Introduction to Game Theory (2)

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Mixed Strategies

		0.2	0.8
		c_1	c_2
0.6	r_1	1, 1	0, 0
0.4	r_2	0, 0	1, 1

$\langle\ (0.6\ ,\ 0.4)\ ,\ (0.2\ ,\ 0.8)\ \rangle$

		0.2	0.5	0.3
		c_1	c_2	c_3
0.3	r_1	1, 2	2, 0	1, 1
0.7	r_2	2, 0	1, 2	0, 2

$\langle\ (0.3\ ,\ 0.7)\ ,\ (0.2\ ,\ 0.5\ ,\ 0.3)\ \rangle$

Mixed Strategies

		1/2	1/2
		c_1	c_2
1/3	r_1	1, 1	0, 0
2/3	r_2	0, 0	1, 1

$\langle\, (1/3\ ,\ 2/3)\ ,\ (1/2\ ,\ 1/2)\,\rangle$

		1/3	1/3	1/3
		c_1	c_2	c_3
1/4	r_1	1, 2	2, 0	1, 1
3/4	r_2	2, 0	1, 2	0, 2

$\langle\, (1/4\ ,\ 3/4)\ ,\ (1/3\ ,\ 1/3\ ,\ 1/3)\,\rangle$

Mixed Strategies and Expected Utility

Definition: Let (N, A, u) be a strategic game. Then:

- $\Delta(A_i)$ is the set of *mixed strategies*, *i.e.*, set of all probability distributions over A_i.

- $\Delta(A) = \Delta(A_1) \times \cdots \times \Delta(A_n)$, set of *mixed strategy profiles*.

- Expected utility of mixed strategy $s \in \Delta(A)$ for player i is defined as:

$$u_i(s) = \sum_{a \in A} \left(u_i(a) \cdot \prod_{j \in N} s_j(a_j) \right)$$

  where a is a pure strategy profile, a_j is the strategy of player j in a, and $s_j(a_j)$ is the probability value assigned to a_j by s_j.

Notes:

- A pure strategy a is identified with the mixed strategy s for which $s(a) = 1$.
- Moreover, $u_i(a)$ is interpreted as the utility of pure strategy a for player i, while $u_i(s)$ is interpreted as the expected utility of mixed strategy s for i.

Mixed Strategies and Expected Utility

$$u_i(s) = \sum_{a \in A} \left(u_i(a) \cdot \prod_{j \in N} s_j(a_j) \right)$$

	A_q	B_{1-q}
A_p	1, 1	0, 0
B_{1-p}	0, 0	1, 1

$$s = \langle (A_p, B_{1-p}) \, , \, (A_q, B_{1-q}) \rangle$$

$$\begin{aligned}
u_{row}(s) \quad &= \quad \textstyle\sum_{a \in A} \left(u_{row}(a) \cdot \prod_{j \in N} s_j(a_j) \right) \\
& \quad\quad 1 * (p * q) \;\; + \\
& \quad\quad 0 * (p * (1 - q)) \;\; + \\
& \quad\quad 0 * ((1 - p) * q) \;\; + \\
& \quad\quad 1 * ((1 - p) * (1 - q) \\
&= \quad 2pq - p - q + 1
\end{aligned}$$

The Prisoner's Dilemma

> *Two suspects are taken into custody and separated. The district attorney is certain that they are guilty of a specific crime, but he does not have adequate evidence to convict them at a trial. He points out to each prisoner that each has two alternatives: to confess to the crime the police are sure they have done, or not to confess. If they will both do not confess, then the district attorney states he will book them on some very minor trumped up charge such as petty larceny and illegal possession of a weapon, and they will both receive minor punishment; if they both confess they will be prosecuted, but he will recommend less than the most severe sentence; but if one confesses and the other does not, then the confessor will receive lenient treatment for turning state's evidence whereas the latter will get "the book" slapped on him.*
>
> (Luce and Raiffa, 1957, p. 95)

The Prisoner's Dilemma

	NotConfess	*Confess*
NotConfess	$2, 2$	$0, 3$
Confess	$3, 0$	$1, 1$

Pareto Efficiency

Definition: A pure strategy profile $a \in A$ is *Pareto efficient* if there is no pure strategy profile that is strictly better for all players, *i.e.*, if

there is no $a' \in A$ such that for all $i \in N$: $\quad u_i(a') > u_i(a)$

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Pareto Efficiency

	NotConfess	*Confess*
NotConfess	2, 2	0, 3
Confess	3, 0	1, 1

Which are the Pareto efficient strategy profiles?

