Introduction to Game Theory

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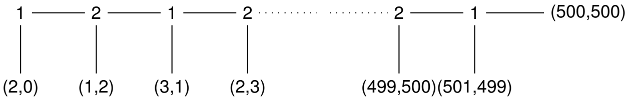
Extensive Games

- Sequential Structure of Games.
- Perfect and Imperfect-Information Extensive Games.
- Strategies and Equilibria for Extensive Games.

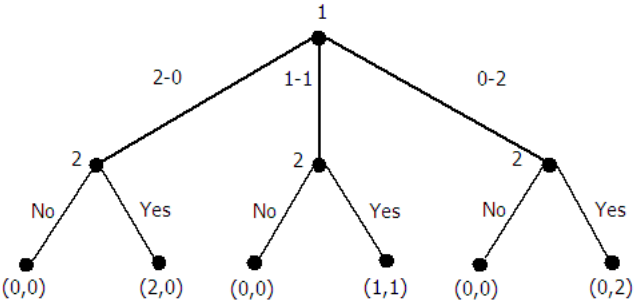
Extensive Games

Situation: There are 1,000 diamonds to divide between two players. They decide to do this by playing a turn-based game where each player can take either one or two diamonds when it is his/her turn. When one player chooses two diamonds, the game ends and the remaining diamonds are destroyed. Otherwise it continues until there are no diamonds left.



Perfect Information Game: Sharing Game

- players 1 and 2 has to divide two indivisible and identical items.
- Player 1 can suggest to keep both, to keep one, or to give both to player 2.
- Player 2 can then accept or reject a proposal.



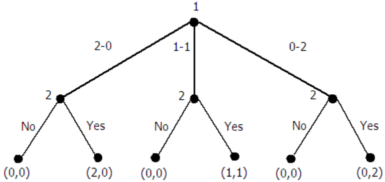
Perfect Information Game

A perfect information game in extensive form is $(N, A, H, Z, \chi, \rho, \sigma, u)$, where

- N is a set of players
- A is a set of actions
- H is a set of nonterminal choice nodes
- Z is a set of terminal nodes, disjoint from H,
- $\chi : H \to 2^A$ is the action function (assigning possible actions to each choice node),
- $\rho : H \to N$ is the player function (assigning players to choice nodes),
- $\sigma : H \times A \to H \cup Z$ is the successor function (assigning to each choice node and an action to a choice node or terminal node),
- $u = (u_1, \ldots, u_n)$, where $u_i : Z \to \mathbb{R}$ is a real-valued utility function for agent i and terminal node Z.

Pure Strategies in Extensive Games

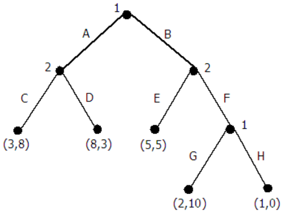
Definition: A pure strategy of player i in extensive game $(N, A, H, Z, \chi, \rho, \sigma, u)$ consists of the Cartesian product $\prod_{h \in H,\ \rho(h)=i} \chi(h)$.



- Strategies of player 1: $S_1 = \{\, 2-0\ ,\ 1-1\ ,\ 0-2\, \}$
- Strategies of player 2:

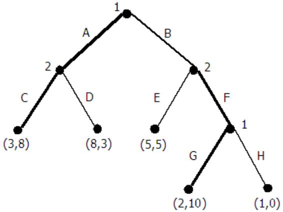
$$S_2 = \{ \quad (yes, yes, yes), \quad (yes, yes, no), \quad (yes, no, yes), \quad (yes, no, no), \\ (no, yes, yes), \quad (no, yes, no), \quad (no, no, yes), \quad (no, no, no) \ \}$$

From Extensive to Normal Form



	CE	*CF*	*DE*	*DF*
AG	3, 8	3, 8	8, 3	8, 3
AH	3, 8	3, 8	8, 3	8, 3
BG	5, 5	2, 10	5, 5	2, 10
BH	5, 5	1, 0	5, 5	1, 0

Equilibria in Extensive Games



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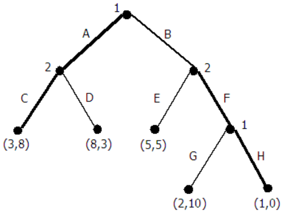
Equilibria in Extensive Games



	CE	CF	DE	DF
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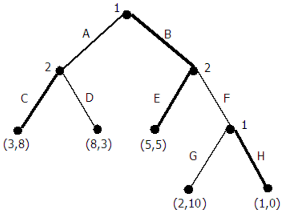
Equilibria in Extensive Games



