Additional Exercises Reasoning Part MAIR

1. Which class of human beings is described by the following concept?

$$\neg\forall child.Male \sqcap \neg\forall child.Female$$

   Answer: The human beings that have at least one daughter and at least one son.

2. Translate the concept in the previous exercise in FOL:

   Answer:

$$(\neg\forall y(child(x, y) \rightarrow Male(y)) \wedge (\neg\forall y(child(x, y) \rightarrow Female(y)).$$

3. Translate the following FOL sentence in DL and describe the concept in natural language.

$$\forall y(neighbour(x, y) \wedge Old(y)) \wedge \forall y(neighbour(x, y) \rightarrow Friendly(y)).$$

   Answer:

$$\forall \mathsf{neighbour.Old} \sqcap \forall \mathsf{neighbour.Friendly}$$

   The concept of "people who only have friendly old neighbours".

4. Translate the following FOL sentence in DL and describe the concept in natural language.

$$\exists y\big(neighbour(x, y) \wedge \forall z(neighbour(y, z) \rightarrow \neg Friendly(z))\big).$$

   Answer:

$$\exists neighbour.\forall neighbour.\neg Friendly$$

   The concept of "people who have a neighbour that only has unfriendly neighbours".

5. Explain for each of the following FOL sentences why they are not expressible in DL (at least not in the standard way).

   (a) $\exists xyzP(x, y, z)$, where P is a 3-ary predicate.

   (b) $\forall x\forall y(P(x, y) \rightarrow P(y, x)$.

   (c) $\forall xP(x, x)$.

   Answer: (a) There are no 3-ary predicates in DL. (b) and (c) Statements in DL are about concepts, that is, unary predicates.

6. What does it mean for a logic to be decidable?

   Answer: See slides Lecture 13.

7. Is propositional logic decidable? And FOL?

   Answer: See slides Lecture 13.

8. Is any extension of a decidable logic decidable?

   Answer: No. FOL is an extension of porpositional logic, which is clearly decidable, while FOL is not.

Answers to Exercises in Book

2.7.1 The properties expressed in (a) and (b) have names: (a) transitivity, (b) being antisymmetric.

In all three cases, let the domain be $D = \{r, s, t\}$.

(a) false, (b) and (c) true: $I(a) = I(b) = t$ and $I(P) = \{(r, s), (s, t)\}$.

(b) false, (a) and (c) true: $I(a) = I(b) = t$ and $I(P) = \{(r, s), (s, r), (r, r)\}$.

(c) false, (a) and (b) true: $I(a) = r$ and $I(b) = t$ and $I(P) = \{(r, s)\}$.

2.7.2 (1) $\forall x \forall y \forall z (f(f(x, y), z) = f(x, f(y, z)))$.
(2) $\forall x (f(x, e) = f(e, x) = x)$.
(3) $\forall x \exists y (f(x, y) = f(y, x) = e)$.

Let Γ consist of the three sentences (1)-(3). To show that

$$\Gamma \models \forall x \forall y \exists z (f(x, z) = y)$$

we consider an arbitrary model $M(D, I)$ of (all formulas in) Γ. We have to show that $\forall x \forall y \exists z (f(x, z) = y)$ holds in M, in other words: for all elements a, b in domain D there is an element c in D such that $I(f)(a, c) = b$. (Side note: a, b are elements of D, while f, e belong to the formal language and are in model M interpreted as function $I(f)$ from D to D and element $I(e)$ of D.)

Because (3) holds there is an element a' in D such that $I(f)(a, a') = I(e)$. Let $c = I(f)(a', b)$. It remains to be shown that $I(f)(a, c) = b$. But this follows from the following equalities, which hold because (1)-(3) hold in M: $I(f)(a, c) = I(f)(a, f(a', b)) = I(f)(f(a, a'), b) = I(f)(e, b) = b$.

2.7.3 (a) For easy of writing, I write $s(x, y)$ for $\mathrm{Sub}(x, y)$ and $e(x, y)$ for $\mathrm{Elt}(x, y)$. T is the conjunction of (I), (II) and (III), where

(I) $\forall x (\neg e(x, x))$.
(II) $\forall x \forall y \big(s(x, y) \leftrightarrow \forall z (e(z, x) \rightarrow e(z, y))\big)$.
($A \leftrightarrow B$ is short for $(A \rightarrow B) \wedge (B \rightarrow A)$)
(III) $\forall x \forall y \forall z \big(e(z, u(x, y)) \leftrightarrow (e(z, x) \vee e(z, y)\big)$.

(b) To show: $T \models \forall x \forall y(s(x, u(x, y)))$.
Consider an arbitrary model $M(D, I)$ of T. We have to show that $\forall x \forall y(s(x, u(x, y))$ holds in M, in other words, that for all elements a, b of D, $M \models s(a, u(a, b))$. By the second formula, (II), in T, it suffices to show that for all elements c in D, $M \models e(c, a) \to e(c, u(a, b))$. By the third formula, (III), in T, it suffices to show that $M \models e(c, a) \to e(c, a) \vee e(c, b)$, but that clearly holds.