Pareto Efficiency

	NotConfess	*Confess*
NotConfess	2, 2	0, 3
Confess	3, 0	1, 1

Which are the Pareto efficient strategy profiles?

Dominance

Definition: A pure strategy a_i for player i strongly dominates another pure strategy a_i' of i if for any strategies of the opponents, a_i is strictly better than a_i', *i.e.*, if:

$$\text{for all } b \in A: \quad u_i(b_1, \ldots, a_i, \ldots, b_n) > u_i(b_1, \ldots, a_i', \ldots, b_n).$$

A pure strategy a_i that strongly dominates all other pure strategies of player i is called a strong dominant pure strategy of player i.

Definition: A pure strategy profile $a = (a_1, \ldots, a_n)$ is called a strongly dominant pure strategy equilibrium if a_i is strongly dominant strategy for player i, for every $i = 1, \ldots, n$.

Dominance

	NotConfess	*Confess*
NotConfess	2, 2	0, 3
Confess	3, 0	1, 1

Which are the strongly dominant strategy profiles?

Dominance

	NotConfess	*Confess*
NotConfess	$2, 2$	$0, 3$
Confess	$3, 0$	$1, 1$

Which are the strongly dominant strategy profiles?

Dominance

	NotConfess	*Confess*
NotConfess	2, 2	0, 3
Confess	3, 0	1, 1

Which are the strongly dominant strategy profiles and which ones are Pareto efficient strategy profiles?

Dominance

Definition: A mixed strategy s_i for player i strongly dominates another mixed strategy s'_i of i if for any mixed strategies of the opponents, s_i has a greater expected utility than s'_i, *i.e.*, if:

$$\text{for all } t_{j \neq i} \in \Delta(A_j) : \quad u_i(t_1, \ldots, s_i, \ldots, t_n) > u_i(t_1, \ldots, s'_i, \ldots, t_n).$$

A mixed strategy s_i of player i that strongly dominates all other mixed strategies of i is called a *strongly dominant* strategy for player i.

Definition: A mixed strategy profile $(s_1, \ldots, s_n)$ is called a strongly dominant mixed strategy equilibrium if s_i is strongly dominant strategy for player i, for every $i = 1, \ldots, n$.

Dominance

	left	*right*
top	$0, 3$	$3, 0$
middle	$3, 0$	$0, 3$
bottom	$1, 1$	$1, 1$

Dominance

		left	*right*
0.5	*top*	0, 3	3, 0
0.5	*middle*	3, 0	0, 3
0.0	*bottom*	1, 1	1, 1

Exercise: Check out other mixed strategies.

Iterated Elimination of Dominated Strategies

Procedure of iterated elimination of dominated strategies:

- Eliminate one after another actions of player that are (weakly or strongly) dominated, until this is no longer possible
- If only one profile remains, we say the game is *dominance solvable*.

Fact: The strategy profiles that survive iterated elimination of weakly dominated strategies may depend on the order of elimination. This is not the case for iterated elimination of strongly dominated strategies.

Exercise

3, 1	0, 0	0, 0
1, 1	1, 2	5, 0
0, 1	4, 0	0, 0

1, 1	1, 1	0, 0
0, 0	1, 2	1, 2
0, 2	0, 0	0, 3

Best Responses

Notation: Given a pure (or mixed) strategy profile $a = (a_1, \dots, a_i, \dots, a_n)$, we use $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ (strategies of i's opponent in a), and $(a_i, a_{-i}) = (a_1, \dots, a_i, \dots, a_n) = a$.

