Introduction to Game Theory (1)

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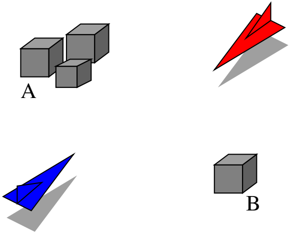
Game Theory

- What is the subject matter of game theory and which phenomena does it help us understand?
- What is the problem of game theory?
- What are the elementary concepts of game theory?
- What is the relevance of game theory to agent research?
- How can game-theoretic concepts be put to use so as to design better systems?

Example: Defence-Attack

Situation: Attacker (Red, column player) can attack either target A or target B, but not both. Defender (Blue, row player) can defend either of two targets but not both. Target A is three times as valuable as Target B.



	A	*B*
A	4, 0	3, 1
B	1, 3	4, 0

Question: *Which target is Red to attack and which target is Blue to defend?*

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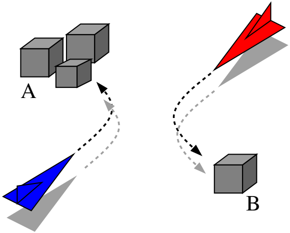

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Battle of the Sexes

John and Mary agreed to go out. They can attend a ballet performance or a box match. Mary would like to go to the ballet performance while John would most of all like to go to the box match. Both prefer to go to the same place rather than different ones. If they cannot communicate, where should they go?

	Fight	*Ballet*
Fight	2, 1	0, 0
Ballet	0, 0	1, 2

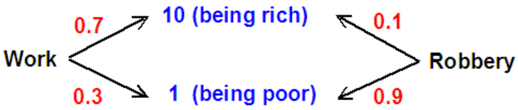
Game of Chicken

Two drivers headed each other from opposite directions. The one to turn aside loses. If neither player turn aside, the result is a deadly collision. The best outcome for each driver is to stay straight while the other turns aside and the worst outcome for both driver is to have a deadly collision. In this situation each player wants to secure his/her best outcome, risking the worst scenario.

	Aside	*Straight*
Aside	$0, 0$	$-5, 5$
Straight	$5, -5$	$-10, -10$

Decision Theory: An Agent Plays Against Environment

- An agent is autonomous if it is capable of *deciding* actions in order to achieve its objectives.
- Classical Decision Theory (Savage 1954)
  - probability and utility functions



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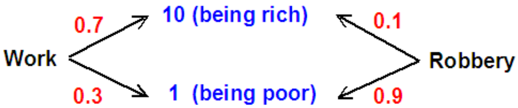

  - Decision rule = *maximum expected utility* for each action "*a*" given the set of outcomes *O*

$$EU(a) = \sum_{o \in O} U(o) * P(o \mid a)$$

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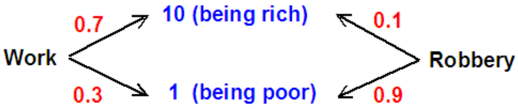

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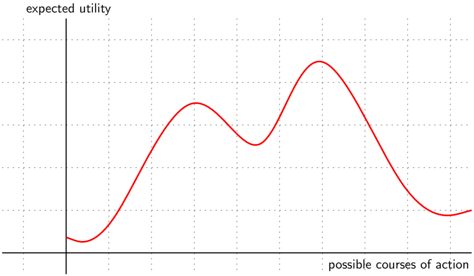
$$\begin{aligned} EU(Work) &= (0.7 * 10) + (0.3 * 1) = 7.3 \\ EU(Robbery) &= (0.1 * 10) + (0.9 * 1) = 1.9 \end{aligned}$$

Decision Theory



Issue: *Find the course of action that maximizes expected utility given particular environmental parameters.*

Utilities and Preferences (1)

- An agent's *Utility* quantifies its degree of *preferences* over a set $O = \{o_1, \ldots, o_n\}$ of outcomes.
- "The agent prefers weakly o_1 to o_2" is denoted by $o_1 \succeq o_2$.
  - $o_1 \succ o_2$ iff $o_1 \succeq o_2$ and not $o_2 \succeq o_1$.
  - $o_1 \sim o_2$ iff $o_1 \succeq o_2$ and $o_2 \succeq o_1$.

