I(b) = t$ and $I(P) = \{(r, s), (s, t)\}$.

(b) false, (a) and (c) true: $I(a) = I(b) = t$ and $I(P) = \{(r, s), (s, r), (r, r)\}$.

(c) false, (a) and (b) true: $I(a) = r$ and $I(b) = t$ and $I(P) = \{(r, s)\}$.

2.7.2 (1) $\forall x \forall y \forall z(f(f(x,y),z) = f(x, f(y,z))$.
(2) $\forall x(f(x,e) = f(e,x) = x)$.
(3) $\forall x \exists y(f(x,y) = f(y,x) = e)$.

Let Γ consist of the three sentences (1)-(3). To show that

$$\Gamma \models \forall x \forall y \exists z(f(x,z) = y)$$

we consider an arbitrary model $M(D,I)$ of (all formulas in) Γ. We have to show that $\forall x \forall y \exists z(f(x,z) = y)$ holds in M, in other words: for all elements a, b in domain D there is an element c in D such that $I(f)(a,c) = b$. (Side note: a, b are elements of D, while f, e belong to the formal language and are in model M interpreted as function $I(f)$ from D to D and element $I(e)$ of D.)

Because (3) holds there is an element a' in D such that $I(f)(a,a') = I(e)$. Let $c = I(f)(a',b)$. It remains to be shown that $I(f)(a,c) = b$. But this follows from the following equalities, which hold because (1)-(3) hold in M: $I(f)(a,c) = I(f)(a, f(a',b)) = I(f)(f(a,a'),b) = I(f)(e,b) = b$.

2.7.3 (a) For easy of writing, I write $s(x,y)$ for $\mathrm{Sub}(x,y)$ and $e(x,y)$ for $\mathrm{Elt}(x,y)$. T is the conjunction of (I), (II) and (III), where

(I) $\forall x(\neg e(x,x))$.

(II) $\forall x \forall y\big(s(x,y) \leftrightarrow \forall z(e(z,x) \to e(z,y))\big)$.
($A \leftrightarrow B$ is short for $(A \to B) \wedge (B \to A)$)

(III) $\forall x \forall y \forall z\big(e(z, u(x,y)) \leftrightarrow (e(z,x) \vee e(z,y)\big)$.

(b) To show: $T \models \forall x \forall y(s(x, u(x,y))$.
Consider an arbitrary model $M(D,I)$ of T. We have to show that $\forall x \forall y(s(x, u(x,y))$ holds in M, in other words, that for all elements a, b of D, $M \models s(a, u(a,b))$. By the second formula, (II), in T, it suffices to show that for all elements c in D, $M \models e(c,a) \to e(c, u(a,b))$. By the third formula, (III), in T, it suffices to show that $M \models e(c,a) \to e(c,a) \vee e(c,b)$, but that clearly holds.

(c) To show: $T \not\models \forall x \forall y(u(x,y) = u(y,x))$.
We could take as a model $M = (D,I)$ with $D = \{a,b\}$ with $I(u)(a,a) = I(u)(a,b) = I(u)(b,b) = a$ and $I(u)(b,a) = b$, and let $I(e)$ be the binary relation that holds for no pair (so the empty 2-ary relation) and let $I(s)$ be the binary relation that holds for all pairs. In other words, we interpret s and e such that

$$M \models \forall x \forall y\big(\neg e(x,y) \wedge s(x,y)\big).$$

Clearly, $I(u)(a,b) \neq I(u)(b,a)$. Hence $M \not\models \forall x \forall y(u(x,y) = u(y,x))$. Thus it remains to show that $M \models T$, that is, that the sentences (I), (II), and (III) hold in M:

(I) Since $e(x, y)$ does not hold for any x, y in D, it certainly follows that $M \models \forall x \neg e(x, x)$.

(II) We have to show $M \models s(x, y) \leftrightarrow \forall z(e(z, x) \to e(z, y))$ for all x, y in D. Since $M \models s(x, y)$ for all x, y, by the interpretation of s in M, it suffices to show that $M \models \forall z(e(z, x) \to e(z, y))$. But this follows from the fact that for all z, $M \models \neg e(z, x)$, by the interpretation of e in M.