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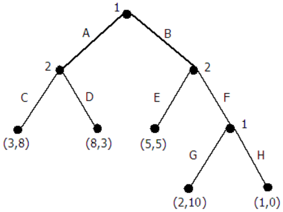
From Extensive to Normal Form



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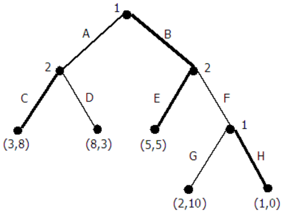
Theorem: *Every (finite) perfect-information game in extensive form has a pure-strategy Nash equilibrium.*

Subgames and Subgame-Perfect Equilibrium



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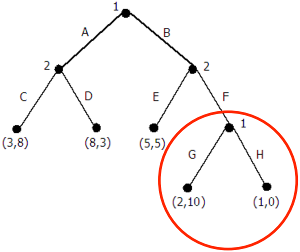
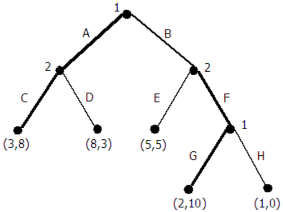
Subgames

Definition: The subgame of an extensive game G with root h is the restriction of G to the descendants of h. The set of subgames of G includes all subgames of G rooted at some node in G.



Subgame-Perfect Equilibrium

Definition: The *subgame-perfect equilibria* of extensive game G are all strategy profiles s such that for any subgame G' of G, the restriction of s to G' is a Nash equilibrium of G'.



(AG,CF)

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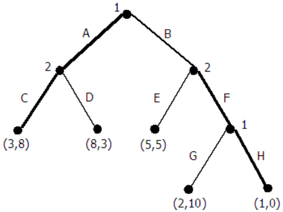

(AG,CF)
(AH,CF)

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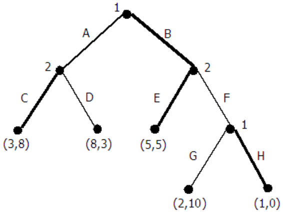

(AG,CF)
(AH,CF)
(BH,CE)

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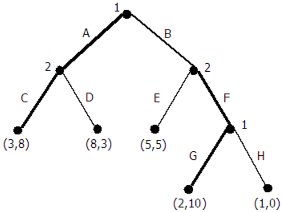
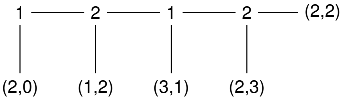


(AG,CF)
(AH,CF)
(BH,CE)

Every perfect-information game in extensive form has at least one subgame-perfect equilibrium.

Extensive Games and Backward Induction

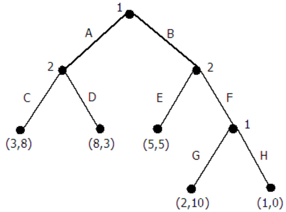


	D	SD	SSD
D	$2, 0$	$2, 0$	$2, 0$
SD	$1, 2$	$3, 1$	$3, 1$
SSD	$1, 2$	$2, 3$	$2, 2$

Problems with backward induction in extensive games

- People do not behave as predicated by this analysis.
- The analysis is somehow contradictory. It predicts the players should go down at each choice point, but does not explain how players can get there.

From Extensive to Normal Form



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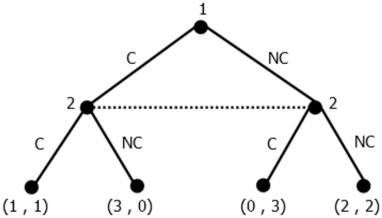
How about going from normal form games to extensive game? Can we model synchronous decisions in Extensive forms?

Imperfect-Information Extensive Games

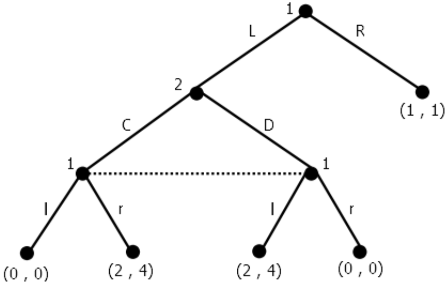
Definition: An imperfect information extensive game is $(N, A, H, Z, \chi, \rho, \sigma, u, I)$, where

- $(N, A, H, \chi, \rho, \sigma, u)$ is a *perfect-information extensive game*,
- $I = (I_1, \ldots, I_n)$ for players $1, \ldots, n$, where $I_i : (I_{i,1}, \ldots, I_{i,k})$ is an equivalence relation on (i.e., a partition of) $\{h \in H \mid \rho(h) = i\}$. If there exists a j for which $h, h' \in I_{i,j}$, then it must be $\chi(h) = \chi(h')$ and $\rho(h) = \rho(h')$.

