Solutions to extra probability exercises

November 19, 2019

(Source: Pearl et al, 2016, solution manual)

Study question 1.3.1.
Identify the variables and events invoked in the lollipop story of Study question 1.2.4

Solution to study question 1.3.1

Variables: Let X indicate Treatment / Drug receipt, Z indicate Lollipop receipt, and Y indicate Recovery Status.
Events: "$X = 1$ *and* $Z = 1$ *and* $Y = 1$" indicates the event where an individual takes the drug, receives a lollipop, and recovers (the same applies for other values of each variable).

Study question 1.3.3.
Consider the casino problem described in Section 1.3.7

(a) *Compute* $P(\text{“craps”}|\text{“}11\text{”})$ *assuming that there are twice as many roulette tables as craps games at the casino.*

(b) *Compute* $P(\text{“roulette”}|\text{“}10\text{”})$ *assuming that there are twice as many craps games as roulette tables at the casino.*

Solution to study question 1.3.3

Part (a)

Assuming that there are twice as many roulette tables as craps games at the casino, we have:

$$P(\text{“}roulette\text{”}) = 2/3$$
$$P(\text{“}craps\text{”}) = 1/3$$

So, by the law of total probability, we can write our target quantity $P(\text{“}11\text{”})$ in terms of what we know:

$$\begin{aligned}
P(\text{“}11\text{”}) &= P(\text{“}11\text{”}|\text{“}craps\text{”})P(\text{“}craps\text{”}) + P(\text{“}11\text{”}|\text{“}roulette\text{”})P(\text{“}roulette\text{”}) \\
&= 1/18 * 1/3 + 1/38 * 2/3 \\
&= 37/1026 \\
&= 0.036 \\
P(\text{“}craps\text{”}|\text{“}11\text{”}) &= P(\text{“}craps\text{”}, \text{“}11\text{”})/P(\text{“}11\text{”}) \\
&= \frac{1/18 * 1/3}{37/1026} \\
&= 0.514
\end{aligned}$$

Part (b)

Assuming that there are twice as many craps games as roulette tables at the casino, we have:

$$P(\text{"}roulette\text{"}) = 1/3$$
$$P(\text{"}craps\text{"}) = 2/3$$

We can use the same tactic as in (a) (the law of total probability) to write our target quantity in terms of what we know:

$$\begin{aligned}
P(\text{"10"}) &= P(\text{"10"}|\text{"}craps\text{"})P(\text{"}craps\text{"}) + P(\text{"10"}|\text{"}roulette\text{"})P(\text{"}roulette\text{"}) \\
&= 1/12 * 2/3 + 1/38 * 1/3 \\
&= 11/171 \\
&= 0.064 \\
P(\text{"}roulette\text{"}|\text{"10"}) &= P(\text{"}roulette\text{"}, \text{"10"})/P(\text{"10"}) \\
&= \frac{1/38 * 1/3}{11/171} \\
&= 0.136
\end{aligned}$$

Study question 1.3.7.

Two fair coins are flipped simultaneously to determine the payoffs of two players in the town's casino. Player 1 wins a dollar if and only if at least one coin lands on head. Player 2 receives a dollar if and only if the two coins land on the same face. Let X stand for the payoff of Player 1 and Y for the payoff of Player 2.

(a) Find and describe the probability distributions

$$P(x), P(y), P(x,y), P(y|x) \text{ and } P(x|y)$$

(b) Using the descriptions in (a), compute the following measures:

$$E[X], E[Y], E[Y|X=x], E[X|Y=y]$$

$$Var(X), Var(Y), Cov(X,Y), \rho_{XY}$$

(c) Given that Player 2 won a dollar, what is your best guess of Player 1's payoff?

(d) Given that Player 1 won a dollar, what is your best guess of Player 2's payoff?

(e) Are there two events, $X = x$ and $Y = y$, that are mutually independent?

Solution to study question 1.3.7

Let X and Y stand for the winnings of Player 1 and Player 2, respectively. We have:

Part (a)

The descriptions of these distributions are as follows:
$P(x)$: The probability that player 1 gets x dollars.
$P(y)$: The probability that player 2 gets y dollars.
$P(x, y)$: The probability that player 1 gets x dollars and player 2 gets y dollars.
$P(y|x)$: The probability that player 2 gets y dollars given that player 1 gets x dollars.
$P(x|y)$: The probability that player 1 gets x dollars given that player 2 gets y dollars.

