# Introduction to Coalitional Game Theory

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# Coalitional Game Theory

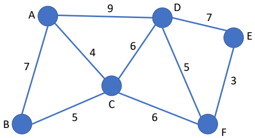
- In non-coalitional games, decisions are made by individual players.
  - Binding agreements between players is not possible
  - Individual players maximise their own utilities

- In coalitional games, decisions are made by groups of players.
  - Binding agreements between players is possible
  - Individual players are utility maximisers but can benefit by cooperating

**Transferable Utility Games:** actions are decided by groups of players and utilities are assigned to the groups. The group utility is then distributed among individual players.

# Coalitional Game Theory

- Players form groups to perform tasks
- Each group of players receives a utility (to be distributed among themselves)
- Examples:
  - Matching games based on weighted graphs




  - Weighted voting game:
    - A, B, C, and D have 45, 25, 15, and 15 votes.
    - 51 votes are required to pass the $100 million bill.

# Modelling Coalitional Games

A coalitional game is $G = (N, v)$, where

- $N$ is a set of players
- $v : 2^N \to \mathbb{R}$ is the characteristic function of the game

Example: $G = (\{1, 2\}, v)$, where

- $v(\emptyset) = 0$
- $v(\{1\}) = v(\{2\}) = 5$
- $v(\{1, 2\}) = 20$

Note :

- $v(S)$ is the value that is assigned to coalition $S \subseteq N$
- $2^N$ is the set of all possible coalitions
- A coalition structure $CS$ is a partition on $N$
- Optimal coalition structure: $\max\limits_{CS} \sum\limits_{S \in CS} v(S)$

Exercise 1: Model weighted voting game as a coalitional game.

# Classes of Coalitional Games

A coalitional game $G = (N, v)$ is (for any coalition $S, S' \subseteq N$):

- **Simple**: $v(S) \in \{0, 1\}$ with $v(N) = 1$ (winning/losing coalitions).
- **Additive**: $v(S \cup S') = v(S) + v(S')$ for $S \cap S' = \emptyset$.
- **Superadditive**: $v(S \cup S') \geq v(S) + v(S')$ for $S \cap S' = \emptyset$.
- **Monotonic**: $v(S) \leq v(S')$ for $S \subseteq S'$.
- **Convex**: $v(S \cup S') \geq v(S) + v(S') - v(S \cap S')$.
- **Constant-sum**: $v(S) + v(N \setminus S) = v(N)$.

Exercise 2: Show that additive implies convex implies superadditive. Show also that additive implies constant-sum, and supperadditive implies monotonic.

# Analysing Coalitional Games

Let $G = (N, v)$ be a coalitional game.

- $v$ is normalised: $v(\emptyset) = 0$
- $v$ is non-negative: $v(S) \geq 0$ for any $S \subseteq N$

An outcome $x = \langle x_1, \ldots, x_k \rangle$ for a coalition $S$ (consisting of $k$ members in game $G$) is a distribution of $v(S)$ to its members such that

$$\sum_{i \in S} x_i \; = \; v(S)$$

.

Given the coalitional game $G = (\{1, \ldots, n\}, v)$, an outcome for the grand coalition is $\langle x_1, \ldots, x_n \rangle$ such that

$$\sum_{i \in N} x_i \; = \; v(N)$$

**Which coalitions can be formed? $\Longrightarrow$ Which coalitions are stable?**

# Coalition Stability: The Core

An outcome for a coalition is **stable** if no subcoalition can object to it.

Is the grand coalition stable?

An outcome $x = \langle x_1, \ldots, x_n \rangle$ for the grand coalition is optimal when each coalition is getting at least what it can make on its own, i.e., no one has any incentive to deviate.

**The core** of a coalitional game $G = (\{1, \ldots, n\}, v)$ consists of all outcomes $x = \langle x_1, \ldots, x_n \rangle$ for the grand coalition for which it holds:

$$\forall S \subseteq N : \sum_{i \in S} x_i \geq v(S)$$

Exercise 3: Determine the Core of the weighted voting game.

# Coalition Stability: The Core

The core of a coalitional game $G = (\{1, \ldots, n\}, v)$ is the set of vectors in $x = \langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n$ that satify the following constraints:

- $x$ is **feasible**: $\sum_{i \in N} x_i \;\leq\; v(N)$
  i.e., $x$ does not allocate more than possible
- $x$ is **efficient**: $\sum_{i \in N} x_i \;=\; v(N)$
  i.e., $x$ allocates everything
- $x$ is **individually rational**: $x_i \;\geq\; v(\{i\})$ for all $i \in N$
  i.e., players do better in $x$ than individually

**Is the grand coalition stable? $\Longrightarrow$ Is the core non-empty?**

# Coalitional Games with Empty Core

- Let $G = (\{1,2,3\}, v)$ with
  $v(\{1\}) = v(\{2\}) = v(\{3\}) = 2$,
  $v(\{1,2\}) = v(\{2,3\}) = v(\{1,3\}) = 5$, and
  $v(\{1,2,3\}) = 6$.