Equal (indistinguishable)

- An agent's *Preference*, denoted by $\succeq$, over a set of outcomes O is a reflexive, transitive, and complete relation on O.
  - Reflexivity: $\forall o \in O : \; o \succeq o$.
  - Transitivity: $\forall o_1, o_2, o_3 \in O : \;$ if $o_1 \succeq o_2$ and $o_2 \succeq o_3$, then $o_1 \succeq o_3$.
  - Completeness: $\forall o_1, o_2 \in O : \; o_1 \succeq o_2$ or $o_2 \succeq o_1$ or $o_1 \sim o_2$.

Utilities and Preferences (2)

- Substitutability (indifference in outcomes implies indifference in actions):

$$\textbf{If } o_1 \sim o_2, \textbf{ then } [p : o_1, \; p_3 : o_3, \ldots, p_k : o_k] \sim [p : o_2, \; p_3 : o_3, \ldots, p_k : o_k]$$

any two actions that behave similarly to all situations with the same outcomes with the same probability, they are indistinguishable

for all outcomes $o_3, \ldots, o_k$ and probabilities $p, \; p_3, \ldots, p_k (p + \sum_{i=3}^{k} p_i = 1)$.

- Decomposability (indifference in actions with similar expected outcomes):

$$\textbf{if } \forall o_i \in O : P(o_i \mid a_1) = P(o_i \mid a_2), \textbf{ then } a_1 \sim a_2$$

- Monotonicity:

$$\textbf{if } o_1 \succ o_2 \textbf{ and } p > q, \textbf{ then } [p : o_1 \,, \; 1 - p : o_2] \succ [q : o_1 \,, \; 1 - q : o_2]$$

- Continuity:

$$\textbf{if } o_1 \succ o_2 \textbf{ and } o_2 \succ o_3, \textbf{ then } \exists p \in [0, 1] \textbf{ such that } o_2 \sim [p : o_1, 1 - p : o_3]$$

Utilities and Preferences (3)

Lemma: If a preference relation $\succeq$ satisfies Completeness, Transitivity, Decomposability, and Monotonicity, and if $o_1 \succ o_2 \succ o_3$, then $\exists p \in [0,1]$ such that

- $\forall p' : p' < p : o_2 \succ [p' : o_1 \; ; \; 1 - p' : o_3]$, and
- $\forall p'' : p'' > p : [p'' : o_1 \; ; \; 1 - p'' : o_3] \succ o_2$

Theorem: (Von Neumann and Morgenstern, 1944) If a preference relation $\succeq$ satisfies Reflexivity, Transitivity, Completeness, Substitutability, Decomposability, Monotonicity, and Continuity, then there exists a utility function $u : O \rightarrow [0,1]$ with the properties that:

- $u(o_1) \geq u(o_2)$ iff $o_1 \succeq o_2$, and
- $u([p_1 : o_1, \ldots, p_k : o_k]) = \sum_{i=1}^{k} p_i u(o_i)$.

Utilities and Preferences (4)

Fact: All preference relations over a countable set O are representable by a utility function. These utility functions are invariant under monotonically increasing functions.

you can't find a utility function over a non countable set

Fact: Let $O = \mathbb{R} \times \mathbb{R}$ and $\succsim$ be the *lexicographic order on* O:

$$(o_1, o_1') \succsim (o_2, o_2') \text{ iff } o_1 > o_2 \text{ or both } o_1 = o_2 \text{ and } o_1' \geq o_2'$$

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Lexicographical Preference Order

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Proof by contradiction

Proof: Assume such a utility function "u" exists. Then, for all positive $r \in \mathbb{R}$, it holds $(r, 2) \succsim (r, 1)$ iff $u(r, 2) > u(r, 1)$, and there exists a rational number $q \in \mathbb{Q}$ such that

$$u(r, 2) > q_r > u(r, 1)$$

Note that there exists always a rational number between any two real numbers. Now take two real numbers r and r' such that $r > r'$. We have $(r, 1) \succsim (r', 2)$ iff $u(r, 1) > u(r', 2)$ and therefore

$$u(r, 2) > q_r > u(r, 1) > q_{r'} > u(r', 2)$$

This means that if $r \neq r'$ then $q_r \neq q_{r'}$. Moreover, it is always the case that if $q_r \neq q_{r'}$ then $r \neq r'$. Together these two facts imply the existence of a one to one mapping between $\mathbb{R}$ and $\mathbb{Q}$ (a bijection between $\mathbb{R}$ and $\mathbb{Q}$). However, such a bijection does not exists.