Definition: Given a_{-i} as the pure strategies of i's opponents, a pure strategy a_i is a pure best response of i to a_{-i} if for all $b_i \in A_i$: $u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i})$

Best Responses

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Definition: Given a_{-i} as the pure strategies of i's opponents, a pure strategy a_i is a pure best response of i to a_{-i} if for all $b_i \in A_i$: $u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i})$

Definition: Given s_{-i} as the mixed strategies of i's opponents, a mixed strategy s_i is mixed best response of a player i to s_{-i} if for all $t_i \in \Delta(A_i)$: $u_i(s_i, s_{-i}) \geq u_i(t_i, s_{-i})$.

Nash Equilibrium

Definition: A pure strategy profile a is a pure Nash equilibrium if no player has an incentive to *unilaterally* deviate from a, *i.e.*, if for all players i:

$$\text{for all } b_i \in A_i : \quad u_i(a) \geq u_i(a_1, \ldots, b_i, \ldots, a_n)$$

Equivalently: A pure strategy profile a is a pure Nash equilibrium if a_i is the best response to a_{-i} for all players i.

2, 2	0, 3
3, 0	1, 1

1, 0	0, 1
0, 1	1, 0

2, 1	0, 0
0, 0	1, 2

Nash Equilibrium

Definition: A mixed strategy profile s is a *Nash equilibrium* if no player has an incentive to *unilaterally* deviate from s, *i.e.*, if for all players i:

$$\text{for all } t_i \in \Delta(A_i): \quad u_i(s) \geq u_i(s_1, \ldots, t_i, \ldots, s_n)$$

Equivalently: A mixed strategy profile s is a mixed Nash equilibrium if s_i is the best response to s_{-i} for all players i.

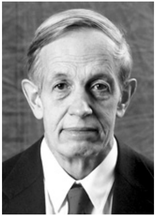
2, 2	0, 3
3, 0	1, 1

1, 0	0, 1
0, 1	1, 0

2, 1	0, 0
0, 0	1, 2

Nash's Theorem

Theorem (*Nash 1950*): Every strategic game with a finite number of pure strategies has a Nash equilibrium in mixed strategies.



Remark: *The proofs are non-constructive and use Brouwer's or Kakutani's fixed point theorems.*

Properties of Nash Equilibrium

- Nash equilibrium is perhaps the most important solution concept for non-cooperative games, for which numerous refinements have been proposed.
- Any combination of dominant strategies is a Nash equilibrium.
- Nash equilibria are not generally Pareto efficient.
- Existence in (pure) strategies is not in general guaranteed.
- Nash equilibria are not in general unique (equilibria selection, focal points).
- Nash equilibria are not generally interchangeable.
- Payoffs in different Nash equilibria may vary.

Finding Mixed-Strategy Nash equilibria

- Genrally, it is tricky to compute mixed-strategy Nash equilibria
- But, easy if the **support** of the mixed-strategies at equilibrium can be identified

Definition: The *support* of a mixed strategy s_i for a player i is the set of pure strategies $\{ a_i \mid s_i(a_i) > 0 \}$.

Finding Mixed-Strategy Nash equilibria

- Let the best response to s_{-i} be a mixed-strategy s_i with a support consisting of more than one action.
- **Observation**: All actions (pure strategies) in the support of strategy of s_i have the same expected utility, i.e., player i is indifferent between the actions in the support of its mixed-strategy at equilibrium.
- **Reason**: If an action a in the support of s_i has a higher expected utility than the other actions, then action a would be a better response than the mixed-strategy s_i.

$2, 1$	$0, 0$
$0, 0$	$1, 2$

For the row player: Suppose column player has the mixed-strategy $(p, 1-p)$ at equilibrium. For the row player holds that $U_{row}(r_1) = U_{row}(r_2)$, i.e.,

$$\begin{aligned} 2 * p + 0 * (1-p) &= 0 * p + 1 * (1-p) \\ 2p &= 1 - p \\ 3p &= 1 \\ p &= 1/3 \end{aligned}$$

Exercise: Find mixed-strategy Nash equilibrium for Rock, Scissors, Paper game.