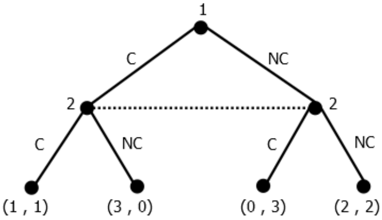


Pure Strategies in Imperfect-Information Extensive Games

Definition: Let $(N, A, H, Z, \chi, \rho, \sigma, u, I)$ be an imperfect-information extensive form game. Then, the **pure strategies** of player i consists of the Cartesian product $\prod_{I_{i,j} \in I_i} \chi(I_{i,j})$, where $\chi(I_{i,j})$ is the set of actions of player i at any node in $I_{i,j}$.



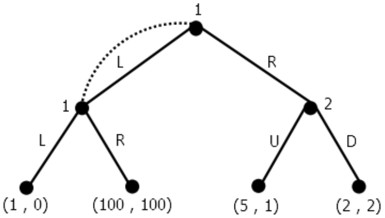
From Normal to Imperfect-Information Extensive Forms



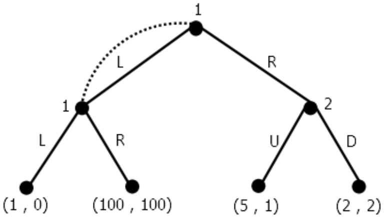
Normal-Form Games (e.g., Prisoner's Dilemma) can be transformed to Imperfect-Information Extensive-Form Games.

Exercise: Transform Battle of Sexes, Matching Pennies, and Game of Chicken in Imperfect-Information Extensive Games and analyse their equilibria.

From Imperfect-Information Extensive to Normal Forms



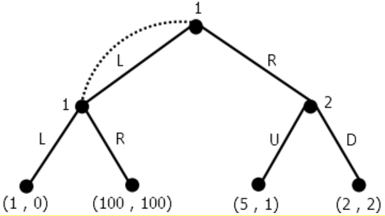
From Imperfect-Information Extensive to Normal Forms



	U	D
L	1, 0	1, 0
R	5, 1	2, 2

Mixed strategies in Imperfect-Information Games

Definition: A *mixed strategy* in an (imperfect-information) extensive game is a distributions over complete pure strategies.



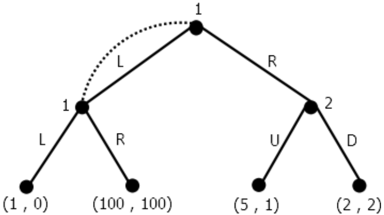
	U	D
L	$1, 0$	$1, 0$
R	$5, 1$	$2, 2$

Mixed Strategy of agent 1: $(\ L : p, \quad R : 1 - p \ )$

Example: $(\ L : 0.6, \quad R : 0.4 \ )$

Behavioral strategies in Imperfect-Information Games

Definition: A *behavioral strategy* in an (imperfect-information) extensive game consists of *independent* randomization at each information set. An agent randomize afresh each time it gets into one and the same information set.



Using the behavioral strategy (p , $1 - p$) by agent 1, the best response to the pure strategy D of agent 2 is to maximize the expected values as follows:

$$1 * p^2 + 100 * p(1 - p) + 2 * (1 - p)$$

Perfect Recall

Definition: Player i has *perfect recall* in an imperfect-information game if for any two nodes h and h' that are in the same information set for player i, for any path $h_0, a_0, h_1, a_1, \ldots, h_n, a_n, h$ from the root of the game to h and for any path$h_0, a'_0, h'_1, a'_1, \ldots, h'_m, a'_m, h'$ from the root to h' it must be the case that:

1. $n = m$,
2. for all $0 \leq j \leq n$, the decision nodes h_j and h'_j are in the same information set for player i, and,
3. for all $0 \leq j \leq n$, if $\rho(h_j) = i$, then $a_j = a'_j$.

A game is a *game of perfect recall* if every player has perfect recall in it.

In a perfect recall game, any mixed strategy of a player can be replaced by an equivalent behavioral strategy, and vice versa. Two strategies are equivalent iff they induce the same probabilities on outcomes, for any fixed strategy profile (mixed or behavioral) of the remaining players.

Exercise: Check if the Imperfect-Information Extensive game corresponding to Prisoner's Dilemma is a perfect recall game. Check the equivalence between the mixed and behavioral strategies in this game.

Normal and Extensive Form Games

- Extensive to Normal Form with Pure Strategies
- Normal to Extensive Form with Pure Strategies
- Extensive to Normal Form with Mixed and Behavioral Strategies
- Normal to Extensive Form with Mixed and Behavioral Strategies