(III) We have to show $M \models e(z, u(x, y)) \leftrightarrow (e(z, x) \vee e(z, y))$ for all x, y, z in D. Since $I(e)$ is the empty relation, $M \models \neg e(z, u(x, y)) \wedge \neg e(z, x) \wedge \neg e(z, y)$. Thus the above equivalence indeed holds in M.

(d) To show: $T \models \exists z\, s(u(A, z), A)$.
Consider an arbitrary model $M(D, I)$ of T. We have to show that there is an element b of D, such that $M \models s(u(A, b), A)$. Take for b the set A. Then we have to show that $M \models s(u(A, A), A)$. Using the fact that (II) holds in M, it suffices to show that every element of $u(A, A)$ is an element of A. Since M is a model of (III), every element of $u(A, A)$ is "an element of A or an element of A", which means that every element of $u(A, A)$ is an element of A, which is exactly what we had to show.

(e) Does $T \models \exists z \forall x\, s(u(x, z), x)$ or $T \not\models \exists z \forall x\, s(u(x, z), x)$?
The sentence says that there is an empty set. But there is no garuantee that such a set is an element of any model of T. Thus $T \not\models \exists z \forall x\, s(u(x, z), x)$. Now the formal argument:
Here is a counter model: $M = (D, I)$, where $D = \{a, b, c\}$ and

$$I(e) = \{(a, b), (b, a), (a, c), (b, c)\} \quad I(s) = \{(a, a), (b, b), (c, c), (a, c), (b, c)\}$$

$$I(u)(a, b) = I(u)(a, c) = I(u)(b, c) = I(u)(c, c) = c \quad I(u)(a, a) = a \quad I(u)(b, b) = b.$$

It is not hard to show that in this model $\exists z \forall x\, s(u(x, z), x)$ does not hold. For if, arguing by contradiction, it would hold, there would be an element d in D such that for all elements d' in D, $M \models s(u(d', d), d')$. But does not hold if $d = a$, nor if $d = b$ nor if $d = c$.

(f) $\forall x \exists z \forall y(e(y, z) \leftrightarrow y = x)$.
Arguing by contradiction, suppose there is a finite model M in which T_1 holds. Let $D = \{a_1, \ldots, a_n\}$. And let c denote the union of all elements, that is

$$c = I(u)(a_1, (I(u)(a_2, (I(u)(a_3, \ldots)))))).$$

Let d be the singleton set that consists of c, which exists because T_1 holds. Thus $M \models \forall y(e(y, d) \leftrightarrow y = c)$. Since c, d are in D there are j, h such that $c = a_h$ and $d = a_j$. Hence $M \models e(a_h, a_j)$. Because (III) and (II) hold in M, $M \models s(a_i, c)$ for all $i = 1, \ldots, n$. In particular, $M \models s(a_j, a_h)$. But then by (II), $M \models e(a_h, a_h)$, which contradicts that (I) holds in M.

(g) T does not entail that there exists an empty set. The example in (e) shows that.

2.7.4 The language: b is a constant that stands for "the barber" and $s(x, y)$ stands for "x shaves y". The two required sentences are

$$\forall x\big(\neg s(x,x) \to s(b,x)\big) \qquad \forall x\big(s(b,x) \to \neg s(x,x)\big).$$

We show that in any model M, the first sentence implies that $M \models s(b,b)$ and the second sentence implies $M \models \neg s(b,b,)$. This proves that there can be no model of both sentences.

Consider a model M of the first sentence, and let $d = I(b)$. This implies that $M \models s(b,b)$. For if $M \models \neg s(b,b)$, then $M \models s(b,b,)$ would follow, a contradiction.

Consider a model M of the second sentence, and let $d = I(b)$. This implies that $M \models \neg s(b,b)$. For if $M \models s(b,b)$, then $M \models \neg s(b,b,)$ would follow, a contradiction.