What is Game Theory Trying to Accomplish?

Point of Departure: Game theory as *interactive* decision theory.

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Hence, the optimality of an action depends on the optimality of the other players' actions. As this holds for all players, a circularity threatens.

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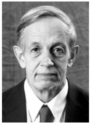
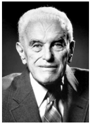
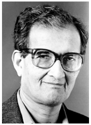
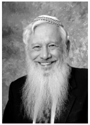
Conclusion: New mathematical concepts required to take over the role of the optimum, *solution concepts*.

Nobel Prizes for Game Theory

1972	Arrow	Welfare theory
1978	Simon	Decision making
1994	Nash, Harsanyi, Selten	Equilibria
1996	Vickrey	Incentives
1998	Sen	Welfare economics
2005	Aumann and Schelling	Conflict and cooperation
2007	Hurwicz, Maskin and Myerson	Mechanism design

Games and Game Forms: The Strategic Form

- Players: Who is involved?
- Rules: What can the players do? What do they know when they act?
- Outcomes: What will happen when the players act in a particular way?
- Preferences: What are the players' preferences over the possible outcomes?

	Fight	*Ballet*
Fight	2, 1	0, 0
Ballet	0, 0	1, 2

Exercise: Describe Defence-Attack, Battle of the Sexes, and Game of Chicken games as strategic games.

Games and Game Forms: The Strategic Form

Definition: A *game form* is a quadruple (N, A, O, g) where:

- N is a set of n players
- $A = A_1 \times \cdots \times A_n$, an n-dimensional space of *strategy profiles* (action profiles), where A_i denote the set of *strategies* (actions) of player i
- O is a set of *outcomes*
- $g \colon A \to O$ is an *outcome function*

N = Player profiles
A = action
O = outcome
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Definition: A *strategic game* is a quintuple (N, A, O, g, u), where:

- (N, A, O, g) is a game form
- $u = (u_1, \ldots, u_n)$, where $u_i : A \to \mathbb{R}$ is a *utility function* for player i.

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We write (N, A, u) instead of (N, A, O, g, u) by assuming $A = O$.

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Preferences and Utilities in Games

Let O be a set of outcomes.

- $\succsim_i \subseteq O \times O$, reflexive, transitive and complete
- $\succsim = (\succsim_1, \ldots, \succsim_n)$

Notations:

- $o_1 \sim_i o_2$ if both $o_1 \succsim_i o_2$ and $o_2 \succsim_i o_1$
- $o_1 \succ_i o_2$ if both $o_1 \succsim_i o_2$ and not $o_2 \succsim_i o_1$
- $a \succsim_i a'$ if $g(a) \succsim_i g(a')$

Definition: A *utility function* $u_i : O \to \mathbb{R}$ represents preferences $\succsim_i$ over outcomes O so that:

$$u_i(o_1) \geq u_i(o_2) \quad \text{iff} \quad o_1 \succsim_i o_2$$

Notation: For a a strategy in a game, we write $u_i(a) = u_i(g(a))$.

Security Level

Notation: Let $A = A_1 \times \ldots \times A_n$ be the strategy space of n players, where A_i is the set of strategies of player i. We use $(s_i, \mathbf{s_{-i}})$ to denote a strategy profile $(s_1, \ldots, s_i, \ldots, s_n) \in A$.

Definition: The *pure security level* of player i is *the least payoff* he can guarantee himself, *no matter what strategies the other players play*, *i.e.*:

$$\max_{s_i} \min_{\mathbf{s_{-i}}} (u_i(s_i, \mathbf{s_{-i}}))$$

the agent considers also other agents possible actions

0, 4	3, 1
1, 2	2, 3

1, 0	0, 1
0, 1	1, 0

Exercise: Determine the security level of Defence-Attack, Battle of the Sexes, and Game of Chicken games.