Alternative Characterization of Nash Equilibria

Lemma: A mixed strategy profile s is a Nash equilibrium iff for all players i

- Given s_{-i}, all actions in the support of s_i yield the same expected utility.
- Given s_{-i} no action not in the support of s_i yields a higher expected utility than any action in the support of s_i.

Alternative Characterization of Nash Equilibria

Lemma: A mixed strategy profile s is a Nash equilibrium iff for all players i

- $u_i(s_1, \ldots, a_i, \ldots, s_n) = u_i(s_1, \ldots, b_i, \ldots, s_n)$, for all actions $a_i, b_i \in A_i$ in the support of s_i.
- $u_i(s_1, \ldots, a_i, \ldots, s_n) \geq u_i(s_1, \ldots, b_i, \ldots, s_n)$, for all actions $a_i, b_i \in A_i$ with a_i in but b_i not in the support of s_i.

Strictly Competitive Games (zero-sum games)

A strategic game $G = (\{1,2\}, A, u)$ is strictly competitive if there exists a constant c such that for each strategy profile a it is the case that $u_1(a) + u_2(a) = c$.

	head	*tail*
head	$1, -1$	$-1, 1$
tail	$-1, 1$	$1, -1$

Lemma: Let $G = (\{1,2\}, A, u)$ be a strictly competitive game. We have:

- $max_x min_y \; u_1(x,y) \;\leq\; min_y max_x \; u_1(x,y)$.
- $max_x min_y \; u_1(x,y) \;=\; -\, min_x max_y \; u_2(x,y)$.

Exercise: Verify the above results in the above matching Pennies game.

Strictly Competitive Games (zero-sum games)

A strategic game $G = (\{1,2\}, A, u)$ is strictly competitive if there exists a constant c such that for each strategy profile a it is the case that $u_1(a) + u_2(a) = c$.

	head	*tail*
head	$1, -1$	$-1, 1$
tail	$-1, 1$	$1, -1$

Lemma: Let $G = (\{1,2\}, A, u)$ be a strictly competitive game. We have:

- If $(x^*, y^*) \in A$ is a Nash equilibria, then x^* is a maxminimizer for player 1 and y^* is a maxminimizer for player 2.
- If $(x^*, y^*) \in A$ is a Nash equilibria, then $max_x min_y u_1(x,y) = min_y max_x u_1(x,y) = u_1(x^*, y^*)$.
- if $max_x min_y u_1(x,y) = min_y max_x u_1(x,y) = u_1(x^*, y^*)$, x^* is a maxminimizer for player 1, and y^* is a maxminimizer for player 2, then (x^*, y^*) is a Nash equilibria.

Exercise: Design a strictly competitive game with Nash equilibria and verify the above results in that game.

Iterated Prisoner's Dilemma

- In Prisoner's dilemma is *defect* the dominant strategy.
- Can self-interested agents cooperate? Why?
- Examples from real world: nuclear arm race, public transport
- Shadow of future: cooperation is possible because the game will be played in future again.
- Iterated Prisoner's dilemma is such a scenario.

Axelrod's Tournament (1980)

Robert Axelrod (a political scientist) held a computer tournament designed to investigate how cooperation emerge among self interested agents.

- Computer programs play iterated prisoner's dilemma games against each other.
- Which strategy results in maximum overall payoff?
- Possible strategies followed by the submitted programs:
  - ALLD: always defect
  - ALLC: always cooperate
  - RANDOM: sometime cooperate sometimes defect
  - TIT-FOR-TAT: 1st round Cooperate. Other rounds do what the opponent did at previous round.
  - MAJORITY: 1st round cooperates. Other rounds examines the history of the opponent's actions, counting its total number of defect and cooperates. If opponent defect more often dan cooperate, then defect; otherwise cooperate.
  - JOSS: As TIT-FOR-TAT, except periodically defect.