- For the Core of $G$ not to be empty, the outcomes $x = \langle x_1, x_2, x_3 \rangle$ should be:

- **feasible**: $\sum_{i \in N} x_i \le v(N)$
  $x_1 + x_2 + x_3 \le 6$
  **efficient**: $\sum_{i \in N} x_i = v(N)$
  $x_1 + x_2 + x_3 = 6$
  **individually rational**: $\forall i \in N : x_i \ge v(\{i\})$
  $x_i \ge 2$

- **stable**: $\forall S \subseteq N : \sum_{i \in S} x_i \ge v(S)$
  $x_1 + x_2 \ge 5$
  $x_2 + x_3 \ge 5$
  $x_1 + x_3 \ge 5$

- Impossible, $G$ has an **empty** Core.

# Coalitional Games with Non-Empty Core

- **Theorem**: Any convex games has a non-empty core.
- **Theorem**: Any simple game with at least one veto player has a non-empty core.
  A player $i$ in a simple game $(N, v)$ is a veto player if for all $S \subseteq N$ we have $v(S) = 0$ if $i \notin S$.
- **Theorem**: Any coalitional game $(N, v)$ has a non-empty core iff for all **balanced sets of weights** $\lambda$, we have:

  $$v(N) \geq \sum_{S \subseteq N} \lambda(S) v(S)$$

  A balanced set of weights $\lambda$ over $2^N$ is a set of non-negative weights such that

  $$\forall i \in N : \sum_{S : i \in S} \lambda(S) = 1$$

# Stability versus Fairness

Outcomes in the Core are stable, but may not be fairly distributed.

Example: $G = (\{1,2\}, v)$, where

- $v(\emptyset) = 0$
- $v(\{1\}) = v(\{2\}) = 5$
- $v(\{1,2\}) = 20$

Every outcome between $(15,5), \ldots, (5,15)$ is in the core, but some outcomes such as $15, 5$ are not fair.

**Which outcomes can be considered as fair?**

# Marginal Contribution

The basic idea is to distribute the utility of a coalition based on the contribution of players in that coalition.

Let $G = (N, v)$ be a coalitional game. The **marginal contribution** of a player $i$ to a coalition $S \subseteq N \setminus \{i\}$ is denoted as $\mu_i(S)$ and defined as follows:

$$\mu_i(S) = v(S \cup \{i\}) - v(S)$$

The **average marginal contribution** of a player $i$ in a game $G$ is defined as follows:

$$\frac{1}{2^{n-1}} \cdot \sum_{S \subseteq N \setminus \{i\}} \mu_i(S)$$

Example: $G = (\{1, 2\}, v)$, where

- $v(\emptyset) = 0$
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# Shapley Value

In some cases, the marginal contribution of player $i$ to coalition $S$ depends on the order in which $S$ is formed.

Let $G = (N, v)$ be a coalitional game where $N = \{1, \ldots, n\}$. The set of possible permutations of players is $\Pi(N)$. Note we have $n!$ permutations.

Let $C_i(\pi)$ denote the set of predecessors of $i$ in the permutation $\pi \in \Pi(N)$.

The **Shapley value** of player $i$ in the game $G$ is defined as follows:

$$sh_i = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \mu_i(C_i(\pi))$$

Exercise 4: determine the Shapley value for all players in $G = (\{1, 2, 3\}, v)$, where

- $v(\emptyset) = 0;\ \ v(\{1\}) = v(\{2\}) = v(\{3\}) = 5$
- $v(\{1, 2\}) = v(\{1, 3\}) = 10;\ \ v(\{2, 3\}) = 20$
- $v(\{1, 2, 3\}) = 25$

# Shapley Value: Properties

Let $G = (N, v)$ be a coalitional game.

- Dummy player: player $i$ is dummy if $v(S) = v(S \cup \{i\})$ for any $S \subseteq N$
- Two players $i$ and $j$ are symmetric if $v(S \cup \{i\}) = v(S \cup \{j\})$ for any $S \subseteq N \subseteq \{i, j\}$.

Properties of the Shapley value:

1. Efficiency: $sh_1 + \ldots + sh_n = v(N)$
2. Dummy: if $i$ is a dummy player, $sh_i = 0$
3. Symmetry: if $i$ and $j$ are symmetric, $sh_i = sh_j$
4. Additivity: $sh_i(G_1 + G_2) = sh_i(G_1) + sh_i(G_2)$ for games $G_1 = (N, v_1)$, $G_2 = (N, v_2)$, and $G_1 + G_2 = (N, v_1 + v_2)$ defined as $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for any $S \subseteq N$.

## Theorem

*Shapley value is the only payoff distribution scheme that has properties 1- 4.*

Exercise 5: Check properties 1 - 4 for the game of previous slide.
Exercise 6: Determine the Shapley values of the players in the Weighted voting game and check if their Shapely values is in the Core of the game.