Part (b)

We'll compute each measure by its definition, using the fact that each coin flip is fair and independent:

First, observe that Player 1 wins a dollar if at least 1 of the coins lands on heads. Another way to think about this scenario is that Player 1 loses if both coins land on tails, which we can subtract from 1 to find the probability of them winning. Specifically:

$$P(X = 1) = 1 - P(X = 0) = 1 - P(tails_1)P(tails_2) = 1 - 1/2 * 1/2 = 3/4$$

Computing the expected value follows from Eq. (1.10), summing over all outcomes and their associated probabilities:

$$E[X] = \sum_x x * P(x) = 1 * P(X = 1) + 0 * P(X = 0) = 3/4$$

We'll use a similar approach to computing the winning probability for Player 2 as well as the expected value of their winnings. Observe that the winning conditions for Player 2 are when both coins land on the same face, specifically:

$$P(Y = 1) = P(heads_1)P(heads_2) + P(tails_1)P(tails_2) = 1/2 * 1/2 + 1/2 * 1/2 = 1/2$$

$$E[Y] = \sum_y y * P(y) = 1 * P(Y = 1) + 0 * P(Y = 0) = 1/2$$

To compute the conditional expected values, we will use Eq. (1.13), which intuitively sums over all possible values of the query and weights by the conditional probability of each:

$$\begin{aligned}
E[Y|X=x] &= \sum_y P(y|X=x) \\
&= 1 * P(Y=1|X=x) + 0 * P(Y=0|X=x) \\
&= P(Y=1|X=x) \\
E[X|Y=y] &= \sum_x P(x|Y=y) \\
&= 1 * P(X=1|Y=y) + 0 * P(X=0|Y=y) \\
&= P(X=1|Y=y)
\end{aligned}$$

Next, we can compute the variances of each variable using Eq. (1.15), their covariance using Eq. (1.16), and their correlation coefficient using Eq. (1.17).

$$\begin{aligned}
Var(X) &= E((X-3/4)^2) \\
&= (1-3/4)^2 * P(X=1) + (0-3/4)^2 * P(X=0) \\
&= 1/16 * 3/4 + 9/16 * 1/4 \\
&= 3/16 \\
Var(Y) &= E((Y-1/2)^2) \\
&= (1-1/2)^2 * P(Y=1) + (0-1/2)^2 * P(X=0) \\
&= 1/4 * 1/2 + 1/4 * 1/2 \\
&= 1/4 \\
Cov(X,Y) &= E[(X-3/4)(Y-1/2)] \\
&= 1/4 * 1/2 * P(X=1,Y=1) - 3/4 * 1/2 * P(X=0,Y=1) \\
&\quad + 1/4 * -1/2 * P(X=1,Y=0) - 3/4 * -1/2 * P(X=0,Y=0) \\
&= 1/4 * 1/2 * 1/4 - 3/4 * 1/2 * 1/4 + 1/4 * -1/2 * 1/2 - 3/4 * -1/2 * 0 \\
&= -1/8 \\
\rho_{XY} &= \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \\
&= \frac{-1/8}{\sqrt{3/16}\sqrt{1/4}} \\
&= -1/\sqrt{3}
\end{aligned}$$

Part (c)

To answer this query, we know that if both $X = 1$ and $Y = 1$, then the outcome of the two coins must have been both heads, meaning that $P(X = 1, Y = 1) = 1/4$. Furthermore, we can phrase our query as $E[X|Y = 1]$, since we are interested in the expectation of Player 1's winnings having observed that Player 2 won a dollar. Combining this knowledge with our solution to each conditional expected value from part (b) above, we have:

$$\begin{aligned} E[X|Y=1] &= P(X=1|Y=1) \\ &= \frac{P(X=1,Y=1)}{Y=1} \\ &= \frac{1/4}{1/2} \\ &= 1/2 \end{aligned}$$

Part (d)

We use the same strategy as in part (c) above, and have:

$$\begin{aligned} E[Y|X=1] &= P(Y=1|X=1) \\ &= \frac{P(X=1,Y=1)}{X=1} \\ &= \frac{1/4}{3/4} \\ &= 1/3 \end{aligned}$$

Part (e)

Consider what we know about the joint events:

$$\begin{aligned} P(X=1,Y=1) &= 1/4 \\ P(X=0,Y=1) &= 1/4 \\ P(X=1,Y=0) &= 1/2 \\ P(X=0,Y=0) &= 0 \end{aligned}$$

Now, examining their priors, we have:

$$\begin{aligned} &P(X=1) = 3/4 \\ &P(X=0) = 1/4 \\ &P(Y=1) = P(Y=0) = 1/2 \end{aligned}$$

Plainly, there are no two values for X and Y such that the product of their priors will equal their joint, i.e., for no two values $X = x, Y = y$ do we have: $P(Y = y, X = x) = P(Y = y) * P(X = x)$. Therefore, we conclude that there are no two mutually independent events.

Correction: what is called 'prior' here should be called 'marginal'