(c) To show: $T \not\models \forall x \forall y(u(x, y) = u(y, x))$.
We could take as a model $M = (D, I)$ with $D = \{a, b\}$ with $I(u)(a, a) = I(u)(a, b) = I(u)(b, b) = a$ and $I(u)(b, a) = b$, and let $I(e)$ be the binary relation that holds for no pair (so the empty 2-ary relation) and let $I(s)$ be the binary relation that holds for all pairs. In other words, we interpret s and e such that

$$M \models \forall x \forall y\big(\neg e(x, y) \wedge s(x, y)\big).$$

Clearly, $I(u)(a, b) \neq I(u)(b, a)$. Hence $M \not\models \forall x \forall y(u(x, y) = u(y, x))$. Thus it remains to show that $M \models T$, that is, that the sentences (I), (II), and (III) hold in M:

(I) Since $e(x, y)$ does not hold for any x, y in D, it certainly follows that $M \models \forall x \neg e(x, x)$.

(II) We have to show $M \models s(x, y) \leftrightarrow \forall z(e(z, x) \to e(z, y))$ for all x, y in D. Since $M \models s(x, y)$ for all x, y, by the interpretation of s in M, it suffices to show that $M \models \forall z(e(z, x) \to e(z, y))$. But this follows from the fact that for all z, $M \models \neg e(z, x)$, by the interpretation of e in M.

(III) We have to show $M \models e(z, u(x, y)) \leftrightarrow (e(z, x) \vee e(z, y))$ for all x, y, z in D. Since $I(e)$ is the empty relation, $M \models \neg e(z, u(x, y)) \wedge \neg e(z, x) \wedge \neg e(z, y)$. Thus the above equivalence indeed holds in M.

(d) To show: $T \models \exists z\, s(u(A, z), A)$.
Consider an arbitrary model $M(D, I)$ of T. We have to show that there is an element b of D, such that $M \models s(u(A, b), A)$. Take for b the set A. Then we have to show that $M \models s(u(A, A), A)$. Using the fact that (II) holds in M, it suffices to show that every element of $u(A, A)$ is an element of A. Since M is a model of (III), every element of $u(A, A)$ is "an element of A or an element of A", which means that every element of $u(A, A)$ is an element of A, which is exactly what we had to show.

(e) Does $T \models \exists z \forall x\, s(u(x, z), x)$ or $T \not\models \exists z \forall x\, s(u(x, z), x)$?
The sentence says that there is an empty set. But there is no garuantee that such a set is an element of any model of T. Thus $T \not\models \exists z \forall x\, s(u(x, z), x)$. Now the formal argument:
Here is a counter model: $M = (D, I)$, where $D = \{a, b, c\}$ and

$I(e) = \{(a, b), (b, a), (a, c), (b, c)\}$ $\quad I(s) = \{(a, a), (b, b), (c, c), (a, c), (b, c)\}$

$I(u)(a,b) = I(u)(a,c) = I(u)(b,c) = I(u)(c,c) = c \quad I(u)(a,a) = a \quad I(u)(b,b) = b.$

It is not hard to show that in this model $\exists z \forall x\, s(u(x,z),x)$ does not hold. For if, arguing by contradiction, it would hold, there would be an element d in D such that for all elements d' in D, $M \models s(u(d',d),d')$. But does not hold if $d = a$, nor if $d = b$ nor if $d = c$.

(f) $\forall x \exists z \forall y (e(y,z) \leftrightarrow y = x)$.

Arguing by contradiction, suppose there is a finite model M in which T_1 holds. Let $D = \{a_1, \dots, a_n\}$. And let c denote the union of all elements, that is

$$c = I(u)(a_1, (I(u)(a_2, (I(u)(a_3, \dots)))))).$$

Let d be the singleton set that consists of c, which exists because T_1 holds. Thus $M \models \forall y(e(y,d) \leftrightarrow y = c)$. Since c, d are in D there are j, h such that $c = a_h$ and $d = a_j$. Hence $M \models e(a_h, a_j)$. Because (III) and (II) hold in M, $M \models s(a_i, c)$ for all $i = 1, \dots, n$. In particular, $M \models s(a_j, a_h)$. But then by (II), $M \models e(a_h, a_h)$, which contradicts that (I) holds in M.

(g) T does not entail that there exists an empty set. The example in (e) shows that.

2.7.4 The language: b is a constant that stands for "the barber" and $s(x,y)$ stands for "x shaves y". The two required sentences are

$$\forall x\big(\neg s(x,x) \rightarrow s(b,x)\big) \qquad \forall x\big(s(b,x) \rightarrow \neg s(x,x)\big).$$

We show that in any model M, the first sentence implies that $M \models s(b,b)$ and the second sentence implies $M \models \neg s(b,b,)$. This proves that there can be no model of both sentences.

Consider a model M of the first sentence, and let $d = I(b)$. This implies that $M \models s(b,b)$. For if $M \models \neg s(b,b)$, then $M \models s(b,b,)$ would follow, a contradiction.

Consider a model M of the second sentence, and let $d = I(b)$. This implies that $M \models \neg s(b,b)$. For if $M \models s(b,b)$, then $M \models \neg s(b,b,)$ would follow, a contradiction.

16.7.1 (a-c) (a) $\mathsf{fisheater(fred)}$

(b) $\mathsf{rodent(stan)}$. The subsumtion $\mathsf{mouse} \sqsubseteq \mathsf{rodent}$ is lost.

(c) Since there is no $\mathsf{Rabbit(sam)}$, the disjunction is $\mathsf{Dog(sam)} \vee \mathsf{Snake(sam)}$. It can be replaced by $\mathsf{Carnivore(sam)}$.